# Generalization of Quartic and Quintic Calabi – Yau Manifolds Fibered by Polarized K3 Surfaces

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**Research Article**

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# Generalization of Quartic and Quintic Calabi – Yau Manifolds Fibered by Polarized K3 Surfaces

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ABSTRACT – I will present Calabi-Yau manifolds emphasizing quartic and quintic variety considering Bogomolov-Gieseker Inequality with two specific types of Clifford inequality for associated fibers by Polarized K3 surfaces with the case considered upon threefolds.

Generalizations have been made to a different form of quintics with coordinates and associated mirrors for the hypersurface being concerned with taking multi-homogeneous polynomials where necessary.

Minimal Calabi-Yau threefold considered for fiber varieties $I_0, I_+, II_0, II_+, III, 0$.

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# 1. INTRODUCTION

In the complex dimension one Calabi-Yau manifolds, there is the existence of elliptic curves for a single topological type but the hint of K3 surfaces can be dominant and found in the complex dimension two being a quartic hypersurface in $\mathbb{P}^3$. These structures having the maximal Picard number for the existence of those K3 surfaces where the transcendental lattice holds for the degree 2 computations can be made through Hasse-Weil Zeta functions provided the K3 is singular. For a Galois representation of weight 3 for the 2D – There exists a weight 4 modular form in the L-series for all rigid CY threefold in the middle cohomology. This indeed suffices for 3D rigid CY manifolds. For the middle cohomology, there exists an L-series of weight $\mathcal{G}_6$ in the case of a 5D CY manifold. Any $n$-dimensional provided that $n$ is of much higher variety there is,

[1] A Kummer construction for an elliptic cone $e^0$ for automorphisms.

[2] This $e^0$ relates to group ^ in quotient form $(E \times \ldots\ldots \times E)/\, \mathbb{Z}_3^n \ or\ \mathbb{Z}_4^n$.

Originating from the Calabi conjecture a compact K$\ddot{a}$hler manifold for $(S, g, \omega)$ where there lies a unique metric $g_c$ with K$\ddot{a}$hler form $\widetilde{\omega}$ one can find the Ricci $R = Ric(\widetilde{\omega})\, \exists [\omega] \equiv [\widetilde{\omega}]$ for the first vanishing Chern class $c_1$ of $S^{\wedge}$ with the $R \in \Omega^{1,1}(S)$ for the $c_1$ taking $c_1(T_S) \in H^{1,1}(S)$ representing the canonical line bundle $\Lambda^{\mathrm{n}}_{\mathrm{T}_S^\star} \cong \sigma_S$ giving a CY n-fold with a holomorphic n-form and a covariantly constant spinor over $S$ [1-3]. The Hodge diamond of a simply connected and compact CY is shown as[12,43],

$$
\begin{array}{ccccccccccc}
 & & & 1 & & & & & & & \\
 & & 0 & & 0 & & & & b_0 & 1 \\
 & 0 & & & & 0 & & & b_1 & 0 \\
 & & & h^{1,1} & & & & & b_2 & h^{1,1} \\
1 & & h^{2,1} & & h^{2,1} & & 1 & \longrightarrow & b_3 & 2+2h^{2,1} \\
 & & & h^{1,1} & & & & & b_4 & h^{1,1} \\
 & 0 & & & & 0 & & & b_5 & 0 \\
 & & 0 & & & & & & b_6 & 1 \\
 & & & & 0 & & & & & \\
 & & & 1 & & & & & &
\end{array}
$$

Where the compact and connected form of CY can be represented easily by $b_0 = 1$ which assets clearly that the top of the Hodge diamond is $h^{0,0} = 1$, so as the same for the bottom with $h^{1,0}$ and $h^{0,1}$ is $0$. Therefore from this diamond one can deduce two non-trivial relations of manifold $S$ in general terms,

[1] $h^{1,1}$ represents the K$\ddot{a}$hler parameters.

[2] $h^{2,1}$ represents the complex structure parameters

The Euler characteristic of CY threefold, the quintic stated as $\mathcal{X}_E = -200$ which can be explicitly defined by the Hodge numbers $h^{1,1}$, $h^{2,1}$ through the relation $h^{1,1} - h^{2,1} = -100$ which gives the value of $\mathcal{X}_E$ for $2h^{1,1}$ and $2h^{2,1}$ as seen in the diamond giving $\mathcal{X}_E = 2(h^{1,1} - h^{2,1}) = -200$. Furthermore the K$\ddot{a}$hler $(K, g, \omega)$ where there is the vanishing Ricci $R(g, \omega)$ is the compact K$\ddot{a}$hler or Calabi-Yau $(S, g, \omega)$ but the K$\ddot{a}$hler has a global holonomy for the nowhere vanishing n-form suffice $SU(n)$ for $\Omega^{(\mathrm{n},0)}$ where in $n = 3$ one gets the spinor $\Theta$ defined via $\Omega^{(3,0)}$ for the manifold $\mathcal{M}$ in K$\ddot{a}$hler $K_{\mathcal{M}}$ gives two relations for $n$ and $n = 3$[43],

$$\mathcal{M} \equiv \begin{cases} \text{cotangent bundle (sheaf) in n} - \text{fold}, & \bigwedge^{n} T_{\mathcal{M}}^{\vee} \\ \Omega^{(3,0)} \text{ and } \Omega_{pqr} \text{ for spinor } \Theta \text{ with } \gamma - \text{matrics}, & \begin{cases} \frac{1}{3!}\Omega_{pqr}\, dz^p \wedge dz^q \wedge dz^r \\ \Theta_{-}^{T}\ \gamma^{[p}\gamma^{q}\gamma^{r]}\ \Theta_{-}^{T} \end{cases} \end{cases}$$

Thus establishing a relation for manifold category with associated holonomy,

| $Holonomy$ | $Manifold$ |
|---|---|
| $K\ddot{a}hler$ | $U(d/2)$ |
| $Calabi-Yau$ | $SU(d/2)$ |

## 2. CONSTRUCTIONS

There exists a one-to-one diffeomorphism on the K3 surfaces. For the vacuum solutions of the Einstein equation there exists a Ricci flat metric over $S$ for the case being $R = 0$ where for the compact K$\ddot{a}$hler cases there exists a SU($n$) holonomy for the CY($n$)-fold. For the complete intersection cases in Calabi-Yau manifolds (CICY) in the case of a CY 3-fold for the CY being the K$\ddot{a}$hler manifold for $(S, g, \omega)$ when one recognizes a projective variety $S_X$ for manifold being considered as $X$ - There exist 4 important conditions[1-8],

[1] For the multi-homogeneous polynomial $k$, the complete intersection holds for,

$$k = \sum_{i=1}^{m} n_i - 3$$

[2] For the multi-degree $q_j^i$ for $i = 1$ and $j = 1, \dots\dots, m$ the generalization of adjunction holds,

$$n_{j+1} = \sum_{i=1}^{k} q_j^i \ \forall j = 1, \dots\dots, m$$

[3] The associated Euler charectaristics being $\mathcal{X} = [-200,0]$ with Hodge pairs $(h^{1,1}, h^{2,1})$.

