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Soft subalgebras and ideals of BCK/BCI-algebras based on $\mathcal{N}$-structures

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Abstract The notions of $\mathcal{N}$-ideal of types $(\in, \in)$ and $(\in, \in \vee q)$, soft $\mathcal{N}_{\in}$-set, soft $\mathcal{N}_q$-set, soft $\mathcal{N}_{\in \vee q}$-set, soft $\mathcal{N}$-subalgebra and soft $\mathcal{N}$-ideal in BCK/BCI-algebra are introduced, and several properties are investigated. Characterizations of $\mathcal{N}$-subalgebra of types $(\in, \in)$ and $(\in, \in \vee q)$, $\mathcal{N}$-ideal of types $(\in, \in)$ and $(\in, \in \vee q)$, soft $\mathcal{N}$-subalgebra and soft $\mathcal{N}$-ideal are discussed.

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1 Introduction

The generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tools. To attain such an object, Jun et al. [6] introduced a new function which is called a negative-valued function and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras, and discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. Jun et al. [7] considered closed ideals in BCH-algebras based on $\mathcal{N}$-structures. Bordbar et al. [3], [8] applied these notions. To obtain a more general form of an $\mathcal{N}$-subalgebra in BCK/BCI-algebras, Jun et al. [5] defined the notions of $\mathcal{N}$-subalgebras of types $(\in, \in)$, $(\in, q)$, $(\in, \in \vee q)$, $(q, \in)$, (q, q) and $(q, \in \vee q)$, and investigated related properties. They provided a characterization of an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$, and considered conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subalgebra of type $(q, \in \vee q)$. Also for more information about soft algebraic structures, please refer to [1], [2] and [9].

In this paper, we introduce $\mathcal{N}$-ideal of types $(\in, \in)$ and $(\in, \in \vee q)$, soft $\mathcal{N}$-set, soft $\mathcal{N}_{\in}$-set, soft $\mathcal{N}_q$-set, soft $\mathcal{N}_{\in \vee q}$-set, soft $\mathcal{N}$-subalgebra and soft $\mathcal{N}$-ideal in BCK/BCI-algebra, and investigate several properties. We consider characterizations of $\mathcal{N}$-subalgebra of types $(\in, \in)$ and $(\in, \in \vee q)$, $\mathcal{N}$-ideal of types $(\in, \in)$ and $(\in, \in \vee q)$, soft $\mathcal{N}$-subalgebra and soft $\mathcal{N}$-ideal.

2 Preliminaries

Let $K(\tau)$ be the class of all algebras with type $\tau = (2, 0)$. By a *BCI-algebra* we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(i) $((x * y) * (x * z)) * (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) $x * y = y * x = 0 \implies x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. We can define a partial ordering $\leq$ by

$$(\forall x, y \in X)\,(x \leq y \iff x * y = 0).$$

In a BCK/BCI-algebra X, the following hold:

(a1) $(\forall x \in X)\ (x * 0 = x)$,

(a2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.

for all $x, y, z \in X$.

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x*y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if

$$0 \in I, \tag{2.1}$$

$$(\forall x, y \in X)\,(x * y \in I,\ y \in I\ \Rightarrow\ x \in I)\,. \tag{2.2}$$

We refer the reader to the books [4] and [10] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

For two real numbers a_1 and a_2, $\bigvee\{a_1, a_2\}$ and $\bigwedge\{a_1, a_2\}$ are also denoted by $a_1 \vee a_2$ and $a_1 \wedge a_2$, respectively.

Denote by $\mathscr{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathscr{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathcal{N}$*-function* on X). By an $\mathcal{N}$*-structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X.

Let (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f(y) = \begin{cases} 0 & \text{if} \quad y \neq x, \\ \alpha & \text{if} \quad y = x \end{cases}$$

where $\alpha \in [-1, 0)$. In this case, f is denoted by x_α and we call $(X,\ x_\alpha)$ a *point* $\mathcal{N}$*-structure*. For any $\mathcal{N}$-structure (X, g), we say that a point $\mathcal{N}$-structure $(X,\ x_\alpha)$ is an $\mathcal{N}_\in$*-subset* (resp., $\mathcal{N}_q$*-subset*) of (X, g) if $g(x) \leq \alpha$ (resp., $g(x) + \alpha + 1 < 0$). If a point $\mathcal{N}$-structure $(X,\ x_\alpha)$ is an $\mathcal{N}_\in$-subset of (X, g) or an $\mathcal{N}_q$-subset of (X, g), we say $(X,\ x_\alpha)$ is an $\mathcal{N}_{\in\vee q}$*-subset* of (X, g).

An $\mathcal{N}$-structure (X, f) is called an *$\mathcal{N}$-subalgebra* of type $(\in, \in)$ (resp., type $(\in, \in \vee q)$) (see [5]) if whenever two point $\mathcal{N}$-structures (X, x_α) and (X, y_β) are $\mathcal{N}_\in$-subsets of (X, f) then the point $\mathcal{N}$-structure $\big(X, (x * y)_{\bigvee\{\alpha,\beta\}}\big)$ is an $\mathcal{N}_\in$-subset (resp., $\mathcal{N}_{\in\vee q}$-subset) of (X, f).

