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Soliton solutions to a (2+1)-dimensional nonlocal NLS equation

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Abstract In this paper, applying the Hirota's bilinear method and the KP hierarchy reduction method, we obtain the general soliton solutions in the forms of $N \times N$ Gram-type determinants to a (2+1)-dimensional nonlocal nonlinear Schrödinger equation with time reversal under zero and nonzero boundary conditions. The general bright soliton solutions with zero boundary condition are derived via the tau functions of two-component KP hierarchy. Under nonzero boundary condition, we first construct general soliton solutions on periodic background, when N is odd. Furthermore, we discuss typical dynamics of solutions analytically, and graphically.

Keywords Kadomtsev-Petviashvili hierarchy · Hirota's bilinear method · the nonlocal nonlinear Schrödinger equation · soliton solutions · periodic background

1 Introduction

In 2013, Ablowitz and Mussilmani [1] considered a nonlocal nonlinear Schrödinger (NLS) equation

$$iu_t(x, t) + u_{xx}(x, t) + 2\sigma u^2(x, t)u^*(-x, t) = 0, \quad (1)$$

where $\sigma = \pm 1$, and first derived its soliton solutions making use of the inverse scattering method (IST). Actually, Eq. (1) comes from a symmetry reduction of the

AKNS hierarchy, and it is a parity-time (PT) symmetry introduced by Bender and Boettcher [2, 3] in 1998, as the equation (1) is invariant under the parity-time (PT) operator, that is, the variable transformation $x \rightarrow -x$, $t \rightarrow -t$ and the complex conjugate.

After that, soliton solutions of the nonlocal NLS equations and various new nonlocal equations appearing from different symmetry reductions of the AKNS hierarchy were paid attention by many authors [4–11]. In addition, people also considered some other nonlocal equations with self-induced PT symmetric potential [12–14]. For example, in 2016, Fokas [5] provided soliton solutions of the nonlocal Davey-Stewartson (DS) equation

$$\begin{cases} iA_t = A_{xx} + \gamma^2 A_{yy} + (\varepsilon V - 2Q)A, \\ Q_{xx} - \gamma^2 Q_{yy} = (\varepsilon A A^*(-x, -y, t))_{xx}, \end{cases} \quad (2)$$

where $\varepsilon = \pm 1$, by using the same method (IST) in the work of Ablowitz and Mussilmani [1]. Later, the general breather and rogue wave solutions were obtained by Rao et al. [15] applying the Hirota bilinear method. An extension of the usual DS II equation involving a PT symmetric potential was considered by Liu et al [16], and the families of n -order rational solutions were obtained. In [5], Fokas also considered soliton solutions of a (2+1)-dimensional NLS equation

$$\begin{cases} iu_t + u_{xx} + uv = 0, \\ v_y = [u(x, y, t)u^*(-x, -y, t)]_x. \end{cases} \quad (3)$$

Cao et al. [17] kept on investigating this multidimensional equation (3) and gave families of rational and semi-rational solutions. Its general soliton solutions with zero and nonzero boundary conditions were studied by Liu and Li [18], they derived the soliton solutions expressing by $N \times N$ Gram-type determinants with even N , however, the odd case is not involved.

It is known that there are a variety of useful and powerful tools to deal with the nonlocal equations, namely,

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the inverse scattering transformation method, the Hirota's bilinear method, the KP hierarchy reduction method, and so on. Combining the Hirota's bilinear method and the KP reduction hierarchy method, very recently, Li et al. [19] discussed the nonlocal Mel'nikov equation and obtained its general soliton solution and (semi-)rational solutions on the periodic background. By employing the Hirota bilinear method, recently, Ma and Zhu [6] constructed the N-soliton solution for an integrable nonlocal discrete focusing nonlinear Schrödinger(dNLS+) equation, and gave the asymptotic analysis of two-soliton solution. By using the KP hierarchy reduction method, Rao et al. [20] found new families of semi-rational solutions termed as lump-soliton solutions to the nonlocal DS I equation.

In this paper, inspired by Liu and Li [18] and Li et al. [19], we study the (2+1)-dimensional nonlocal nonlinear Schrödinger equation with time reversal instead of space reversal,

$$\begin{cases} iu_t + u_{xx} + uv = 0, \\ v_y = [u(x, y, t)u(x, y, -t)]_x. \end{cases} \quad (4)$$

We also present the general soliton solutions of (4) in terms of $N \times N$ Gram-type determinants with zero and nonzero boundary conditions. Under the nonzero boundary condition, when N is odd, the general soliton solutions on the periodic background are constructed.

The rest of this article is organized as follows. In Sec.2 and 3, we present our main theorem for soliton solutions of the multi-dimensional nonlocal NLS equation with zero and nonzero boundary conditions, respectively. The construction of explicit soliton solutions appears in Sec.3.1. In Sec.4, we illustrate the dynamics of the two-, four- and multi-soliton solutions on different backgrounds. The conclusion is given in Sec.5.

2 Solution of the (2+1)-dimensional nonlocal nonlinear Schrödinger equation with zero boundary condition

In this section, we consider general bright soliton solutions to the (2+1)-dimensional nonlocal nonlinear Schrödinger equation (4) with zero boundary condition.