[4] There exists an elliptic fibration for concerned morphism on the CY threefold where for the base $b$ the fibration is defined over,

$$\pi: S \longrightarrow b$$

With $S$ being the CY threefold where the fibers can be distinguished over $\mathbb{P}^2$ through a smooth cubic equation,

$$y_0^3 - y_1^3 = (y_0 - y_1)(y_0^2 + y_0 y_1 + y_1^2)\ \exists (1{:}-1{:}\,0) \in \mathbb{P}^2$$

$$\forall the\ bi-homogeneous\ deg(1,3)\ where\ there\ exists\ S\ \subset \mathbb{P}^2 \times \mathbb{P}^2$$

## 3. POLARIZED K3 SURFACE

Going back to the fibrations of the smooth curve $b$ and smooth projective therefore $S - \pi: S \longrightarrow b$, there exists a Néron-Severi group $NS(\sigma_{\mathcal{F}})$ for the fiber of the manifold $S$ having point $\mathcal{F} \in b$ where there is a choice of $M_2$ over $M_1$ being taken here as explained[9,11,13],

[1] K3 surface having $M_1 -$ polarized family there is a need to control the monodromy around the single fibers of $S$ for an extreme need of homological invariant.

[2] In the $M_2 -$ polarized family the functional invariants worked due to the back of automorphisms thus,

$$M_1; H \oplus E_8 \oplus E_8 \oplus \langle -2 \rangle$$

Is replaced by,

$$M_2 \cong NS(\sigma_{\mathcal{F}}) \coloneqq H \oplus E_8 \oplus E_8 \oplus E_8 \oplus \langle -4 \rangle$$

Thus, the K3 surface with a quasi-projective base $b_Q$ denoted by the L-lattice in the smooth projective variety[9,14],

$$\pi_{b_Q}: S_{b_Q} \longrightarrow b_Q \; \forall L - polarized\, family$$

Here, for the primitive sub-lattice of $\ell_p$ on general point $p$ of NS$\left(\sigma_p\right)$ being isomorphic of the L-lattice there exists the K3 fiber[10-14],

$$\ell_p \subseteq H^2\left(\sigma_p, \mathbb{Z}\right)$$

Therefore, taking the K$\ddot{a}$hler metric $g_c$ there is a generalized invariant map of Manifold $\mathcal{M}$ for every point $p \in b_Q$ the associated map is,

$$\pi_\ell: b_Q \longrightarrow \mathcal{M}_{M_2} \; \forall M_2 \; - polarized\, K3\, surface$$

$$\text{for Hodge h}^{2,1} \text{ in CY threefolds of toric varieties}$$

This suffices an interesting point for Quartic K3 surfaces in $M_2$ – polarized family where for the associated compact K$\ddot{a}$hler there is still not any complete intersection for all the toric varieties. Further generalizations can be made by taking the same generalized invariant map $\pi_\ell$ for manifold $\mathcal{M}$ where $\ell$ can have a degree,

$$deg(\ell) = i + j \equiv n \; \exists i, j \in \{1,2,3,4\}$$

For this $deg(\ell)$ there exists unitarity in the Hodge $h^{2,1}$ for its association with $(S_\ell)$ where $S$ is threefold for $\ell$ appearing in the generalized invariant map $\pi_\ell$ where 3 relations can be found,

$$\pi_\ell: \mathbb{P}^1 \longrightarrow \pi_{M_2} \simeq \begin{cases} h^{2,1}(S_\ell) = k = r = 1 \\ h^{2,1}(S_\ell) = k = 1 = 20 + (2y^2 + 1) + c_1 + c_2 = |0| \\ \exists there\, is\, an\, unramified\, field\, \gamma \in \{0, 256^{-1}, \infty\} \end{cases}$$

## 4. INEQUALITY AND POLYNOMIALS WITH STABILITY

The stability condition on the CY threefold can be presented by Bogomolov-Gieseker inequality of Type $B_{2,4}$ and $B_{2,2,4}$ where the former describes the stability of the concerned objects. This when suffice to the Quadrics of $\mathbb{P}^5$ of order 3 along with a Clifford inequality of Type $C_{2,2,2,4}$ then a Quartic intersection can be found while for the Clifford inequality $C_{2,2,5}$ one can get a quintic variety[15,16].

For the Quintic threefold the Clifford inequality of $C_{2,2,5}$ also proceed towards a stronger Bogomolov-Gieseker inequality for a Quintic threefold $\mathbb{P}^4$. Taking a sheaf $S^{\wedge}$ on a smooth quintic threefold $(S^{\wedge}, H)$ for a related torsion $\rho$ over a $\rho -$ free slope $\alpha_H$ making $S^{\wedge}$ semistable there is a stronger bond on the second Chern class $c_2$ provided there lies the inequality of the $\rho -$ free for the same sheaf of threefold $(S^{\wedge}, H)$ such that[15,16],

$$H\,\mathrm{ch}_2(\rho) \le \begin{cases} -\frac{1}{2}|H^2\,\mathrm{ch}_1(\rho)| & \exists \left|\frac{H^2\,\mathrm{ch}_1(\rho)}{H^3\,\mathrm{rk}(\rho)}\right| \in \left[0, \frac{1}{4}\right] \\ \frac{1}{2}|H^2\,\mathrm{ch}_1(\rho)| - \frac{5}{4}rk(E) & \exists \left|\frac{H^2\,\mathrm{ch}_1(\rho)}{H^3\,\mathrm{rk}(\rho)}\right| \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ \frac{3}{2}|H^2\,\mathrm{ch}_1(\rho)| - 5rk(E) & \exists \left|\frac{H^2\,\mathrm{ch}_1(\rho)}{H^3\,\mathrm{rk}(\rho)}\right| \in \left[\frac{3}{4}, 1\right] \quad -H^2\,\mathrm{ch}_1(\rho) \end{cases}$$