3 Soft $\mathcal{N}$-subalgebras and soft $\mathcal{N}$-ideals

In what follows, let X denote a BCK/BCI-algebra unless otherwise specified. For a subset Δ of $[-1, 0]$, a pair $(\mathcal{A}, \Delta)$ is called a *soft $\mathcal{N}$-set* over X, where $\mathcal{A} : \Delta \to \mathcal{P}(X)$ is a mapping.

Given an $\mathcal{N}$-structure (X, f) and $\Delta \subseteq [-1, 0]$, we define two mappings:

$$\mathcal{A}_\in : \Delta \to \mathcal{P}(X),\ \alpha \mapsto \{x \in X \mid (X, x_\alpha) \text{ is an } \mathcal{N}_\in\text{-subset of } (X, f)\} \tag{3.1}$$

$$\mathcal{A}_q : \Delta \to \mathcal{P}(X),\ \alpha \mapsto \{x \in X \mid (X, x_\alpha) \text{ is an } \mathcal{N}_q\text{-subset of } (X, f)\} \tag{3.2}$$

Then $(\mathcal{A}_\in, \Delta)$ and $(\mathcal{A}_q, \Delta)$ are soft $\mathcal{N}$-sets over X. If $\mathcal{A}_\in(\alpha) \neq \emptyset$ (resp., $\mathcal{A}_q(\alpha) \neq \emptyset$) for $\alpha \in \Delta$, then we say $(\mathcal{A}_\in, \Delta)$ (resp., $(\mathcal{A}_q, \Delta)$) is a *soft $\mathcal{N}_\in$-set* (resp., *soft $\mathcal{N}_q$-set*) over X. A *soft $\mathcal{N}_{\in\vee q}$-set* over X is defined to be the union of a soft $\mathcal{N}_\in$-set and a soft $\mathcal{N}_q$-set over X, and is denoted by $(\mathcal{A}_{\in\vee q}, \Delta)$ where $\mathcal{A}_{\in\vee q}(\alpha) = \mathcal{A}_\in(\alpha) \cup \mathcal{A}_q(\alpha)$ for all $\alpha \in \Delta$.

Definition 3.1. A soft $\mathcal{N}$-set $(\mathcal{A}, \Delta)$ over X is called a *soft $\mathcal{N}$-subalgebra* over X if it satisfies:

$$(\forall \alpha \in \Delta)\,(\mathcal{A}(\alpha) \neq \emptyset \ \Rightarrow\ \mathcal{A}(\alpha) \text{ is a subalgebra of } X)\,. \tag{3.3}$$

Theorem 3.2. *Given an $\mathcal{N}$-structure (X, f) and $\Delta = [-1, 0)$, the soft $\mathcal{N}_\in$-set $(\mathcal{A}_\in, \Delta)$ is a soft $\mathcal{N}$-subalgebra over X if and only if (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$.*

Proof. Assume that $(\mathcal{A}_\in, \Delta)$ is a soft $\mathcal{N}$-subalgebra over X. If (X, f) is not an $\mathcal{N}$-subalgebra of type $(\in, \in)$, then there exist $a, b \in X$ and $t \in \Delta$ such that (X, a_t) and (X, b_t) are $\mathcal{N}_\in$-subsets of (X, f), but $(X, (a * b)_t)$ is not an $\mathcal{N}_\in$-subset of (X, f). Hence $a, b \in \mathcal{A}_\in(t)$ and $a * b \notin \mathcal{A}_\in(t)$, which shows that $\mathcal{A}_\in(t)$ is not a subalgebra of X. This is a contradiction, and therefore (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$.

Conversely, suppose that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$ and let $\alpha \in \Delta$ be such that $\mathcal{A}_\in(\alpha) \neq \emptyset$. If $x, y \in \mathcal{A}_\in(\alpha)$, then (X, x_α) and (X, y_α) are $\mathcal{N}_\in$-subsets of (X, f). Thus $(X, (x * y)_\alpha) = (X, (x * y)_{\bigvee\{\alpha,\alpha\}})$ is an $\mathcal{N}_\in$-subset of (X, f), that is, $x * y \in \mathcal{A}_\in(\alpha)$. Hence $\mathcal{A}_\in(\alpha)$ is a subalgebra of X for all $\alpha \in \Delta$, and thus $(\mathcal{A}_\in, \Delta)$ is a soft $\mathcal{N}$-subalgebra over X. $\square$

Lemma 3.3 ([5]). *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$ if and only if the following inequality is valid.*

$$(\forall x, y \in X)\left(f(x * y) \leq \bigvee\{f(x), f(y)\}\right). \tag{3.4}$$

Theorem 3.4. *Given an $\mathcal{N}$-structure (X, f) and $\Delta = [-1, 0)$, the soft $\mathcal{N}_q$-set $(\mathcal{A}_q, \Delta)$ is a soft $\mathcal{N}$-subalgebra over X if and only if (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$.*