Theorem 1 *The (2+1)-dimensional nonlocal nonlinear Schrödinger equation (4) has soliton solutions with the zero boundary condition,*

$$\begin{cases} u(x, y, t) = \sqrt{2} \frac{g(x, y, t)}{f(x, y, t)}, \\ v(x, y, t) = 2(\ln f(x, y, t))_{xx}, \end{cases} \quad (5)$$

where

$$\begin{aligned} f(x, y, t) &= |A_{0,ij} + A_{1,ij}|_{N \times N}, \\ g(x, y, t) &= \begin{vmatrix} A_{0,ij} + A_{1,ij} & e^{p_i x - i p_i^2 t} \\ -e^{-\bar{q}_j y + \bar{\eta}_{j0}} & 0 \end{vmatrix}_{(N+1) \times (N+1)}, \\ A_{0,ij} &= \frac{1}{p_i + \bar{p}_j} e^{(p_i + \bar{p}_j)x + i(p_i^2 - \bar{p}_j^2)t}, \\ A_{1,ij} &= \frac{1}{q_i + \bar{q}_j} e^{-(q_i + \bar{q}_j)y + \eta_{i0} + \bar{\eta}_{j0}} \end{aligned}$$

with the parameters $p_i, q_i, \bar{p}_i, \bar{q}_i$ satisfying following constraints:

$$p_i = \bar{p}_i, \quad q_i = \bar{q}_i, \quad i = 1, 2, \dots, N. \quad (6)$$

2.1 Derivation of Solution with zero boundary condition

Under the transformation

$$u = \sqrt{2} \frac{g(x, y, t)}{f(x, y, t)}, \quad v = 2(\log f(x, y, t))_{xx}, \quad (7)$$

(4) can be converted into the following bilinear forms

$$\begin{cases} (D_x^2 + iD_t)(g \cdot f)(x, y, t) = 0, \\ D_x D_y (f \cdot f)(x, y, t) = 2g(x, y, t)g(x, y, -t). \end{cases} \quad (8)$$

where D is the Hirota's bilinear differential operator defined by

$$D_x^m D_t^n (f \cdot g)(x, t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n [f(x, t)g(x', t')] \Big|_{x'=x, t'=t}.$$

We start with the Gram-type determinant expression of the tau functions,

$$\tau_0 = |A|, \quad \tau_1 = \begin{vmatrix} A & \Phi^T \\ -\bar{\Psi} & 0 \end{vmatrix}, \quad \tau_{-1} = \begin{vmatrix} A & \Psi^T \\ -\bar{\Phi} & 0 \end{vmatrix}. \quad (9)$$

where $A = (a_{ij})_{N \times N}$ with

$$\begin{cases} a_{ij} = \frac{1}{p_i + \bar{p}_j} e^{\xi_i + \bar{\xi}_j} + \frac{1}{q_i + \bar{q}_j} e^{\eta_i + \bar{\eta}_j}, \\ \xi_i = p_i x_1 + p_i^2 x_2, \quad \bar{\xi}_j = \bar{p}_j x_1 - \bar{p}_j^2 x_2, \\ \eta_i = q_i y_1 + \eta_{i0}, \quad \bar{\eta}_j = \bar{q}_j y_1 + \bar{\eta}_{j0}. \end{cases} \quad (10)$$

and the $\Phi, \bar{\Phi}, \Psi, \bar{\Psi}$ are row vectors defined by

$$\Phi = (e^{\xi_1}, \dots, e^{\xi_N}), \quad \Psi = (e^{\eta_1}, \dots, e^{\eta_N}). \quad (11)$$

According to the Sato theory, it is clear that the tau functions given above satisfy the following two bilinear equations

$$\begin{cases} (D_{x_1}^2 - D_{x_2}) \tau_1 \cdot \tau_0 = 0, \\ D_{x_1} D_{x_{-1}} \tau_0 \cdot \tau_0 = -2\tau_1 \tau_{-1}. \end{cases} \quad (12)$$

Indeed, assume that $x_1 = x, x_2 = -it, x_{-1} = -y$, making $p_i = \bar{p}_i, q_i = \bar{q}_i$, we have

$$\xi_i + \bar{\xi}_j = (\xi_i + \bar{\xi}_j)(x, y, -t),$$

$$\eta_i + \bar{\eta}_j = (\eta_i + \bar{\eta}_j)(x, y, -t).$$

Then the tau function $\tau_0(x, y, t)$ can be rewritten as follow:

$$\tau_0(x, y, t) = |a_{ij}| \quad (13)$$

$$= |a_{ij}(x, y, -t)|. \quad (14)$$

Obviously, we obtain $\tau_0(x, y, t) = \tau_0(x, y, -t)$. Besides, we can also claim that $\tau_1(x, y, t) = \tau_{-1}(x, y, -t)$. Actually, it is not hard to see that

$$\tau_1(x, y, t) = \begin{vmatrix} a_{ij} & e^{\xi_i} \\ -e^{\bar{\eta}_j} & 0 \end{vmatrix} = \begin{vmatrix} a_{ij} & e^{\bar{\eta}_j} \\ -e^{\xi_i} & 0 \end{vmatrix} \quad (15)$$

$$= \begin{vmatrix} a_{ij}(x, y, -t) & e^{\eta_i(x, y, -t)} \\ -e^{\bar{\xi}_j(x, y, -t)} & 0 \end{vmatrix}. \quad (16)$$

