Random World and Quantum Mechanics

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Abstract Quantum mechanics (QM) predicts probabilities on the fundamental level which are, via Born probability law, connected to the formal randomness of infinite sequences of QM outcomes. Recently it has been shown that QM is algorithmic 1-random in the sense of Martin-Löf. We extend this result and demonstrate that QM is algorithmic ω-random and generic, precisely as described by the 'miniaturisation' of the Solovay forcing to arithmetic. This is extended further to the result that QM becomes Zermelo–Fraenkel Solovay random on infinite-dimensional Hilbert spaces. Moreover, it is more likely that there exists a standard transitive ZFC model M, where QM is expressed in reality, than in the universe V of sets. Then every generic quantum measurement adds to M the infinite sequence, i.e. random real $r \in 2^\omega$, and the model undergoes random forcing extensions $M[r]$. The entire process of forcing becomes the structural ingredient of QM and parallels similar constructions applied to spacetime in the quantum limit, therefore showing the structural resemblance of both in this limit. We discuss several questions regarding measurability and possible practical applications of the extended Solovay randomness of QM. The method applied is the formalization based on models of ZFC; however, this is particularly well-suited technique to recognising randomness questions of QM. When one works in a constant model of ZFC or in axiomatic ZFC itself, the issues considered here remain hidden to a great extent.

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1 Introduction – randomness, formalization and quantum probability

Randomness and quantum probability are inherent features of quantum mechanics (QM). Even though the statement like this is substantiated by numerous observational evidences and QM is one of the most successful physical theories, the very nature and origins of QM probability remain unclear. This and related measurement problem are often considered as fundamental issues in QM. The purpose of this work is to shed light on these two cavities by recognising the role which random infinite sequences, known and widely analysed in mathematics, play in QM formalism.

Reasoning based on quantum phenomena allows one to predict the future experimental results with probabilities generically smaller than unity. There are probabilities rather than unique results to be predicted along the celebrated Born rule applied to state vectors in a complex vector space, e.g. wavefunctions, of a quantum system. This feature is not the matter of our biased or uncertain information about the world and quantum systems, but rather unavoidable, irreducible to non-probabilistic phenomena necessity, independent of how much and how exact information we would collect about the system. This leads to the exceptional phenomenon of randomness, which can be assigned even to individual objects, not necessarily in the ensemble of other objects-states. In other words, this kind of randomness is not reducible to (classical) statistical probabilities in the ensemble description and quantum mechanics yields only probabilistic predictions for individual events. In general, one might speculate whether there exists a kind of randomness, more basic than probabilistic (e.g. ensemble) description in QM. On the one hand it is well-understood where probability comes from in QM mathematically, and how to calculate it: this is the Hilbert space formalism, which explains the computational side of QM. On the other hand many deep, interpretational questions remain open, like the role and existence of (local) hidden variables ((L)HV), reduction of the wave function during the measurement and their relations to entanglement, nonlocality, contextuality etc. Despite the fact that much has been told about these unavoidable probabilities within the years of vivid disputes since the birth of QM, we believe that there is still much to uncover, especially when the interplay of quantum systems with classical spacetime is considered and spacetime would enter the quantum realm in a nontrivial way (or QM enters spacetime).

From the point of view of mathematics, randomness has been a vast and vividly developed research topic. In particular, mathematicians try to understand randomness from the standpoint of descriptive set theory, where formal tools specific to set theory (like forcing) are naturally incorporated to grasp the very notion of random infinite sequences. Various notions of random $\{0,1\}$-sequences, along with the formal criteria of detecting them as random, have been proposed since the celebrated definitions of Martin-Löf and Solovay (see e.g. [31]).

Over the years, despite a substantial mathematicians' effort to understand randomness, there is a relatively modest number of research papers devoted to this kind of strong randomness in QM. Nevertheless, it has recently changed and there have appeared important works in the field. This has been partly motivated by an increasing interest in the possibility to operate with efficient, true random number generators instead of pseudo-random ones and in possible experimental discrimination of both on the ground of QM (see e.g. [1,57]). Another reason is the fundamental investigations performed on the role of strong randomness in the very structure of QM and the possibility of finding its experimental evidences (e.g. [68]). Certainly the latter, perhaps in a longer perspective, is aiming at improvement of our manageability of strong random generators in computer science as well.

In the present work we focus on fundamental aspects of random infinite sequences in QM, although we try to approach the experimental verification problem for such sequences as well. It is fair to say that from the purely formal point of view, the proof that a random binary sequence is indeed random is a kind of incomplete task for first order formal systems containing Peano arithmetic (PA). One can not formally prove the randomness of each of almost all random sequences separately. However, similarly as for Gödel's unprovability in PA of certain true PA sentences, in order to infer their truth we turn to another, usually richer formal systems (e.g. [27,68]). This shows that experimental verification might be a hard task (if possible at all). Another aspect which matters here is the fact that infinite binary sequences which represent infinite process of fair coin tossing is impossible classically and its realisation requires nondeterministic theory like QM [68]. Thus any verification of randomness as above should presumably be designed as a quantum process. On the other hand, the strong randomness of binary sequences seems to carry intrinsic QM properties and suitable studying of them can unravel still hidden aspects of QM.

The method adopted in this work relies on formal power of models of set theory and on showing that they are *necessary* in some essential respect in QM. The fundamental role of randomness of binary sequences in QM has been already analysed from different perspectives in a variety of works [67,68, 57,1,24,88,95]. Our concern here is to uncover even more fundamental role of models of set theory underlying both QM and randomness of the sequences. As such, the present work develops and extends the approach and methods from our recent publication [60]. It is well-known that any consistent formal theory has a model and mathematicians are typically working in a variety of models (which are sets or proper classes) of Zermelo–Fraenkel (ZF) set theory, possibly with the axiom of choice (ZFC). ZFC is a first order axiomatic theory (all its axioms are written in the first order language) and, if consistent, it has also countable models. This does not mean that ZFC realized in a countable model M is weaker, limited or truncated — even though there are sets, like real numbers $\mathbb{R}$, of cardinality strictly bigger than the cardinality of $\mathbb{N}$, i.e. $\aleph_0$ — the smallest infinite cardinality. In fact, every provable sentence (theorem) of ZFC is still provable in M, as well as in any other model of ZFC. However, it does not mean that the same sentences are true in all models of ZFC. Actually, there are plenty of sentences holding true in one model, but not in another — these are the sentences independent of ZFC axioms. For instance, the celebrated continuum hypothesis (CH), or the axiom of choice (AC) itself (in ZF) hold in some models and do not hold in some others [49, 7, 65]. The subtle point here is the fact that ZFC cannot prove its own consistency; still, most of the researchers assume the consistency of ZF and proceed to work in models of ZFC such that the procedure of changing models yields additional information, which can be of special importance also from the point of view of physics, e.g. [60, 62, 61]. One technique of passing between models of ZFC is the set-theoretic forcing, which is both the method of proving theorems and the recipe how to extend models of ZFC [49]. Paul Cohen invented the method of forcing to prove independence of CH from the ZFC axioms and the AC from the ZF axioms. Since then the plethora of various forcing methods have been recognised and developed in mathematics and this is still one of the most vivid topics in set theory. Proving theorems and investigating relative consistency of certain theories (e.g. extending ZF) is a dominant way of thinking about forcing, while the other one, important in this paper, is to trace relative changes of structures in different models. An example would be the fate of real line in e.g. a countable transitive ZFC model M and its (nontrivial) forcing extension $M[G]$, i.e. R_M and $R_{M[G]}$ respectively. From the point of view of an 'observer' external to both models, living in the real universe of sets V (with the real line $\mathbb{R}$), it holds that $R_M \subsetneq R_{M[G]} \subsetneq \mathbb{R}$. However, for an observer in any model all properties provable in ZFC are the same as in any other model of ZFC. Still there are certain properties or constructions absolute for certain class of ZFC models, like countable and transitive ones, which 'survive' as they are the change of a model within the class. Again this is a vast and well-recognized part of set theory. What is important here is that even though there are many kinds of forcing in mathematics, a particular one is clearly distinguished in the context of QM: that is a random forcing, developed soon after Cohen and described by Robert Solovay [86] in terms of Boolean valued models of ZFC. It is the relative phenomena between ground and extended models, that gives a formal tool to grasp higher random phenomena, attributed to the sequences of outcomes of QM on infinite-dimensional Hilbert spaces. The pioneering work about the possible role of Cohen forcing in the context of random sequences in QM consists of two papers by Paul Benioff [9, 10] published just a few years after the breakthrough work of Solovay. Benioff also performed an analysis of what is (not) absolute when one builds the Hilbert space formalism of QM in different models of ZFC.

From the very general standpoint, almost all mathematics can be formulated in the language of set theory $\mathcal{L}_{\mathrm{ZFC}}$ (it does not mean that all theorems in this language, which could be believed to be mathematically true are in fact ZFC-provable — the example are independent sentences) and different models of ZFC capture different sentences as the true ones in the models, thus one can look for a set-theoretic model as small as possible, in which *all the mathematics required by QM* can be formulated. Such a minimal natural model indeed exists, as demonstrated many years ago by Paul Benioff [9,10], and this allows to look for the description of randomness of sequences of experimental outcomes of QM related to this particular model. Moreover, the model can additionally be countable, so we propose the method of a "microscope" based on set theory, which traces variations of countable models of ZFC within QM represented by systems in spacetime. This leads to a new formal perspective on QM based on countable models of ZFC. However, we do not assume that QM in the real world is formulated in a countable transitive model (CTM) of ZFC; rather we consider such an option as a method of investigation (see the Methods section and discussion below). We also present arguments in favour of the situation that such an option could be indeed realised in nature.