Proof. Assume that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$ and let $\alpha \in \Delta$ be such that $\mathcal{A}_q(\alpha) \neq \emptyset$. If $x, y \in \mathcal{A}_q(\alpha)$, then (X, x_α) and (X, y_α) are $\mathcal{N}_q$-subsets of (X, f), and so $f(x) + \alpha + 1 < 0$ and $f(y) + \alpha + 1 < 0$. It follows from (3.4) that

$$f(x * y) + \alpha + 1 \leq \bigvee\{f(x), f(y)\} + \alpha + 1 < 0$$

and so that $(X, (x * y)_\alpha) = \left(X, (x * y)_{\bigvee\{\alpha,\alpha\}}\right)$ is an $\mathcal{N}_q$-subset of (X, f). Hence $x * y \in \mathcal{A}_q(\alpha)$, and thus $\mathcal{A}_q(\alpha)$ is a subalgebra of X for all $\alpha \in \Delta$ with $\mathcal{A}_q(\alpha) \neq \emptyset$. Therefore $(\mathcal{A}_q, \Delta)$ is a soft $\mathcal{N}$-subalgebra over X.

Conversely, suppose that the soft $\mathcal{N}_q$-set $(\mathcal{A}_q, \Delta)$ is a soft $\mathcal{N}$-subalgebra over X, and assume that $f(a * b) > \bigvee\{f(a), f(b)\}$ for some $a, b \in X$. Then there exists $t \in \Delta$ such that

$$f(a * b) + t + 1 \geq 0 \text{ and } \bigvee\{f(a), f(b)\} + t + 1 < 0.$$

It follows that (X, a_t) and (X, b_t) are $\mathcal{N}_q$-subsets of (X, f) but $(X, (a * b)_t)$ is not an $\mathcal{N}_q$-subset of (X, f). This is a contradiction, and hence $f(x * y) \leq \bigvee\{f(x), f(y)\}$ for all $x, y \in X$. Therefore (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in)$ by Lemma 3.3. □

Theorem 3.5. *Given an $\mathcal{N}$-structure (X, f) and the soft $\mathcal{N}_\in$-set $(\mathcal{A}_\in, \Delta)$ with $\Delta = [-1, -0.5)$, the following are equivalent:*

(1) *$(\mathcal{A}_\in, \Delta)$ is a soft $\mathcal{N}$-subalgebra over X.*

(2) $(\forall x, y \in X)\left(\bigwedge\{f(x * y), -0.5\} \leq \bigvee\{f(x), f(y)\}\right)$.

Proof. Assume that the soft $\mathcal{N}_\in$-set $(\mathcal{A}_\in, \Delta)$ is a soft $\mathcal{N}$-subalgebra over X. Then $\mathcal{A}_\in(\alpha)$ is a subalgebra of X for all $\alpha \in \Delta$ with $\mathcal{A}_\in(\alpha) \neq \emptyset$. If there exist $a, b \in X$ such that

$$\bigwedge\{f(a * b), -0.5\} > t := \bigvee\{f(a), f(b)\},$$

then $t \in \Delta$, and (X, a_t) and (X, b_t) are $\mathcal{N}_\in$-subsets of (X, f), that is, $a, b \in \mathcal{A}_\in(t)$, but $(X, (a * b)_t)$ is not an $\mathcal{N}_\in$-subset of (X, f), i.e., $a * b \notin \mathcal{A}_\in(t)$. This is a contradiction, and so $\bigwedge\{f(x * y), -0.5\} \leq \bigvee\{f(x), f(y)\}$ for all $x, y \in X$.

Conversely, suppose that (2) is valid. Let $x, y \in \mathcal{A}_{\in}(\alpha)$ for every $\alpha \in \Delta$. Then (X, x_α) and (X, y_α) are $\mathcal{N}_{\in}$-subsets of (X, f), and so

$$\bigwedge\{f(x * y), -0.5\} \leq \bigvee\{f(x), f(y)\} \leq \alpha < -0.5.$$

It follows that $(X, (x * y)_\alpha)$ is an $\mathcal{N}_{\in}$-subset of (X, f), that is, $x * y \in \mathcal{A}_{\in}(\alpha)$. Thus $\mathcal{A}_{\in}(\alpha)$ is a subalgebra of X, and therefore $(\mathcal{A}_{\in}, \Delta)$ is a soft $\mathcal{N}$-subalgebra over X. □

Lemma 3.6 ([5]). *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$ if and only if it satisfies:*

$$(\forall x, y \in X)\left(f(x * y) \leq \bigvee\{f(x), f(y), -0.5\}\right). \tag{3.5}$$

Theorem 3.7. *Given an $\mathcal{N}$-structure (X, f) and a soft $\mathcal{N}_{\in}$-set $(\mathcal{A}_{\in}, \Delta)$, the following assertions are equivalent:*

(1) *(X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$.*

(2) *$(\mathcal{A}_{\in}, \Delta)$ is a soft $\mathcal{N}$-subalgebra over X for $\Delta = [-0.5, 0)$.*

Proof. Assume that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$. Let $x, y \in X$ and $\alpha \in \Delta$ be such that $x, y \in \mathcal{A}_{\in}(\alpha)$. Then (X, x_α) and (X, y_α) are $\mathcal{N}_{\in}$-subsets of (X, f). It follows from (3.5) that

$$(\forall x, y \in X)\left(f(x * y) \leq \bigvee\{f(x), f(y), -0.5\} \leq \bigvee\{\alpha, -0.5\} = \alpha\right).$$

This shows that $(X, (x * y)_\alpha)$ is an $\mathcal{N}_{\in}$-subset of (X, f). Thus $x * y \in \mathcal{A}_{\in}(\alpha)$, and so $(\mathcal{A}_{\in}, \Delta)$ is a soft $\mathcal{N}$-subalgebra over X.