and

$$\tau_{-1}(x, y, t) = \begin{vmatrix} a_{ij} & e^{\eta_i} \\ -e^{\bar{\xi}_j} & 0 \end{vmatrix} \quad (17)$$

$$= \begin{vmatrix} a_{ij}(x, y, -t) & e^{\eta_i(x, y, -t)} \\ -e^{\bar{\xi}_j(x, y, -t)} & 0 \end{vmatrix}, \quad (18)$$

thus, the claim $\tau_1(x, y, t) = \tau_{-1}(x, y, -t)$ is proved. Now, if we set

$$f(x, y, t) = \tau_0, \quad g(x, y, t) = \tau_1, \quad g(x, y, -t) = \tau_{-1},$$

then (12) can be converted to (8), therefore, we complete the proof of Theorem 1.

2.2 Soliton solutions with zero boundary condition

Below, we focus on the one- and two-bright-soliton solutions.

By taking $N = 1$, we have

$$f(x, y, t) = \frac{1}{2p_1} e^{2p_1 x} + \frac{1}{2q_1} e^{-2q_1 y + 2\eta_{10}}, \quad (19)$$

$$g(x, y, t) = e^{p_1 x - ip_1^2 t - \bar{q}_1 y + \bar{\eta}_{10}}, \quad (20)$$

therefore

$$u = \sqrt{2} \frac{2p_1 q_1 e^{p_1 x - ip_1^2 t - \bar{q}_1 y + \bar{\eta}_{10}}}{q_1 e^{2p_1 x} + p_1 e^{-2q_1 y + 2\eta_{10}}} \quad (21)$$

Then the one-soliton solution can be plotted with parameters $p_1 = 3, q_1 = 3, \eta_{10} = 0$ (see Fig. 1).

Fig. 1: One-soliton solution on zero boundary with parameter values $\eta_{10} = 0, p_1 = 3, q_1 = 3$.

By taking $N = 2$, we can express two-soliton solutions on zero-boundary condition as

$$f(x, y, t) = \begin{vmatrix} A_{0,11} + A_{1,11} & A_{0,12} + A_{1,12} \\ A_{0,21} + A_{1,21} & A_{0,22} + A_{1,22} \end{vmatrix},$$

$$g(x, y, t) = \begin{vmatrix} A_{0,11} + A_{1,11} & A_{0,12} + A_{1,12} & e^{p_1 x - ip_1^2 t} \\ A_{0,21} + A_{1,21} & A_{0,22} + A_{1,22} & e^{p_2 x - ip_2^2 t} \\ -e^{-\bar{q}_1 y + \bar{\eta}_{10}} & -e^{-\bar{q}_2 y + \bar{\eta}_{20}} & 0 \end{vmatrix}.$$

Then, we plot the two-soliton solution on zero-boundary condition with parameter values $\eta_{10} = 0, \eta_{20} = 0, p_1 = 1, p_2 = 2, q_1 = 1, q_2 = 2$ in Fig. 2.

Fig. 2: Two-soliton solution on zero boundary with parameter values $\eta_{10} = 0, \eta_{20} = 0, p_1 = 1, p_2 = 2, q_1 = 1, q_2 = 2$.

3 Solution of the (2+1)-dimensional nonlocal nonlinear Schrödinger equation with nonzero boundary condition

In this section, we would like to present the general soliton solutions of the (2+1)-dimensional nonlocal nonlinear Schrödinger equation with nonzero boundary conditions.

Theorem 2 Let N be a positive integer and δ_{ij} be the Kronecker delta. The (2+1)-dimensional nonlocal nonlinear Schrödinger equation (4) has soliton solutions

$$u = \sqrt{2} e^{ikt} \frac{g}{f}, \quad v = k + 2(\ln f)_{xx}, \quad (22)$$

where $f = \det_{1 \leq i, j \leq N} (M_{ij}^{(0)})$ and $g = \det_{1 \leq i, j \leq N} (M_{ij}^{(1)})$, with

$$\begin{cases} M_{ij}^{(n)} = \left| c_i \delta_{ij} e^{-(\xi_i + \eta_j)} + \frac{1}{p_i + q_j} \left(-\frac{p_i}{q_j} \right)^n \right|, \\ \xi_i = -\frac{1}{p_i} y + p_i x + ip_i^2 t + \xi_{i0}, \\ \eta_j = -\frac{1}{q_j} y + q_j x - iq_j^2 t + \eta_{j0}. \end{cases} \quad (23)$$

The complex parameters $c_i, p_i, q_i, \xi_{i0}, \eta_{j0}$ satisfy following constraints:

I) when N is even, i.e. $N=2M$,

$$\begin{aligned} c_j &= c_{M+j}, & p_j &= q_{M+j}, \\ q_j &= p_{M+j}, & j &= 1, 2, \dots, M, \end{aligned} \quad (24)$$

II) when N is odd, i.e. $N=2M+1$,

$$\begin{aligned} c_i &= c_{M+i}, & p_j &= q_{M+j}, & q_j &= p_{M+j}, \\ p_{2M+1} &= q_{2M+1}, & j &= 1, 2, \dots, M. \end{aligned} \quad (25)$$

