Stability Analysis of Periodic Solutions of Bonhoeffer Van-der-pol System With Applied Impulse

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STABILITY ANALYSIS OF PERIODIC SOLUTIONS OF BONHOEFFER VAN-DER-POL SYSTEM WITH APPLIED IMPULSE

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Abstract

In this paper, the stability analysis of periodic solutions of Bonhoeffer Van der Pol system with applied impulse was investigated using Lyapunov direct method. Through the use of appropriate values of the control parameters, three equilibria points and periodic solutions of the system were obtained. A Lyapunov candidates which depended on the parameters were constructed. Hence, we concluded that the equilibria points have different regions of stability and instability of the system in which the two regions were found to be stable. Furthermore, Mathcad software was used to analyze the behavior of the system, thereby improving known results in literature.

Keywords: Stability, Lyapunov method, Bonhoeffer Van-der pol equation

Mathematics Subject Classification (2010): 34B15, 34C15, 34C25, 34K13

Introduction

Consider the system

$$\left.\begin{aligned} \dot{x} &= c(x + Y - \frac{x^3}{3} + z) \\ \dot{y} &= \frac{1}{c}(-x - bY + A) \end{aligned}\right\} \qquad (1)$$

where x, Y are the dependent variables, $\dot{x}$ and $\dot{Y}$ denote the derivatives with respect to time, and $A, b, c,$ and z are constant parameters where $c > 0$. Here A, b, c are the control parameters while z represent an applied impulse. Since $A, b,$ and c are constant, it can be removed by the following transformations

$$y = Y + z, \quad a = A + bz \qquad (2)$$

Equation (1) becomes

$$\left.\begin{aligned} \dot{x} &= c(x + y - \frac{x^3}{3}) \\ \dot{y} &= \frac{1}{c}(-x - by + a) \end{aligned}\right\} \qquad (3)$$

with periodic initial conditions

$$x(0) = x(2\pi) \quad \dot{x}(0) = \dot{x}(2\pi)$$

Bonhoeffer Van der Pol equation is a nonlinear system of differential equations with two or more control parameters. The equation was introduced by [1] as a simplified version (two-dimensional) representation of the four-dimensional Hodgen-Huxley system. This system describes how nerve impulses travel down the axon of a nerve cell. The Bonhoeffer Van der Pol (BVP) equation is a physiological model which describes the electrical activity of certain nerve cells. However, from the mathematical perspective, the BVP system is one of a class of periodically forced nonlinear relaxation oscillators. The equation is used in the modeling of cardiac pulses from electrocardiographic signals, normal cardiac pulses which account for the adaptive response for every single heartbeat, manipulation of cell division in a state of inactivity or dormancy and formation of a multicellular fruiting body. In [2], BVP equation is used to simulate neuron cells and control the dynamics of the neuron cells from chaotic to periodic states. Under proper parameters, BVP equation describes the action potentials and regulates heart rate pulse. It is highly preferred over others because of its possibility to facilitate the classification of normal and pathological heart beats for diagnosing purposes.

Furthermore, its applications are seen in the modeling of a cell cycle which describes the lipid peroxidation in an open membrane and the corresponding production of free radicals. Due to the wide range of applications of BVP equation in different fields of endeavor, many researchers have worked on the Bonhoeffer Van der Pol equation using different methods which yielded amazing results. See [3] [4] [5] [6] and [7]

Stability is a qualitative property of a differential equation that is important in linear and nonlinear analysis. It describes the behavior of the system when the system undergoes small changes. Analytically, the stability is determined by the interval placed on the total derivative of the system formed by the given differential equation. For the linear systems with one parameter, stability is assessed only at one equilibrium point. However, for a nonlinear systems with more than two parameters, the search for the stability analysis of the equilibrium point becomes an issue. Emphasis on stability has been discussed by many authors. For instance, see [8], [9], [10], [11] [12], [13] and there references therein. Researchers who have worked on the stability of nonlinear systems are [14], [15], [16] ,[17] and [18].

Motivated by the above literature, the objective of this paper is to investigate the stability analysis of periodic solutions of Bonhoeffer van der Pol equation. The paper further investigated the effect of the parameters on the system and analyzed the behavior of the system using Mathcad software.

2. Preliminaries

Definition 2.1 Consider a real valued function $V: \mathbb{R}^n \to \mathbb{R}$ which is continuously differentiable then the continuous function $V(x)$ is said to be positive definite if $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$.

Definition 2.2 The continuous function $V(x)$ with respect to definition 2.1 is said to be negative definite if $V(x) < 0$ for all $x \neq 0$ and $V(0) = 0$

Theorem 2.3 (Lyapunov Stability) Let $x = 0$ be an equilibrium of $\dot{x} = f(x)$ and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V: D \to \mathbb{R}$ be a continuously differentiable function such that (i) $V(0) = 0$ (ii) $V(x) > 0$ in $D - \{0\}$ (iii) $\dot{V}(x) \leq 0$ in $D - \{0\}$ Thus $x = 0$ is stable in the sense of Lyapunov.

Theorem 2.4 (Asymptotic Stability) Let $V: D \to \mathbb{R}$ be a continuously differentiable function such that (i) $V(0) = 0$ (ii) $V(x) > 0$ in $D - \{0\}$ (iii) $\dot{V}(x) < 0$ in $D - \{0\}$ Thus $x = 0$ is asymptotically stable in the sense of Lyapunov.

Theorem 2.5 (Global Asymptotic Stability) Let $V: D \to \mathbb{R}$ be a continuously differentiable function such that (i) $V(0) = 0$ (ii) $V(x) > 0$ in $D - \{0\}$ (iii) $V(x)$ is 'radially' bounded (iv) $\dot{V}(x) < 0$ in $D - \{0\}$. Thus $x = 0$ is globally asymptotically stable in the sense of Lyapunov.

Definition 2.6 (**Periodic Solution**) It is the solution $y = f(x)$ of a differential equation with the property that there exist a positive real number $k \neq 0$ such that $f(x + k) = f(x)$. k is called the period of the function.