Conversely, suppose that the soft $\mathcal{N}_{\in}$-set $(\mathcal{A}_{\in}, \Delta)$ with $\Delta = [-0.5, 0)$ is a soft $\mathcal{N}$-subalgebra over X. Assume that (3.5) is not valid. Then

$$f(a * b) > t \geq \bigvee\{f(a), f(b), -0.5\}$$

for some $t \in \Delta$ and $a, b \in X$. It follows that (X, a_t) and (X, b_t) are $\mathcal{N}_{\in}$-subsets of (X, f), and so that $a, b \in \mathcal{A}_{\in}(t)$. But $f(a * b) > t$ induces that $(X, (a * b)_t)$ is not an $\mathcal{N}_{\in}$-subset of (X, f). This is a contradiction, and thus $f(x * y) \leq \bigvee\{f(x), f(y), -0.5\}$ for all $x, y \in X$. Using Lemma 3.6, we know that (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$. □

Theorem 3.8. *Let $(\mathcal{A}_{\in}, \Delta)$ be a soft $\mathcal{N}_{\in}$-set over X. If $\Delta = [-0.5, 0)$, then for any subalgebra S of X there exists an $\mathcal{N}$-subalgebra (X, f) of type $(\in, \in \vee q)$ such that $\mathcal{A}_{\in}(\alpha) = S$ for all $\alpha \in \Delta$.*

Proof. Take an $\mathcal{N}$-subalgebra (X, f) in which f is given as follows:

$$f : X \to [-1, 0],\ x \mapsto \begin{cases} \alpha \in \Delta & \text{if} \quad x \in S, \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

Obviously, $\mathcal{A}_{\in}(\alpha) = S$ for all $\alpha \in \Delta$. Assume that

$$f(a * b) > \bigvee\{f(a), f(b), -0.5\}$$

for some $a, b \in X$. Then $f(a * b) = 0$ and $\bigvee\{f(a), f(b), -0.5\} = \alpha$ since $|\mathrm{Im}(f)| = 2$. It follows that $f(a) = \alpha = f(b)$ so that $a, b \in S$. But $a * b \notin S$ since $f(a * b) = 0$. This is impossible, and so

$$f(x * y) \leq \bigvee\{f(x), f(y), -0.5\}$$

for all $x, y \in X$. Therefore (X, f) is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee q)$ by Lemma 3.6. □

Definition 3.9. An $\mathcal{N}$-structure (X, f) is called an $\mathcal{N}$-*ideal* of type $(\in, \in)$ (resp., type $(\in, \in \vee q)$) if the following assertions are valid.

(1) If a point $\mathcal{N}$-structure $(X,\ x_\alpha)$ is an $\mathcal{N}_{\in}$-subset of (X, f), then $(X,\ 0_\alpha)$ is an $\mathcal{N}_{\in}$-subset (resp., $\mathcal{N}_{\in \vee q}$-subset) of (X, f).

(2) If two point $\mathcal{N}$-structures $(X,\ (x * y)_\alpha)$ and $(X,\ y_\beta)$ are $\mathcal{N}_{\in}$-subsets of (X, f) then the point $\mathcal{N}$-structure $(X, x_{\alpha \vee \beta})$ is an $\mathcal{N}_{\in}$-subset (resp., $\mathcal{N}_{\in \vee q}$-subset) of (X, f).

Lemma 3.10. *Let (X, f) be an $\mathcal{N}$-structure. Then $f(0) \leq f(x)$ for all $x \in X$ if and only if $(X, 0_\alpha)$ is an $\mathcal{N}_{\in}$-subset of (X, f) whenever (X, x_α) is an $\mathcal{N}_{\in}$-subset of (X, f) for all $x \in X$ and $\alpha \in \Delta = [-1, 0)$.*

Proof. Assume that $f(0) \leq f(x)$ for all $x \in X$ and let $\alpha \in \Delta$ be such that (X, x_α) is an $\mathcal{N}_{\in}$-subset of (X, f). Then $f(0) \leq f(x) \leq \alpha$, and so $(X, 0_\alpha)$ is an $\mathcal{N}_{\in}$-subset of (X, f).