The result by Benioff is rather formal and seems to be purely mathematical: there exists a (countable, transitive, minimal) model of ZFC where the mathematics of QM (on Hilbert spaces) can be formulated. The model, however, can not contain among its sets strongly random infinite sequences which could arise as outcomes of quantum measurements [9]. Moreover, none of such strongly random sequence can be represented as a set in any of the so-called Cohen forcing extensions of the model [10]. Thus fundamental questions arise: (A1) does QM live in any model of ZFC in the real world? (A2) Can we answer in any reasonable way the question like that? (B) Can we find any evidences in real world for this kind of enquires? (C) Moreover, even though the answer for the question (A1) would be YES, is it true that the existence of a model expressing mathematics of QM and containing no strong random sequence are relevant to better understanding of QM itself?

In this paper we show that (A1,2) can be answered affirmatively in the following way. The recent important result by Landsman is that the Born probability rule in QM enforces 1-randomness of certain infinite sequences of QM outcomes [68]. Based on this we show that

The Born probability rule enforces arithmetic 1-weak randomness and Solovay ω-generic randomness; (1)

and additionally for infinite dimensional Hilbert spaces, the Born probability rule and the structure of the lattice $\mathbb{L}$ enforce that

QM can be formulated in a certain model of ZFC which has to be extended by the Solovay forcing.
The model can be standard, transitive and countable.
(2)

We can state this result as: *if QM realises infinite sequences of outcomes as strongly random (Set theory Solovay generic)* then QM on infinite-dimensional Hilbert spaces has to be formulated in a model of ZFC such that the Solovay forcing is a part of the QM formalism and the carrier for randomness in QM.

The above results open the possibility and validate the technique of using formal models of ZFC in the context of QM. The technique is extensively applied in the present work. The clarification of 1-, weak 1- or Solovay generic randomnesses will be given in the following sections. At this point let us note that a natural need for formal models of set theory arises also in studying formal aspects of random sequences. The point is that truly random infinite sequence (r_i) with $\{0,1\}$-entries should not be characterised by any specific (non-random, algorithmic computable) procedure like e.g. "take r_{i+1} to be $r_i + 1$ mod 2" or "take r_{i+1} to be r_i itself". Generally speaking, a sequence will be called random if it omits *all* Borel sets of (continuous) measure 0 (i.e. the sequence "omits" any specific characterisation) and if it belongs to all Borel sets of full measure (measure 1). However, it is well known that such sequences do not exist, since every $\{0,1\}$-sequence (r_i) is just a binary representation for some unique real number r and obviously it cannot omit the null-measure singleton $\{r\}$. To avoid this difficulty, one usually truncates the allowed class of measure zero sets into certain effective subclasses to be omitted by random sequences. One such widely accepted possibility is to restrict those to the arithmetic and algoritmically computable sets of measure zero. Another possibility, which is followed in this paper, is to restrict our formal considerations to a model M of ZFC such that a random sequence r has to live in the generic extension $M[r]$ and omits *all* Borel sets of measure zero (coded) in M. This last approach is substantiated by the result (2) stating that such model M indeed exists for QM and for r strongly random, *relative to the model* M. Thus having (2) established, to determine the proper notion of randomness for QM, we proceed twofold. First, it is to grasp and describe a model of ZFC which would cover the mathematical structure of QM, and second, it is to find a notion of genericity, hence forcing in M, suitable for capturing the randomness of QM.

One could wonder why not to take just entire universe V of sets. This is a deep question with respect to the problem of existence of random sequences and random forcing in V. If one stipulates on the existence of r in V to be random generic then such r does not exist. This follows from the provable property that in V there are no generic filters on an atomless Boolean algebra B [93, p.70 Remark 4.1.8]. The random algebra is atomless, which means there are no random sequences corresponding to generic ultrafilters in V. However, one can still perform the random forcing on V in the sense of Boolean valued models which are definable in V [93, p.70 Remark 4.1.8]. However, in the case of CTM M there always exist random sequences, given by generic ultrafilters G in the generic extension $M[G]$. That is why the existence of such G favours the model M. In section 3.3 we have collected the arguments showing that random forcing is a structural part of QM formalism on infinite-dimensional Hilbert spaces. It is important to note that in the present context, random forcing distinguishes random sequences relative to a model M rather than serves as a procedure of proving theorems in set theory.

Finally, and it is yet another important result of the paper: with such formal, set-theoretic "microscope" we will look at the basic conceptual issues of QM like measurement problem or existence of suitable HV theories through their relation with random infinite sequences. In particular, analysing the measurement in QM we find that it is natural (in the formal context developed here) to admit additional intermediate Boolean mixture of quantum states. This Boolean state generalizes the classical (2-valued) mixture and at the same time it is a (partial, Boolean) reduction of a fully quantum interference state. From the set-theoretic point of view it determines *all* random forcing extensions of the base model M. Another finding is that, during the measurement, the change of a model of ZFC takes place, as justified by (2), with addition of a random sequence (in fact, a set of full measure of such sequences in the extended real line). The formalism allows also for further decomposition into classes, expressing different degrees of random sequences in QM.

In terms of a fair coin tossing, we could restate the result regarding the role of forcing in QM as follows. Classically, i.e. deterministically, the 50/50-chance coin tossing is prohibited. This would require the existence of deterministic local hidden variables in QM, compatible with the Born rule [68]. To get access to fully random coin-tossing, one needs quantum phenomena, in particular the Born measure on 2^ω. Full strength of such randomness is an ω-randomness of the sequences. The results in the paper show that the truly random sequences *on infinite-dimensional Hilbert spaces extend this arithmetic randomness and are irreducible to the finite-dimensional case.* The tools properly suited for this enhanced randomness are forcing and change of models of ZFC where, in the extended model, the random sequences create the set of measure 1 while nonrandom, relative to M, sequences — the set of measure 0. Consequently, we are able to reformulate many properties of random phenomena in QM on infinite-dimensional Hilbert spaces in terms of forcing relation in set theory.

The issue of fundamental importance here is the problem of distinguishing the finite- and infinite-dimensional cases via corresponding outcomes sequences. First of all, it is known that for some pairs of self-adjoint operators, e.g. position and momentum, commutation relations giving rise to Heisenberg's uncertainty principle demand the underlying Hilbert space to be infinite-dimensional. Furthermore, there are projections defined on such Hilbert spaces that do not appear in a finite-dimensional case. The true challenge is, however, whether one can distinguish both cases on the level of sequences of outcomes in QM or, even more generally, in any experimental setup. If that were not be the case, it would be plausible that any enhanced randomness assigned to infinite-dimensional spaces could not be experimentally verified. On the other hand, if an experiment distinguishing both finite- and infinite-dimensional spaces by statistical results was possible, it would give experimentally distinguishable $\{0, 1\}$-sequences, characterizing these two situations. In other words, not all $\{0, 1\}$-sequences in QM could be reduced to sequences generated in finite-dimensional systems. It seems that this highly important question cannot be answered unambiguously at present, although there are some indications that the problem can be experimentally approached. In particular, we mention a recent work [28], where the authors use a concept of the so-called dimension witness [23] to design an experiment, where dimensionality of a Hilbert space is decidable. Namely, to obtain the lower bound for a dimension of a Hilbert space describing measured physical systems, one settles a correlation based on a particular variant of Clauser–Horne–Shimony–Holt (CHSH) game. While it is a well-known fact that Bell inequalities might be desirably violated in the usual setup of CHSH game through an appropriate entanglement of subsystems, it can be also rigorously proved that there exist correlations, which cannot be realized in any finite-dimensional setting, even if the number of questions and outcomes in the experiment is finite. As promising as it sounds, there is also the other side of the coin: above results can be reproduced in principle with a large enough dimensionality to any desired precision of the measurement. In other words, one could imagine that for any given precision there is a finite-dimensional representation of the problem that gives exactly the same answer (within that precision) as an infinite-dimensional representation. Therefore one could find an analogy with the problem of an experimental verification of uncertainty principle for position-momentum pair of operators, where one also studies finite-dimensional variants of canonical commutation relations (CCR), that cannot be distinguished from the exact, infinite-dimensional ones for large enough dimensionality of a Hilbert space [85]. Again, as in the dimension witness case, one proves rigorously the necessity of infinite-dimensionality for exact position-momentum CCR to hold. Therefore the question of devising an experiment testing the infinite-dimensionality of an underlying Hilbert space remains open; however, note that further work on random number generators based on operators of continuous spectrum may shed some light on that issue, cf. [70].

Another possibility would be to merge ingeniously various approaches to infinite-dimensionality of Hilbert space in QM and designing a kind of an experimental cross-check on the common domain of the approaches. In fact, it might happen that approximate, finite-dimensional results (e.g. in a CHSH-like game) are much more uncertain (or approximated differently) in finite-dimensional results based on Heisenberg uncertainty relation in spacetime. Such program would require a natural notion of coexistence of both situations. We are not able to give any decisive resolution to this experimental problem currently, though in the section 4 we will consider certain hints coming from extreme regimes of curvature and energies at the cosmological scale of our Universe. In brief, in a quantum regime of spacetime there might be spatial regions that are quantum correlated (e.g. [84]) or entangled, such that the finite-dimensional approximation of what happens in a flat spacetime is not enough to reconstruct these spatial correlations. So far this is just a bold assumption which would require a better understanding of common domain of QM and spacetime in extreme conditions.