Definition 2.7 (**Stability**) An equilibrium solution x_e of an autonomous system is said to be stable if for every $\varepsilon > 0$ there exist $\delta > 0$ such that every solution $x(t)$ having an initial conditions within δ ie $\|x(t_0) - x_e\| < \delta$ of the equilibrium remain within ε is $\|x(t) - x_e\| < \varepsilon$ for all $t \geq t_0$.

3. Results

3.1 Stability of the Equilibrium Point

Consider the system defined by equation (3)

$$\left.\begin{aligned} \dot{x} &= c(x + y - \tfrac{x^3}{3}) \\ \dot{y} &= \tfrac{1}{c}(-x - by + a) \end{aligned}\right\}$$

where a, b, c are constant parameters and $c > 0$.

Equation (3) can be combined to give a Lienard equation of the form:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \qquad (4)$$

where $f(x) = c(x^2 - 1) + \frac{b}{c}$ and $g(x) = x(1 - b) + \frac{1}{3}bx^3 - a$. By setting $c = 0$, $b = 0$ and $a = 0$, equation (4) reduces to

$$\ddot{x} + x = 0 \qquad (5)$$

Equation (5) is oscillatory in $\mathbb{R}$. Hence, the solution obtained in equation (5) is periodic.

The equilibrium points are obtained by setting the right hand side of equation (3) to zero which gives;

$$y = \frac{1}{3}x^3 - x \quad \text{and} \quad y = \frac{a}{b} - \frac{x}{b} \qquad (6)$$

Eliminating y from equation (6) we obtain the cubic equation

$$x^3 - 3\left(1 - \frac{1}{b}\right)x - \frac{3a}{b} = 0 \qquad (7)$$

To solve the cubic equation in (7), we let $a = 0$ and $b = 3$. Equation (7) becomes

$$x\left(x^2 - 3\left(1 - \frac{1}{b}\right)\right) = 0 \quad (8)$$

The roots of equation (8) are $x_1 = 0\,, x_2 = \sqrt{2}$ and $x_3 = -\sqrt{2}$

Substituting x_1, x_2, x_3 into equation (6) we have $y_1 = 0\,, y_2 = -\sqrt{\frac{2}{3}}$ $and\ y_3 = \sqrt{\frac{2}{3}}$

Therefore, the equilibrium points are (0,0), $(\sqrt{2}, -\sqrt{\frac{2}{3}})$ and $(-\sqrt{2}, \sqrt{\frac{2}{3}})$

Linearized form of equation (4) is given by:

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -f(x)y - g(x) \end{aligned}\right\} \quad (9)$$

where f and g are polynomials of degree two and three respectively. The Lyapunov function is calculated as the total energy of the system which is given by

$$V(x, y) = KE + PE \quad (10)$$

where KE is the kinetic energy of the system, and PE is the potential energy of the system. $V(x, y)$ is a function defined by $V(x, y)\colon \mathbb{R}^2 \to \mathbb{R}$. Equation (10) can further be written as

$$V(x, y) = \frac{1}{2}y^2 + G(x) \quad (11)$$

where $G(x) = \int_0^x g(s)ds = \int_0^x (s(1 - b) + \frac{1}{3}bs^3 - a)ds = b\frac{x^4}{12} - b\frac{x^2}{2} + \frac{x^2}{2} - ax$

Therefore,

$$V(x, y) = \frac{1}{2}y^2 + b\frac{x^4}{12} - b\frac{x^2}{2} + \frac{x^2}{2} - ax \quad (12)$$

Equation (12) is the Lyapunov function for the system.

Clearly, $V(x, y)$ is continuous at (x_0, y_0) since $\lim\limits_{(x,y)\to(x_0,y_0)} V(x, y) = V(x_0, y_0)$. (x_0, y_0) is an element of the domain of $V(x, y)$.

For $V(0,0)$ we have

$$V(0,0) = \frac{1}{2}0^2 + b\frac{0^4}{12} - b\frac{0^2}{2} + \frac{0^2}{2} - a(0) = 0 \quad (13)$$

Evaluating for other values of the equilibrium point we have

$$V\left(\sqrt{2}, -\sqrt{\frac{2}{3}}\right) = \frac{1}{2}(-\sqrt{\frac{2}{3}})^2 + b\frac{\sqrt{2}^4}{12} - b\frac{\sqrt{2}^2}{2} + \frac{\sqrt{2}^2}{2} - a(\sqrt{2}) = \frac{8}{9} > 0 \quad (14)$$

$$V\left(-\sqrt{2}, \sqrt{\frac{2}{3}}\right) = \frac{1}{2}(\sqrt{\frac{2}{3}})^2 + b\frac{(-\sqrt{2})^4}{12} - b\frac{(-\sqrt{2})^2}{2} + \frac{(-\sqrt{2})^2}{2} - a(-\sqrt{2}) = \frac{8}{9} > 0 \quad (15)$$

This shows that $V(x,y) > 0$ for the equilibrium points except at the origin. Hence

$$V(x,y) = \frac{1}{2}y^2 + \ b\left(\frac{x^4}{12} - \frac{x^2}{2}\right) + \frac{x^2}{2} - ax > 0 \tag{16}$$

Therefore, $V(x,y)$ is positive definite.