3.1 Derivation of Solutions

3.1.1 Bilinearization

In order to construct soliton solutions, we first introduce the dependent variable transformations

$$u = \sqrt{2}e^{ikt}\frac{g}{f}, \quad v = k + 2(\ln f)_{xx}, \quad (26)$$

where f and g are functions with respect to three variables x , y and t . Similar to the case in zero boundary condition, plugging Eq.(26) into Eq.(4), we have

$$\begin{cases} (D_x^2 + iD_t)(g \cdot f)(x, y, t) = 0, \\ (D_x D_y + 2)(f \cdot f)(x, y, t) = 2g(x, y, t)g(x, y, -t), \end{cases} \quad (27)$$

3.1.2 Construction of solutions

In this section, we start with tau functions of single component KP hierarchy

$$\tau_n = \det_{1 \leq i, j \leq N} (m_{ij}^{(n)}), \quad (28)$$

with

$$\begin{cases} m_{ij}^{(n)} = c_i \delta_{ij} + \frac{1}{p_i + q_j} \left(-\frac{p_i}{q_j} \right)^{(n)} e^{\xi_i + \eta_j}, \\ \xi_i = p_i^{-1} x_{-1} + p_i x_1 + p_i^2 x_2 + \xi_{i0}, \\ \eta_j = q_j^{-1} x_{-1} + q_i x_1 - q_i^2 x_2 + \eta_{j0} \end{cases}$$

where p_i , q_j , c_i , ξ_{i0} and η_{j0} are arbitrary complex constants, $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ elsewhere. Due to the Sato theory, we know that the tau functions τ_n satisfy the bilinear equations,

$$\begin{cases} (D_{x_1}^2 - D_{x_2}) \tau_{n+1} \cdot \tau_n = 0, \\ (D_{x_1} D_{x_{-1}} - 2) \tau_n \cdot \tau_n = -2 \tau_{n+1} \tau_{n-1}. \end{cases} \quad (29)$$

By taking the variables transformation, $x_1 = x$, $x_2 = it$ and $x_{-1} = -y$, we can rewrite τ_n as

$$\tau_n = \prod_{j=1}^N e^{\xi_j + \eta_j} \left| c_i \delta_{ij} + \frac{1}{p_i + q_j} \left(-\frac{p_i}{q_j} \right)^n \right|, \quad (30)$$

with

$$\xi_i = -\frac{1}{p_i} y + p_i x + i p_i^2 t + \xi_{i0}$$

and

$$\eta_j = -\frac{1}{q_j} y + q_i x - i q_i^2 t + \eta_{j0}.$$

Furthermore, if we set

$$f(x, y, t) = \frac{\tau_0}{C}, \quad g(x, y, t) = \frac{\tau_1}{C}, \quad h(x, y, t) = \frac{\tau_{-1}}{C}, \quad (31)$$

where $C = \prod_{j=1}^N e^{\xi_j + \eta_j}$, then the bilinear equations (29) can be as the following,

$$\begin{cases} (D_x^2 + iD_t) g(x, y, t) \cdot f(x, y, t) = 0, \\ (D_x D_y + 2) f(x, y, t) \cdot f(x, y, t) = 2g(x, y, t)h(x, y, t). \end{cases} \quad (32)$$

After simple algebra calculation, it follows that

$$f = |M_{ij}^{(0)}|_{N \times N}, \quad g = |M_{ij}^{(1)}|_{N \times N}, \quad h = |M_{ij}^{(-1)}|_{N \times N}. \quad (33)$$

By substituting the parameters constraints (24) and (25), we can obtain

$$M_{i,j}^{(n)}(x, y, t) = M_{M+j, M+i}^{(-n)}(x, y, -t), \quad (34)$$

$$M_{i, M+j}^{(n)}(x, y, t) = M_{j, M+i}^{(-n)}(x, y, -t), \quad (35)$$

$$M_{M+i, j}^{(n)}(x, y, t) = M_{M+j, i}^{(-n)}(x, y, -t), \quad (36)$$

$$M_{M+i, M+j}^{(n)}(x, y, t) = M_{j, i}^{(-n)}(x, y, -t), \quad (37)$$

$$M_{2M+1, M+i}^{(n)}(x, y, t) = M_{i, 2M+i}^{(-n)}(x, y, -t), \quad (38)$$

$$M_{M+j, 2M+1}^{(n)}(x, y, t) = M_{2M+1, j}^{(-n)}(x, y, -t), \quad (39)$$

$$M_{2M+1, 2M+1}^{(n)}(x, y, t) = M_{2M+1, 2M+1}^{(-n)}(x, y, -t). \quad (40)$$

Thus, we have

$$f(x, y, t) = f(x, y, -t), \quad g(x, y, -t) = h(x, y, t)$$

and then Eq. (32) reduces into Eq. (27). Therefore, the proof of Theorem 2 is completed.

4 Dynamics of the soliton solutions

In this section, we would like to present the concrete form of the one-soliton solution with the periodic background and the explicit form of the two-soliton solutions on constant and periodic backgrounds. Furthermore, we also give the asymptotic behaviour of the two-soliton solution.

