A Novel Approach to the Fuzzification of Fields

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Research Article

Keywords: L-fuzzy subfield degree, L-fuzzy field homo-morphism, L-fuzzy convexity, L-fuzzy convexity preserving mapping, L-fuzzy convex-to-convex mapping

Posted Date: November 17th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-197870/v1>

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A novel approach to the fuzzification of fields

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Received: date / Accepted: date

Abstract In this paper, a novel approach to the fuzzification of field is introduced. its characterizations are given. An L -fuzzy convexity can be induced by an L -fuzzy subfield degree, and an L -fuzzy field homomorphism is exactly an L -fuzzy convexity preserving mapping and an L -fuzzy convex-to-convex mapping.

Keywords L -fuzzy subfield degree, L -fuzzy field homomorphism, L -fuzzy convexity, L -fuzzy convexity preserving mapping, L -fuzzy convex-to-convex mapping

1 Introduction

In algebra, the notion of field is one of the most basic and important concepts. The notion of fuzzy fields was first introduced in [13]. Later it was redefined in [1, 3, 9]. However, given a fuzzy set in a field, we know that it is either a fuzzy subfield or not. Only one of both cases is true. It does not have any fuzziness.

In Section 3, we shall present a novel approach to the fuzzification of subfield. We introduce the notion of degree to which an L -fuzzy set is L -fuzzy subfield. Thus in a field, each L -fuzzy set can be regarded as an L -fuzzy subfield to a certain degree.

Convexity theory has been accepted to be of increasing importance in recent years in the study of extremum problems in many areas of applied mathematics. As the axiomatization of the properties that usual convex sets fulfill, a

(abstract) convexity on a set was proposed. In fact, convexity exists in so many mathematical areas, such as lattices, algebras, metric spaces, graphs, topological spaces and so on. Some more details can be found in [19].

In 1994, the notions of fuzzy convex spaces and convex closure operators were first proposed by Rosa [14]. Later, Maruyama [10] generalized Rosa's definition to completely distributive lattice-valued setting, and investigated some combinatorial properties of lattice-valued fuzzy convex sets in Euclidean spaces. In a completely different direction, Shi and Xiu [17] provided a new approach to fuzzification of convex structures and presented the notion of M -fuzzifying convex structures. Later, Shi and Xiu [18] proposed the concept of (L, M) -fuzzy convex structures, which contains L -convex structures and M -fuzzifying convex structures as special cases. There is a close relationship between fuzzy algebra and fuzzy convex structure. For example, fuzzy convexity exists in so many mathematical areas, such as hazy lattices, hazy group, fuzzy algebras, and so on. Some more details can be found in [4–8, 21].

In Section 4, we shall prove that an L -fuzzy subfield degree can exactly induce an L -fuzzy convexity. Moreover an field homomorphism is exactly an L -fuzzy convexity preserving mapping and an L -fuzzy convex-to-convex mapping.

2 Preliminaries

Throughout this paper, L is a completely distributive lattice [2, 20], the smallest element and the largest element in L are denoted by $\top$ and $\perp$, respectively.

An element a in L is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. a in L is called co-prime if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$ [2]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $J(L)$.

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The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [20]. $\{a \in L \mid a \prec b\}$ is called the greatest minimal family of b , denoted by $\beta(b)$. Moreover, define a binary relation $\prec^{op}$ as follows: for all $a, b \in L$, $b \prec^{op} a$ if and only if for every subset $D \subseteq L$, $\bigwedge D \leq b$ implies $d \leq a$ for some $d \in D$. The set $\{a \in M \mid b \prec^{op} a\}$, denoted by $\alpha(b)$, is called the greatest maximal family of a in the sense of [20]. In a completely distributive lattice L , α is a $\bigwedge$ - $\bigcup$ map, β is a union-preserving map [20].

Theorem 1 ([15]) Let $A \in L^X$ and $a \in L$. Define

$$\begin{aligned} A^{[a]} &= \{x \in X \mid a \notin \alpha(A(x))\}, \\ A^{(a)} &= \{x \in X \mid A(x) \not\leq a\}, \\ A_{[a]} &= \{x \in X \mid a \leq A(x)\}, \\ A_{(a)} &= \{x \in X \mid a \in \beta(A(x))\}. \end{aligned}$$

In a complete Heyting algebra L , there exists an implication operator $\rightarrow: L \times L \rightarrow L$ as the right adjoint for the meet operation $\wedge$ by

$$a \rightarrow b = \bigvee \{c \in L \mid a \wedge c \leq b\}.$$

We list some properties of the implication operation in the following lemma.