Since $V(x,y) > 0 \ \forall \ x, y \neq 0$, the time derivative of the energy of the system is given by

$$\dot{V}(x,y) = \frac{\partial v}{\partial x}\ \dot{x} + \frac{\partial v}{\partial y}\ \dot{y}$$

$$= (\frac{1}{3}bx^3 - bx + x - a)\dot{x} + y\dot{y} \tag{17}$$

$$= (\frac{1}{3}bx^3 - bx + x - a)\dot{x} + y(-f(x)y - g(x))$$

$$= (\frac{1}{3}bx^3 - bx + x - a)y + y(-cx^2y + cy - \frac{b}{c}y - x + bx - \frac{1}{3}bx^3 + a)$$

$$= \left(\frac{1}{3}bx^3y - bxy + xy - ay\right) + (-cx^2y^2 + cy^2 - \frac{b}{c}y^2 - xy + bxy - \frac{1}{3}bx^3y + ay)$$

$$=(-cx^2y^2 + cy^2 - \frac{b}{c}y^2)$$

$$\dot{V}(x,y) = -y^2(cx^2 - c + \frac{b}{c}) \tag{18}$$

With the condition placed on c in equation (1), $\dot{V}(x,y) < 0 \ \ \forall \ x, y \neq 0$. Hence, from theorem (2.4) the equilibrium point is asymptotically stable. If $c < 0$, then the equilibrium point is unstable. This shows that the occurrence of different regions of stability and instability is based on parameter variations.

3.2 Stability of the Equilibria

The equilibria of system (3) which is determined by the following system

$$c\left(x + y - \frac{x^3}{3}\right) = 0$$

$$\frac{1}{c}(-x - by + a) = 0 \tag{19}$$

is given by $C_1(0,0)$, $C_2(\sqrt{2}, -\sqrt{\frac{2}{3}})$ and $C_3(-\sqrt{2}, \sqrt{\frac{2}{3}})$.

We shall now investigate the stability property of the above equilibria. The variational matrix of system (3) is

$$J(x,y) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{20}$$

where $a_{11} = c - x^2$, $a_{12} = c$, $a_{21} = -\frac{1}{c}$ and $a_{22} = -\frac{3}{c}$

Theorem 3.2 $C_1(0{,}0)$ is unstable.

Proof. From equation (20), the Jacobian matrix of the system about the equilibrium point $C_1(0{,}0)$ is given by

$$J(0.0) = \begin{bmatrix} c & c \\ -\frac{1}{c} & -\frac{3}{c} \end{bmatrix} \tag{21}$$

The characteristics equation of equation (21) is given as

$$\lambda^2 + \lambda\left(-c + \frac{3}{c}\right) - 2 = 0 \tag{22}$$

The eigenvalues are $\lambda_1 = \frac{-\alpha+\sqrt{\alpha^2+8}}{2}$ and $\lambda_2 = \frac{-\alpha-\sqrt{\alpha^2+8}}{2}$ where $\alpha = \frac{3}{c} - c$. Both eigenvalues are real and nonzero. Since the eigenvalues have opposite signs, $C_1(0{,}0)$ is a saddle and thus unstable.

Theorem 3.2 $C_2\left(\sqrt{2}, -\sqrt{\frac{2}{3}}\right)$ is stable.

Proof. From equation (20), the Jacobian matrix of the system about the equilibrium point

$C_2\left(\sqrt{2}, -\sqrt{\frac{2}{3}}\right)$ is given by

$$J\left(\sqrt{2}, -\sqrt{\frac{2}{3}}\right) = \begin{bmatrix} c-2 & c \\ -\frac{1}{c} & -\frac{3}{c} \end{bmatrix} \tag{23}$$

The characteristics equation of equation (23) is given by

$$\lambda^2 + \lambda\left(\frac{3}{c} + 2 - c\right) + \frac{6}{c} - 2 = 0 \tag{24}$$

The eigenvalues are $\lambda_1 = \frac{-\beta+\sqrt{\beta^2-4\mu}}{2}$ and $\lambda_2 = \frac{-\beta-\sqrt{\beta^2-4\mu}}{2}$ where $\beta = 2 + \frac{3}{c} - c$ and

$\mu = \frac{6}{c} - 2$. The eigenvalues are real, nonzero and negative. Hence $C_2\left(\sqrt{2}, -\sqrt{\frac{2}{3}}\right)$ is stable.

Theorem 3.3 $C_2\left(-\sqrt{2}, \sqrt{\frac{2}{3}}\right)$ is stable.

Proof. From equation (20), the Jacobian matrix of the system about the equilibrium point

$C_2\left(-\sqrt{2}, \sqrt{\frac{2}{3}}\right)$ is given by

$$J\left(-\sqrt{2}, \sqrt{\frac{2}{3}}\right) = \begin{bmatrix} c-2 & c \\ -\frac{1}{c} & -\frac{3}{c} \end{bmatrix} \tag{25}$$

The characteristics equation of equation (25) is given by

$$\lambda^2 + \lambda\left(\frac{3}{c} + 2 - c\right) + \frac{6}{c} - 2 = 0 \tag{26}$$

The eigenvalues are $\lambda_1 = \frac{-\beta+\sqrt{\beta^2-4\mu}}{2}$ and $\lambda_2 = \frac{-\beta-\sqrt{\beta^2-4\mu}}{2}$ where $\beta = 2+\frac{3}{c}-c$ and $\mu = \frac{6}{c}-2$. The eigenvalues are real, nonzero and negative. Consequently, $C_2\left(-\sqrt{2},\sqrt{\frac{2}{3}}\right)$ is stable

4. NUMERICAL SIMULATION OF THE RESULTS

We illustrate the numerical simulation of the results using the MATHCAD software;

Simulation-1

$\alpha := 0.7 \quad \beta := 0.8 \quad \gamma := 3$

Define a function that determines a vector of derivative values at any solution point (t,Y):

$$D(t,Y) := \begin{bmatrix} \gamma \cdot \left[Y_0 + Y_1 - \frac{(Y_0)^3}{3} \right] \\ \frac{-1}{\gamma} \cdot \left(Y_0 + \beta \cdot Y_1 - \alpha \right) \end{bmatrix}$$

$T1 = 50$ Endpoint of solution interval

Define additional arguments for the ODE solver:

$t0 := 0$ Initial value of independent variable

$Y_0 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Vector of initial function values

$num := 1 \times 10^3$ Number of solution values on [t0, T1]

Solution matrix:

$S1 := \text{Rkadapt}\left(Y_0, t0, T1, num, D\right)$

$t := S1^{\langle 0 \rangle}$ Independent variable values

$x := S1^{\langle 1 \rangle}$ First solution function values

$y := S1^{\langle 2 \rangle}$ Second solution function values

S1 =

	0	1	2
0	0	1	1
1	0.05	1.245	0.98
2	0.1	1.468	0.956
3	0.15	1.652	0.929
4	0.2	1.791	0.9
5	0.25	1.887	0.869
6	0.3	1.949	0.837
7	0.35	1.986	0.805
8	0.4	2.005	0.773
9	0.45	2.014	0.741
10	0.5	2.016	0.709
11	0.55	2.013	0.678
12	0.6	2.008	0.647
13	0.65	2.001	0.617
14	0.7	1.993	0.588
15	0.75	1.984	...