4.1 The periodic background

When $N = 1$, Theorem 2 yields the following solution

$$\begin{aligned} u(x, y, t) &= \sqrt{2}e^{ikt} \frac{M_{11}^{(1)}}{M_{11}^{(0)}} \\ &= \sqrt{2}e^{ikt} \frac{c_1 e^{-(\xi_1 + \eta_1)} + \frac{1}{p_1 + q_1} \left(-\frac{p_1}{q_1} \right)}{c_1 e^{-(\xi_1 + \eta_1)} + \frac{1}{p_1 + q_1}}. \end{aligned}$$

If we set $p_1 = m + ni$, $c_1 = r + di$, where m , n , r , d are real numbers, then (41) is expressed as

$$u(x, y, t) = \sqrt{2}e^{ikt} \frac{e^{-(i\alpha + \beta)} - \frac{1}{2} \frac{m+ni}{m^2+n^2} \frac{r-di}{r^2+d^2}}{e^{-(i\alpha + \beta)} + \frac{1}{2} \frac{m+ni}{m^2+n^2} \frac{r-di}{r^2+d^2}}, \quad (41)$$

and $\alpha = \frac{2ny}{m^2+n^2} + 2nx$, $\beta = -2mx + \frac{2m}{m^2+n^2}y + \xi_{01} + \eta_{01}$. Clearly, the solutions are periodic in both x and y with period $2n$ and $\frac{2n}{m^2+n^2}$ respectively when p_1 is purely imaginary. In this paper, regular solutions (4.1) provide the periodic background for higher-order soliton solution (see Fig. 3).

Fig. 3: Periodic solution with parameter values $\xi = 0, \eta = 0, k = 1, p_1 = i, q_1 = i, c_1 = 2 + i$.

4.2 Two-soliton solutions on both constant and periodic backgrounds

By taking $N = 2$, we can express two-soliton solutions on the constant background as

$$u = \sqrt{2}e^{ikt} \frac{g}{f}, \quad (42)$$

where

$$f = \begin{vmatrix} M_{1,1}^{(0)} & M_{1,2}^{(0)} \\ M_{2,1}^{(0)} & M_{2,2}^{(0)} \end{vmatrix}, \quad g = \begin{vmatrix} M_{1,1}^{(1)} & M_{1,2}^{(1)} \\ M_{2,1}^{(1)} & M_{2,2}^{(1)} \end{vmatrix}.$$

Let $p_1 = \rho e^{i\theta}$, $\rho \neq 0$, then

$$\begin{aligned} f &= c_1^2 e^{-(\xi_1 + \xi_2 + \eta_1 + \eta_2)} + \frac{c_1}{p_1 + p_2} e^{-(\xi_1 + \eta_1)} \\ &\quad + \frac{c_1}{p_1 + p_2} e^{-(\xi_2 + \eta_2)} - \frac{1}{(p_1 + p_2)^2} - \frac{1}{4p_1 p_2}, \\ g &= c_1^2 e^{-(\xi_1 + \xi_2 + \eta_1 + \eta_2)} + \frac{c_1}{p_1 + p_2} \left(-\frac{p_2}{p_1} \right) e^{-(\xi_1 + \eta_1)} \\ &\quad + \frac{c_1}{p_1 + p_2} \left(-\frac{p_1}{p_2} \right) e^{-(\xi_2 + \eta_2)} - \frac{1}{(p_1 + p_2)^2} - \frac{1}{4p_1 p_2}, \end{aligned}$$

where

$$\begin{aligned} -(\xi_1 + \eta_1) &= -2\rho \cos \theta x + \frac{2 \cos \theta}{\rho} y \\ &\quad + 2\rho^2 \sin 2\theta t - \xi_{10} - \eta_{10}, \\ -(\xi_2 + \eta_2) &= -2\rho \cos \theta x + \frac{2 \cos \theta}{\rho} y \\ &\quad - 2\rho^2 \sin 2\theta t - \xi_{10} - \eta_{10}. \end{aligned}$$

We define the soliton moving along the line

$$-2\rho \cos \theta x + \frac{2 \cos \theta}{\rho} y + 2\rho^2 \sin 2\theta t - \xi_{01} - \eta_{01}$$

as Soliton 1, and the soliton moving along the line

$$-2\rho \cos \theta x + \frac{2 \cos \theta}{\rho} y - 2\rho^2 \sin 2\theta t - \xi_{02} - \eta_{02}$$

as Soliton 2. Without loss of generality, we assume that $0 < \theta < \frac{\pi}{2}$ and $2\rho \cos \theta > 0$. Then these two solitons possess the following asymptotic forms:

I) Before collision ($t \rightarrow -\infty$)

Soliton 1 ($-(\xi_1 + \eta_1) \approx 0, -(\xi_2 + \eta_2) \rightarrow -\infty$)

$$u_1^-(\xi_1 + \eta_1) \simeq -\sqrt{2}e^{ikt} \frac{c_1 d_1 e^{-(\xi_1 + \eta_1)} + \frac{\sin^2 \theta}{2\rho \cos \theta}}{c_1 e^{-(\xi_1 + \eta_1)} + \frac{\sin^2 \theta}{2\rho \cos \theta}}.$$

Soliton 2 ($-(\xi_2 + \eta_2) \approx 0, -(\xi_1 + \eta_1) \rightarrow -\infty$)

$$u_2^-(\xi_2 + \eta_2) \simeq -\sqrt{2}e^{ikt} \frac{c_1 d_2 e^{-(\xi_2 + \eta_2)} + \frac{\sin^2 \theta}{2\rho \cos \theta}}{c_1 e^{-(\xi_2 + \eta_2)} + \frac{\sin^2 \theta}{2\rho \cos \theta}}.$$