Lemma 1 ([16]) Let L be a complete Heyting algebra and let $\rightarrow$ be the implication operator corresponding to $\wedge$. Then for all $a, b, c \in L$, $\{a_i\}_{i \in I} \subseteq L$, the following statements hold:

- (1) $\top \rightarrow a = a$;
- (2) $c \leq a \rightarrow b \Leftrightarrow a \wedge c \leq b$;
- (3) $a \rightarrow b = \top \Leftrightarrow a \leq b$;
- (4) $a \rightarrow \left(\bigwedge_{i \in I} a_i\right) = \bigwedge_{i \in I} (a \rightarrow a_i)$, hence $a \rightarrow b \leq a \rightarrow c$ whenever $b \leq c$;
- (5) $\left(\bigvee_{i \in I} a_i\right) \rightarrow b = \bigwedge_{i \in I} (a_i \rightarrow b)$, hence $b \rightarrow c \leq a \rightarrow c$ whenever $a \leq b$;
- (6) $(a \rightarrow c) \wedge (c \rightarrow b) \leq a \rightarrow b$;
- (7) $(a \rightarrow b) \wedge (c \rightarrow d) \leq a \wedge c \rightarrow b \wedge d$.

Lemma 2 ([16]) Let $f: S \mapsto S'$ be a set mapping and let $\{\mu_i\}$ and $\{\eta_i\}$ be the families of L -fuzzy subsets in S and S' respectively. Then we have

- (1) $f_L^{-\rightarrow}(f_L^{\leftarrow}(\eta_i)) = \eta_i$ if f is surjective;
- (2) $f_L^{\leftarrow}(f_L^{-\rightarrow}(\mu_i)) = \mu_i$ if f is injective,

where $f_L^{-\rightarrow}: L^S \rightarrow L^{S'}$ and $f_L^{\leftarrow}: L^{S'} \rightarrow L^S$ are defined by

$$f_L^{-\rightarrow}(\mu)(y) = \bigvee \{\mu(x) \mid f(x) = y\}, f_L^{\leftarrow}(\eta) = \eta \circ f.$$

Definition 1 ([18]) A mapping $\mathcal{C}: L^X \rightarrow M$ is called an (L, M) -fuzzy convexity on X if it satisfies the following conditions:

(LMC1) $\mathcal{C}(\chi_\emptyset) = \mathcal{C}(\chi_X) = \top_M$;

(LMC2) if $\{A_i \mid i \in \Omega\} \subseteq L^X$ is nonempty, then

$$\bigwedge_{i \in \Omega} \mathcal{C}(A_i) \leq \mathcal{C}\left(\bigwedge_{i \in \Omega} A_i\right);$$

(LMC3) if $\{A_i \mid i \in \Omega\} \subseteq L^X$ is nonempty and totally ordered, then $\bigwedge_{i \in \Omega} \mathcal{C}(A_i) \leq \mathcal{C}\left(\bigvee_{i \in \Omega} A_i\right)$.

The pair $(X, \mathcal{C})$ is called an (L, M) -fuzzy convex space. An (L, L) -fuzzy convex space is called an L -fuzzy convex space for short.

Definition 2 ([18]) Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M) -fuzzy convexity spaces. A mapping $f: X \rightarrow Y$ is called

- (1) an (L, M) -fuzzy convexity preserving mapping provided $\mathcal{D}(\lambda) \leq \mathcal{C}(f_L^{\leftarrow}(\lambda))$ for all $\lambda \in L^Y$.
- (2) an (L, M) -fuzzy convex-to-convex mapping provided $\mathcal{C}(\lambda) \leq \mathcal{D}(f_L^{\rightarrow}(\lambda))$ for all $\lambda \in L^S$.
- (3) A mapping $f: X \rightarrow Y$ is called an (L, M) -fuzzy isomorphism provided f is bijective, (L, M) -fuzzy convexity preserving and (L, M) -fuzzy convex-to-convex.

An (L, L) -fuzzy convexity preserving mapping is called an L -fuzzy convexity preserving mapping, an (L, L) -fuzzy convex-to-convex mapping is called an L -fuzzy convex-to-convex mapping, and an (L, L) -fuzzy isomorphism is called an L -fuzzy isomorphism for short.

Definition 3 ([1,3,9]) An L -fuzzy set μ of a field F is said to be an L -fuzzy subfield of F if for any $x, y \in F$,

$$\begin{aligned} \mu(xy^{-1}) &\geq \mu(x) \wedge \mu(y), \quad y \neq 0 \\ \mu(x - y) &\geq \mu(x) \wedge \mu(y). \end{aligned}$$

3 A novel definition of L -fuzzy subfield

According to Definition 3, we know that an L -fuzzy set in a field F is either an L -fuzzy subfield or not. Only one of both cases is true. It does not have any fuzziness. Next we shall present a novel approach to the fuzzification of subfields.

Definition 4 Let μ be an L -fuzzy subset in a field F . Then L -fuzzy subfield degree $FD(\mu)$ is defined as

$$\begin{aligned} FD(\mu) &= \bigwedge_{x, y \in F} ((\mu(x) \wedge \mu(y)) \mapsto \mu(xy^{-1})) \\ &\quad \wedge \bigwedge_{x, y \in F} ((\mu(x) \wedge \mu(y)) \mapsto \mu(x - y)). \end{aligned}$$

It is obvious that μ is an L -fuzzy subfield of F if and only of $FD(\mu) = \top$.