Table 1.1 Table of values for $\alpha = 0.7,\ \beta = 0.8,\ \gamma = 3$

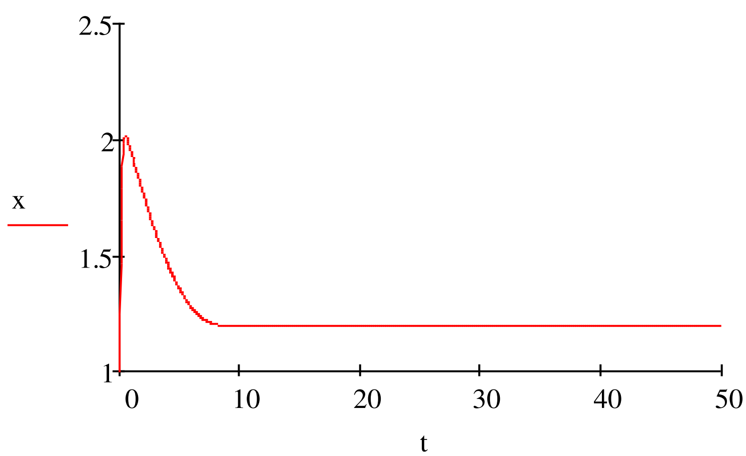

Figure1a. Shows the trajectory profile of the Bonhoeffer Van der Pol equation which starts out from the origin and moves away from the equilibrium point on the positive axis. This shows that the equilibrium point is unstable and the solution is not periodic.

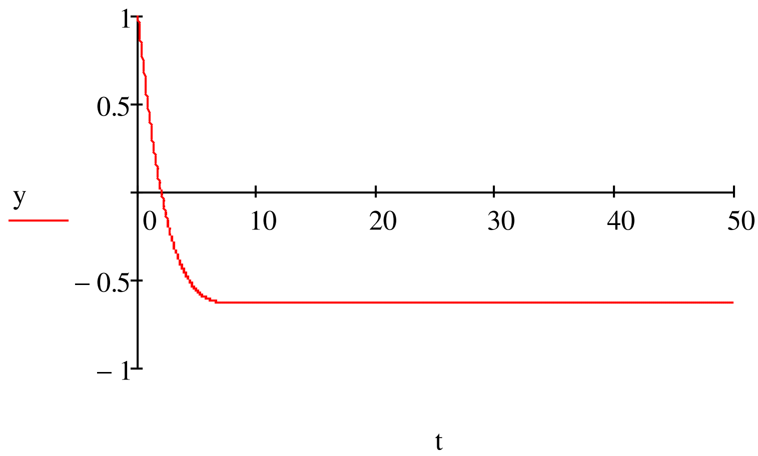

Figure 1b. Shows the trajectory profile of the Bonhoeffer Van der Pol equation which moves away from the equilibrium point on the positive axis. This shows the instability and aperiodic nature of the system.

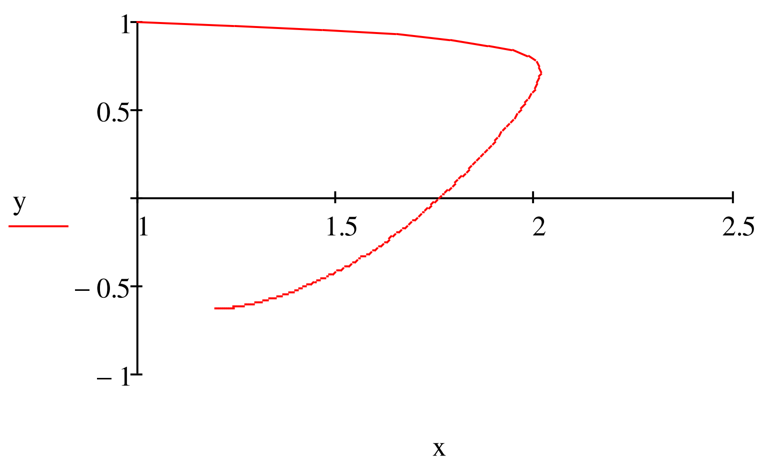

Figure1c. A phase diagram of BVP equation showing the instability nature of the system..

Simulation-2

$\alpha := 0.5 \quad \beta := 0.1 \quad \gamma := 3$

S1 =

	0	1	2
0	0	1	1
1	0.05	1.246	0.988
2	0.1	1.47	0.972
3	0.15	1.657	0.953
4	0.2	1.798	0.93
5	0.25	1.896	0.906
6	0.3	1.96	0.881
7	0.35	1.999	0.855
8	0.4	2.02	0.828
9	0.45	2.03	0.802
10	0.5	2.034	0.775
11	0.55	2.033	0.748
12	0.6	2.029	0.721
13	0.65	2.023	0.695
14	0.7	2.016	0.668
15	0.75	2.009	...