II) After collision ($t \rightarrow +\infty$)

Soliton 1 ($-(\xi_1 + \eta_1) \approx 0, -(\xi_2 + \eta_2) \rightarrow +\infty$)

$$u_1^+(\xi_1 + \eta_1) \simeq -\sqrt{2}e^{ikt} R \frac{c_1 d_1 e^{-(\xi_1 + \eta_1) + \Delta} + \frac{\sin^2 \theta}{2\rho \cos \theta}}{c_1 e^{-(\xi_1 + \eta_1) + \Delta} + \frac{\sin^2 \theta}{2\rho \cos \theta}}.$$

Soliton 2 ($-(\xi_2 + \eta_2) \approx 0, -(\xi_1 + \eta_1) \rightarrow +\infty$)

$$u_2^+(\xi_2 + \eta_2) \simeq -\frac{\sqrt{2}e^{ikt}}{R} \frac{c_1 d_2 e^{-(\xi_2 + \eta_2) + \Delta} + \frac{\sin^2 \theta}{2\rho \cos \theta}}{c_1 e^{-(\xi_2 + \eta_2) + \Delta} + \frac{\sin^2 \theta}{2\rho \cos \theta}},$$

where $e^\Delta = \rho^2 \sin^2 \theta$, $d_1 = 2 \sin^2 \theta + i \sin 2\theta - 1$, $d_2 = 2 \sin^2 \theta - i \sin 2\theta - 1$ and $R = \frac{\cos \theta - i \sin \theta}{\cos \theta + i \sin \theta}$.

Based on the above asymptotic analysis, we conclude that these two solitons undertake the elastic collisions. As their amplitudes satisfy $|u^-(\xi_i + \eta_i)| = |u^+(\xi_i + \eta_i + \Delta)|$, $i = 1, 2$, and two solitons undertake a phase shift $-\Delta$, under some parameter choices, three types of two-soliton, that is, anti-dark-anti-dark soliton, anti-dark-dark soliton and dark-dark soliton on the constant background can be obtained (see Figs. 4 (a)-(c)).

Next, two-soliton solutions on the periodic background can be derived when $N = 3$ in Theorem 2. By selecting the different parameters, the same three types of two-soliton appear on the periodic background (see Figs. 4 (d)-(f)).

4.3 Four-soliton solutions on both constant and periodic backgrounds

Likewise, we construct the four-soliton solutions on the constant and periodic background when $N = 4$ and $N = 5$ respectively. Four-soliton solutions describe the superposition of two-soliton. As illustrated in Fig. 4, two-soliton solutions are presented as three patterns, so four-soliton solutions have more different types. By adjusting different parameter values, five types of solutions can be obtained, that is, 4-dark soliton, 3-dark-1-anti-dark soliton, 2-dark-2-anti-dark soliton, 1-dark-3-anti-dark soliton, and 4-anti-dark soliton (see Fig. 5). Similarly, when $N = 5$, we also have five types of solutions

Fig. 4: Two-soliton solutions on the constant and periodic backgrounds with parameter values $\xi = 0, \eta = 0, k = 1$ and (a) $p_1 = 1 - i, p_2 = 1 + i, q_1 = 1 + i, q_2 = 1 - i, c_1 = i, c_2 = i$, (b) $p_1 = 2 - i, p_2 = 2 + i, q_1 = 2 + i, q_2 = 2 - i, c_1 = -1 + i, c_2 = -1 + i$, (c) $p_1 = 2 - i, p_2 = 2 + i, q_1 = 2 + i, q_2 = 2 - i, c_1 = -1 + i, c_2 = -1 + i$, (d) $p_1 = 2 + i, p_2 = 2 - i, p_3 = 2i, q_1 = 2 - i, q_2 = 2 - i, q_3 = 2i, c_1 = 5i, c_2 = 5i, c_3 = 4i$, (e) $p_1 = 5 + i, p_2 = 5 - i, p_3 = 2i, q_1 = 5 - i, q_2 = 5 + i, q_3 = 2i, c_1 = 2 - i, c_2 = 2 - i, c_3 = -3i$, (f) $p_1 = 3 + i, p_2 = 3 - i, p_3 = 2i, q_1 = 3 - i, q_2 = 3 + i, q_3 = 2i, c_1 = -1 + i, c_2 = -1 + i, c_3 = i$.

(see Fig. 6). Next, we give the dynamical behavior of multi-soliton solutions in general cases.

- i) When $N = 2M$, $N + 1$ types of N -soliton solutions on the constant background can be obtained, which contains k -dark- $(N - k)$ -anti-dark solutions, where $0 \leq k \leq N$.
- ii) When $N = 2M + 1$, $N + 1$ types of N -soliton solutions on the periodic background can be obtained.