Example 1 Let $\mathbb{R}$ be the field of real numbers, and $L = 2^{\{a, b\}}$ with $a \neq b$. Define $\mu, \nu, \lambda: \mathbb{R} \rightarrow L$ by

$$\mu(x) = \begin{cases} \{a, b\}, & \text{if } x = 1, \\ \emptyset, & \text{if } x \neq 1; \end{cases}$$

$$v(x) = \begin{cases} \{a, b\}, & \text{if } x \text{ is integer,} \\ \{b\}, & \text{if } x \text{ is not integer;} \end{cases}$$

$$\lambda(x) = \begin{cases} \{a\}, & \text{if } x \text{ is a rational number,} \\ \emptyset, & \text{if } x \text{ is an irrational number.} \end{cases}$$

Then we have

$$FD(\mu) = \perp, \quad FD(v) = \{b\} \text{ and } FD(\lambda) = \top.$$

(1) In fact, if $x = y = 1$, then $x - y = 0$, in this case,

$$\mu(x) \wedge \mu(y) \mapsto \mu(0) = \{a, b\} \mapsto \emptyset = \emptyset,$$

therefore we get $FD(\mu) = \emptyset (= \perp)$.

(2) If $x = 2, y = 3$, then xy^{-1} is not integer, in this case,

$$v(x) \wedge v(y) \mapsto v(xy^{-1}) = \{a, b\} \mapsto \{b\} = \{b\},$$

therefore we get $FD(v) = \{b\}$.

(3) If both x and y are rational numbers, then $x - y, xy^{-1}$ are rational numbers, in this case,

$$\begin{aligned} \lambda(x) \wedge \lambda(y) &\mapsto \lambda(x - y) \wedge \lambda(xy^{-1}) \\ &= \{a, b\} \mapsto \{a, b\} = \{a, b\} (= \top). \end{aligned}$$

If one of x and y is irrational number, then both $x - y$ and xy^{-1} are irrational numbers, in this case,

$$\begin{aligned} \lambda(x) \wedge \lambda(y) &\mapsto \lambda(x - y) \wedge \lambda(xy^{-1}) \\ &= \emptyset \mapsto \emptyset = \{a, b\} (= \top). \end{aligned}$$

However we can get $FD(\lambda) = \top$.

Example 2 Let $\mathbb{R}$ be the field of real numbers, and $L = [0, 1]$, define $\mu, v, \lambda : \mathbb{R} \longrightarrow L$ by

$$\mu(x) = \begin{cases} 0, & \text{if } x = 0, 1, \\ 1, & \text{if } x \neq 0, 1. \end{cases}$$

$$v(x) = \begin{cases} 0.5, & \text{if } x \text{ is a rational number,} \\ 1, & \text{if } x \text{ is an irrational number.} \end{cases}$$

$$\lambda(x) = 0.5, \quad \forall x \in \mathbb{R}.$$

Then we have

$$FD(\mu) = 0, \quad FD(v) = 0.5 \text{ and } FD(\lambda) = 1.$$

(1) In fact, if $x = y \neq 0, 1$, then $x - y = 0$ and $xy^{-1} = 1$, in this case,

$$\mu(x) \wedge \mu(y) \mapsto \mu(0) \wedge \mu(1) = 1 \mapsto 0 = 0.$$

So we can get $FD(\mu) = 0$.

(2) If $x = y$ are irrational numbers, then $x - y = 0, xy^{-1} = 1$ are rational numbers, in this case,

$$v(x) \wedge v(y) \mapsto v(0) \wedge v(1) = 1 \mapsto 0.5 = 0.5.$$

So $FD(v) = 0.5$.

(3) $FD(\lambda) = 1$ is obvious.

Theorem 2 Let μ be an L -fuzzy set in a field F . Then

$$\begin{aligned} FD(\mu) &= \bigwedge_{x, y \in F} (\mu(x) \wedge \mu(y) \mapsto \mu(xy)) \\ &\wedge \bigwedge_{y \in F, y \neq 0} (\mu(y) \mapsto \mu(y^{-1})) \\ &\wedge \bigwedge_{x, y \in F} (\mu(x) \wedge \mu(y) \mapsto \mu(x + y)) \\ &\wedge \bigwedge_{y \in F} (\mu(y) \mapsto \mu(-y)). \end{aligned}$$

Proof From Definition 4 we can obtain that $\forall x, y \in F, (y \neq 0$ when considering $y^{-1})$

$$FD(\mu) \wedge \mu(x) \wedge \mu(y) \leq \mu(xy^{-1}),$$

$$FD(\mu) \wedge \mu(x) \wedge \mu(y) \leq \mu(x - y),$$

$$FD(\mu) \wedge \mu(y) \leq \mu(1),$$

$$FD(\mu) \wedge \mu(y) \leq \mu(0).$$

In particular we have that

$$FD(\mu) \wedge \mu(y) \leq FD(\mu) \wedge \mu(e) \wedge \mu(y) \leq m(y^{-1}),$$

$$FD(\mu) \wedge \mu(y) \leq FD(\mu) \wedge \mu(0) \wedge \mu(y) \leq m(-y),$$

$$FD(\mu) \wedge \mu(y) = FD(\mu) \wedge m(y^{-1}),$$

$$FD(\mu) \wedge \mu(y) = FD(\mu) \wedge m(-y),$$

$$FD(\mu) \wedge \mu(x) \wedge \mu(y) = FD(\mu) \wedge \mu(x) \wedge \mu(y^{-1}),$$