Conversely, suppose that $(X, 0_\alpha)$ is an $\mathcal{N}_{\in}$-subset of (X, f) when (X, x_α) is an $\mathcal{N}_{\in}$-subset of (X, f) for all $x \in X$ and $\alpha \in \Delta = [-1, 0)$. If we take $\beta = f(x)$ for any $x \in X$, then (X, x_β) is an $\mathcal{N}_{\in}$-subset of (X, f), and thus $(X, 0_\beta)$ is an $\mathcal{N}_{\in}$-subset of (X, f). Hence $f(0) \leq \beta = f(x)$ for all $x \in X$. □

Lemma 3.11. *Given an $\mathcal{N}$-structure (X, f), the following are equivalent.*

(1) *$f(x) \leq \bigvee\{f(x * y), f(y)\}$ for all $x, y \in X$.*

(2) *For any $x, y \in X$ and $\alpha, \beta \in \Delta = [-1, 0)$, if $(X, (x * y)_\alpha)$ and (X, y_β) are $\mathcal{N}_{\in}$-subsets of (X, f), then $(X, x_{\alpha \vee \beta})$ is an $\mathcal{N}_{\in}$-subset of (X, f).*

Proof. Assume that $f(x) \leq \bigvee\{f(x*y), f(y)\}$ for all $x, y \in X$. Let $x, y \in X$ and $\alpha, \beta \in \Delta$ be such that $(X, (x*y)_\alpha)$ and (X, y_β) are $\mathcal{N}_\in$-subsets of (X, f). Then $f(x*y) \leq \alpha$ and $f(y) \leq \beta$, which imply that

$$f(x) \leq \bigvee\{f(x*y), f(y)\} \leq \alpha \vee \beta.$$

Hence $(X, x_{\alpha\vee\beta})$ is an $\mathcal{N}_\in$-subset of (X, f).

Conversely, suppose (2) is valid. If we take $\alpha = f(x*y)$ and $\beta = f(y)$, then $(X, (x*y)_\alpha)$ and (X, y_β) are $\mathcal{N}_\in$-subsets of (X, f). It follows that $(X, x_{\alpha\vee\beta})$ is an $\mathcal{N}_\in$-subset of (X, f) and so that

$$f(x) \leq \alpha \vee \beta = \bigvee\{f(x*y), f(y)\}.$$

This completes the proof. □

Combining Lemmas 3.10 and 3.11, we have the following theorem.

Theorem 3.12. *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal of type $(\in, \in)$ if and only if the following inequality is valid.*

$$(\forall x, y \in X)\left(f(0) \leq f(x) \leq \bigvee\{f(x*y), f(y)\}\right). \tag{3.6}$$

Definition 3.13. A soft $\mathcal{N}$-set $(\mathcal{A}, \Delta)$ over X is called a *soft $\mathcal{N}$-ideal* over X if it satisfies:

$$(\forall \alpha \in \Delta)\,(\mathcal{A}(\alpha) \neq \emptyset \;\Rightarrow\; \mathcal{A}(\alpha) \text{ is an ideal of } X). \tag{3.7}$$

Example 3.14. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the binary operation $*$ in Table 1.
Given $\Delta = [-1, 0]$, let $(\mathcal{A}, \Delta)$ be a soft $\mathcal{N}$-set over X in which $\mathcal{A}$ is given as follows:

$$\mathcal{A} : \Delta \to \mathcal{P}(X),\ \alpha \mapsto \begin{cases} \{0\} & \text{if } \alpha = -1, \\ \{0,3\} & \text{if } \alpha \in (-1, -0.8), \\ \{0,4\} & \text{if } \alpha \in (-1, -0.8], \\ \{0,2,3\} & \text{if } \alpha \in (-0.8, -0.6], \\ \{0,3,4\} & \text{if } \alpha \in (-0.6, -0.4], \\ \{0,1,2,3\} & \text{if } \alpha \in (-0.4, -0.2], \\ \{0,1,3,4\} & \text{if } \alpha \in (-0.2, 0] \end{cases}$$

It is routine to verify that $(\mathcal{A}, \Delta)$ is a soft $\mathcal{N}$-ideal over X.

Table 1: Tabular representation of the binary operation $*$

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Theorem 3.15. *Given an $\mathcal{N}$-structure (X, f) and the soft $\mathcal{N}_\in$-set $(\mathcal{A}_\in, \Delta)$, the following are equivalent:*

(1) *$(\mathcal{A}_\in, \Delta)$ is a soft $\mathcal{N}$-ideal over X for $\Delta = [-1, 0)$.*

(2) *(X, f) is an $\mathcal{N}$-ideal of type $(\in, \in)$.*

Proof. Assume that $(\mathcal{A}_\in, \Delta)$ is a soft $\mathcal{N}$-ideal over X for $\Delta = [-1, 0)$. If there exists $a \in X$ such that $f(0) > f(a)$, then we can take $\alpha \in \Delta$ such that $f(0) > \alpha \geq f(a)$. Thus $(X, 0_\alpha)$ is not an $\mathcal{N}_\in$-subset of (X, f), and so $0 \notin \mathcal{A}_\in(\alpha)$. This is a contradiction, and so $f(0) \leq f(x)$ for all $x \in X$. Suppose that there exist $a, b \in X$ such that $f(a) > \bigvee\{f(a*b), f(b)\}$. Taking $\beta = \bigvee\{f(a*b), f(b)\}$ implies that $(X, (a*b)_\beta)$ and (X, b_β) are $\mathcal{N}_\in$-subsets of (X, f), that is, $a*b \in \mathcal{A}_\in(\beta)$ and $b \in \mathcal{A}_\in(\beta)$. Since $\mathcal{A}_\in(\beta)$ is an ideal of X, it follows that $a \in \mathcal{A}_\in(\beta)$. Hence (X, a_β) is an $\mathcal{N}_\in$-subset of (X, f), and so $f(a) \leq \beta$. This is a contradiction, and therefore $f(x) \leq \bigvee\{f(x*y), f(y)\}$ for all $x, y \in X$. Hence (X, f) is an $\mathcal{N}$-ideal of type $(\in, \in)$ by Theorem 3.12.