Where $\left|\frac{H^2\,\mathrm{ch}_1(\rho)}{H^3\,\mathrm{rk}(\rho)}\right| \in \left[\frac{3}{4}, 1\right] - H^2\,\mathrm{ch}_1(\rho)$ represents the strong bound. For the Clifford curve $C_{2,2,5}$ of rank $r$ and slope $\alpha_H$ of curve $C$ there are complete intersections for both [Quadratic and Quintic] hypersurface in $\mathbb{P}^4$ where the bounds are expressed by slope $\alpha_H$ takes the inequality quotient of $\frac{h^0(\rho)}{r} \le \alpha_H^{-2} + 1$ where for the slope $\alpha_H \in [5,10]$ and $\alpha_H \in [2,5)$ there exists a net bound for the relation[16],

$$\begin{cases} \frac{40}{41} + \frac{\alpha}{41}, & \exists\alpha \quad \in (0,2) \\ max\left\{\frac{24}{55} + \frac{4}{125}\alpha, \frac{241}{248} + \frac{33}{1240}\alpha\right\}, & \exists\alpha \quad \in \left[2, \frac{5}{2}\right) \\ max\left\{\frac{12}{13} + \frac{3}{65}\alpha, \frac{193}{204} + \frac{7}{170}\alpha\right\}, & \exists\alpha \quad \in \left[\frac{5}{2}, \frac{10}{3}\right] \\ max\left\{\frac{4}{5} + \frac{2}{25}\alpha, \frac{193}{204} + \frac{7}{170}\alpha\right\}, & \exists\alpha \quad \in [10,5) \quad \forall\alpha \in (0,10] \cup [30,40] \\ max\left\{\frac{1}{5}\alpha, \frac{55}{62} + \frac{9}{124}\alpha\right\}, & \exists\alpha \quad \in [5,10] \\ \frac{2}{5}\alpha - 4, & \exists\alpha \quad \in [30,37] \\ \frac{11}{15}\alpha - \frac{49}{3}, & \exists\alpha \quad \in [37,40] \end{cases}$$

Satisfying the Bridgeland Stability Conditions on Bogomolov-Gieseker inequality Type $B_{2,4}$ in CICY for $\rho^{\times}$ such that $ch_0(\rho) \neq 0$ for $H = N_{B_{2,4(1)}}$ there is an $\alpha_H^0$ semistable slope for the Clifford variety being considered $C_{2,2,2,4}$ for the Bogomolov inequality considered here for sheaf $\rho^{\times}$ (*ignoring the notational ambiguity*) proof can be found in $B_{2,4} \subset \mathbb{P}^5$ representing complete intersections through [Quadratic and Quartic] over CY threefolds[20-22].

If any smooth projective threefold $S^{\wedge}$ over $\mathbb{C}$ be defined on the ample $\mathcal{A}$ taking the base $b$ with $\omega$ from the compact Kähler manifold $(S, g, \omega)$then for the bounded structure there exists $H^2(S^{\wedge}, \mathbb{C})$ for every $\mathcal{A}_{b,\omega} \subset Coh(S)$ where for the Picard norm one might able to find $H$ through a generator $\mu_S(H)$ for the dimensions $dim|H| \geq \frac{7}{6}(H^3 - 3)$ making equality with $E = \mu_S(H)$[17-19].

The smooth quintic threefold $S \subset \mathbb{P}^4$ can be taken through the first Chern class $c_1$ over the dimensions $c_1(E) = |H|$ where there are polynomials $ch_0(E).ch_1(E), ch_2(E)$ bounds the $ch_3(E)$ for the sheaf rank $r_S \equiv E \geq 2$. The bounding over $ch_3(E)$ for $E \in coh(S)$ makes the torsion $(\rho) -$ free slope over the extension of[18,21],

$$\mathcal{O} \longrightarrow E \times T^1(E, \mu_S)^{\vee} \otimes_{\mathbb{C}} \mu_S \longrightarrow E' - E \longrightarrow \mathcal{O}$$

For a stable slope taking $(S, E) \subset \mathbb{P}^4$.

Considering the fact of the Kodaira vanishing $h^{i>0}\big(\mu_S(H)\big) = 0$ the existence of Riemann-Roch can be obtained for the bound Chern character $ch_3(E)$ for $dim|H|$ one gets[17-22],

$$\sum_{i \geq 0} (-1)^i h^i(E) = ch_3(E) + dim|H| - \frac{1}{6}(H^{3+1})$$

Giving the Bogomolev-Gieseker inequality through an equivalence for $ch_i(E)$ for $i = 0,1,2$ suffice,

$$ext^1(\rho^{\times}, \mu_S) \leq \frac{H^3}{2ch_2(E)H} - ch_0(\rho^{\times}) \equiv \Big(ext^1(\rho^{\times}, \mu_S) + H^2 2\big(ch_0(\rho^{\times})\big)ch_2(E)\Big)H \geq 0$$

Where for even $dim|H|$ suffice the old relation once more in CICY,

$$B_{2,4} \subset \mathbb{P}^5 \;\forall \frac{2}{3}(H^3 - 3)$$

Therefore, one gets a detailed relation for the Chern character $ch_i$ for $i = 0,1,2$ the $\mu_{B2,4}$ to suffice for all $H^J$ for $J = 2$[23,24],