Table 1.2 Table of values for $\alpha = 0.5,\ \beta = 0.1,\ \gamma = 3$

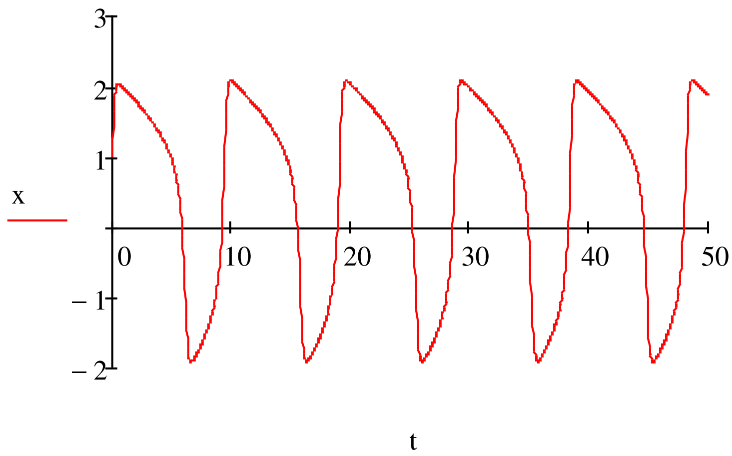

Figure 2a: A trajectory profile of the BVP equation showing the periodic nature of the solution.

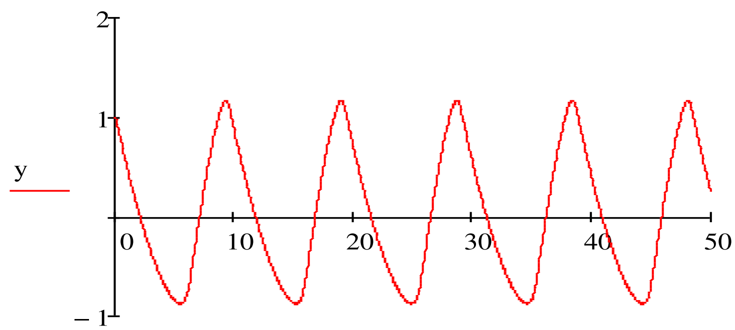

Figure 2b: A trajectory profile of the BVP equation which shows that the solution is periodic.

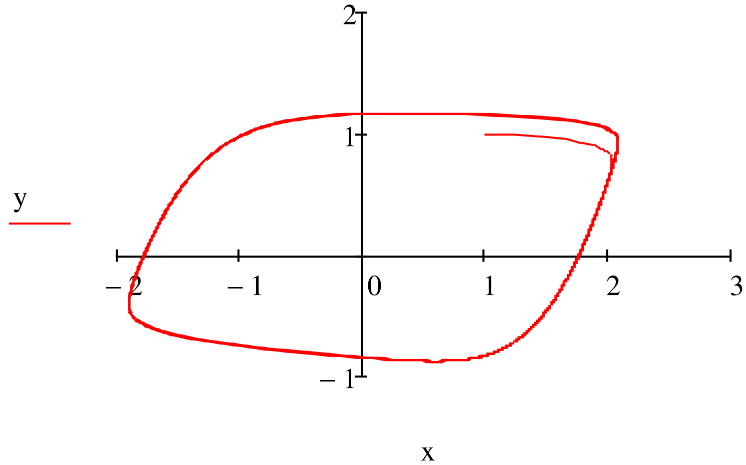

Figure 2c: A phase portrait of BVP equation depicting the stability of the solution within a bounded region.

Simulation-3

$\alpha := 0.1 \quad \beta := 0.1 \quad \gamma := 1$

- S1 =

	0	1	2
0	0	1	1
1	0.05	1.082	0.948
2	0.1	1.16	0.892
3	0.15	1.235	0.833
4	0.2	1.304	0.771
5	0.25	1.368	0.705
6	0.3	1.426	0.637
7	0.35	1.478	0.566
8	0.4	1.523	0.494
9	0.45	1.562	0.419
10	0.5	1.594	0.343
11	0.55	1.62	0.266
12	0.6	1.641	0.189
13	0.65	1.656	0.11
14	0.7	1.666	0.032
15	0.75	1.672	...

Table 1.3 Table of values for $\alpha = 0.1,\ \beta = 0.1,\ \gamma = 1$

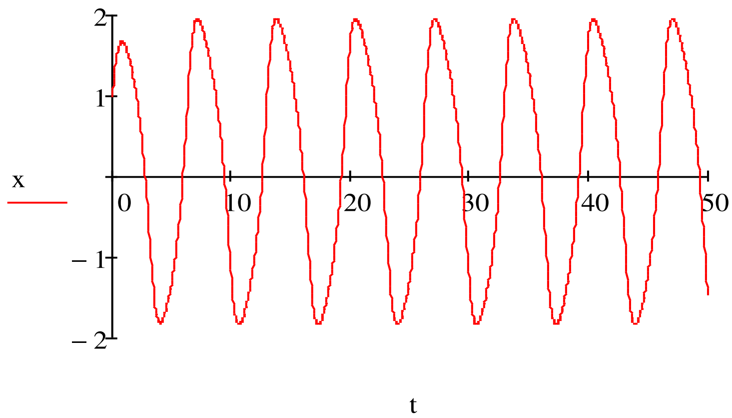

Figure 3a: A trajectory profile of the BVP equation showing the periodic nature of the solution

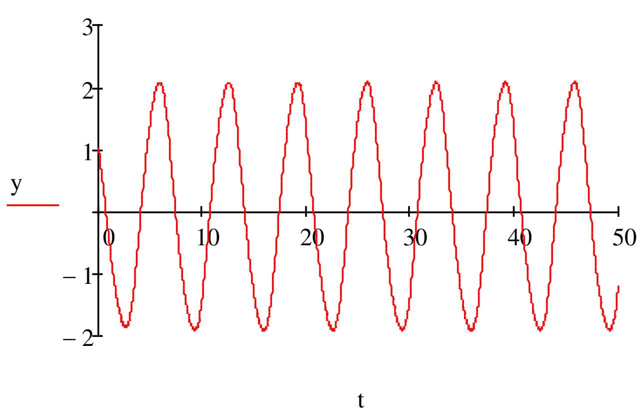

Figure 3b: A trajectory profile of the BVP equation showing the periodic nature of the solution.

