Teleparallel Stringy Black Holes and Topological Nieh-Yan Charges

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Teleparallel Stringy Black Holes and Topological Nieh-Yan Charges

By L.C. Garcia de Andrade¹

Abstract

Recently Banerjee [Class Quantum Gravity 2010] has investigated the Nieh-Yan (NY) topological charges in static black holes (BH) of Schwarzschild type. In this paper we extend Banerjee computations to a static BH with a cosmic string inside it and check how this modifies the NY anomaly. In orthonormal coordinates it is shown that NY topological invariant does not produce any contribution to anomaly. Nevertheless since other topological invariants as the Pontryagin density may appear in teleparallel gravity, since there the full Riemann-Cartan (RC) tensor vanishes and the Riemann tensor can be expressed in terms of torsion. Hence, Pontryagin topological charge may be computed in terms of torsion. The horizons of black holes and singularities are examined. The vanishing of torsion flux along Strings inside BH indicates that the string is confined inside the BH. This similarity is between the NY topological invariant $N = d(T^i \wedge e_i) \sim T^i \wedge T_i$ and the torsion scalar defined here as $T^2 = T_{ijk} T^{ijk}$ where T represents torsion differential forms and tensors. It is also shown that the Kerr BH can pursue a NY form invariant. the same problem in some metric forms.

Key-words: Torsion invariants, teleparallelism, Black Holes, strings, Nieh-Yan charge

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1 Introduction

Recently K Banerjee [1], has computed the NY form topological charges in the case of static BHs. The mathematical and topological techniques to investigate BHs and cosmic singularities has been investigated mainly by R Penrose [2]. Poplawski recent discovered that Cartan torsion [3], has real significance from experimental point of view, is only strong enough inside BHs. This makes it more interesting yet to develop topological invariants in BHs and torsion. We may also build the so-called topological Nieh-Yan invariant [4] in teleparallelism to compute the NY invariant in the case of rotating BHs. The static BHs with strings are given by only non-totally skew-symmetric torsion [5] and we are not able to investigate this exotic astrophysical object in terms of this topological torsion invariant. However, we are left with another type of torsion scalar invariant similar to the Riemannian scalar curvature called the Kretschmann curvature invariant given by $R_{ijkl}R^{ijkl} = K$ where R_{ijkl} is the Riemannian curvature tensor where $(i, j = 0, 1, 2, 3)$. In the case of Schwarzschild static BH the Kretschmann scalar $K = \frac{2m}{r^3}$ where the newtonian gravitational constant $G = 1$ throughout the paper. This scalar invariant given by T^2 in the abstract, hence shows that the BH singularity is at its center. Different to what happens in the so-called event horizon. In the present paper we shall compute the torsion components of the string metric inside the static black hole [6] and examine what the torsion invariant tells us about horizon and torsion singularity along the BH string. This paper is organised as follows: In section 2 we show that in teleparallel gravity the Nieh Yan topological invariant only depends upon the exterior product of the torsion totally skew-symmetric form, and show the need of another scalar torsion invariant for static BHs [7]. In section 3, we compute the torsion scalar invariant in the case of a string inside a static BH. In section 4 the torsion flux is computed along the string. Section 5 shows similar problems which happens in Kerr rotating BHs. Section 6 is left for conclusions and discussions.

2 Nieh-Yan form in teleparallel gravity

The NY form is given by

$$N = T_i \wedge T^i - R_{ij} \wedge e^i \wedge e^j \quad (1)$$

where T^i is totally skew differential form of torsion which in general is given by

$$T^i = de^i + \omega^i_j \wedge e^j \quad (2)$$

which is the first Cartan equation of differential calculus. Here e^i is an orthonormal basis. Here also R^i_j is the Riemann-Cartan curvature 2-form given by the second Cartan equation

$$R^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j \quad (3)$$

which shows that for teleparallel torsion gravity where the full Riemann-Cartan curvature given by

$$R^i_j = R^i_{jkl} e^k \wedge e^l \quad (4)$$

vanishes, one is left with the following expression for NY topological invariant

$$N = -T_i \wedge T^i \quad (5)$$

and since in the examples of black holes addressed here we do not find totally skew-symmetric torsion the NY form N also vanishes in this example of teleparallel gravity. In the next sections we shall show by computation that this is exactly the case and compute the scalar torsion invariant to analyse the behaviour of the BH horizon and truly singularities to obtain the behaviour of torsion there too. Nevertheless in the last section of this paper we present a Kerr like metric which gives rise to torsion skew-symmetric components of torsion two-forms which leads to NY topological invariant in teleparallel gravity.