$$\frac{ch_2(\rho^{\times})}{H^2\, ch_0(\rho^{\times})} \leq \begin{cases} \dfrac{H\, ch_1(\rho^{\times})}{H^2\, ch_0(\rho^{\times})} - \dfrac{H\, ch_1(\rho^{\times})}{H^2\, ch_0(\rho^{\times})} & \exists \dfrac{H\, ch_1(\rho^{\times})}{H^2\, ch_0(\rho^{\times})} \in \left[0, \dfrac{4}{3} - \dfrac{\sqrt{13}}{3}\right] \\ \dfrac{5}{8}\left(\dfrac{H\, ch_1(\rho^{\times})}{H^2\, ch_0(\rho^{\times})}\right)^2 - \dfrac{1}{8} & \exists \dfrac{H\, ch_1(\rho^{\times})}{H^2\, ch_0(\rho^{\times})} \in \left[\dfrac{4}{3} - \dfrac{\sqrt{13}}{3}, \dfrac{1}{2}\right] \\ \dfrac{5}{8}\left(\dfrac{H\, ch_1(\rho^{\times})}{H^2\, ch_0(\rho^{\times})}\right)^2 - \dfrac{1}{4}\dfrac{H\, ch_1(\rho^{\times})}{H^2\, ch_0(\rho^{\times})} & \exists \dfrac{H\, ch_1(\rho^{\times})}{H^2\, ch_0(\rho^{\times})} \in \left[\dfrac{1}{2}, \dfrac{\sqrt{13}}{3} - \dfrac{1}{3}\right] \\ \left(\dfrac{H\, ch_1(\rho^{\times})}{H^2\, ch_0(\rho^{\times})}\right)^2 - \dfrac{1}{2} & \exists \dfrac{H\, ch_1(\rho^{\times})}{H^2\, ch_0(\rho^{\times})} \in \left[\dfrac{\sqrt{13}}{3} - \dfrac{1}{3}, 1\right] \end{cases}$$

For the condition $\rho^{\times}(-nH) \forall n \in \mathbb{Z}$ iff $\frac{H\, ch_1\left(\rho^{\times}(-nH)\right)}{H^2\, rk\left(\rho^{\times}(-nH)\right)} \in [0,1)$. Hence the Bill-Noether semistable relation for $ch\left(B_{2,4}\right)$ the equality will hold for $\left|\frac{H^2\, ch_1(\rho^{\times})}{H^3\, rk\ (\rho^{\times})}\right| =$ taking the value $0, \frac{1}{5}, \frac{1}{2}, \frac{4}{5}, \frac{1}{4}, \frac{10}{11}, 1$ where for the interval of $\frac{H^2\, ch_1(\rho^{\times})}{H^3\, rk\ (\rho^{\times})} \in [-1,1]$ there is[23,24],

$$\frac{H\, ch_2(\rho^{\times})}{H^3\, rk\ (\rho^{\times})} \leq \begin{cases} -\frac{1}{2}\left|\frac{H\, ch_2(\rho^{\times})}{H^3\, rk\ (\rho^{\times})}\right| & where \quad \left|\frac{H\, ch_2(\rho^{\times})}{H^3\, rk\ (\rho^{\times})}\right| \in \left[0, \frac{1}{5}\right] \\ \frac{7}{16}\left|\frac{H\, ch_2(\rho^{\times})}{H^3\, rk\ (\rho^{\times})}\right| - \frac{3}{16} & where \quad \left|\frac{H\, ch_2(\rho^{\times})}{H^3\, rk\ (\rho^{\times})}\right| \in \left[\frac{1}{5}, \frac{1}{2}\right] \\ \frac{9}{16}\left|\frac{H\, ch_2(\rho^{\times})}{H^3\, rk\ (\rho^{\times})}\right| - \frac{1}{4} & where \quad \left|\frac{H\, ch_2(\rho^{\times})}{H^3\, rk\ (\rho^{\times})}\right| \in \left[\frac{1}{2}, \frac{4}{5}\right] \\ \frac{51}{54}\left|\frac{H\, ch_2(\rho^{\times})}{H^3\, rk\ (\rho^{\times})}\right| - \frac{8}{11} & where \quad \left|\frac{H\, ch_2(\rho^{\times})}{H^3\, rk\ (\rho^{\times})}\right| \in \left[\frac{4}{5}, \frac{10}{11}\right] \\ \frac{21}{11}\left|\frac{H\, ch_2(\rho^{\times})}{H^3\, rk\ (\rho^{\times})}\right| - \frac{31}{32} & where \quad \left|\frac{H\, ch_2(\rho^{\times})}{H^3\, rk\ (\rho^{\times})}\right| \in \left[\frac{10}{11}, 1\right] \end{cases}$$

Other varieties of CICY threefolds for $\mathbb{P}^N$ when $N$ is large the Bogomolov-Gieseker inequality Type can be altered for the value of $H\frac{ch_1}{rk}$ over Clifford curve $C_{2,2,5}$ being replaced by $|2H_Y|\ \exists Y \in |2H|$ for a smooth variety to generalize CY threefolds.

## 5. COORDINATES WITH GENERALIZATIONS AND MIRROR

If we consider the Ricci flat metric that is already employed on the CY manifold where for sheaves $S$ with the metric class $(S, g, \omega)$ when we take the $n$-fold say $S^n$ for the ample $\mathcal{A}$ being considered here before with the form $\mathcal{A}_{b,\omega}$ there is a line bundle $\mathcal{L}$ for the $m^{th}$ power being the holomorphic line bundle as $\mathcal{L}^m$ which with the $n$-fold considered here $S^n$ makes the K$\ddot{a}$hler potential $K$ for $p-$ parameters to make the Ricci flat metric where the relation takes as[25-27],

$$K_{m,p} = \frac{1}{m\pi} In\ \left(\bar{\epsilon}_{\bar{\imath}}{}^{m^{\bar{\imath}j}} \epsilon_j\right)$$

Where $\epsilon_j \in (S, \mathcal{L}^m)$ where for $p \longrightarrow \infty$ the vanishing Ricci ot the Ricci flat would be prominent as the limit of $p$ tends to infinity.

For this same $n$-fold there is the same degree 5 quintic hypersurface of projective variety $\mathbb{P}^4$ where for the polydiscs $d_i$ the ambient homogeneous coordinates of the $n$-fold $\mathbb{P}^{n+1}$ for the vanishing locus $\ell^0$ with the polynomial $k$, the $\ell^0_k$ one gets a $n+2$ degree form of the $S^n$ represented through a bilinear relation taking the polydiscs $d_i$ being amalgamated with the $S^n$ for the same ambient coordinates of $\mathbb{P}^4$ being taken here as $\mathbb{P}^{n+1}$ in $S^n$ suffice[26,28],

$$\left[\bigcup_{i=0}^{N} S_j^n \middle| \, n+2\right] \equiv \left[\bigcup_{i=0}^{N} d_i = \mathbb{P}^{n+1}\right]$$