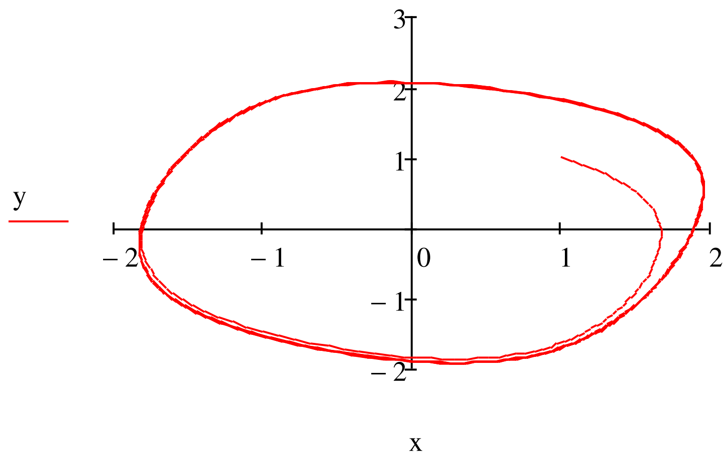

Figure 3c A phase portrait of BVP equation depicting the stability of the solution within a bounded region.

Simulation-4

$\alpha := 0.1 \quad \beta := 0.1 \quad \gamma := 0.4$

S1 =

	0	1	2
0	0	1	1
1	0.05	1.032	0.874
2	0.1	1.062	0.745
3	0.15	1.088	0.615
4	0.2	1.112	0.483
5	0.25	1.134	0.35
6	0.3	1.152	0.216
7	0.35	1.168	0.081
8	0.4	1.181	-0.053
9	0.45	1.191	-0.187
10	0.5	1.199	-0.321
11	0.55	1.203	-0.454
12	0.6	1.205	-0.586
13	0.65	1.205	-0.716
14	0.7	1.202	-0.844
15	0.75	1.196	...

Table 1.4 Table of values for $\alpha = 0.1,\ \beta = 0.1,\ \gamma = 0.4$

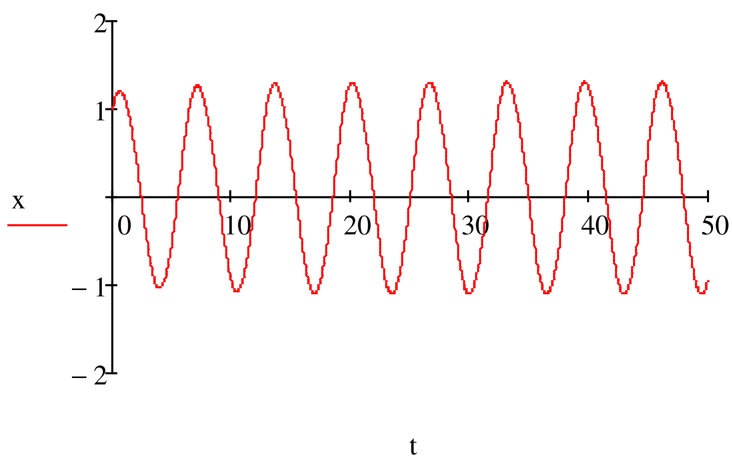

Figure 4a A trajectory profile of the BVP equation depicting the periodic nature of the solution.

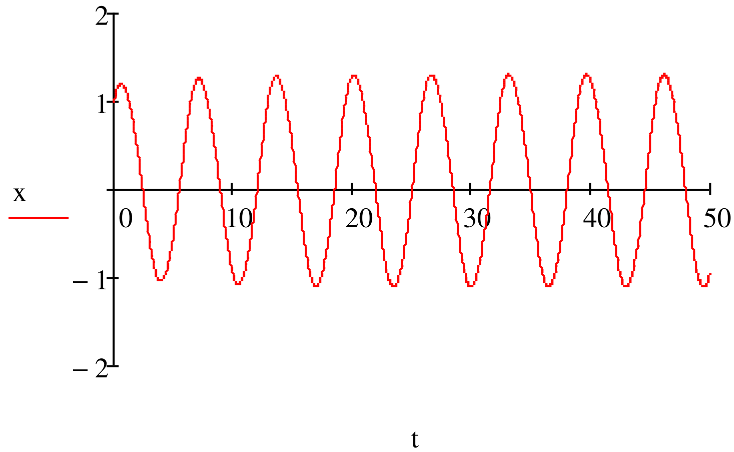

Figure 4a A trajectory profile of the BVP equation depicting the periodic nature of the solution.

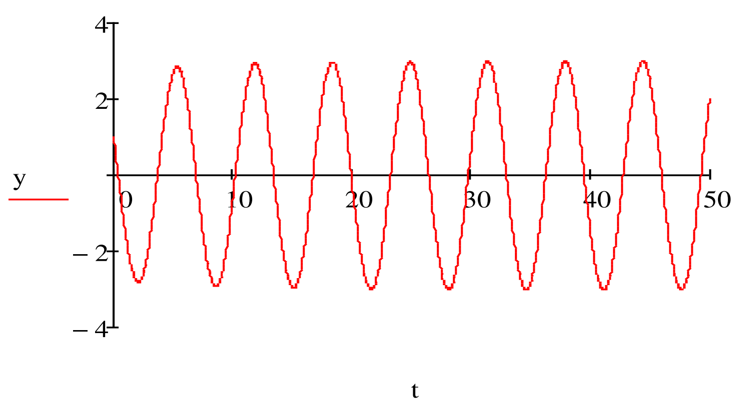

Figure 4b: A trajectory profile of the BVP equation showing the periodic nature of the solution.

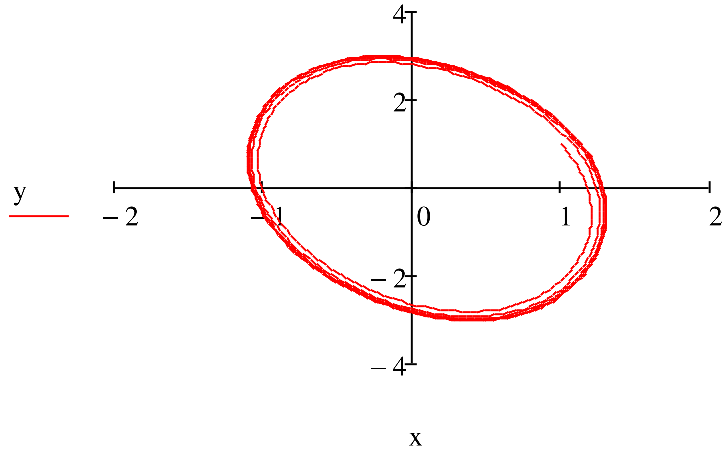

Figure 4c: A phase portrait of BVP equation depicting stability of the system.