Conversely, assume that (X, f) is an $\mathcal{N}$-ideal of type $(\in, \in)$ and let $\alpha \in \Delta$ be such that $\mathcal{A}_\in(\alpha) \neq \emptyset$. Then there exists $x \in \mathcal{A}_\in(\alpha)$, and so (X, x_α) is an $\mathcal{N}_\in$-subset of (X, f). It follows that $f(0) \leq f(x) \leq \alpha$ and so that $(X, 0_\alpha)$ is an $\mathcal{N}_\in$-subset of (X, f), that is, $0 \in \mathcal{A}_\in(\alpha)$. Let $x*y \in \mathcal{A}_\in(\alpha)$ and $y \in \mathcal{A}_\in(\alpha)$. Then $(X, (x*y)_\alpha)$ and (X, y_α) are $\mathcal{N}_\in$-subsets of (X, f). Thus $f(x*y) \leq \alpha$ and $f(y) \leq \alpha$. It follows from (3.6) that $f(x) \leq \bigvee\{f(x*y), f(y)\} \leq \alpha$. Hence $(X, x_\alpha) = (X, x_{\alpha\vee\alpha})$ is an $\mathcal{N}_\in$-subset of (X, f), and so $x \in \mathcal{A}_\in(\alpha)$. Thus $\mathcal{A}_\in(\alpha)$ is an ideal of X for all $\alpha \in \Delta$, and therefore $(\mathcal{A}_\in, \Delta)$ is a soft $\mathcal{N}$-ideal over X for $\Delta = [-1, 0)$. □

Theorem 3.16. *The soft $\mathcal{N}_q$-set $(\mathcal{A}_q, \Delta)$ is a soft $\mathcal{N}$-ideal over X for $\Delta = [-1, 0)$ if and only if the $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-ideal of type $(\in, \in)$.*

Proof. Assume that (X, f) is an $\mathcal{N}$-ideal of type $(\in, \in)$ and let $\alpha \in \Delta$ be such that $\mathcal{A}_q(\alpha) \neq \emptyset$. Then there exists $x \in \mathcal{A}_q(\alpha)$, and so (X, x_α) is an $\mathcal{N}_q$-subset of (X, f). If $0 \notin \mathcal{A}_q(\alpha)$, then $(X, 0_q)$ is not an $\mathcal{N}_q$-subset of (X, f) and so $f(0) + \alpha + 1 \geq 0$. It follows from (3.6) that

$$f(x) + \alpha + 1 \geq f(0) + \alpha + 1 \geq 0,$$

and so that (X, x_α) is not an $\mathcal{N}_q$-subset of (X, f). This is a contradiction, and thus $0 \in \mathcal{A}_q(\alpha)$. Let $x * y \in \mathcal{A}_q(\alpha)$ and $y \in \mathcal{A}_q(\alpha)$. Then $(X, (x * y)_\alpha)$ and (X, y_α) are $\mathcal{N}_q$-subsets of (X, f). If (X, x_α) is not an $\mathcal{N}_q$-subset of (X, f), then $f(x) + \alpha + 1 \geq 0$. It follows from (3.6) that

$$\bigvee\{f(x * y), f(y)\} + \alpha + 1 \geq f(x) + \alpha + 1 \geq 0.$$

Hence $f(x * y) + \alpha + 1 \geq 0$ or $f(y) + \alpha + 1 \geq 0$, that is, $(X, (x * y)_\alpha)$ is not an $\mathcal{N}_q$-subset of (X, f) or (X, y_α) is not an $\mathcal{N}_q$-subset of (X, f). This is a contradiction, and thus $x \in \mathcal{A}_q(\alpha)$. Therefore $(\mathcal{A}_q, \Delta)$ is a soft $\mathcal{N}$-ideal over X for $\Delta = [-1, 0)$.

Conversely, suppose that the soft $\mathcal{N}_q$-set $(\mathcal{A}_q, \Delta)$ is a soft $\mathcal{N}$-ideal over X for $\Delta = [-1, 0)$. If $f(0) > f(a)$ for some $a \in X$, then there exists $\beta \in \Delta$ such that $f(0) + \beta + 1 \geq 0$ and $f(a) + \beta + 1 < 0$. Thus (X, a_β) is an $\mathcal{N}_q$-subset of (X, f), and so $(X, 0_\beta)$ is an $\mathcal{N}_q$-subset of (X, f). This is a contradiction, and therefore $f(0) \leq f(x)$ for all $x \in X$. Suppose that there exist $a, b \in X$ such that $f(a) > \bigvee\{f(a * b), f(b)\}$. Then $f(a) + \beta + 1 \geq 0$ and $\bigvee\{f(a*b), f(b)\} + \beta + 1 < 0$ for some $\beta \in \Delta$. Thus $f(a*b) + \beta + 1 < 0$ and $f(b) + \beta + 1 < 0$, that is, $(X, (a * b)_\beta)$ and (X, b_β) are $\mathcal{N}_q$-subsets of (X, f). Hence $a * b \in \mathcal{A}_q(\beta)$ and $b \in \mathcal{A}_q(\beta)$. Since $\mathcal{A}_q(\beta)$ is an ideal of X, we have $a \in \mathcal{A}_q(\beta)$, that is, (X, a_β) is an $\mathcal{N}_q$-subset of (X, f). This is a contradiction, and hence $f(x) \leq \bigvee\{f(x * y), f(y)\}$ for all $x, y \in X$. Using Theorem 3.12, (X, f) is an $\mathcal{N}$-ideal of type $(\in, \in)$. □