For the local coordinates of $d_i$ represented through an equivalence for the previously considered ambient homogeneous coordinates $\mathbb{P}^{n+1}$,

$$\{\mathcal{K}_i\}, i = 0, \dots \dots, n+1$$

With the relation of equivalence,

$$d_i = \{[z_0 : \dots \dots : z_{i-1} : z_{i+1} : \dots \dots : z_N] | |z_i| \leq 1\} \simeq d^{n+1}$$

For the suitable coordinates of Calabi-Yau on $S_j^n$ for any value of $n$ (being generalized for this purpose) with the polynomial $k$, for $m = 0, \dots \dots, 0+1$ in the relation $z_m = \frac{\mathcal{K}_m}{\mathcal{K}_i}$ expressed through[28,29],

$$\mathcal{K}(z_0, \dots \dots, z_{i-1}, z_{i+1}, \dots \dots, z_N) = 0$$

For the $\mathbb{CP}^{n+1}$ there is a hypersurface of the smooth toric varieties $S$ which can only be recognized as a CY for $d = n+2$ where the ambient homogeneous coordinates $\mathbb{P}^{n+1}$ there exists a relation[30],

$$\mathcal{O}_{\mathbb{P}^{n+1}}\big(-(n+2)\big)|_S \cong \Omega_{\mathrm{S}}^{\mathrm{n}} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(-d)|_{\mathrm{S}} \implies \Omega_{\mathrm{S}}^{\mathrm{n}} \cong \mathcal{O}$$

Such that considering $\mathbb{CP}^{n+1}$ for rank $r < n = 3$, there is $H_r(S) \tilde{\rightarrow} H_r(\mathbb{CP}^4)$ for the quintic $\mathbb{P}^4$ as the non-trivial CY threefold where for $h^{1,0} = 0$ and $h^{2,0} = 0$ for the argument $h^{1,1} = 1$ a conclusion can be taken for the Euler charecteristics $\mathcal{X}$ through the relation[33,35],

$$1 + 1 - \deg H_3(S) + 1 + 1 = \mathcal{X} \, \exists \mathcal{X} = -200$$

For the quintic threefolds of CY variety having sheave $S$ in $\mathbb{P}^4$ for the generic variety $\theta$ the mirror can be constructed via singular $S_\theta/G$ for the relation[31-33],

$$\frac{S_\theta \equiv \{y_0, \dots \dots, y_4\} \in \mathbb{P}^4 | G_\theta^{inv} = y_0^5 + _{\dots\dots} + y_4^5 - 5\{y_0, y_1, y_2, y_3, y_4 = 0\}}{G \equiv \{m_0 \dots \dots m_4\} \in (\mathbb{Z}/5\mathbb{Z})^5 \rfloor \sum m_j = 0 \ / \ (\mathbb{Z}/5\mathbb{Z})^5 = \{(m, m, m, m, m)\}}$$

Thus for $S_\theta/G$ where $G \cong (\mathbb{Z}/5\mathbb{Z})^3$ there are $10 - curves$ meeting at point $\tilde{p}$,

$$\tilde{p}_{1,2,3} \begin{cases} \widehat{curve}_{\,1,2} & \exists \widehat{curve}_{\,1,2} \quad = \quad \widehat{curve}_{\,1,2} / (\mathbb{Z}/5\mathbb{Z})^3 \cong \mathbb{P}^1 \\ \widehat{curve}_{\,2,3} \\ \widehat{curve}_{\,3,1} \end{cases}$$

Thus for the $curve_{1,2}$ the canonical bundle $\eta$ takes the complete construction of $\eta_{S_\theta^\vee} = \pi^\star \eta_{S_{\vee/G}}$ for the smooth $S_\theta^\vee$ taking $curve_{1,2}$ over $\frac{1}{\pi}\big(\cup \, curve_{1,2}\big)$ being the result of an outside isomorphism satisfying the map[32,35],

$$\Psi : S_\theta^\vee \xrightarrow{\pi} S_{\theta/G}$$

Making a complete construction of a Calabi-Yau threefold through $S_\theta^\vee$.

As mentioned earlier about the quintic $\mathbb{P}^4$ with the group $G$ (not to be confused with the group notation ^ at the very beginning of the paper; this has been taken for generalization and to remove confusion while considering the mirror and Schoen's quintic of standard family of curves) where the coordinate action for $S_{\bar{\wedge}}\mathbb{C}\mathbb{P}^4$ with $\cdot_{\bar{\wedge}}$ as the $5^{th}$ root of unity the action $G \simeq (\mathbb{Z}/5\mathbb{Z})^3$ gives the $zeta\, function$ for$S_{\bar{\wedge}}$ through the transformation in $\xi_5$ for the same $5^{th}$ root with the quotient $S_{\bar{\wedge}}/G$ over[33,35],

$$\{y_0 : y_1 \cdot \xi_5^{\psi_1} : y_2 \cdot \xi_5^{\psi_2} : y_3 \cdot \xi_5^{\psi_3} : y_4 \cdot \xi_5^{\psi_4}\}$$

For $\psi_1 + \psi_2 + \psi_3 + \psi_4 \equiv 0 \ mod\, 5$ the Euler characteristic of $\hat{S}$ being a small resolution of $S$ gives $-200 + 2 \cdot 125$ for $125\, nodes$ of $S_{\bar{\wedge}}$ having the corresponding points in the quadratic for the $mod\, 5$ in $p \equiv 4$ there is $I_p(\hat{S})$ denoting the modularity principle with the relation,

$$I_p(\hat{S}) \begin{cases} p^3 + 25, & p^2 + 100p + 1 - \#S_{p,} \quad p \equiv 1 \quad mod\, 5 \\ p^3, & p^2 + 1 - \#S_{p,} \quad p \equiv 4 \quad mod\, 5 \\ p^3, & p^2 + 2p + 1 - \#S_{p,} \quad p \equiv 2,3 \quad mod\, 5 \end{cases}$$

For further generalizations to the quintic $\mathbb{P}^5$ over the relation $S^n \subset \mathbb{P}^5$ for $n$ (being generalized in any number being concerned with the association of projective variety) with the $zeta\, function$ taken before (again giving for clarity and relational significance)[32-35],

$$\{y_0 : y_1 \cdot \xi_5^{\psi_1} : y_2 \cdot \xi_5^{\psi_2} : y_3 \cdot \xi_5^{\psi_3} : y_4 \cdot \xi_5^{\psi_4}\} \longleftarrow (y_i) \text{ for } i = 0:1:2:3:4:5$$