3 Teleparallel stringy static BH and torsional scalar invariants

In this section we shall make use of the cosmic string inside the BH through the spacetime static metric

$$ds^2 = (e^0)^2 - [(e^1)^2 + (e^2)^2 + (e^3)^2] \quad (6)$$

where the differential basis one-forms are given by

$$e^0 = (1 - \frac{2m}{r})^{\frac{1}{2}} dt \quad (7)$$

where dt is the cosmic time, and

$$e^1 = (1 - \frac{2m}{r})^{-\frac{1}{2}} dr \quad (8)$$

$$e^2 = r d\theta \quad (9)$$

$$e^3 = (1 - 8\mu)^{\frac{1}{2}} \sin\theta d\phi \quad (10)$$

Let us now compute the teleparallel gravity equations by doing the simple ansatz namely, that the affine connection Riemann-Cartan one forms ω^i_j vanishes. This reduces the Cartan torsion equations to a much simpler form,

$$T^k = de^k \quad (11)$$

which allows us immediately to compute the torsion forms and components. Let us start by computing T^0 as

$$T^0 = de^0 = (1 - \frac{2m}{r})^{-\frac{1}{2}} \frac{2m}{r} e^1 \wedge e^0 \quad (12)$$

Component T^1 vanishes while T^2 and T^3 are given respectively by

$$T^2 = (1 - \frac{2m}{r})^{\frac{1}{2}} e^1 \wedge \frac{1}{r} e^2 \quad (13)$$

and

$$T^3 = de^3 = \frac{1}{r} \cot\theta e^2 \wedge e^3 \quad (14)$$

where the string mass μ does not appear explicitly. But actually shall appear in the torsion flux below. Now by making use of the expression for the torsion components

$$T^i = T^i_{jk} e^j \wedge e^k \quad (15)$$

By comparing this expression with the above torsion two-form components one obtains that T^1_{jk} all vanish and the non-vanishing ones are given by

$$T^0_{10} = (1 - \frac{2m}{r})^{-\frac{1}{2}} \frac{2m}{r} \quad (16)$$

by expressing the vorticity perturbed term as

$$T^2_{12} = (1 - \frac{2m}{r})^{\frac{1}{2}} \frac{1}{r} \quad (17)$$

$$T^3_{23} = \frac{\cot\theta}{r} \quad (18)$$

From these expressions we are able already to conclude the physical behaviour of torsion components at the Schwarzschild horizon where $r_s = 2m$ and at truly singularity or $r = 0$. Thus we see that at the horizon T^2_{12} vanishes and the torsion component T^3_{23} is

$$T^3_{23} = (1 - \frac{2m}{r})^{-\frac{1}{2}} \frac{2m}{r} \quad (19)$$

which shows that for extremely massive BHs we see that this component may be neglected. Finally the component T^0_{10} goes to infinity even away from the truly singularity however, we need now to define the scalar torsion invariant

$$T^2 = T_{ijk} T^{ijk} \quad (20)$$

now by making use of the tetrad Minkowski metric $\eta_{ij} = \text{diag}[+1, -1, -1, -1]$ is this torsion invariant reduces to

$$T^2 = -2[(T^0_{10})^2 + (T^2_{12})^2 + (T^3_{23})^2] \quad (21)$$

From the torsion components given above one obtains

$$T^2 = -2[(1 - \frac{2m}{r})^{-1} \frac{4m^2}{r^2} + (\frac{\cot\theta}{r})^2 + (1 - \frac{2m}{r})] \quad (22)$$

At the horizon the second and third terms on the last expression are finite the last zero but, the first expression is not finite and tends to infinity like in the true singularity in the singularity at the center of BH T^2 fastly diverges as in the Riemannian Kretschmann scalar.

4 Spin Polarised particles inside Black Holes from torsion strings

Before we compute the torsion scalar for the Kerr metric we shall address in this section the computations of the flux of torsion across BH surface. This

allows us to better understand the role of torsion strings inside the BH. Let us consider the torsion flux as

$$\int T^i = \int e^i \quad (23)$$

here they integrated over a surface Σ across which the torsion flux is computed and the second integral is the line integral due to Stokes theorem. It is easy to note that from expressions of the torsion forms the only non-vanishing torsion flux would be

$$\int T^i = \int e^3 = (1 - 8\mu)^{\frac{1}{2}} \sin\theta \int d\phi \quad (24)$$