$$FD(\mu) \wedge \mu(x) \wedge \mu(y) = FD(\mu) \wedge \mu(x) \wedge \mu(-y).$$

This shows that

$$FD(\mu) \leq \bigwedge_{x, y \in F} (\mu(x) \wedge \mu(y) \mapsto \mu(xy))$$

$$\wedge \bigwedge_{y \in F} (\mu(y) \mapsto \mu(y^{-1})),$$

$$FD(\mu) \leq \bigwedge_{x, y \in F} (\mu(x) \wedge \mu(y) \mapsto \mu(x + y))$$

$$\wedge \bigwedge_{y \in F} (\mu(y) \mapsto \mu(-y)).$$

So

$$FD(\mu) \leq \bigwedge_{x, y \in F} (\mu(x) \wedge \mu(y) \mapsto \mu(xy))$$

$$\wedge \bigwedge_{y \in F} (\mu(y) \mapsto \mu(y^{-1}))$$

$$\wedge \bigwedge_{x, y \in F} (\mu(x) \wedge \mu(y) \mapsto \mu(x + y))$$

$$\wedge \bigwedge_{y \in F} (\mu(y) \mapsto \mu(-y)).$$

can be proved.

Analogously we can prove

$$\begin{aligned}
 FD(\mu) &\geq \bigwedge_{x,y \in F} (\mu(x) \wedge \mu(y) \mapsto \mu(xy)) \\
 &\quad \wedge \bigwedge_{y \in F} (\mu(y) \mapsto \mu(y^{-1})) \\
 &\quad \wedge \bigwedge_{x,y \in F} (\mu(x) \wedge \mu(y) \mapsto \mu(x+y)) \\
 &\quad \wedge \bigwedge_{y \in F} (\mu(y) \mapsto \mu(-y)).
 \end{aligned}$$

To sum up, it is available.

The following lemma is obvious.

Lemma 3 Let μ be an L -fuzzy set in a field F . Then $FD(\mu) \geq a$ if and only if for any $x, y \in F$,

$$\begin{aligned}
 \mu(x) \wedge \mu(y) \wedge a &\leq \mu(xy^{-1}), \quad y \neq 0, \\
 \mu(x) \wedge \mu(y) \wedge a &\leq \mu(x-y).
 \end{aligned}$$

The next theorem presents some equivalent descriptions of L -fuzzy subfield degree.

Theorem 3 Let μ be an L -fuzzy set in a field F . Then

- (1) $FD(\mu) = \bigvee \left\{ a \in L \mid \begin{array}{l} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y), \\ \forall x, y \in F \end{array} \right\}.$
- (2) $FD(\mu) = \bigvee \{ a \in L \mid \forall b \leq a, \mu_{[b]} \text{ is a subfield of } F \}.$
- (3) $FD(\mu) = \bigvee \{ a \in L \mid \forall b \notin \alpha(a), \mu^{[b]} \text{ is a subfield of } F \}.$
- (4) $FD(\mu) = \bigvee \left\{ a \in L \mid \begin{array}{l} \forall b \in P(L), b \not\geq a, \\ \mu^{(b)} \text{ is a subfield of } F \end{array} \right\}.$
- (5) $FD(\mu) = \bigvee \{ a \in L \mid \forall b \in \beta(a), \mu_{(b)} \text{ is a subfield of } F \}$
if $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for any $a, b \in L$.

Proof (1) $\iff$ (2) Suppose that $\mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1})$, $\mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y)$, $\forall x, y \in F$. Then for any $b \leq a$ and for any $x, y \in \mu_{[b]}$, we have

$$\begin{aligned}
 \mu(xy^{-1}) &\geq \mu(x) \wedge \mu(y) \wedge a \geq \mu(x) \wedge \mu(y) \wedge b \geq b, \\
 \mu(x-y) &\geq \mu(x) \wedge \mu(y) \wedge a \geq \mu(x) \wedge \mu(y) \wedge b \geq b,
 \end{aligned}$$

this shows $xy^{-1} \in \mu_{[b]}$, $x-y \in \mu_{[b]}$. Therefore $\mu_{[b]}$ is a subfield of F . Hence

$$\begin{aligned}
 FD(\mu) &= \bigvee \left\{ a \in L \mid \begin{array}{l} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y), \\ \forall x, y \in F \end{array} \right\} \\
 &\leq \bigvee \{ a \in L \mid \forall b \leq a, \mu_{[b]} \text{ is a subfield of } F \}.
 \end{aligned}$$

Conversely, assume that $a \in L$ and $\forall b \leq a$, $\mu_{[b]}$ is a subfield of F . For any $x, y \in F$, let $b = \mu(x) \wedge \mu(y) \wedge a$. Then $b \leq a$ and $x, y \in \mu_{[b]}$, thus $xy^{-1}, x-y \in \mu_{[b]}$, i.e.

$$\begin{aligned}
 \mu(xy^{-1}) &\geq b = \mu(x) \wedge \mu(y) \wedge a, \\
 \mu(x-y) &\geq b = \mu(x) \wedge \mu(y) \wedge a.
 \end{aligned}$$