One gets the Barth-Nieto Quintic,

$$\left\{\sum_{i=0}^{5} y_i = \sum_{i=0}^{5} \frac{1}{y_i} = 0\right\} \subset \mathbb{P}^5 \text{ for Barth} - \text{Nieto Quintic suffice } \mathcal{M}_{1,0}$$

## 6. NON-RIGID CALABI YAU THREEFOLD AND $M_\Delta$ POLARIZED K3 FIBRES

Considering a generalized functional invariant map here taken as $f$ and a quasi-projective curve $\mathbb{Q}$ concerning the CY manifold $S$ if one takes a moduli parameter for the sheave $\widetilde{M}$ then through a fiber $F$ with the relation to point $p$ there can be an implicit parameter $\widetilde{M}_p^F$ where for the polarized K3 fiber $M_\Delta \ \exists\ \Delta \geq 2$ the quasi − projective curve $\mathbb{Q}$ Neron-Sevari group for the isomorphism of the fiber $F^{\mathbb{Q}}$ to $M_\Delta$ in the map[38-40],

$$f : S^{\mathbb{Q}} \longrightarrow \widetilde{M}_{M_\Delta} \quad \Longrightarrow \quad \text{generalized functional invariant map for CY S}$$

$$L^{\mathbb{Q}} : S^{\mathbb{Q}} \longrightarrow \widetilde{M}_{M_\Delta} \quad \Longrightarrow \quad M_\Delta - \text{Polarized family of K3 for afiine } L \text{ with CY threefold } S$$

For the projective $\mathbb{P}^1$ with $\geq 2$ when the $M_\Delta - Polarized\, K3$ is considered for the inclusion set of $\Delta = \{2,3,4,5,6,7,8,9,10,11\}$ the generalized invariant map $f$ taking the form $f : \mathbb{P}^1 \longrightarrow \widetilde{M}_{M_\Delta}$ gives two orbifolds for the parameters $\beta$ for a ramification profile of $f$ takes the for the cusp $\beta = c$ and two orbifolds satisfying the ramification through the projection of the rank $r$ (as previously concerned throughout the paper) one gets a summation for the ramification index $R$ with point $p$ over $R_p$ $\exists p \in \mathbb{P}^1$ suffice[38-40],

$$\begin{matrix} \tilde{1} - orbifold & \longrightarrow & \Delta = \alpha \\ \tilde{2} - orbifold & \longrightarrow & \Delta = \Delta_1 \dots \dots \Delta_j \end{matrix} \quad \Rightarrow r \coloneqq \sum_{p \in \mathbb{P}^1} \left( R_p - 1 \right) \text{ for} \begin{cases} \Delta \in \{\alpha, c, \Delta_1 \dots \dots \Delta_j\} \\ \quad f(p) \notin \Delta \end{cases}$$

There can be evidence of $S_{\Delta,f}$ being smooth with a canonical sheaf when the recently resolved case is satisfied for the smooth curves resulting in the fibration for the singularities $\sigma$ in $S_{\Delta,f}$ for,

[1] $Exclusion\ elements\ with\ unramification - \Delta \in \{2,3,4,5,6,8,9,10\}$

[2] $Preimage\ for\ the\ concerned\ points\ for\ the\ singularities\ in\ S_{\Delta,f}$ $-$

| $\Delta$ | $\overline{\sigma},\ \tilde{\sigma}$ |
|---|---|
| 2 | $6 \times \overline{\sigma}_3$ |
| 3 | $3 \times \overline{\sigma}_1,\ 6 \times \overline{\sigma}_2$ |
| 4 | $12 \times \overline{\sigma}_1$ |
| 5 | $3 \times \overline{\sigma}_1,\ 3 \times \tilde{\sigma}_4$ |
| 6 | $3 \times \overline{\sigma}_1,\ 2 \times \overline{\sigma}_2,\ 2 \times \overline{\sigma}_3$ |
| 7 | $1 \times \overline{\sigma}_1,\ 3 \times \overline{\sigma}_2,\ 1 \times \overline{\sigma}_3,\ 1 \times \overline{\sigma}_4$ |
| 8 | $6 \times \overline{\sigma}_1,\ 3 \times \overline{\sigma}_2$ |
| 9 | $3 \times \overline{\sigma}_1,\ 3 \times \overline{\sigma}_2,\ 1 \times \tilde{\sigma}$ |
| 11 | $2 \times \overline{\sigma}_1,\ 1 \times \overline{\sigma}_2,\ 2 \times \overline{\sigma}_3$ |

The CY manifold having the Hodge numbers $h^{1,1}$ and $h^{2,1}$ we will denote a correction coefficient $\nabla$ where for the $S_{\Delta,f}$ $(with\ \Delta = correction\ coefficient\ \nabla = n)$ for the case being concerned here (again ignoring the notiational ambiguity where $\Delta$ appears before in this section with $M_\Delta$ being the degree of the polarized fiber family of K3) a table can be constructed for $h^{2,1}(S_n)|_\nabla$ with the associated Fricke involution[38,39,41,42],

| | | | | |
|---|---|---|---|---|
| $\Delta = 2$ | $\Delta = 3$ | $\Delta = 5$ | | |
| $[1,3], [2,3], [3,4], [5], [6], [8]$ | $[5]$ | $[3]\ or\ [4]$ | | |
| $\left[q_{1,\dots\dots,}q_l\right] \in [5,6,7,8]$ | $\left[q_{1,\dots\dots,}q_l\right] \in [5]$ | $\left[q_{1,\dots\dots,}q_l\right] \in [3,4]$ | | |
| $\left[q_{1,\dots\dots,}q_l\right] = \left[4, \tilde{1} - 4\right]$ | $\left[q_{1,\dots\dots,}q_l\right] = [2,3]$ | $\left[q_{1,\dots\dots,}q_l\right] = \left[2, \tilde{1} - 2\right]$ | | |
| $\Delta = 2$ | $\Delta = 3$ | $\Delta = 4$ | $\Delta \neq 4$ | $\Delta > 3$ |
| $\cup_2 \Longrightarrow order\ 2 - orbifold\ point$ | $\cup_3 \Longrightarrow order\ 3 - orbifold\ point$ | $\cup_4 \Longrightarrow order\ 4 - orbifold\ point$ | $cusp\ absent$ | $elliptic\ orbifold\ point\ absent$ |