The relation of randomness and genericity in the theory of random sequences is a widely recognised subject and, in particular, it follows that Cohen genericity is not related directly to algorithmic randomness (e.g. [31]). At the same time there exists the class of genericity of infinite sequences, indicating its direct applicability to algorithmic random sequences (like Martin-Löf sequences). This kind of genericity can be called arithmetic since it corresponds to the arithmetic classes of formulas expressible in the hereditary finite model of ZF without the axiom of infinity (HF_ω). The corresponding algorithmic randomness of infinite sequences from 2^ω characterise the randomness of QM on both finite- and infinite-dimensional Hilbert spaces. However, in the infinite-dimensional case we have special enhancement extending the arithmetic hierarchy, namely the set-theoretic Solovay genericity (it was proposed and analysed by Solovay in his celebrated work on models of set theory in ref. [86]). As stated already, the Solovay forcing is realised by QM on infinite-dimensional Hilbert space of states.

An important indication follows, relating above with the structure of spacetime at quantum regime. This structure should somehow respect the existence of both ground and extended models of ZFC [60]. In the present work we find that the pair of models $(L_\alpha, L_\alpha[r])$ emerges on the 'generic' QM side, where r is a random sequence. The way this pair matches the structure of spacetime and gravity at the quantum level will be further studied in a separate work. Note that L_α is the minimal (countable) model of ZFC, where the mathematics of QM is expressed, and r is a random generic real (see the section 'Minimal model L_α as the carrier for mathematics of QM').

The analysis presented in the paper shows that, as long as the probabilistic Born rule remains valid, the arithmetic genericity always holds, which is to be extended toward the Solovay genericity in the case of infinite-dimensional spaces of states. We believe that, in a sense, Solovay genericity might underlie the Born rule in QM such that infinite-dimensionality is respected as well, although we do not analyse that possibility in depth here (cf. [96,38]). Finally, is the randomness of binary sequences in QM a phenomenon, that could be used in practice? Presumably, there are certain additional conditions needed to characterise quantum systems in spacetime, which would ensure strong randomness to be accessible experimentally in a real world. In the following sections we will formulate proposals toward this direction.

2 Key Terminologies

Since the approach is interdisciplinary and touches certain interpretational questions in QM, to fix the notation and avoid potential confusion, in this section we will collect and explain main definitions and properties of variety of concepts appearing in the paper.

2.1 Quantum systems in spacetime

Our concern here is to approach a quantum system S which interacts with spacetime and decoheres, leading to the classical description. Let us quote the description given in ref. [14]

> The emergence of the macroscopic classical world from the microscopic quantum world is commonly understood to be a consequence of the fact that any given quantum system is open, unavoidably interacting with unobserved environmental degrees of freedom that will cause initial

quantum superposition states of the system to decohere, resulting in classical mixtures of either/or alternatives ...

We will see below (the Measurement section) that any measurement performed on a quantum system, where the apparatus is a quantum system as well, will lead to the 2-stage process; the first stage is precisely the reduction of the superposition (pure) state to the mixture state of either/or alternatives (von Neumann). The case of classical spacetime, enforcing decoherence of the quantum system, agrees with the Copenhagen measurement interpretation, where the two stages of such measurement process are still present. More precisely, such spacetime decoherence can be considered as gravitationally induced, where weak gravitational fluctuations enforce the classicallity of S as in [4]. One reason for considering this mechanism could be the overwhelming presence of gravitational bath for any quantum system, from the very beginning of the cosmological evolution of the Universe [14]. Here we do not develop the particular mechanism that causes the decoherence of quantum systems in classical spacetime; rather, we take it for granted as a basic fact. What we really need is to grasp the formal counterpart of such spacetime-driven decoherence in the Hilbert space formulation of QM, such that the results would become universal for QM on infinite-dimensional state spaces. The role of spacetime can be thus reduced to the opening of a classical window onto quantum systems. As we will see, this process always factors through the choice of the maximal set of commuting observables, containing position (or momentum) operator in a spacetime, and it appears as equivalent to the choice of maximal Boolean algebra of projections from the QM lattice of projections $\mathbb{L}$. Then we will observe the Boolean stage assigned to the mixture state stage, which in the case of infinite-dimensional Hilbert spaces generates *nontrivial* random forcing. For the completeness of the presentation, we are explaining main ingredients of the QM formalism on (infinite-dimensional) Hilbert spaces in the subsection below. In the next subsection (Measurement) we describe general form of measurement procedure in QM, including the Boolean stage.

The subsequent section explains briefly models of set theory and forcing relation and then randomness *à la* Martin-Löf which will be needed in the course of the paper. Key Terminologies section is closed by discussing aspects of hidden variable theories in QM and generic filters defining forcing relation.

2.1.1 Hilbert space formalism of QM

For the purpose of the present work, let us start with a specific setup of a quantum-mechanical system, namely the one that is both general and matches further discussion. It is not our goal to give the full picture here; instead, the introduction is narrowed down to the path to logical structure of QM, i.e. a lattice of projections on a space of states.

To begin with, a quantum mechanical system S is considered together with the Hilbert space of states $\mathcal{H}_S$, which means that one is allowed to take linear combinations and scalar products of states and this space is complete in appropriate sense. Usually, $\mathcal{H}_S$ is demanded to be separable (i.e. it contains a countable dense subset). Then, to describe properties of S, one represents physical quantities — observables — by operators on $\mathcal{H}_S$; the eigenvalue problem

$$a\psi = \lambda\psi \tag{3}$$

represents the measurement of an observable a in a state ψ with an outcome λ, thus requiring $\lambda \in \mathbb{R}$ (which is the most obvious premise for an observable to be self-adjoint, i.e. $a^\dagger = a$). Already at this point one comes across the question of dimensionality of $\mathcal{H}_S$; while it is perfectly valid to study particular quantum-mechanical systems in finite-dimensional representations, one has to bear in mind that the canonical commutation relations for position q_x and momentum p_x operators

$$[q_x, p_x] = \mathrm{i}\hbar \tag{4}$$

require $\mathcal{H}_S$ to be strictly infinite-dimensional, since the trace of a commutator in finite-dimensional setting is always equal to 0). Since any meaningful incorporation of a smooth spacetime likely involves the operators with continuous spectrum, we concentrate on the case $\dim(\mathcal{H}_S) = \infty$. This, together with a separability condition, makes all such Hilbert spaces isomorphic to the space of square-integrable functions $L_2(\mathbb{R}, \mathrm{d}x)$.

There is another important reason to require the observables to be self-adjoint: such operators always have a countable orthonormal basis of eigenvectors (for simplicity, let us limit ourselves to discrete spectrum here). This way, any state $\psi \in \mathcal{H}_S$ can be expanded in such a basis as

$$\psi = \sum_{i=0}^{\infty} c_i \phi_i \tag{5}$$

and a pair (a, ψ) defines a probability distribution $p_i = |c_i|^2$ for obtaining ϕ_i after measuring the physical quantity a. This generalizes to operators with a continuous spectrum, giving continuous probability distributions [18].

Among the observables there is a special class of operators, that play the role of building blocks for the algebra of observables: these are projections (recall that a self-adjoint operator p is a projection whenever $p^2 = p$). Since all projections have only two eigenvalues: $\{0, 1\}$, they can be viewed either as yes-no propositions about the system S or, through an isomorphism between projections and their ranges, as closed subspaces of $\mathcal{H}_S$. The latter, together with subspace operations of closed linear span of union (join $\vee$), intersection (meet $\wedge$) and orthogonal complement (negation $\neg$), give the logical structure of a lattice $\mathbb{L}(\mathcal{H}_S)$ — a partial order such that for any two elements $p_1, p_2 \in \mathbb{L}(\mathcal{H}_S)$, their join and meet are defined. This structure resembles the mathematics of classical physics to some extent, despite the fact that projections are replaced there by characteristic functions of measurable subsets of a phase space. As a consequence, in classical physics the above lattice becomes a Boolean algebra, which enjoys all properties we would recognize as characterizing classical logic, such as the law of excluded middle or distributivity.

On the quantum side, one should keep in mind the following [73]

Lemma 1 *Whenever* $\dim(\mathcal{H}_S) \geqslant 2$*, the lattice* $\mathbb{L}(\mathcal{H}_S)$ *is not distributive and therefore fails to be a Boolean algebra.*

It is easy to give an example of such a failure [29]: consider three one-dimensional, pairwise non-orthogonal subspaces p_1, p_2, p_3 on a plane. It is easy to see that

$$p_1 \wedge (p_2 \vee p_3) = p_1 \neq 0 = (p_1 \wedge p_1) \vee (p_1 \wedge p_3). \tag{6}$$

This fact allows to speak about truly quantum, non-classical logic of $\mathbb{L}(\mathcal{H}_S)$.

To cope with more general observables, every self-adjoint operator on $\mathcal{H}_S$ can be decomposed into elements of $\mathbb{L}(\mathcal{H}_S)$ in the sense of a spectral resolution. Actually, one proves [90]:

Theorem 1 *For every family* $\{a_i\}_{i \in I}$ *of self-adjoint and pairwise commuting operators, there exists a complete Boolean algebra of projections* B *such that, given the spectral decompositions of each* a_i

$$a_i = \int \lambda \mathrm{d}e^i_\lambda, \tag{7}$$

it holds that $\forall i \in I(\mathrm{d}e^i_\lambda \in B)$*. In such case we say that* $\{a_i\}_{i \in I}$ *is contained in* B*.*

Therefore, there is a strict correspondence between classical (i.e. commuting) families of observables in $\mathcal{B}(\mathcal{H}_S)$ and classical (i.e. Boolean) lattices in $\mathbb{L}(\mathcal{H}_S)$. As a consequence, there are (at least) two parallel programs to conceive QM through "classical glasses": to study (maximal) abelian subalgebras of $\mathcal{B}(\mathcal{H}_S)$ (aka MASA's) or (maximal) Boolean subalgebras of $\mathbb{L}(\mathcal{H}_S)$ (known as blocks). While the former is a mature and vast research topic, both in pure mathematics [37] and physics [46], considerably less attention has been paid to the latter (but see e.g. [89]). Nevertheless, it is the language of Boolean subalgebras lurking behind $\mathbb{L}(\mathcal{H}_S)$ that is necessary to construct model-theoretic extensions in the following sections, thus we state the basic facts about this matter below.