This means that

$$\begin{aligned}
 FD(\mu) &= \bigvee \left\{ a \in L \mid \begin{array}{l} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y), \\ \forall x, y \in F \end{array} \right\} \\
 &\geq \bigvee \{ a \in L \mid \forall b \leq a, \mu_{[b]} \text{ is a subfield of } F \}.
 \end{aligned}$$

So (1) $\iff$ (2) is clearly established;

(1) $\iff$ (3) Suppose that $\mu(x) \wedge \mu(y) \wedge a \leq \mu(xy)$, $\mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y)$ for any $x, y \in F$. Then for any $b \notin \alpha(a)$ and $x, y \in \mu_{[b]}$, we have

$$b \notin \alpha(\mu(x)) \cup \alpha(\mu(y)) \cup \alpha(a) = \alpha(\mu(x) \wedge \mu(y) \wedge \mu(a)).$$

By

$$\begin{aligned}
 \mu(x) \wedge \mu(y) \wedge a &\leq \mu(xy^{-1}), \\
 \mu(x) \wedge \mu(y) \wedge a &\leq \mu(x-y),
 \end{aligned}$$

we know

$$\begin{aligned}
 \alpha(\mu(xy^{-1})) &\subseteq \alpha(\mu(x) \wedge \mu(y) \wedge a), \\
 \alpha(\mu(x-y)) &\subseteq \alpha(\mu(x) \wedge \mu(y) \wedge a).
 \end{aligned}$$

Hence $b \notin \alpha(\mu(xy^{-1}))$, $b \notin \alpha(\mu(x-y))$, i.e. $xy^{-1}, x-y \in \mu^{[b]}$. This means that $\mu^{[b]}$ is a subfield of F and

$$a \in \{ a \in L \mid \forall b \notin \alpha(a), \mu^{[b]} \text{ is a subfield of } F \}.$$

This shows that

$$\begin{aligned}
 FD(\mu) &= \bigvee \left\{ a \in L \mid \begin{array}{l} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y), \\ \forall x, y \in F \end{array} \right\} \\
 &\leq \bigvee \left\{ a \in L \mid \begin{array}{l} \forall b \notin \alpha(a), \\ \mu^{[b]} \text{ is a subfield of } F \end{array} \right\}.
 \end{aligned}$$

Conversely, assume that

$$a \in \{ a \in L \mid \forall b \notin \alpha(a), \mu^{[b]} \text{ is a subfield of } F \}.$$

Now we prove for any $x, y \in F$,

$$\begin{aligned}
 \mu(x) \wedge \mu(y) \wedge a &\leq \mu(xy^{-1}), \\
 \mu(x) \wedge \mu(y) \wedge a &\leq \mu(x-y).
 \end{aligned}$$

Suppose that $b \notin \alpha(\mu(x) \wedge \mu(y) \wedge a)$. By

$$\alpha(\mu(x) \wedge \mu(y) \wedge a) = \alpha(\mu(x)) \cup \alpha(\mu(y)) \cup \alpha(a),$$

we know that $b \notin \alpha(a)$ and $x, y \in \mu^{[b]}$. Since $\mu^{[b]}$ is a subfield of F , it holds that $xy^{-1}, x-y \in \mu^{[b]}$, i.e., $b \notin \alpha(\mu(xy^{-1}))$ and $b \notin \alpha(\mu(x-y))$. This shows that

$$\begin{aligned}
 \mu(x) \wedge \mu(y) \wedge a &\leq \mu(xy^{-1}), \\
 \mu(x) \wedge \mu(y) \wedge a &\leq \mu(x-y).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 FD(\mu) &= \bigvee \left\{ a \in L \mid \begin{array}{l} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y), \\ \forall x, y \in F \end{array} \right\} \\
 &\geq \bigvee \left\{ a \in L \mid \begin{array}{l} \forall b \in P(L), b \not\geq a, \\ \mu^{(b)} \text{ is a subfield of } F \end{array} \right\}.
 \end{aligned}$$

So (1) $\iff$ (3) is clearly established;

(1) $\iff$ (4) Suppose that $a \in L$ and for any $x, y \in F$,

$$\begin{cases} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y). \end{cases}$$

Let $b \in P(L)$, $b \not\leq a$ and $x, y \in \mu^{(b)}$. Now we prove $xy^{-1}, x-y \in \mu^{(b)}$. If $xy^{-1}, x-y \notin \mu^{(b)}$, i.e., $\mu(xy^{-1}) \leq b$ and $\mu(x-y) \leq b$, then

$$\mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}) \leq b,$$

$$\mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y) \leq b.$$

By $b \in P(L)$ and $x, y \in \mu^{(b)}$, we have $a \leq b$, which contradicts $b \not\leq a$. Hence $xy^{-1}, x-y \in \mu^{(b)}$. This shows that $\mu^{(b)}$ is a subfield of F . Therefore

$$\begin{aligned} FD(\mu) &= \bigvee \left\{ a \in L \mid \begin{array}{l} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y), \\ \forall x, y \in F \end{array} \right\} \\ &\leq \bigvee \left\{ a \in L \mid \begin{array}{l} \forall b \in P(L), b \not\leq a, \\ \mu^{(b)} \text{ is a subfield of } F \end{array} \right\}. \end{aligned}$$

Conversely, assume that

$a \in \{a \in L : \forall b \in P(L), b \not\leq a, \mu^{(b)} \text{ is a subfield of } F\}$.