Theorem 3.17. *Given an $\mathcal{N}$-structure (X, f) and the soft $\mathcal{N}_\in$-set $(\mathcal{A}_\in, \Delta)$, the following are equivalent:*

(1) *(X, f) is an $\mathcal{N}$-ideal of type $(\in, \in \vee q)$.*

(2) *$(\mathcal{A}_\in, \Delta)$ is a soft $\mathcal{N}$-ideal over X for $\Delta = [-0.5, 0)$.*

Proof. Assume that (X, f) is an $\mathcal{N}$-ideal of type $(\in, \in \vee q)$. We first show that

$$(\forall x \in X)\left(f(0) \leq \bigvee\{f(x), -0.5\}\right). \tag{3.8}$$

Suppose that $f(0) > f(x) > -0.5$. Then $f(0) > \alpha \geq f(x)$ for some $\alpha \in (-0.5, 0)$, which implies that (X, x_α) is an $\mathcal{N}_\in$-subset of (X, f), but $(X, 0_\alpha)$ is not an $\mathcal{N}_\in$-subset of (X, f). Also, $(X, 0_\alpha)$ is not an $\mathcal{N}_q$-subset of (X, f) since $f(0) + \alpha + 1 \geq 0$. Thus $(X, 0_\alpha)$ is not an $\mathcal{N}_{\in\vee q}$-subset of (X, f), a contradiction. Hence $f(0) \leq f(x)$ for all $x \in X$. Now if $f(x) \leq -0.5$, then $(X, x_{-0.5})$ is an $\mathcal{N}_\in$-subset of (X, f) and so $(X, 0_{-0.5})$ is an $\mathcal{N}_{\in\vee q}$-subset of (X, f). It follows that $f(0) \leq -0.5$ or $f(0) - 0.5 + 1 < 0$ and so that $f(0) \leq -0.5$ because if not, then $f(0) - 0.5 + 1 > 0$, a contradiction. Consequently, the condition (3.8) is valid. Let $\alpha \in \Delta = [-0.5, 0)$. The condition (3.8) implies that $f(0) \leq \bigvee\{f(x), -0.5\}$ for all $x \in \mathcal{A}_\in(\alpha)$, and so

$$f(0) \leq \bigvee\{f(x), -0.5\} \leq \bigvee\{\alpha, -0.5\} = \alpha,$$

that is, $(X, 0_\alpha)$ is an $\mathcal{N}_\in$-subset of (X, f). Hence $0 \in \mathcal{A}_\in(\alpha)$. Now we show that

$$(\forall x, y \in X)\left(f(x) \leq \bigvee\{f(x * y), f(y), -0.5\}\right). \tag{3.9}$$

If $\bigvee\{f(x * y), f(y)\} > -0.5$, then $f(x) \leq \bigvee\{f(x * y), f(y)\}$. Otherwise, there exists $\beta \in (-0.5, 0)$ such that $f(x) > \beta \geq \bigvee\{f(x * y), f(y)\}$. It follows that $(X, (x * y)_\beta)$ and (X, y_β) are $\mathcal{N}_\in$-subsets of (X, f), but (X, x_β) is not an $\mathcal{N}_\in$-subset of (X, f). Also (X, x_β) is not an $\mathcal{N}_q$-subset of (X, f) because $f(x) + \beta + 1 > 2\beta + 1 > 0$. Hence (X, x_β) is not an $\mathcal{N}_{\in\vee q}$-subset of (X, f), a contradiction. If $\bigvee\{f(x * y), f(y)\} \leq -0.5$, then $(X, (x * y)_{-0.5})$ and $(X, y_{-0.5})$ are $\mathcal{N}_\in$-subsets of (X, f). Thus $(X, x_{-0.5})$ is an $\mathcal{N}_{\in\vee q}$-subset of (X, f), and so $f(x) \leq -0.5$ or $f(x) - 0.5 + 1 < 0$. It follows that $f(x) \leq -0.5$ because if $f(x) > -0.5$, then $f(x) - 0.5 + 1 > 0$ which is a contradiction. Therefore $f(x) \leq \bigvee\{f(x * y), f(y), -0.5\}$ for all $x, y \in X$. Let $x, y \in X$ be such that $x * y \in \mathcal{A}_\in(\alpha)$ and $y \in \mathcal{A}_\in(\alpha)$. Then $(X, (x * y)_\alpha)$ and (X, y_α) are $\mathcal{N}_\in$-subsets of (X, f), and so $f(x * y) \leq \alpha$ and $f(y) \leq \alpha$. Using (3.9), we have

$$f(x) \leq \bigvee\{f(x * y), f(y), -0.5\} \leq \bigvee\{\alpha, -0.5\} = \alpha.$$

Thus (X, x_α) is an $\mathcal{N}_\in$-subset of (X, f), and hence $x \in \mathcal{A}_\in(\alpha)$. Therefore $(\mathcal{A}_\in, \Delta)$ is a soft $\mathcal{N}$-ideal over X for $\Delta = [-0.5, 0)$.