Let us observe that, from the lattice-theoretic point of view, Boolean subalgebras of $\mathbb{L}(\mathcal{H}_S)$ build up the structure known as semilattice, which determines $\mathbb{L}(\mathcal{H}_S)$ up to isomorphism[45]. By Zorn's lemma, it is easy to see that every self-adjoint operator, together with a Boolean subalgebra which it belongs to, is contained in a block, although the extension might be non-unique (take e.g. $B = \{0, 1\}$). Thus we can speak of a cover for $\mathbb{L}(\mathcal{H}_S)$, built by the family of blocks. While $\mathbb{L}(\mathcal{H}_S)$ is always obviously atomic (i.e. there are elements — one-dimensional subspaces — with no lower elements greater than 0), in infinite-dimensional $\mathcal{H}_S$ one finds blocks that are atomless. As an example, one may consider the range of a spectral measure of the position operator [8]. This fact will be of crucial importance in the application of the forcing method below.

It is worth noting that for each MASA $A \subseteq \mathcal{B}(\mathcal{H}_S)$, projections in A generate a Boolean lattice and therefore A admits a two-valued homomorphism. Such a valuation is at the same time a linear functional on A and can be interpreted as the so-called dispersion-free state (i.e. a state with no uncertainty with respect to measurement of observables in A) [43]. Hence each MASA gives a classical (and usually partial) perspective on the system; in the following, Boolean homomorphisms set the ultrafilters and model extensions as a result, thus making a bridge between the classical-quantum border and model-theoretic approach to randomness in QM.

2.1.2 Measurement

Traditional quantum measurement theory as formulated by von Neumann [73] predicts that both quantum system S and the measuring apparatus A follow the same QM rules and one distinguishes two stages of the measurement process. Let Ψ_S and Ψ_A be initial states of S and A, respectively, and let $\Psi_{SA} = \Psi_S \otimes \Psi_A = \Psi_S\Psi_A$ be the product state of the quantum system and apparatus, thus belonging to the Hilbert space $\mathcal{H} \otimes \mathcal{H_A}$ of the system and apparatus composed together. The first stage reflects the interaction of S and A such that the combined system SA is isolated from interaction with any other system. This means that the dynamics of Ψ_{SA} is initially described by the Schrödinger equation and even if the initial states were pure, they become mixtures of states parameterized by the eigenstates of $\mathcal{O}_S$ and the pointer states of the apparatus. Thus in the first stage of the measurement process the eigenstates $\{|s_i\rangle\}_{i=1}^{\infty}$ of the observable $\mathcal{O}_S$ of S are correlated uniquely with the pointer-states $\{|a_i\rangle\}_{i=1}^{\infty}$ of the apparatus, such that the off-diagonal terms of Ψ_{SA} are absent (in the base $\{|s_i\rangle\,|a_j\rangle\}_{i,j\in\mathbb{N}}$) and the superposition state becomes a mixed state. In the case of discrete spectra we can represent the stage 1 as the following decomposition, see e.g. ref. [94,67]

$$\Psi_{SA} = \sum_i d_i\,|s_i\rangle\,\Psi_A \xrightarrow{1.} \sum_i d_i\,|s_i\rangle\,|a_i\rangle\,, \quad d_i \in \mathbb{C},\ \sum_i |d_i|^2 = 1\,. \tag{8}$$

Then the stage 2 relies on the unpredictable choice of a single state $|s_k\rangle$, hence also $|a_k\rangle$, for certain $k \in \mathbb{N}$

$$\sum_i d_i\,|s_i\rangle\,|a_i\rangle \xrightarrow{2.} |s_k\rangle\,|a_k\rangle \tag{9}$$

with the probability $|d_k|^2$. From the point of view of the lattice of projections, we will show in the Results section that this measurement 2-stage process should be complemented by adding the intermediate 'Boolean-valued' (not 2-valued) stage. The point is that while the stage 1. of the measurement procedure destroys quantum superposition and transposes the pure state to the mixture of states corresponding to eigenstates of $\mathcal{O}_S$, this mixture requires additional Boolean reduction before reaching the final single eigenstate. The QM formalism indicates the existence of Boolean-valued model of ZFC assigned to the mixture state such that the further QM reduction to a single state is due to the random forcing leading to the 2-valued, though extended, model containing this final state.

$$\sum_i d_i |s_i\rangle |a_i\rangle \in B\text{-valued } L_\alpha^B \xrightarrow{\text{forcing}} \text{2-valued } L_\alpha[r] \ni |s_k\rangle |a_k\rangle \ . \tag{10}$$

This is also where the random sequence r is assigned to the measurement process. Grasping together the 'generic' measurement process of an observable of a quantum system with an infinite-dimensional Hilbert space of states, this is characterized by the 2-stage process, where the second stage is augmented by the nontrivial (random) forcing

$$\sum_i d_i |s_i\rangle |a_i\rangle \quad \xrightarrow{L_\alpha^B \xrightarrow{\text{forcing}} L_\alpha[r]} \quad |s_k\rangle |a_k\rangle \ . \tag{11}$$

The left-hand side of the above diagram should be rather considered as expressed in the Boolean-valued model L_α^B, while the right-hand side in the 2-valued model $L_\alpha[r]$; this Boolean reduction can not be therefore neglected in the measurement process. In other words, one can say there is still some residual Boolean interference after the first stage of the measurement, destroying the non-classical and non-Boolean quantum superposition. (Here the word 'quantum' refers to the logic which is non-Boolean i.e. based on an entire, nondistributive lattice $\mathbb{L}$.) This Boolean interference state is further reduced by forcing to the 2-valued, classical window in the stage 2. Such scenario is explicit in the lattice of projections approach to QM for infinite-dimensional Hilbert spaces, remaining obscure without the reference to $\mathbb{L}$. However, this is unavoidable component of the QM formalism on infinite-dimensional Hilbert spaces. In the Results section we will also discuss the eventual experimental side of the above scenario, where the random sequence r extending the model L_α appears.

If one follows the Copenhagen interpretation of a measuring apparatus as a classical device, there is a measurement process again, where in the first stage the reduction of a pure state to the corresponding mixture takes place, and then in the second stage a single eigenstate is realised with the corresponding Born's probability ([67, Ch. 11]). Moreover, in the case of infinite-dimensional Hilbert spaces the mixture state is assigned again to the Boolean-valued model L_α^B, which is further reduced to the 2-valued, extended model $L_\alpha[r]$ in the second stage of the measurement process.

We will see that the presence of random forcing in the QM formalism is the feature, which allows to describe an overlapping domain of a quantum system in spacetime and spacetime in a quantum regime (e.g. [60]). The decoherence process, driven by classical spacetime, follows the measurement-like procedure with random forcing.

2.2 Random sequences and models of ZFC

As we saw in the Introduction section, the analysis of randomness in QM requires reference to certain formal, mathematical constructions like real numbers, probability (measure) on σ-algebras of sets, subsets of real numbers e.g. Borel sets, sets of reals of Lebesgue measure zero, and the like. Recall that Borel sets $BOR(X)$ on a topological space X comprises the smallest σ-algebra containing all open subsets of X. (A σ-algebra of subsets of X is a family $\mathcal{F}$ of subsets of X ($\mathcal{F} \subset \mathcal{P}(X)$) such that 1) $\emptyset, X \in \mathcal{F}$; 2) if $A \in \mathcal{F}$ then $X \setminus A \in \mathcal{F}$; 3) if $A_n \in \mathcal{F}, n \in \mathbb{N}$ then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.)

A function $\mu : BOR(X) \to [0,1] \subset \mathbb{R}$ is a (probability) measure if 1) $\mu(\emptyset) = 0, \mu(X) = 1$; 2) if $A_n \in BOR(X), n \in \mathbb{N}$ are Borel sets which are pairwise disjoint, i.e. $\forall_{i\neq j, i,j\in\mathbb{N}} (A_i \cap A_j = \emptyset)$, then $\mu(\bigcup_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} \mu(A_n)$; we also require μ to be nonatomic i.e. 4) for every set $A \in BOR(X)$ with $\mu(A) > 0$ there exists $B \in BOR(X)$ such that $B \subset A$ and $0 < \mu(B) < \mu(A)$. If X is a topological group, e.g. in the case $X = (\mathbb{R}, +)$, the translational invariance is additionally required: 5) $\mu(A) = \mu(A + t)$ for all $A \in BOR(X)$ and $t \in X$. An example for such a measure is the Lebesgue measure on $\mathbb{R}$ i.e. on $BOR(\mathbb{R})$. We call a subset $A \subset X$ to be *measurable* if there exists a Borel set $B \in BOR(X)$ such that $A \triangle B := (A \setminus B) \cup (B \setminus A)$ is a set of measure zero, i.e. $\mu(A \triangle B) = 0$.