Now we prove that for any $x, y \in F$,

$$\begin{cases} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y). \end{cases}$$

Let $b \in P(L)$ and $\mu(x) \wedge \mu(y) \wedge a \not\leq b$. Then $\mu(x) \not\leq b, \mu(y) \not\leq b$ and $a \not\leq b$, i.e., $x, y \in \mu^{(b)}$. Since $\mu^{(b)}$ is a subfield of F , it holds that $xy^{-1} \in \mu^{(b)}, x-y \in \mu^{(b)}$, i.e., $\mu(xy^{-1}) \leq b, \mu(x-y) \leq b$. This shows that

$$\mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}),$$

$$\mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y).$$

Therefore

$$\begin{aligned} FD(\mu) &= \bigvee \left\{ a \in L \mid \begin{array}{l} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y), \\ \forall x, y \in F \end{array} \right\} \\ &\geq \bigvee \left\{ a \in L \mid \begin{array}{l} \forall b \in P(L), b \not\leq a, \\ \mu^{(b)} \text{ is a subfield of } F \end{array} \right\}. \end{aligned}$$

So (1) $\iff$ (4) is clearly established;

(1) $\iff$ (5) Suppose that

$$a \in \left\{ a \in L \mid \begin{array}{l} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y), \\ \forall x, y \in F \end{array} \right\}.$$

Then for any $b \in \beta(a)$ and for any $x, y \in \mu^{(b)}$, it holds that

$$b \in \beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a)$$

$$= \beta(\mu(x) \wedge \mu(y) \wedge a) \subseteq \beta(\mu(xy)),$$

$$b \in \beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a)$$

$$= \beta(\mu(x) \wedge \mu(y) \wedge a) \subseteq \beta(\mu(x-y)),$$

i.e., $xy^{-1}, x-y \in \mu^{(b)}$. This shows that $\mu^{(b)}$ is a subfield of F . This means that

$$\begin{aligned} FD(\mu) &= \bigvee \left\{ a \in L \mid \begin{array}{l} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y), \\ \forall x, y \in F \end{array} \right\} \\ &\leq \bigvee \left\{ a \in L \mid \begin{array}{l} \forall b \in \beta(a), \\ \mu^{(b)} \text{ is a subfield of } F \end{array} \right\}. \end{aligned}$$

Conversely, assume that

$a \in \{a \in L \mid \forall b \in \beta(a), \mu^{(b)} \text{ is a subfield of } F\}$.

Now we prove that for any $x, y \in F$,

$$\mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}),$$

$$\mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y).$$

Let $b \in \beta(\mu(x) \wedge \mu(y) \wedge a)$. By

$$\beta(\mu(x) \wedge \mu(y) \wedge a) = \beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a),$$

we know that $x, y \in \mu^{(b)}$ and $b \in \beta(a)$. Since $\mu^{(b)}$ is a subfield of F , it holds that $xy^{-1}, x-y \in \mu^{(b)}$, i.e., $b \in \beta(\mu(xy^{-1}))$ and $b \in \beta(\mu(x-y))$. This shows that

$$\mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}),$$

$$\mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y).$$

Therefore

$$\begin{aligned} FD(\mu) &= \bigvee \left\{ a \in L \mid \begin{array}{l} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y), \\ \forall x, y \in F \end{array} \right\} \\ &\geq \{a \in L \mid \forall b \in \beta(a), \mu^{(b)} \text{ is a subfield of } F\}. \end{aligned}$$

So (1) $\iff$ (5) is clearly established.

In conclusion, (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) $\iff$ (5).

4 The relation between L -fuzzy subfield degree and L -fuzzy convexity

In this section, we will investigate the relation between L -fuzzy subfield degree and L -fuzzy convexity. We will prove that a field homomorphism is exactly an L -fuzzy convexity preserving mapping and an L -fuzzy convex-to-convex mapping.

For each $\mu \in L^F$, $FD(\mu)$ can be naturally considered as a mapping $FD : L^F \rightarrow L$ defined by $\mu \mapsto FD(\mu)$.

The following theorem shows that FD is an L -fuzzy convexity on F .

Theorem 4 Let F be a field. Then the mapping $FD : L^F \rightarrow L$ defined by $\mu \mapsto FD(\mu)$ is an L -fuzzy convexity on F , which is called the L -fuzzy convexity induced by L -fuzzy subfield degree on F .