$\forall Q - Partition\ cover\ \left[q_{1,\dots\dots,}q_l\right]$

$smooth$
$degeneration$

$\forall Fricke\ involution\ \cup_I$ for $I \in 2,3,4$

For the CY manifolds $S_n$ $or$ $S$[36-39] which in this section is considered as $S_{\Delta,f}$ over the choice of parameters of $n$ and $\Delta$ being trivial here the correction takes the measuring function for the moduli of $S_n$ where there can be the existence of a smooth FANO variety over the ramification index of $R_1$ and $R_2$ over Tyurin degeneration that gives the Quasi-FANO which smooths to a mirror of the CY manifold $S_n$ in the hypersurface of $degree \geq 2$. For the CY threefold a simply connected structure (being almost trivial as regards to the K3 fiber) where for $S_n{}^{\sim}$ the $\sim$ denoting the simply connected structure for the elliptic curve $e^0$ with the canonical form $\bar{\bar{S}}$ for the compact Kählerr $K$ in CY threefold admits a $II_e$ for $e = 0$ with other varieties[41-43],

$$K_{\bar{\bar{S}}} \begin{cases} II_e \quad \text{for } e = 0, + & \qquad \{ \; c_2 \\ I_+ \qquad \rightarrow \qquad K3 & \\ I_0 \qquad \rightarrow \qquad \text{Abelian} & \Big\} \; \mathbb{P}^1 \end{cases}$$

Considering the case of a minimal Calabi-Yau threefold as $S_m$ then the $\mathbb{Q}-$factorial projective complex satisfying the second Chern class $c_1$ in the following way $c_1(S_m) \neq 0$ there is the presense of terminal singularities for the Kählerr $K$ with the minimal CY variety $K_{S_m} \simeq \sigma_{S_m}$ the presence of algebraic fibers for $deg_D$ can be found in the hyperplane $\mathcal{H}$ where $\mathcal{H}_+ \ni \mathcal{H}$ and there exists a pull back of fibers from $\mathcal{H}_-$ from $\mathcal{H}_+$ where for every rational curve $q$ there exis the presence of an elliptic fibration of algebraic category $\pi_1^{\text{alg.}} = \{1\} \equiv h^1\left(\sigma_{K_{S_m}}\right) = 0$ with one singular fiber for non-Gorenstein $G_p \in \mathcal{H}_+$ one can find a morphism $\vartheta$ through a projection satisfying $1 - \textbf{non} - \textbf{trivial}$ attachments[39,41-43],

- Let $Q$ be a nef effective divisor of semi-ample $S_m$ for $|P^+Q|$ with the morphism $\vartheta$ establishing a relation with the non-Gorenstein $G_p$ with minimal CY threefold $S_m$ for the positive $P^+$ in $|P^+Q|$ over $\vartheta \neq \textbf{equi} - \textbf{dimensional}$ relating $dim_{\frac{1}{\vartheta}}(G_p) = 2$ with $\vartheta$ being surjective throughout; the connectivity of general fiber of $\vartheta$ is represented in semi-ample $S_m$ for the projective hypersurface $\mathbb{P}^{dim\,|P^+Q|}$ under the conditions $\vartheta_\star \sigma_{S_m} = \sigma_{G_p}$ for a normal $S_m$ and $\mathcal{H}_+$,

$$\vartheta := \vartheta_{|P^+Q|} : S_m \longrightarrow \mathcal{H}_{+_{inv \rightarrow res\ \tau : S_m' \rightarrow S_m \text{ of } S_m(S_m, G_p)\ and\ G_p \cdot c_2(S_m)}} \subset \mathbb{P}^{dim\,|P^+Q|} \begin{cases} \text{Type II}_0 \longrightarrow & dim_{\frac{1}{\vartheta}}(G_p \in \mathcal{H}_+) = 2 \\ \text{Type II}_+ \longrightarrow & \vartheta|_{S_m} : S_m \longrightarrow \mathcal{H} \end{cases}$$

Thus fiber types being properly classified as[12,19,42,43],

- $I_0 \Longrightarrow \vartheta : S_m \to \mathbb{P}^1\ \forall \text{fib}(\vartheta) \simeq \text{Abelian}\ \exists res\ \tau : S_m^{'} \longrightarrow$ $S_m$ of $S_m(S_m, G_p)$ $and$ $G_p \cdot c_2(S_m) \cong 1$ and $0$ resp.
- $I_+ \Longrightarrow \vartheta : S_m \to \mathbb{P}^1\ \forall \text{fib}(\vartheta) \simeq \text{K3}\ \exists res\ \tau : S_m^{'} \longrightarrow S_m$ of $S_m(S_m, G_p)$ $and$ $G_p \cdot$ $c_2(S_m) \cong 1$ and $+$ resp.
- $II_0 \Longrightarrow \text{fib}(\vartheta) \simeq \text{e}^0$ or elliptic where $\mathcal{H}_+ =$ rational $G_p \forall\ 1\ Quo(\sigma)\ for$ positive $\varepsilon$ in $K_{S_m}^{\mathcal{H}_+} \sim 0\ \exists res\ \tau : S_m^{'} \longrightarrow$ $S_m$ of $S_m(S_m, G_p)$ $and$ $G_p \cdot c_2(S_m) \cong 2$ and $0$ resp.
- $II_+ \Longrightarrow \text{fib}(\vartheta) \simeq \text{e}^0$ or elliptic $\exists \varepsilon K_{S_m}^{\mathcal{H}_+} \sim d_{cr} >$ $0$ where $cr$ is the Cartier divisor $\forall\ 1\ Quo(\sigma)$ in rational $\mathcal{H}_+$ for positive $\varepsilon$ in $K_{S_m}^{\mathcal{H}_+}$ $\exists res\ \tau :$ $S_m^{'} \longrightarrow S_m$ of $S_m(S_m, G_p)$ $and$ $G_p \cdot c_2(S_m) \cong 2$ and $+$ resp.
- $III \Longrightarrow \vartheta \simeq birational\ \forall threefold\ \mathcal{H}_+\ in\ Can_{I=1_{K_{S_m}^{\mathcal{H}_+} \sim 0}}\ \exists res\ \tau : S_m^{'} \longrightarrow$ $S_m$ of $S_m(S_m, G_p)$ $and$ $G_p \cdot c_2(S_m) \cong 3$ and $\geq 0$ resp.
- $0 \Longrightarrow G_P \simeq \sigma_{K_{S_m}} : S_m \to \{1\}\ \exists res\ \tau : S_m^{'} \longrightarrow S_m$ of $S_m(S_m, G_p)$ $and$ $G_p \cdot$ $c_2(S_m) \cong 0$