Mathematicians found a suitable representation for $(\mathbb{R}, BOR(\mathbb{R}))$ by another topological spaces and their Borel subsets, that have simplified proofs and have allowed to grasp important properties of subsets of $\mathbb{R}$. This is precisely analogous to set theory, where working in different models of ZF(C) has exhibited deep invariant properties of the line of real numbers (real line) along with its regularity properties (e.g. [6, 49]). Choosing suitable model of ZF(C) leads to in a sense minimal representations for $\mathbb{R}$, where important and demanded properties of reals are highlighted and formally grasped. At the same time one does not loose any important information about objects under studies. Such possibility is based on Borel isomorphism of topological spaces. Given two topological spaces X and Y, we say they are *Borel isomorphic* if there exists a function $f : X \to Y$ such that for each Borel set $B \in BOR(Y)$ the set $f^{-1}(B)$ is Borel in X, i.e. $f^{-1}(B) \in BOR(X)$. The Borel isomorphism between spaces allows to work with any such space when dealing with regularity properties, such as Lebesgue measurability, Baire property, measure zero sets and others, due to the following

Lemma 2 *Let $f : X \to Y$ be a Borel isomorphism of X and Y. A set A is Lebesgue measurable in X (of first category, with Baire property, measure zero) $\iff$ $f(A)$ is Lebesgue measurable (of first category, with Baire property, measure zero) in Y.*

In fact, all Polish spaces (i.e. topological spaces homeomorphic to a complete metric space with no isolated points) are pairwise Borel isomorphic up to certain measure zero sets ([6, Th. 1.3.19]). An example of a Polish space could be the real line $\mathbb{R}$, and to replace it by another Polish space, we need a rather strict Borel isomorphism of $\mathbb{R}$.

To understand the importance of this fact let us observe, that any infinite sequence of possible outcomes in QM can be represented by a $\{0,1\}$-sequence and, thus being a member of the Cantor space $2^\omega = \{x : \omega \to 2\}$. This space is equipped with the topology, generated by basic open sets $[s] = \{x \in 2^\omega \mid s < x\}$ for any $s \in 2^{<\omega}$, where $2^{<\omega}$ is the space of all finite sequences of $\{0,1\}$. Here $s < x$ is the extension of a finite sequence $s \in 2^{<\omega}$ by an infinite $x \in 2^\omega$. In this way the Cantor space 2^ω is a Polish space with the product measure defined on basic opens by $\mu([s]) = 2^{-|s|}$, where $|s|$ is the length of the finite sequence s.

The Baire space ω^ω of all infinite sequences of natural numbers with a natural topology is also a Polish space. Both Cantor and Baire spaces represent the space of all subsets of the set of natural numbers $\mathbb{N}$, hence they are equinumerous with the set of all real numbers $\mathbb{R}$. But there is even stronger resemblance of these spaces with $\mathbb{R}$:

Lemma 3 *2^ω, ω^ω and $\mathbb{R}$ are pairwise Borel isomorphic Polish spaces. In fact, they are homeomorphic up to a countable sets.*

In the case of $\mathbb{R}$, the countable set can be taken as $\mathbb{Q}$ and for 2^ω and ω^ω those are the sets of sequences eventually equal to zero (see [6, p. 8]).

In set theory we frequently work with 2^ω or ω^ω rather than the real line $\mathbb{R}$. At the same time, these spaces contain also *all* random sequences with $\{0,1\}$- or $\mathbb{N}$-entries, and that fact allows to approach randomness in QM better. 2^ω can be described as a product probability space of countably many independent coin tossings, giving the 0 or 1 at every trial. Thus such form of randomness, generated by probability, underlies the Cantor space construction. However, among sequences represented by points in 2^ω, there are sequences that are random or not, and to separate or distinguish them, one requires something more than probability based on coin tossing. In fact we need enhancement of probability concept by referring to set theory independence on ZFC axioms. This is the way towards understanding randomness in QM followed here. That is also the reason why we are going to work with models of ZFC rather than just ZFC axiomatic system itself. By this refinement we have also ZF(C)-independent constructions at our disposal: certain statements are not true as theorems of ZFC (are not provable in ZFC), but become provable in particular models. (On the other hand, all theorems of ZF(C) are provable in *every* model of ZF(C).) Thus the use of forcing to describe random sequences indicates that such randomness is carried by sequences constructed as a whole in a ZFC-independent way. QM randomness exceeds the ZFC provability and rather ocuppies the space in-between two models of ZFC: M and the one extended by forcing — $M[r]$ (it belongs to $M[r]$ but is random relative to M).

To understand better the statement above, even before giving proper definitions of random sequences, let us take a closer look at forcing involved here. First, it is really possible to add new real numbers into many ZFC models M. Working in a model M, it's real line R_M is usually represented (this can be achieved from the outside perspective of the model) by certain subtree $2^{\omega}_{|M}$ of the full Cantor tree, i.e. $2^{\omega}_{|M} \subset 2^{\omega}$ (taking Cantor space representation of the real numbers). For example, if M is a countable transitive model (CTM) of ZFC, the real line in the model R_M is merely a subtree of 2^{ω} with countably many branches. This last statement is provable only from the outside of M and it cannot be proved inside M. Inside M, the set R_M is uncountable and is equal to ω_1 as long as the continuum hypothesis holds in M, i.e. the set of all subsets of ω has cardinality of the continuum: $2^{\omega} = \omega_1$. So there are many remaining real numbers-branches to be potentially added from the outside to M as new reals. Actually, we can produce such extensions by forcings, having the control over the provability relation in the extended model $M[r]$; at the same time, we can not grasp r by ZFC provability as a real number in M (we do that in $M[r]$ naturally).

Second, there exists a big variety of different forcing relations constructed for various purposes. In this paper we are mainly interested in the random forcing and occasionally refer to the Cohen forcing, famously invented by Paul Cohen in 1963 in order to show that the continuum hypothesis is independent of the axioms of ZFC. The new reals added to M by a forcing are called generic in $M[r]$. There are usually plenty (infinitely many) generic reals in $M[r]$, even though one adds a single real r to M. For example, adding a single real r means adding whole perfect tree of generic reals to M in the case of Cohen forcing and (with certain additional condition, like the existence of an amoeba real) for random forcing (e.g. [22, 51]). As the result, the sets of generic reals (Cohen or random) have full measure in $M[r]$, while the set of reals from the ground model M — measure zero in $M[r]$.

Third, to understand how forcing works we can start with a partially ordered set $\langle P, \leqslant\rangle$, where $\leqslant$ is a transitive, reflexive and antisymmetric relation on the set P without any minimal element in P. The purpose is to find canonical objects (in this context generic ultrafilters G) which, as sets, are not in a transitive model M of ZFC, but they belong to the extended model $M[G]$. One defines topology on a poset by taking *open sets* to be $O \subseteq P$, for which if $p \in O$, $q \in P$ and $q \leqslant p$, then $q \in O$. A subset $D \subseteq P$ is *dense* if for every $p \in P$ there exists $d \in D$ such that $d \leqslant p$. A *filter* on $\langle P, \leqslant\rangle$ is a subset $F \subset P$ which is 1) nonempty; 2) If $p \in F$ and $q \in F$, then there is $r \in F$ such that $r \leqslant p$ and $r \leqslant q$; 3) if $p \in F$, $q \in P$ and $p \leqslant q$, then $q \in F$. Two elements (called conditions) $p, q \in P$ of a partial order P are *compatible*, if there exists $r \in P$ such that $r \leqslant p \wedge r \leqslant q$. Otherwise we say p, q are incompatible. An *antichain* in P is a set of pairwise incompatible elements. Partial order $\langle P, \leqslant\rangle$ fulfills the *countable chain condition (CCC)* if every antichain in P is countable.

Recall that $\aleph_0$ is the cardinality (cardinal number) of ω (or $\mathbb{N}$). Then the *forcing axiom* (Martin axiom) states that for a partial order P satisfying the countable chain condition and for any family $\mathcal{D} \subset \mathcal{P}(P)$ of fewer than $2^{\aleph_0}$ of dense subsets of P, there exists a filter $G \subset P$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$

(here $\mathcal{P}(P)$ is the power set of P). The forcing axiom has to be assumed for higher cardinalities, although for countable families of dense subsets it holds

Lemma 4 *For any partially ordered set $\langle P, \leqslant\rangle$ and for a family $\mathcal{D}$ of countably many dense subsets of P and $p \in P$, there exists a filter G on P such that $p \in G$ and $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.*

Such filters G are called *generic ultrafilters* and their existence is crucial for forcing procedure as the following theorem shows.

Theorem 2 *[49, Th. 14.5] Let M be a transitive model of ZFC and $\langle P, \leqslant\rangle$ a partial order in M. Then for any generic ultrafilter $G \subset P$ (meeting all dense subsets of P in M) there exists a transitive model $M[G]$ such that*

$$M[G] \text{ is a model of ZFC;} \tag{12}$$

$$M \subset M[G] \text{ and } G \in M[G]; \tag{13}$$

$$\text{Ordinal numbers in } M \text{ are the same as in } M[G]. \tag{14}$$

Note that if the ground model M is countable (and transitive), then from Lemma 4 it follows that a generic ultrafilter G over P always exists, since $\mathcal{D}$ is a countable family of dense subsets of P seen from the outside of M. The particularly important case of adding new sets to the model M is the addition of a new subset of $\mathbb{N}$. This is represented by a $\{0,1\}$-sequence, hence a number $r \in \mathbb{R}$, where the sequence refers to the characteristic function of this subset of $\mathbb{N}$. We will see below that Cohen forcing is precisely of that type.