Conversely, suppose that (2) is valid. If $f(0) > \bigvee\{f(a), -0.5\}$ for some $a \in X$, then there exists $\alpha \in \Delta$ such that $f(0) > \alpha \geq \bigvee\{f(a), -0.5\}$. Then $\alpha \in \Delta$ and (X, a_α) is an $\mathcal{N}_\in$-subset of (X, f). But $(X, 0_\alpha)$ is not an $\mathcal{N}_\in$-subset of (X, f), i.e., $0 \notin \mathcal{A}_\in(\alpha)$. This is a contradiction, and so $f(0) \leq \bigvee\{f(x), -0.5\}$ for all $x \in X$. Let $x \in X$ and $\alpha \in \Delta$ be such that (X, x_α) is an $\mathcal{N}_\in$-subset of (X, f). Then $f(x) \leq \alpha$. Suppose that $(X, 0_\alpha)$ is not an $\mathcal{N}_\in$-subset of (X, f). Then $f(0) > \alpha$. If $f(x) > -0.5$, then $f(0) \leq \bigvee\{f(x), -0.5\} = f(x) \leq \alpha$

which is impossible. Thus $f(x) \leq -0.5$ and so

$$f(0) + \alpha + 1 < 2f(0) + 1 \leq 2\bigvee\{f(x), -0.5\} + 1 = 0,$$

that is, $(X,\ 0_\alpha)$ is an $\mathcal{N}_q$-subset of (X, f). Hence $(X,\ 0_\alpha)$ is an $\mathcal{N}_{\in\vee q}$-subset of (X, f). Assume that there exist $a, b \in X$ such that

$$f(a) > \bigvee\{f(a * b), f(b), -0.5\}.$$

Taking $\alpha = \bigvee\{f(a * b), f(b), -0.5\}$ implies that $\alpha \in \Delta$, and $(X, (a * b)_\alpha)$ and (X, b_α) are $\mathcal{N}_\in$-subsets of (X, f). But (X, a_α) is not an $\mathcal{N}_\in$-subset of (X, f). This is a contradiction, and thus $f(x) \leq \bigvee\{f(x * y), f(y), -0.5\}$ for all $x, y \in X$. Let $x, y \in X$ and $\alpha, \beta \in \Delta$ be such that $(X,\ (x * y)_\alpha)$ and $(X,\ y_\beta)$ are $\mathcal{N}_\in$-subsets of (X, f), and suppose that $(X,\ x_{\alpha\vee\beta})$ is not an $\mathcal{N}_\in$-subset of (X, f). Then $f(x * y) \leq \alpha$, $f(y) \leq \beta$ and $f(x) > \alpha \vee \beta$. If $\bigvee\{f(x * y), f(y)\} > -0.5$, then

$$f(x) \leq \bigvee\{f(x * y), f(y), -0.5\} = \bigvee\{f(x * y), f(y)\} \leq \alpha \vee \beta.$$

which is a contradiction. Thus $\bigvee\{f(x * y), f(y)\} \leq -0.5$, and so

$$f(x) + (\alpha \vee \beta) + 1 < 2f(x) + 1 \leq 2\bigvee\{f(x * y), f(y), -0.5\} + 1 = 0,$$

which shows that $(X, x_{\alpha\vee\beta})$ is an $\mathcal{N}_q$-subset of (X, f). Thus $(X, x_{\alpha\vee\beta})$ is an $\mathcal{N}_{\in\vee q}$-subset of (X, f). Consequently, (X, f) is an $\mathcal{N}$-ideal of type $(\in,\ \in\vee q)$. □

Conclusion

We have introduced $\mathcal{N}$-ideal of types $(\in, \in)$ and $(\in, \in\vee q)$, soft $\mathcal{N}_\in$-set, soft $\mathcal{N}_q$-set, soft $\mathcal{N}_{\in\vee q}$-set, soft $\mathcal{N}$-subalgebra and soft $\mathcal{N}$-ideal in BCK/BCI-algebra.

We have considered characterizations of $\mathcal{N}$-subalgebra of types $(\in, \in)$ and $(\in, \in\vee q)$, $\mathcal{N}$-ideal of types $(\in, \in)$ and $(\in, \in\vee q)$, soft $\mathcal{N}$-subalgebra and soft $\mathcal{N}$-ideal.

4 Compliance with Ethical Standards

Conflict of Interest: Author declares that he has no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by the author.

Informed consent: Informed consent was obtained from all individual participants included in the study.

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