Simulation-5

$\alpha := 0.1 \quad \beta := 0.1 \quad \gamma := 0.1$

S1 =

	0	1	2
0	0	1	1
1	0.05	1.007	0.51
2	0.1	1.012	0.042
3	0.15	1.014	-0.406
4	0.2	1.014	-0.832
5	0.25	1.013	-1.237
6	0.3	1.009	-1.621
7	0.35	1.003	-1.984
8	0.4	0.996	-2.325
9	0.45	0.987	-2.647
10	0.5	0.976	-2.947
11	0.55	0.964	-3.228
12	0.6	0.95	-3.489
13	0.65	0.936	-3.73
14	0.7	0.92	-3.951
15	0.75	0.903	...

Table 1.5 Table of values for $\alpha = 0.1,\ \beta = 0.1,\ \gamma = 0.1$

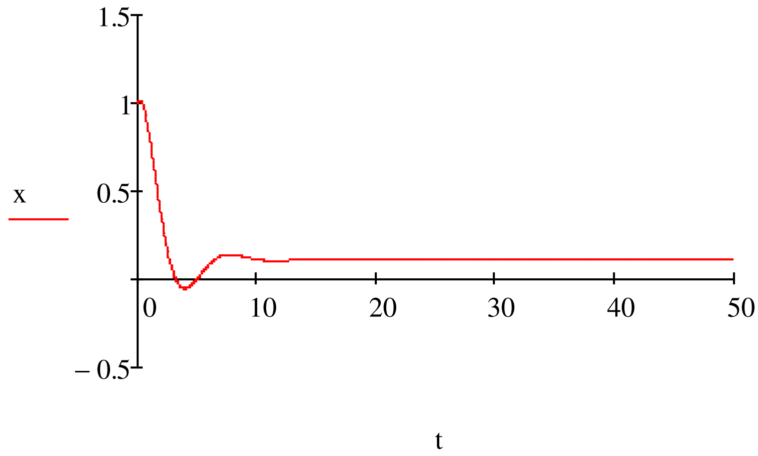

Figure 5a. A trajectory profile of BVP equation showing the aperiodic nature of the solution.

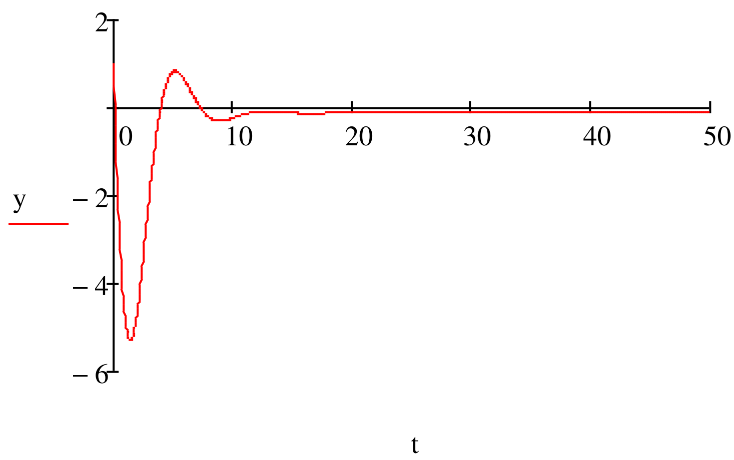

Figure 5b. A trajectory profile of BVP equation showing aperiodic nature of the solution.

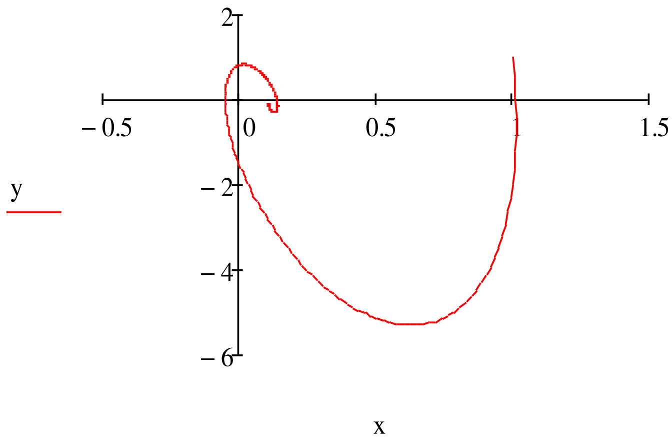

Figure 5c. A phase portrait of BVP equation depicting the asymptotic stability of solution close to the point of origin.

4. Conclusions

From our results, Lyapunov direct method is very effective in constructing suitable Lyapunov functions that depend on the parameters. The advantage of this method is that it is general and can be used to estimate the region of attraction for an equilibrium point. We observed that the solution of BVP equation is periodic by setting value of the control parameters equal to zero. Three equilibria points are obtained due to the cubic nature of the BVP equation. The equilibrium point was found to be asymptotically stable because the time derivative is negative definite. Multiple equilibria shows that some regions of stability and instability exist. The increase in the control parameters leads to stability while the decrease in the control parameters leads to instability. Hence, we conclude that the stability analysis of BVP equation strongly depends on the parameters. Application of our results can be seen in the process of heartbeat where certain conditions can alter the equilibrium state of the body. The numerical behaviors are explained as follows:

In Figure 1a, the trajectory profile of BVP equation was shown by the relationship between the first solution function values and independent variable values. The not periodic nature is unstable since the trajectory does not start from the equilibrium point.