Proof (LMC1) It is straightforward that

$$FD(\chi_\emptyset) = FD(\chi_F) = \top.$$

(LMC2) Let $\{\mu_i | i \in \Omega\} \subseteq L^F$ be nonempty. To prove

$$\bigwedge_{i \in \Omega} FD(\mu_i) \leq FD\left(\bigwedge_{i \in \Omega} \mu_i\right).$$

For any $a \leq \bigwedge_{i \in \Omega} FD(\mu_i)$, by Theorem 3, for all $i \in \Omega$, we have

$$\mu_i(x) \wedge \mu_i(y) \wedge a \leq \mu_i(xy^{-1}),$$

$$\mu_i(x) \wedge \mu_i(y) \wedge a \leq \mu_i(x-y),$$

for all $x, y \in F$. This implies

$$\begin{aligned} & \left(\bigwedge_{i \in \Omega} \mu_i(x)\right) \wedge \left(\bigwedge_{j \in \Omega} \mu_j(y)\right) \wedge a \\ & \leq \bigwedge_{i \in \Omega} (\mu_i(x) \wedge \mu_i(y) \wedge a) \bigwedge_{i \in \Omega} \mu_i(xy^{-1}), \\ & \left(\bigwedge_{i \in \Omega} \mu_i(x)\right) \wedge \left(\bigwedge_{j \in \Omega} \mu_j(y)\right) \wedge a \\ & \leq \bigwedge_{i \in \Omega} (\mu_i(x) \wedge \mu_i(y) \wedge a) \bigwedge_{i \in \Omega} \mu_i(x-y). \end{aligned}$$

By Lemma 3, we have $a \leq FD\left(\bigwedge_{i \in \Omega} \mu_i\right)$. From the arbitrariness of a , we obtain

$$\bigwedge_{i \in \Omega} FD(\mu_i) \leq FD\left(\bigwedge_{i \in \Omega} \mu_i\right).$$

(LMC3) Let $\{\mu_i | i \in \Omega\} \subseteq L^F$ be nonempty and totally ordered. To prove

$$\bigwedge_{i \in \Omega} FD(\mu_i) \leq FD\left(\bigvee_{i \in \Omega} \mu_i\right).$$

For any $a \leq \bigwedge_{i \in \Omega} FD(\mu_i)$. By Lemma 3, for all $i \in \Omega$, we have

$$\mu_i(x) \wedge \mu_i(y) \wedge a \leq \mu_i(xy^{-1}),$$

$$\mu_i(x) \wedge \mu_i(y) \wedge a \leq \mu_i(x-y),$$

for all $x, y \in F$. Let $b \in J(L)$ such that

$$b \prec \left(\bigvee_{i \in \Omega} \mu_i(x)\right) \wedge \left(\bigvee_{i \in \Omega} \mu_i(y)\right) \wedge a,$$

then we have

$$b \prec \bigvee_{i \in \Omega} \mu_i(x), b \prec \bigvee_{i \in \Omega} \mu_i(y), b \leq a.$$

Hence there exists some $i, j \in \Omega$ such that $b \leq \mu_i(s), b \leq \mu_j(t), b \leq a$. Since $\{\mu_i | i \in \Omega\}$ is totally ordered, we assume $\mu_j \leq \mu_i$, it follows that

$$b \leq \mu_i \wedge \mu_i(y) \wedge a.$$

By

$$\mu_i \wedge \mu_i(y) \wedge a \leq \mu_i(xy^{-1}),$$

$$\mu_i \wedge \mu_i(y) \wedge a \leq \mu_i(x-y),$$

we obtain $b \leq \mu_i(xy^{-1})$ and $b \leq \mu_i(x-y)$.

$$\text{Hence } b \leq \bigwedge_{i \in \Omega} \mu_i(xy^{-1}) \text{ and } b \leq \bigwedge_{i \in \Omega} \mu_i(x-y).$$

From the arbitrariness of b , we have

$$\left(\bigvee_{i \in \Omega} \mu_i(x)\right) \wedge \left(\bigvee_{i \in \Omega} \mu_i(y)\right) \wedge a \leq \bigvee_{i \in \Omega} \mu_i(xy^{-1}),$$

$$\left(\bigvee_{i \in \Omega} \mu_i(x)\right) \wedge \left(\bigvee_{i \in \Omega} \mu_i(y)\right) \wedge a \leq \bigvee_{i \in \Omega} \mu_i(x-y).$$

Combining Lemma 3, we have $a \leq FD\left(\bigvee_{i \in \Omega} \mu_i\right)$. By the

arbitrariness of a , we obtain $\bigwedge_{i \in \Omega} FD(\mu_i) \leq FD\left(\bigvee_{i \in \Omega} \mu_i\right)$. Therefore FD is an L -fuzzy convexity on F .

Now we consider the L -fuzzy subfield degrees of homomorphic image and preimage of L -fuzzy sets.

Theorem 5 Let $f : F \rightarrow F'$ be a field homomorphism, $\mu \in L^F$ and $\eta \in L^{F'}$. Then

- (1) $FD(f_L^{\rightarrow}(\mu)) \geq FD(\mu)$; if f is injective, then $FD(f_L^{\rightarrow}(\mu)) = FD(\mu)$.
- (2) $FD(f_L^{\leftarrow}(\eta)) \geq FD(\eta)$; if f is surjective, then $FD(f_L^{\leftarrow}(\eta)) = FD(\eta)$.