## 7. DISCUSSIONS

For the CY manifolds emphasizing threefolds when the existence of quartic hypersurface is found in complex dimension 2 under projective $\mathbb{P}^3$ there can be found the existence of Hasee-Weil zeta function priming the category of singular K3. That CY when goes through the vacuum solutions of Einstein equations with a Ricci flat metric or vanishing Ricci additionally being compact then the transformation occurs from that K$\ddot{a}$hler to CY manifold. That K$\ddot{a}$hler for the specified metric class can be extended to CY$(n)$-folds through the existence of SU$(n)$ holonomy. CY being the CI through the multi-homogeneous polynomial is essential for a further moderate approach to prove the results of being quadratic and quartic also quadratic and quintic being the sole purpose of this paper over a bi-homogeneous degree (1,3) when the concern sheaf $S$ of the structure metric-class $(S, g, \omega)$ is existent through $S \subset \mathbb{P}^2 \times \mathbb{P}^2$. This being computed there opens up a proper channel through the N$\acute{e}$ron-Severi group where the $M_2 -$ polarized domain is much more convenient and mathematically non-trivial for the computation of K3 fibers for the same sheaf $S$ through a quasi-projective base in an L-polarized family for the equivalence of that $M_2$ with the N$\acute{e}$ron-Severi group.

Summing over to the Bogomolov-Gieseker inequality of Type $B_{2,4}$ and $B_{2,2,4}$ one can get a quartic intersection for the Clifford inequality of Type $C_{2,2,2,4}$ while the quintic variety for Type $C_{2,2,5}$ (other inequality Types are not considered in this paper as the relationship can be easily chalked out over these two inequalities of Clifford Types). In the CICY for a $(\rho)-$ Torsion free a strong bound can be observed in the relation $\left|\frac{H^2\,\text{ch}_1(\rho)}{H^3\,\text{rk}(\rho)}\right| \in \left[\frac{3}{4}, 1\right] - H^2\,\text{ch}_1(\rho)$ in projective hypersurface $\mathbb{P}^4$ sufficing the quadratic and quintic CICY for the Clifford curve $C$ over the second Chern class $c_2$ making a stronger bound. Then moving on to the projective hypersurface $\mathbb{P}^5$ in the Bogomolov-Gieseker inequality $B_{2,4}$ an existence of threefolds can be found with the Clifford Type $C_{2,2,2,4}$ suffice a nice relation $B_{2,4} \subset \mathbb{P}^5$.

For the first Chern class $c_1$ with the bounded Polynomial $ch_3(E)$ a stable slope can be taken in sheaf $S$ for the quintic variety $\mathbb{P}^4$ giving the threefold by considering the Kodaira vanishing through the existence of Riemann-Roch for the bounded Chern character $ch_3(E)$ such that in the even dimensions of $|H|$ one can get the same complete intersection Calabi-Yau (CICY) sufficing the old relation computed at the very beginning of the paper. Further extension of the semi-stable relation of Bill-Noether is shown with the Chen $ch$ taking over the Bogomolov-Gieseker inequality Type $B_{2,4}$ for the slope $\mu_{B_{2,4}}$. Having the scope of extendable to higher varieties of CICY threefolds in $\mathbb{P}^N$ the necessary alterations that are required for the same Clifford curve $C$ computed for the quintic variety through the inequality Type $C_{2,2,5}$. However, the extension onto the $\mathbb{P}^N$ with detailed calculations is omitted for not being the scope and purpose of this paper.

For the degree 5 hypersurface of projective variety $\mathbb{P}^4$ there exists a bilinear relation with the polydics $d$ where the hypersurface $S$ can be represented through the ambient homogeneous coordinates $\mathbb{P}^{n+1}$ for the projective $\mathbb{P}^4$ where the suitable coordinates of the CY have been mentioned via the form $z_m = \mathcal{K}_m/\mathcal{K}_i$ with the polynomial $k$ for the concerned threefolds in arguments $h^{1,1} = 1$ showing the Euler characteristics $-200$. The group $G$ being taken here suffice the structure through the zeta function for $\psi_1 + \psi_2 + \psi_3 + \psi_4 \equiv 0 \ mod\, 5$ where one gets the necessary curves in repeated permutations through points $\tilde{p}_{1,2,3}$ giving the Barth-Nieto Quintic along with the complete construction of the Calabi-Yau manifold emphasizing on threefold via the canonical line bundle $\eta$ through the form $S_\theta^\vee$ for the $5^{th}$ roots of the sheave $S$ as $._{\tilde{\wedge}}$ in the quotient $S_{\tilde{\wedge}}/G$ suffice the mirrors and modularity of the CY relations.

Two types of orbifold proved to be non-trivial in respect of the non-rigid CY threefolds when taken over $M_\Delta - Polarized\, K3$ fibers where again the Neron-Sevari groups give two sets of mapping parameters which distinguished through several segregations of the sheaf property through ramifications of two kinds, the singularities with the concerned set being taken. The Hodge number $h^{2,1}$ proves essential for the correction coefficient for the moduli space measuring with a further extension to smooth degeneration, partition cover, and Fricke involution. Other than that Tyurin degeneration is considered for invoking two beautiful properties as the Quasi-FANO for smoothly connected hypersurface and the mirror. Then the canonical norm implies the fibers through 4 kinds being distinct for K3, Abelian, and the elliptic curve concerned for projective $\mathbb{P}^1$ and second Chern class $c_2$.

And at the end for the morphism map due to singularities in K$\ddot{a}$hler $K$ for minimal CY giving $K_{S_m} \simeq \sigma_{S_m}$ the presence of algebraic fibers with $deg_D$ via projection map $\vartheta := \vartheta_{|P^+Q|} : S_m \longrightarrow \mathcal{H}_{+_{inv \to res\ \tau : S_m' \to S_m}} \subset \mathbb{P}^{dim\,|P^+Q|}$ having fiber variety $I_0, I_+, II_0, II_+, III, 0$.

## 8. DECLARATION OF INTERESTS

The author of this paper holds no conflicting interests.

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