which yields

$$\int T^i = 2\pi(1 - 8\mu)^{\frac{1}{2}} \sin\theta \quad (25)$$

Note from this expression there is no torsion flux along the cosmic string where θ takes values either 0 or 2π . Hence there is no torsion flux along the direction around the cosmic string torsion axis, as happens in the torsion vortex in the case of superfluids [8]. Note also that for a torsion string of mass $\mu = \frac{1}{8}$ then there is no torsion flux at this direction and no torsion flux at all! We could imagine this case as a system of BH and torsion strings would shield totally the effects of torsion like a superconductor makes with magnetic fields. Let us now make use of Cartan's equation relating the spin 3-form with the torsion two-form to investigate the polarisation effect of the cosmic string on the spinning particles of BHs as in Einstein-Cartan gravity. There is a controversy whether the teleparallel gravity should possess spin density or not [8], but here we adopt the point of view that spin density is allowed in T_4 gravity. From Cartan's equation we obtain

$$S_{ij} = \frac{-1}{8\pi} [T^k \wedge e^l \epsilon_{ijkl}] \quad (26)$$

By making use of expression (14) and substituting that expression into the expression (19) one obtains the component of spin we wish as

$$S_1 = S_{23} = \frac{1}{8\pi} \left[\frac{\cot\theta}{r} e^2 \wedge e^3 \wedge e^0 \epsilon_{0123} \right] \quad (27)$$

and spinning particles are polarised along the radial direction. The spin density is singular at the BH singularity, it is interesting to note that against

Trautmann [9] idea here torsion strings are not able to avoid BH singularity by simply changing BH topology [2]. This does not imply a torsion singularity as a delta distribution since there is spinning matter outside the string.

5 Torsion scalar invariant in Kerr BHs

In this section we shall compute the torsion scalar invariant in teleparallel gravity without strings inside. We shall examine the horizon and singularities properties again with the aid of torsion invariant. Let us start by considering the Kerr metric in spheroidal coordinates where the basis one-form is given by

$$e^0 = dt \quad (28)$$

where dt is the cosmic time, and

$$e^1 = a\Sigma d\epsilon \quad (29)$$

$$e^2 = a\Sigma d\theta \quad (30)$$

$$e^3 = a\cosh\epsilon\cos\theta d\phi \quad (31)$$

which allows us immediately to compute the torsion forms and components. Let us start by computing T^0 as

$$T^0 = de^0 = 0 \quad (32)$$

Component T^1 yields while T^2 and T^3 are given respectively by

$$T^1 = a\Sigma\sin 2\theta e^2 \wedge e^3 \quad (33)$$

where $\Sigma = [\sin^2\theta + \sinh^2\epsilon]^{\frac{1}{2}}$. With these mathematical tools at hand is easy after some trivial algebra obtain the following non-vanishing torsion two-form components as Component T^1 vanishes while T^2 and T^3 are given respectively by

$$T^2 = a^{-1}\Sigma^{-3}\sinh\epsilon\cosh\epsilon e^1 \wedge e^2 \quad (34)$$

and

$$T^3 = de^3 = -a^{-1}\Sigma^{-1}[tg\theta e^2 \wedge e^3 + tg\hbar e^1 \wedge e^3] \quad (35)$$

Note that at $\theta = 0$ and $\epsilon = 0$ represents the singularity and that Σ vanishes which shows that all the last T^i torsion two-form components diverges. Let

us now compute the NY topological charge in the case of this rotating BH metric. From the torsional Nieh-Yan charge one realizes that the only non-vanish NY term is

$$N_1 = d(T_1 \wedge e^1) \quad (36)$$

Hence the NY factor N_1 given by

$$N_1 = a \Sigma \sin 2\theta e^2 \wedge e^3 \wedge e^1 \quad (37)$$

which may contributes to the investigate the Holst and Immirzi terms [10]. Note from this expression, that when the θ -angle vanishes the full NY term vanishes as well this is exactly the position of black holes jet emission. We still looking for the physical reasons for this result [11]. This is a work under progress.

6 Conclusions

This paper is based on ideas by Letelier [12] and P Tod [13] to add torsion to topological defects in spacetimes as the other has also applied to torsion walls [12] to investigate for example cosmic strings in BHs with strings as in section 2. The discussion of the need to define topological invariants with torsion and to compute this form in T_4 teleparallel spacetime where the Riemann-Cartan U_4 curvature vanishes, is given by scalar torsion invariants. Investigation of topological and scalar torsion invariants may be useful to investigate the horizon and singular behaviour of torsion and polarised particles inside BH s and even in accretion discs. In the future we also compute the torsion scale invariants in Kerr-Newman metric which is a electric version of the Kerr metric. These scalar torsion invariants are also useful to define Lagrangeans in the teleparallel theory of gravity as shown by Bamba et al [14] in the case of magnetic dynamos. Chiral dynamos can also be used in the context of KN black hole solution of Einstein-Cartan Maxwell gravity-electrodynamics to investigate magnetic dynamos in Einstein-Cartan-Maxwell BHs. This may appear shortly.

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