Proof (1) can be proved from Theorem 3 and the following fact.

$$\begin{aligned} & FD(f_L^{\rightarrow}(\mu)) \\ &= \bigvee \left\{ a \in L \mid \begin{array}{l} f_L^{\rightarrow}(\mu)(x') \wedge f_L^{\rightarrow}(\mu)(y') \wedge a \leq f'_L(\mu)(x'y'^{-1}), \\ f_L^{\rightarrow}(\mu)(x') \wedge f_L^{\rightarrow}(\mu)(y') \wedge a \leq f'_L(\mu)(x'-y'), \\ \forall x', y' \in F' \end{array} \right\} \\ &= \bigvee \left\{ a \in L \mid \begin{array}{l} \bigvee_{f(x)=x'} \mu(x) \wedge \bigvee_{f(y)=y'} \mu(y) \wedge a \leq \bigvee_{f(z)=x'y'^{-1}} \mu(z), \\ \bigvee_{f(x)=x'} \mu(x) \wedge \bigvee_{f(y)=y'} \mu(y) \wedge a \leq \bigvee_{f(z)=x'-y'} \mu(z), \\ \forall x', y' \in F' \end{array} \right\} \\ &\geq \bigvee \left\{ a \in L \mid \begin{array}{l} \mu(x) \wedge \mu(y) \wedge a \leq \mu(xy^{-1}), \\ \mu(x) \wedge \mu(y) \wedge a \leq \mu(x-y), \\ \forall x, y \in F \end{array} \right\} \\ &= FD(\mu). \end{aligned}$$

If f is injective, the above $\leq$ can be replaced by $=$. Hence $FD(\mu) = m(f_L^{\rightarrow}(\mu))$.

(2) can be proved from the following fact.

$$\begin{aligned}
 & FD(f_L^{\leftarrow}(\eta)) \\
 &= \bigwedge_{x,y \in G} ((f_L^{\leftarrow}(\eta)(x) \wedge f_L^{\leftarrow}(\eta)(y)) \mapsto f_L^{\leftarrow}(\eta)(xy^{-1})) \\
 &\wedge \bigwedge_{x,y \in G} ((f_L^{\leftarrow}(\eta)(x) \wedge f_L^{\leftarrow}(\eta)(y)) \mapsto f_L^{\leftarrow}(\eta)(x-y)) \\
 &= \bigwedge_{x,y \in G} ((\eta(f(x)) \wedge \eta(f(y))) \mapsto \eta(f(x)f(y^{-1}))) \\
 &\wedge \bigwedge_{x,y \in G} ((\eta(f(x)) \wedge \eta(f(y))) \mapsto \eta(f(x)-f(y))) \\
 &\geq \bigwedge_{x',y' \in G'} ((\eta(x') \wedge \eta(y')) \mapsto \eta(x'y'^{-1})) \\
 &\wedge \bigwedge_{x',y' \in G'} ((\eta(x') \wedge \eta(y')) \mapsto \eta(x'-y')) \\
 &= m(\eta).
 \end{aligned}$$

If f is surjective, the above $\leq$ can be replaced by $=$. Thus we can obtain that $m(\eta) = m(f_L^{\leftarrow}(\eta))$.

By Theorem 5 (1) and (2), we obtain the following theorem.

Theorem 6 Let FD_F and $FD_{F'}$ be the L -fuzzy convexities induced by L -fuzzy subfield degrees on F and F' . If $f: F \rightarrow F'$ is a field homomorphism, then $f: (F, FD_F) \rightarrow (F', FD_{F'})$ is an L -fuzzy convexity preserving and L -fuzzy convex-to-convex mapping.

The following corollary is obvious.

Corollary 1 Let FD_F and $FD_{F'}$ be the L -fuzzy convexities induced by L -fuzzy subfield degrees on F and F' . If $f: F \rightarrow F'$ is a field isomorphism, then $f: (F, FD_F) \rightarrow (F', FD_{F'})$ is an L -fuzzy isomorphism.

5 Conclusions

Given a fuzzy set in a field, we know that it is either a fuzzy subfield or not. Only one of both cases is true. It does not have any fuzziness. Now we presented a novel approach to the fuzzification of fields, i.e., we introduced the notion of L -fuzzy subfield degree. The relation between L -fuzzy subfield degree and L -fuzzy convex structure is established. Most importantly, this method can be applied to different algebra systems, such as groups, rings, ideals, and so on.

Declarations

Funding Not applicable.

Conflicts of interest The authors declare that they have no conflict of interest.

Availability of data and material Not applicable.

Code availability Not applicable.

Authors' contributions All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by Ming-yi Zeng, Fu-Gui Shi and Lan Wang. The first draft of the manuscript was written by Ming-yi Zeng, Fu-Gui Shi revised the manuscript, and Lan Wang commented on the previous versions of the manuscript. All authors read and approved the final manuscript.

Ethics approval Compliance with Ethical Standards: This article does not contain any studies with human participants or animals performed by any of the authors. Informed consent was obtained from all individual participants included in the study.

Consent to participate Not applicable.

Consent for publication All authors agree to publish.

Summary of requirements The authors did not receive support from any organization for the submitted work. The authors have no relevant financial or non-financial interests to disclose.

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