5 Conclusion

In this paper, under the zero and nonzero boundary conditions, we derive the soliton solutions of the (2+1)-dimensional nonlocal nonlinear Schrödinger equation with time reversal. By applying the reduction method of KP hierarchy and the Hirota's bilinear method, we describe the concrete solutions in terms of $N \times N$ Gram-type determinants. With some parameter restrictions, we illustrate the regularity of solutions, two- and four-soliton solutions on constant and periodic backgrounds. For example, when $N = 1$, Fig. 3 presents the regular periodic solutions. We also consider the classical

Fig. 5: Four-soliton solutions on the constant backgrounds with parameter values $\xi = 0, \eta = 0, k = 1$ and (a) $p_1 = 3 + i, p_2 = 2 + i, p_3 = 3 - i, p_4 = 2 - i, q_1 = 3 - i, q_2 = 2 - i, q_3 = 3 + i, q_4 = 2 + i, c_1 = 1 + i, c_2 = 2 + i, c_3 = 1 + i, c_4 = 2 + i$, (b) $p_1 = 1 + i, p_2 = 2 + i, p_3 = 1 - i, p_4 = 2 - i, q_1 = 1 - i, q_2 = 2 - i, q_3 = 1 + i, q_4 = 2 + i, c_1 = -2 - i, c_2 = 3 + i, c_3 = -2 - i, c_4 = 3 + i$, (c) $p_1 = 1 + i, p_2 = 2 + i, p_3 = 1 - i, p_4 = 2 - i, q_1 = 1 - i, q_2 = 2 - i, q_3 = 1 + i, q_4 = 2 + i, c_1 = -2 - i, c_2 = -3 - i, c_3 = -2 - i, c_4 = -3 - i$, (d) $p_1 = 1 + i, p_2 = 2 - i, p_3 = 1 - i, p_4 = 2 + i, q_1 = 1 - i, q_2 = 2 + i, q_3 = 1 + i, q_4 = 2 - i, c_1 = -2 + 4i, c_2 = 3 - 2i, c_3 = -2 + 4i, c_4 = 3 - 2i$, (e) $p_1 = 1 + i, p_2 = 2 - i, p_3 = 1 - i, p_4 = 2 + i, q_1 = 1 - i, q_2 = 2 + i, q_3 = 1 + i, q_4 = 2 - i, c_1 = -1 + 3i, c_2 = -2 + i, c_3 = -1 + 3i, c_4 = -2 + i$.

dynamical behaviors of two-soliton solutions and the elasticity of the collisions of two-solitons. Typical two- and four-soliton solutions on different backgrounds can be referred to as Figs. 4 - 6. Finally, we conclude two dynamical behaviours of multi-soliton solutions. One is $N + 1$ types of N -soliton solutions on the constant background with even N ; another is the N -soliton solutions on the periodic background with odd one.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Fig. 6: Four-soliton solutions on the periodic backgrounds with parameter values $\xi = 0, \eta = 0, k = 1$ and (a) $p_1 = 3 + i, p_2 = 2 + i, p_3 = 3 - i, p_4 = 2 - i, p_5 = 3i, q_1 = 3 - i, q_2 = 2 - i, q_3 = 3 + i, q_4 = 2 + i, q_5 = 3i, c_1 = 1 + i, c_2 = 2 + i, c_3 = 1 + i, c_4 = 2 + i, c_5 = 3i$, (b) $p_1 = 1 - i, p_2 = 2 + i, p_3 = 1 - i, p_4 = 2 - i, p_5 = 3i, q_1 = 1 - i, q_2 = 2 - i, q_3 = 1 + i, q_4 = 2 + i, q_5 = 3i, c_1 = -2 - i, c_2 = 3 + i, c_3 = -2 - i, c_4 = 3 + i, c_5 = i$, (c) $p_1 = 1 + i, p_2 = 2 + i, p_3 = 1 - i, p_4 = 2 - i, p_5 = -i, q_1 = 1 - i, q_2 = 2 - i, q_3 = 1 + i, q_4 = 2 + i, q_5 = -i, c_1 = -2 - i, c_2 = -3 - i, c_3 = -2 - i, c_4 = -3 - i, c_5 = 3i$, (d) $p_1 = 1 + i, p_2 = 2 - i, p_3 = 1 - i, p_4 = 2 + i, p_5 = 3i, q_1 = 1 - i, q_2 = 2 + i, q_3 = 1 + i, q_4 = 2 - i, q_5 = 3i, c_1 = -2 + 4i, c_2 = 3 - 2i, c_3 = -2 + 4i, c_4 = 3 - 2i, c_5 = 5i$, (e) $p_1 = 1 + i, p_2 = 2 - i, p_3 = 1 - i, p_4 = 2 + i, p_5 = -5i, q_1 = 1 - i, q_2 = 2 + i, q_3 = 1 + i, q_4 = 2 - i, q_5 = -5i, c_1 = -1 + 3i, c_2 = -2 + i, c_3 = -1 + 3i, c_4 = -2 + i, c_5 = i$.

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Figures

Figure 1

One-soliton solution on zero boundary with parameter values $\eta_{10} = 0$, $p_1 = 3$, $q_1 = 3$.

Figure 2

Two-soliton solution on zero boundary with parameter values $\eta_{10} = 0$, $\eta_{20} = 0$, $p_1 = 1$, $p_2 = 2$, $q_1 = 1$, $q_2 = 2$.

Figure 3

Periodic solution with parameter values $\xi = 0, \eta = 0, k = 1, p_1 = i, q_1 = i, c_1 = 2 + i$.