The elements of P are finite $\{0,1\}$-sequences, hence they are the members of $2^{<\omega}$. The condition p is stronger than the condition q (written $p < q$), if p extends q. Let M be a ground model, $\langle P, \leqslant\rangle \in M$ and let $G \subset P$ be generic over M (G intersects all subsets of P that are dense in M. For every $n \in \mathbb{N}$ the sets $D_n = \{p \in P \,|\, n \in \mathrm{dom}(p)\}$ (p is a finite sequence, thus a function, and $\mathrm{dom}(p)$ is its domain) are dense in P and hence they intersect G. Crucially, the function $\bigcup G$ is now in 2^ω which is not in M (Theorem 2) and it is a characteristic function of a certain subset of $\mathbb{N}$. Thus it is represented by an infinite $\{0,1\}$-sequence, hence a real number r. This r is contained in $M[G]$, although it is not present in M, and it is precisely a Cohen generic real.

Finally, there exists a presentation of forcing especially useful for our purposes, via so-called Boolean-valued models of ZFC. In the Measurement section this kind of Boolean-valued model was assigned to the QM measurement process at the emergence of the mixture of states stage. The precise construction of a Boolean-valued model of ZFC is presented in a Methods section 5. Here we give the relation of the forcing extension $M[G]$ in terms of the Boolean-valued model M^B. Such a model M^B is constructed in M, where B is a complete Boolean algebra B in the model M (see also section 5). Let $\mathcal{U}$ be an ultrafilter in B and let $P \subset B$ be a partial order completed to B; thus $\mathcal{U}$ is also a generic ultrafilter G in P (in M). The model M^B assigns Boolean values $\|\phi(x)\| \in B$ to formulas of the ZFC language interpreted in M^B. Given the ultrafilter G, one defines the equivalence relation $\equiv_G$ on the B-terms $x, y \in M^B$ by

$$x \equiv_G y \text{ if and only if } \|x = y\| \in G,$$

so that the corresponding $\in$-relation E on equivalence classes $[x], [y] \in M^B/G$ emerges

$$[x]E[y] \text{ if and only if } \|x \in y\| \in G\,.$$

Then it holds [49, Ex. 14.15, p.224; Cor. 14.30, p. 217]

Lemma 5

$$(M^B/G, E) \textit{ is a 2-valued model of ZFC;} \tag{15}$$

$$M^B/G \textit{ is isomorphic to } M[G]; \tag{16}$$

$$M^{\{0,1\}} = M^2 \textit{ is isomorphic to } M. \tag{17}$$

In the Results section we will see that forcing and genericity are well-suited for handling randomness in QM, and all the above mathematical notions will find their place in our understanding of this phenomenon.

2.2.1 Arithmetic hierarchy, Martin-Löf randomness, n-randomness, weak randomness

In this subsection we will present the formal definition of n-randomness, extending the one proposed by Martin-Löf.

Usually one can distinguish three main approaches to randomness of infinite sequences: patternlessness (the measure-theoretic paradigm including Martin-Löf and Solovay approaches), incompressibility (the computational paradigm including Kolmogorov's work) and unpredictability (which refers to martingale's constructions) [68,31]. They are equivalent in the lowest degree and constitute 1-randomness; however, in higher degrees they typically diverge (see below for formal definitions). We distinguish the fourth case how randomness is to be described, namely *genericity*, which plays a special role in exploring QM. In general there is plenty of interconnectedness, characterising one notion of randomness by the others [74,31]. The generic class contains Solovay (random) forcing (see [86] and [49, p. 243]), Cohen forcing and many others, like the variety of class forcings [65,49]. An excellent exposition of the role of Cohen and random forcings for random phenomena and explanation of their connections to other approaches to randomness can be found in [53]. The Martin-Löf (ML) approach belongs to the third class and ML celebrated definition requires a random sequence to omit only those sets of measure zero which are *algorithmically* computable, i.e. belonging to 1-arithmetic class [53, 31] and still the sequence lives in every full set of 1-random sequences.

One way of characterising the complexity of sets (and sequences) in formal set theory is by complexity of the formulas defining these sets. This is where the *arithmetic hierarchy* of sets plays crucial role.

1. An arbitrary set A is in $\sum_0^0$ ($\prod_0^0$) if the characteristic function of A is computable by a Turing machine.

2. $A \in \sum_n^0$, $n \geq 1$, if there exists a computable relation $R(x, y_1, y_2, ..., y_n)$ such that

$$x \in A \Longleftrightarrow \exists_{y_1} \forall_{y_2} \exists_{y_3} ... \exists_{y_n, (n \text{ is odd})} (\forall_{y_n, (n \text{ is even})}) R(x, y_1, y_2, ..., y_n)$$

where the last quantifier is $\exists$ (an existential one) whenever n is odd or $\forall$ (an universal one) if n is even. Similarly $A \in \prod_n^0, n \geq 1$ when

$$x \in A \Longleftrightarrow \forall_{y_1} \exists_{y_2} \forall_{y_3} ... \forall_{y_n,(n \text{ is odd})} (\exists_{y_n,(n \text{ is even})}) R(x, y_1, y_2, ..., y_n)\,.$$

Now the last quantifier remains $\exists$ if n is even and $\forall$ otherwise.

3. If $A \in \sum_n^0 \cap \prod_n^0$ then $A \in \Delta_n^0$.

4. If $A \in \bigcup_n (\sum_n^0 \cup \prod_n^0)$ then A is *arithmetical.*

Given an infinite binary sequence $\sigma \in 2^\omega$ and the product measure μ on the Cantor space 2^ω, the randomness of σ can be defined as (Martin-Löf e.g. [31]):

1. ML test: A sequence $\{A_n, n \in \mathbb{N}\}$ of uniformly computably enumerable (c.e.) (i.e. c.e. together with the set of its indices [31, p.11]) of $\sum_1^0$ classes ($\sum_1^0$ subsets of sequences from 2^ω) such that $\forall_{n\in\mathbb{N}}\ (\mu(A_n) < 2^{-n})$.

2. $A \subset 2^\omega$ is ML-null when there exists a ML test $\{A_n, n \in \mathbb{N}\}$ such that $A \subseteq \bigcap_{n\in\mathbb{N}} A_n$.

3. $\sigma \in 2^\omega$ is ML-random if $\{\sigma\}$ is not ML-null (for each ML test).

4. A ML test $\{A_n, n \in \mathbb{N}\}$ is *universal* when $\bigcap_{n\in\mathbb{N}} B_n \subset \bigcap_{n\in\mathbb{N}} A_n$ for all ML-tests $\{B_n, n \in \mathbb{N}\}$.

Lemma 6 *There exists a universal ML test.*

ML-random sequence $\sigma \in 2^\omega$ is known to be 1-random. The following modification of the above definition of ML randomness explains the hierarchy of n-random sets, for all $n \geq 1$.

i. ML_n test: A sequence $\{A_k,\ k \in \mathbb{N}\}$ of uniformly c.e. of $\sum_n^0$ classes ($\sum_n^0$ subsets of sequences from 2^ω) such that $\forall_{k\in\mathbb{N}}\ \big(\mu(A_k) < 2^{-k}\big)$.

ii. $A \subset 2^\omega$ is ML_n-null when there exists a ML_n test $\{A_k,\ k \in \mathbb{N}\}$ such that $A \subseteq \bigcap_{k\in\mathbb{N}} A_k$.

iii. $\sigma \in 2^\omega$ is n-random if $\{\sigma\}$ is not ML_n-null (for each ML_n test).

Thus a sequence σ considered as n-random should omit all sequences of sets of arbitrary small μ-measure from n-arithmetic class (which are uniformly computable by Turing machines). This means that such σ should pass all ML n-tests. It can be also rephrased by saying that σ omits all measure zero sets belonging to the n-arithmetic class. n-random sequences belong to every n-arithmetic set of measure 1. This motivates the following definition [53, Def. II.3.1]

$$\begin{aligned} &\text{A sequence } \sigma \in 2^\omega \text{ is } \textit{weakly } n\textit{-random} \text{ if} \\ &\sigma \text{ is a member of every } n\text{-arithmetic set of measure 1.} \end{aligned} \tag{18}$$

Working with the arithmetic hierarchy of sets and formulas is in fact working in the hereditarily finite model of ZF — HF_ω — which is a model of ZF without the axiom of infinity ([49, p. 480]). In the von Neumann hierachy (the universe V) it is just V_ω level which is L_ω in the constructible Gödel's universe L. Thus omitting n-arithmetic sets of measure 0 for any $n \in \mathbb{N}$ is precisely omitting the measure 0 sets definable in L_ω and this process is called the miniaturization of the Solovay forcing [31]. This arithmetical random forcing is marked as $L_\omega[r]$ where $r \in 2^\omega$ omits all n-measure 0 sets in L_ω. We will see in the Results section that such r characterise QM on *finite* dimensional Hilbert spaces and that QM is ω-Solovay random.

2.3 Hidden variables in QM

In the paper we also refer to a kind of hidden variables (HV) for QM in the context of randomness. The strongest version of HV for QM, called local hidden variables (LHV), has been already refuted by von Neumann [73]. Still, there is some space for weaker, nonlocal HV theories (e.g. [41]). We write LHV whenever we are referring to these strong local hidden variables, while HV means in general another form of hidden variables, possibly including nonlocal, context dependent, measuring-device dependent or nondeterministic ones.

The weakest form of variables that allow for classical descriptions of a quantum system S localized in spacetime are variables given by contexts (context-dependent) and thus are not measured simultanously in general; the examples could be the position and momentum of S. The contextuality, i.e. the fact that results of measurement depend on the 'context' comprising observables measured simultaneously, can be used to define randomness in QM in general. However, such a procedure is measuring-device dependent, as HV result this way. For instance the Bohm's theory[15–17] can be considered as non-local HV and inherent randomness of QM can be problematic in that perspective [11]. However, randomness in the above case is due to the random character of initial conditions for equations describing trajectories[83]. In other words, the Bohm's trajectories are spacetime entities not that much localised; in fact, they are rather context-dependent, nonlocal ones.