In Figure 1b, the solution is not periodic using the second solution function values. The starting point of the trajectory was far from the origin, hence showing instability of the system.

In Figure 1c, the trajectory was far away from the origin for $\alpha = 0.7, \beta = 0.8, \gamma = 3$. This shows that the equilibrium point is highly unstable for the relationship between the two function values.

In Figure 2a, the trajectory profile of BVP equation was shown for $\alpha = 0.5, \beta = 0.1, \gamma = 3$. The trajectory shows that the solution is periodic and unstable with an increased maximum displacement.

In Figure 2b, the trajectory profile of BVP equation shows that the solution is periodic for the second solution function values. The maximum displacement is closer to the origin when compared with Figure 2a.

In Figures 2c and 3c, the phase portrait of BVP equation was shown for the two function values. The portrait shows that the equilibrium is stable in a bounded region. This region is defined as the interval where the bounds of the solution will occur.

In Figure 3a, the trajectory profiles exhibits a perfect oscillation for $\alpha = 0.1, \beta = 0.1$ and $\gamma = 1$. As the trajectory moves away from the origin, the maximum displacement increased, hence showing an unstable nature of the equilibrium point.

In Figure 3b, the trajectory of BVP equation was shown for $\alpha = 0.1, \beta = 0.1$ and $\gamma = 1$. As the values of the parameters are closer to the origin, a periodic solutions exist.

In Figure 4a and Figure 4b, the trajectory shows that the solution is periodic with a reduction in the maximum displacement.

In Figure 4c, a phase portrait of BVP equation was shown. The phase portrait was closed to the origin when compared with Figure 3c. This shows that the solution is periodic with a stable equilibrium point.

In Figure 5a and Figure 5b, the trajectory of BVP equation was parallel to the t axis. This shows that the solution is not periodic with an unstable equilibrium point.

In Figure 5c, a phase portrait of BVP equation for the two function values was shown. The portrait shows that the equilibrium point is asymptotically stable.

Competing Interests

The authors declared that they have no competing interests.

Authors Contributions

EE wrote the title and Introduction. UO carried out the stability analysis. TU used Mathcad to describe the numerical behavior of the system. All authors read and approved the final manuscript.

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References

[1] R. Fitzhugh, Impulses and Physiological States in Theoretical Model of Nerve Membrane. *Biophysical Journal* 1(1961):445-466.

[2] I.M. Kyprianidis, V. Papachristou and I.N. Stouboulos, Dynamics of Coupled Chaotic Bonhoeffer van der Pol Oscillators. *Aristotle University of Thessaloniki Greece*. 11(**9**) (2012):516.

[3] M. Alghassab, A. Mahmoud and M. Zohdy, Nonlinear Control of Chaotic Forced Duffing and Van der Pol Oscillators. Journal of Modern Nonlinear Theory and Application, 6(1) (2017):26-37.

[4] K. Shimizu and N. Inaba, Piecewise-linear Bonhoeffer-van der Pol Dynamics Explaining mixed-mode oscillation-incrementing bifurcations. Progress of Theoretical and Experimental Physics, 3(2016) 033A01.

[5] R.M. Pennifold, H.S. Sidhu and G.N. Mercer, Investigation of forced Bonhoeffer van der Pol equations via continuation and other methods. *Journal of Austral Mathematical Society.* 49(2008):626-641.

[6] B. Barnes and R. Grimshaw, Numerical Studies of the Periodically Forced Bonhoeffer van Der Pol System. *International journal of bifurcation and chaos,* 7**(12)** (1997)**:**2653-2689.

[7] P.E. Kloeden, E. Platen and H. Shurz, Numerical Solution of Stochastic Differential Equations through Computer Experiments. *Springer, Berlin*. 23(1994):600-632.

[8] S. Rajasekar, Controlling Unstable Periodic Orbits in a Bonhoeffer van der Pol Equation. *Chaos, Solitons and Fractals*. 11(1995):2135-2144.

[9] V. Meigoli and S.K.Y Nikravesh, Stability Analysis of Nonlinear System using Higher Order Derivatives of Lyapunov Function Candidate. *Journal of Systems and Control Letters Science Direct,* 61(10) (2012):973-979.

[10] T. Herbert and S. Klaus, Noise Induced Limit Cycles of the Bonhoeffer van der Pol Model of Neural Pulses. *Bunsen Society of Physical Chemistry,* 89 (1985):710-718.

[11] B. Barnes and R. Grimshaw, Analytical and Numerical studies of the Bonhoeffer van der Pol System. *Journal Australian Mathematical Society.* Series B, 38 (1995):427-453.

[12] E.O. Eze, U.E. Obasi and E.U. Agwu, Stability Analysis of Periodic Solutions of some Duffing's Equations. *Open Journal of Applied sciences*, 9 (2019):198-214.

[13] E.O Eze, E. Ukeje and H.M Ogbu, The Stable, Bounded and Periodic Solutions in a Non-Linear Second Order Differential Equation. *American Journal of Engineering Research,* 4**(8)** (2015**):** 14-18.

[14] E.O. Eze. On Application of Lyapunovand Yoshizawa`s Theorems on Stability, Asymptotic Stability, Boundaries and Periodicity of Solution of Duffing Equation. *ASIAN, Journal of Applied Science, 2*(6) (2014)**:**630-635.

[15] A.C Lazer and P.J. McKenna, On the Existence of Stable Periodic Solutions of Differential Equations of Duffing type. Proceeding of American Mathematical Society, 110 (1990), 125-133.

[16] B.S. Ogundare, Stability and Boundedness Properties of Solution to certain Fifth Order Nonlinear Differential Equation. *Matematicki Vesnik,* 4(2009):257-268.

[17] J.T Edwards, Boundedness and Stability of Solution to Difference Equation. *Journal of Computational and Applied Mathematics,* 1(40) (2002)**:**275-289.

[18] J.M. Alonso, Optimal Interval of Stability of a Forced Oscillator. *Proceedings of the American Mathematical society, 123*(1995):1-7.