Figure 4

Two-soliton solutions on the constant and periodic back-grounds with parameter values $\xi = 0, \eta = 0, k = 1$ and (a) $p_1 = 1 - i, p_2 = 1 + i, q_1 = 1 + i, q_2 = 1 - i, c_1 = i, c_2 = i$, (b) $p_1 = 2 - i, p_2 = 2 + i, q_1 = 2 + i, q_2 = 2 - i, c_1 = -1 + i, c_2 = -1 + i$, (c) $p_1 = 2 - i, p_2 = 2 + i, q_1 = 2 + i, q_2 = 2 - i, c_1 = -1 + i, c_2 = -1 + i$, (d) $p_1 = 2 + i, p_2 = 2 - i, p_3 = 2i, q_1 = 2 - i, q_2 = 2 - i, q_3 = 2i, c_1 = 5i, c_2 = 5i, c_3 = 4i$, (e) $p_1 = 5 + i, p_2 = 5 - i, p_3 = 2i, q_1 = 5 - i, q_2 = 5 +$

$i, q_3 = 2i, c_1 = 2 - i, c_2 = 2 - i, c_3 = -3i, (f) p_1 = 3 + i, p_2 = 3 - i, p_3 = 2i, q_1 = 3 - i, q_2 = 3 + i, q_3 = 2i, c_1 = -1 + i, c_2 = -1 + i, c_3 = i.$

Figure 5

Four-soliton solutions on the constant backgrounds with parameter values $\xi = 0, \eta = 0, k = 1$ and (a) $p_1 = 3 + i, p_2 = 2 + i, p_3 = 3 - i, p_4 = 2 - i, q_1 = 3 - i, q_2 = 2 - i, q_3 = 3 + i, q_4 = 2 + i, c_1 = 1 + i, c_2 = 2 + i, c_3 = 1 + i, c_4 = 2 + i$, (b) $p_1 = 1 + i, p_2 = 2 + i, p_3 = 1 - i, p_4 = 2 - i, q_1 = 1 - i, q_2 = 2 - i, q_3 = 1 + i, q_4 = 2 + i, c_1 = -2 - i, c_2 = 3 + i, c_3 = -2 - i, c_4 = 3 + i$, (c) $p_1 = 1 + i, p_2 = 2 + i, p_3 = 1 - i, p_4 = 2 - i, q_1 = 1 - i, q_2 = 2 - i, q_3 = 1 + i, q_4 = 2 + i, c_1 = -2 - i, c_2 = -3 - i, c_3 = -2 - i, c_4 = -3 - i$, (d) $p_1 = 1 + i, p_2 = 2 - i, p_3 = 1 - i, p_4 = 2 + i, q_1 = 1 - i, q_2 = 2$

$+i, q_3 = 1+i, q_4 = 2-i, c_1 = -2+4i, c_2 = 3-2i, c_3 = -2+4i, c_4 = 3-2i$, (e) $p_1 = 1+i, p_2 = 2-i, p_3 = 1-i, p_4 = 2+i, q_1 = 1-i, q_2 = 2+i, q_3 = 1+i, q_4 = 2-i, c_1 = -1+3i, c_2 = -2+i, c_3 = -1+3i, c_4 = -2+i$.

Figure 6

Four-soliton solutions on the periodic backgrounds with parameter values $\xi = 0, \eta = 0, k = 1$ and (a) $p_1 = 3+i, p_2 = 2+i, p_3 = 3-i, p_4 = 2-i, p_5 = 3i, q_1 = 3-i, q_2 = 2-i, q_3 = 3+i, q_4 = 2+i, q_5 = 3i, c_1 = 1+i, c_2 = 2+i, c_3 = 1+i, c_4 = 2+i, c_5 = 3i$, (b) $p_1 = 1-i, p_2 = 2+i, p_3 = 1-i, p_4 = 2-i, p_5 = 3i, q_1 = 1-i, q_2 = 2-i, q_3 = 1+i, q_4 = 2+i, q_5 = 3i, c_1 = -2-i, c_2 = 3+i, c_3 = -2-i, c_4 = 3+i, c_5 = i$, (c) $p_1 = 1+i, p_2 = 2+i, p_3 = 1-i, p_4 = 2-i, p_5 = -i, q_1 = 1-i, q_2 = 2-i, q_3 = 1+i, q_4 = 2+i, q_5 = -i, c_1 = -2-i, c_2 = -3-i, c_3 = -2-i, c_4 = -3-i, c_5 = 3i$, (d)

$p_1 = 1 + i, p_2 = 2 - i, p_3 = 1 - i, p_4 = 2 + i, p_5 = 3i, q_1 = 1 - i, q_2 = 2 + i, q_3 = 1 + i, q_4 = 2 - i, q_5 = 3i, c_1 = -2 + 4i, c_2 = 3 - 2i, c_3 = -2 + 4i, c_4 = 3 - 2i, c_5 = 5i, (e) p_1 = 1 + i, p_2 = 2 - i, p_3 = 1 - i, p_4 = 2 + i, p_5 = -5i, q_1 = 1 - i, q_2 = 2 + i, q_3 = 1 + i, q_4 = 2 - i, q_5 = -5i, c_1 = -1 + 3i, c_2 = -2 + i, c_3 = -1 + 3i, c_4 = -2 + i, c_5 = i..$