Many worlds interpretation of QM (MWI) also does not allow for the kind of randomness relying on unpredictable reduction of wave function collapse, since there are no reductions at all [11]. However, all contexts are equally real within MWI, thus realising HV in such context-dependent form.

If one adopts the (traditional) point of view, that the source of randomness is the interaction of a quantum system and measuring apparatus with environment, then QM can be intrinsically random but nonlocal (e.g. [96, 11]). In this case, nonlocal HV describing the measurement outcomes are possible, although they would be context- and device-dependent. In general, assuming both that HV theory reproduces all the outcomes of QM such that all QM observables can have definite values at any given time in a putative pseudoclassical state ψ_{psc}, and these values are independent on measuring device settings, contradicts the Kochen–Specker theorem [56]. Thus the theorem makes those HV impossible to exist, as long as the Hilbert space of states is of dimension greater than 2. This fact also reflects the inability to embed the QM lattice of projections in a Boolean contexts for $\dim \mathcal{H} > 2$ (see [26]).

In the approach emerging through this paper we are able to define structural (hence intrinsic) randomness for quantum systems (with $\dim \mathcal{H} = \infty$) in a spacetime, such that we shift the structure of QM responsible for randomness a bit closer to the lattice of projections, thus more distant from measurement outcomes. This is reflected by the Boolean stage of the measurement process as mentioned in the section 2.1.2. To properly grasp the structure, we have decided to work in the models of set theory, although this is in fact mandatory for QM as the results of the subsection 3.2 show. In the same time we will be able to determine intrinsic characterisation of HV in terms of the models of ZFC in the subsection 3.4.2.

3 Results

The 'Results' section is structured as follows. Below we explain the Benioff's result about the possibility to express mathematics of QM in the minimal constructible countable ZFC model L_α. Here, the phrase 'the mathematics of QM is expressible in a model M' is adopted, although eventually random outcomes of QM experiments will not belong to M, due to the existence of random infinite binary sequences. Thus we find a model M where QM formalism is expressible, but the random infinite binary sequences of outcomes are not. This points at some differences in terminology, compared to the original work by Benioff. The subsection 3.2 and the subsequent one are in a sense the central sections that contain the characterisation of the n-randomness of QM and its set-theoretic Solovay genericity. These results are then used in subsection 3.4 to characterise measurement procedure and hidden variables from the new perspective. We close the Results section by discussing the more practical aspects of randomness in QM characterised in this way.

3.1 Models of set theory as carriers for the mathematics of QM

In this section we recapitulate briefly the main aspects of argumentation, following [9], that the minimal countable model of ZFC encloses mathematics of QM, without random outcomes of experiments. This original and pioneering result by Benioff was ingeniously invented in the early stage of the development of the field of randomness in mathematics and QM. This result depends essentially on the definition of randomness used [9], which seems to be rather a free parameter of the approach. Even today the question about the intrinsic notion of randomness for QM appears to be of urgent importance. In the sections which follow, we will address this particular issue extensively and show also that there exists a model of ZFC such that QM in our world *has to* be formulated in the model. The model determines the random infinite binary sequences as not belonging to it, and has to change during the quantum measurement process.

Let M be a CTM of ZFC and $\mathcal{H}_M$ be a Hilbert space in M. What does it mean that a Hilbert space *is* in M and what is its relation with the Hilbert space $\mathcal{H}$ in the entire universe of sets V? Recall that a formula $\phi(x_1, ..., x_n)$ in the language of ZF set theory with n free variables is M-absolute, whenever proving ϕ in M ($M \models \phi$) is equivalent to proving it in V, that is

$$M \models \phi() \iff V \models \phi() \iff \phi() \,.$$

If a set is defined by an M-absolute formula, it is called an M-absolute set. If a formula (or a set defined by it) is M-absolute relative to every standard transitive ZFC model M, then the formula is absolute. Many concepts in set theory are absolute while many are not; however, each ZFC theorem remains true in every M. Among absolute constructs we have the formulas expressing 'to be a natural, rational or real number', 'to be the set of all natural, rational numbers', 'to be an ordinal number' while the examples of non-absolute notions (sets) are 'to be the set of all real, complex numbers', 'to be a cardinal number'. Let the formula ϕ express 'to be a Hilbert space' in the language of set theory; therefore ϕ, when written in M, defines a set with relations and functions which comprise the Hilbert space $\mathcal{H}_M$ in M. The basic fact is that $\mathcal{H}_M$ is not absolute and thus depends on the model. However, it holds

> [Benioff [9], Theorem 1] For each transitive ZFC model M there exists a Hilbert space $\mathcal{H}$ (in V) and a monomorphic isometry $h_M : \mathcal{H}_M \to \mathcal{H}$ of $\mathcal{H}_M$ into $\mathcal{H}$. Moreover, if $M \models$ ($\mathcal{H}_M$ is separable) then $V \models$ ($\mathcal{H}$ is separable).

Note that this statement holds true outside M since $h_M \notin M$, even though everything we can prove about $\mathcal{H}$ in ZFC we prove about $\mathcal{H}_M$ in M (since M is a model of ZFC). This is similar to 'not knowing' in M that certain reals from $\mathbb{R}$ are not in $R_M \subsetneq \mathbb{R}$ and thus we do not have any monomorphism of R_M into $\mathbb{R}$ in M. The fact $R_M \subset \mathbb{R}$ follows from the absoluteness of being a real and $R_M \subsetneq \mathbb{R}$ from the nonabsoluteness of being the set of all reals.

Similarly it holds

> [Benioff [9], Theorem 8] There exists an isometric monomorphism of $H_M : \mathcal{B}(\mathcal{H}_M) \to \mathcal{B}(\mathcal{H})$. Here $\mathcal{B}(\mathcal{H}_M)$, $\mathcal{B}(\mathcal{H})$ are the spaces of all bounded operators on $\mathcal{H}_M$ in M and on $\mathcal{H}$ in V, respectively.

Note that this monomorphism H_M above sends projections in M to projections in V, unitaries to unitaries, selfadjoints to selfadjoints and density operators in M to density operators in V [9, Corollary 9]. Moreover, eigenvalues are preserved and trace in M agrees with the trace on $\mathcal{B}(\mathcal{H})$ in V. Finally, taking into account the sets of preparation procedures for measurements S, set of (asking procedures for) measurements Q, their outcomes (infinite binary sequences), the product measures on $Bor(2^{<\omega})$ and on $Bor(2^{\omega})$ all these constructions can be performed in M and outside of M. One finds that [9, Eq. 3]

$$\forall_{a\in S_M, b\in Q_M} \mathrm{Tr}_M(\psi_M(a)\phi_M(b)) = \mathrm{Tr}(\psi(a)\phi(b)) \text{ in } V \,. \tag{19}$$

In the above, S_M is the set of state preparation procedures, Q_M the set of asking procedures (both in M) and they are transferred to V by the monomorphisms H_M and h_M. One concludes that, living in M, an observer performing measurements in M finds precisely the same experimental results as the observer from the outside (living in V) provided the initial data are truncated to those from M.

The definition of randomness for infinite binary sequences proposed by Benioff and based on definability property (the same property on which the Gödel's constructible universe L is built on [49]) allows to conclude, that random sequences can not live in L_α (L_α is the minimal countable constructible model, so thus α is the minimal ordinal not belonging to it). Benioff writes [9]

> It is important to recognize that the proof that L_α is not a suitable mathematical universe for quantum mechanics depends on how one defines randomness.

In particular, the conclusions about QM in V vs. QM in M appear. Namely, an intuitive version of QM used in everyday practise by physicists, is equivalent to QM formulated formally in V [9]. Given the version of QM in any standard transitive model M, one could find differences between both formulations *only from the outside of* M. Living in M the observer would find no ZFC provable differences of what can be observed in M and V (see (19)). It seems reasonable to ask about the possibility that QM in the real world be in M (in certain circumstances) and how to distinguish this from QM in V.

$$\text{QM in a standard (C)TM } M \overset{??}{\equiv} \text{ QM in } V \equiv \text{informal QM as a set of procedures}\,.$$

In the second paper Benioff showed [10] that no random sequence (by the definability condition) belongs to any Cohen extensions $L_\alpha[r_C]$.Here randomness is understood as given strong definition by Benioff in ref. [9].

We will refer to the results above and extend them by showing in the course of the paper that

1. QM distinguishes the intrinsic Solovay generic (Sg) randomness and
2. QM can be formulated in any standard CTM ZFC model M and random Sg-sequences are not in M.
3. QM favours the situation where it is formulated in M rather than in V.
4. None of the Cohen forcing extensions $M[r_C]$ contains any Sg random sequence.
5. When QM is formulated on infinite-dimensional separable Hilbert spaces, the Boolean local contexts in $\mathbb{L}$ realise Sg randomness, which is a way toward understanding the overlapping region of QM and classical spacetime (cf. [62]).

In 1. above, the intrinsic Solovay genericity is understood as the miniaturisation of forcing to the arithmetic context for finite-dimensional Hilbert spaces or as full set theory random forcing in the infinite-dimensional case. The points 2., 3., 4. and 5. focus on infinite-dimensional Hilbert spaces, thus we show that for infinite-dimensional Hilbert spaces there is a proper definition of randomness, intrinsically realised by QM, which extends the finite-dimensional case, i.e. $L_\omega[r_1] \to L_\alpha[r_2]$. Here r_1 is Sg with respect to arithmetic ML sequences (finite-dimensional $\mathcal{H}$) but r_2 is Sg in the set-theoretic sense in the minimal model L_α (infinite-dimensional $\mathcal{H}$).

3.2 The Born rule and Solovay generic randomness — the finite dimensional case

We have shown so far that it is a very tiny and probably only formal difference between the two excluding situations, from which only one holds true in the real world: 1. QM is in a model of ZFC (QM has to be formulated in M in reality) and 2. QM is rather not confined to a model but has to be formulated in the entire set theory universe V (or there is no reason at all that QM is in a model M). In this and subsequent sections we show that the first option is distinguished by QM formalism on infinite-dimensional Hilbert spaces (where Sg randomness of outcome sequences is allowed). This gives rise to some serious consequences, like the appearance of additional physical degrees of freedom connected with models encompassing QM, which can also be imprinted in the large scale structure of the Universe (see [60]).

Landsman [68] gave deep analysis of how 1-randomness may arise in QM. In this section we show that QM supports n-randomness and hence ω-randomness. It follows that QM supports Solovay generic ω-randomness. The main point of the Landsman's argumentation was the observation, that the measure μ on Cantor space 2^ω is in fact induced by the Born measure μ_A, resulting from a single measurement of A on a Hilbert space $\mathcal{H}$ in a state $\psi \in \mathcal{H}$. The Born's measure μ_A is the measure which gives rise to the probabilities of obtaining an outcome $a \in \sigma(A)$ in the state ψ, where $\sigma(A) \subset \mathbb{R}$ is the spectrum of a self-adjoint (bounded) operator $A \in \mathcal{B}(\mathcal{H})$. The spectral measure λ_A was assigned already to A in the formulation of Theorem 1, in the context of maximal Boolean algebras of projections comprising all projections, which are assigned to every self-adjoint operator from a commuting family in $\mathcal{B}(\mathcal{H})$. The measure is defined on the spectrum $\sigma(A)$ with its extension to $\mathbb{R}$ by 0's on $\mathbb{R} \setminus B$ such that $\lambda_A(\mathbb{R}) = 1$. This projection-valued (spectral) measures allow to define the Born measures for Borel subsets B of $\mathbb{R}$, $B \in BOR(\mathbb{R})$ (for every pair (A, ψ))

$$\mu_A = \mu_A^\psi(B) := \langle \psi, \lambda_A(B)\psi \rangle \quad \text{where } \langle\ ,\ \rangle \text{ is the scalar product in } \mathcal{H}.$$

According to μ_A one calculates the probability that an observable A takes a value $a \in B \subset \sigma(A)$ as

$$P(A \mapsto a \in B|\psi) = \mu_A^\psi(B)\,. \tag{20}$$

In the particular case of the discrete nondegenerate spectrum of A with the eigenvectors $\{e_i\}, i = 1, 2, 3, ...$ and the unit pure state $\psi \in \mathcal{H}$, in which the system is declared to be before the measurement, the measurement of A in ψ gives as the result one of the eigenvalues $\lambda_i, i = 1, 2, 3, ...$, with the probability $|c_i|^2$. Here $\psi = \sum_i c_i e_i, c_i \in \mathbb{C}$ such that $\sum_i |c_i|^2 = 1$, $A\psi = \lambda_i e_i, \lambda_i \in \mathbb{R}, i = 1, 2, 3, ...$ The probability $|c_i|^2$ of obtaining λ_i is the specialisation of the previous general formula (20) to the discrete case, since the projection measure $\lambda_A(A \mapsto \lambda_i) =$ (projection on the 1-dimensional eigenspace spanned by e_i) $= P_i$, so that

$$\mu_A(A \mapsto \lambda_i) = \langle \psi, P_i\psi \rangle_{\mathcal{H}} = \langle c_i e_i, c_i e_i \rangle = |c_i|^2\,.$$

Let us recapitulate briefly the reasoning that led Landsman to the conclusion, that the Born rule (i.e. the existence of the Born measure $\mu_A = \mu_A^\psi$ as in (20)) gives rise to the 1-randomness of QM.

Lemma 7 *The Born measure μ_A determines the infinite-product measure μ^∞ on the Cantor space 2^ω of infinite binary sequences of possible outcomes of QM measurements.*

This is quite a strong result showing how to extend the Born measure μ_A 1) over infinite binary sequences and 2) over the entire Cantor space of all such sequences, despite the fact that it is typically assigned to a single measurement. We present a more detailed argumentation, following [68], in the section 5.

Next, almost all infinite binary sequences in 2^ω are 1-random according to the measure generated by quantum-mechanical Born measure.

Lemma 8 *The set of 1-random sequences in 2^ω has μ^∞ measure 1.*

Hence the Born measure probability of the event that one has an infinite 1-random binary sequence of QM outcomes is equal to 1 and the Born measure probability that one has a nonrandom (infinite, binary) sequence in the sequel of repeating QM measurements, is equal to 0. If the (μ^∞ measure generated from the) Born measure were not supporting any 1-random sequence, it would not allow for the infinite fair coin-tossing sequence, similarly as classical probability measure does not allow for that. Note that $\mu^\infty(r) = 0$, $r \in 2^\omega$ does not mean that $\mu_A(r_{|i}) = 0$, for all $i = 1, 2, 3, ...$ where $r_{|i}$ is the i-th place of the binary extension of r. Therefore we do not have probability measure zero (impossible) events at every i-th trial of the infinite sequence occurring with the μ^∞-probability 0. Nonrandom sequences can be obtained as outcomes of the quantum sequence of experiments. The example is an unfair coin-tossing, which do not produce any random sequence (which is of course possible anyway). Similarly, $\mu^\infty(r) = 1$ does not mean that each stage of the sequence, $r_{|i}, i = 1, 2, 3, ...,$ is obtained with the probability 1. In the case of a fair coin-tossing, this is the probability equal to $\frac{1}{2}$ rather than 1 and is independent of the previously obtained result. However, if one imagines that reality operates somehow on infinite binary sequences of outcomes, not just on single quantum events, the probability μ^∞ would replace the μ_A. In such a case the extended Born's probability $\mu^\infty(r) = 0$ would mean that the sequence r is impossible to obtain and does not exist.

> [n-RAND] *We say that a theory assigning probabilities to outcomes of experiments is n-random, $n \geq 1, n \in \mathbb{N}$, if the theory can be defined on infinite sequences of outcomes and the probability measure extends uniquely onto a probability measure on 2^ω, according to which* almost all *infinite sequences of the outcomes of experiments described within the theory, are n-random.*

Let QM* be the QM allowing for the infinite sequences of the QM measurement outputs. We are going to show that QM* hence QM is ω-random. We follow the proof in ref. [68] of the fact that QM is 1-random and show QM is n-random according to [n-RAND] above. Finally we show that QM is arithmetic Solovay generic random (also on finite-dimensional Hilbert spaces).

Theorem 3 (Landsman [68]) *QM is 1-random.*

As noted above this result is based on a way how probability measure determined inherently by QM (Born measure) extends over spaces of infinite binary sequences of the outcomes of QM. First thing is to show that $(\mathrm{QM}^*, \mu_A^\infty)$ uniquely extends (QM, μ_A), hence the Born's measure μ_A is uniquely μ_A^∞, when defined on 2^ω (here μ_A^∞ is the product measure extending μ_A). Then, the Landsman's Theorem 5.1 [68] shows the *equivalence* of two experimental realisations of infinite repeatedness of identical measurements (of observable A over the system S in a state ψ):

1) QM deals with the entire infinite sequence of the trials and at the end the outcomes are recorded classically. Thus the infinite sequence of the outcomes $r \in 2^\omega$ results in a quantum measurement in $(\mathrm{QM}^*, \mu_A^\infty)$;
2) QM is applied to single experiments and the outcomes are classically recorded one by one and the classical probability theory describes their combination into an infinite sequence $r \in 2^\omega$. One works essentially in (QM, μ_A) and classical probabilities.

The proof of this equivalence refers to the probabilistic Bernoulli process, where the probability it generates gives rise to the unique extension of μ_A to μ_A^∞. The complete proof of the above equivalence can be found in [68]. Next, note that extending each μ_A, $A \in \mathcal{A}$ to the product measure μ_A^∞ gives the same result as extending to the product measure μ^∞ on 2^ω. See also section 5.1 in the Methods section.

In the similar way we will show that QM is n-random for $n \in \mathbb{N}, n > 1$.

Theorem 4 *Given $n \geq 1$, QM is n-random.*

Namely taking the product measure μ^∞ it holds true

Lemma 9 *The set of all n-random sequences in $2^\omega, n \geq 1$, has μ^∞-measure 1.*

This lemma is a consequence of the definition of n-randomness *à la* Martin-Löf as in subsection 2.2.1 of the Results section 3. Namely, any n-random sequence $r \in 2^\omega$ omits every *measure zero* set in the n-th arithmetic class.

But again from the equivalence 1) $\iff$ 2) above ([68, Th. 5.1]), it follows that QM probability of that arbitrary infinite binary sequence $s \in 2^\omega$ (of QM outcomes) which is *non n-random*, is zero, since any random sequence omits every measure zero set in n-th arithmetic class and belongs to every set of measure 1. Thus $\mu_A^\infty(S) = 0$ where S is the set of all non n-random sequences and Born's probability μ_A supports n-randomness in the same sense as for 1-random case, i.e. μ_A^∞-probability of choosing in QM n-random infinite sequence of outcomes is unity. The measure μ_A^∞ extends uniquely the Born measure in the same way as for 1-random case.

Corollary 1 *QM is ω-random.*

This is just the reformulation of the above lemma, since ($\forall_{n\in\mathbb{N}}$ QM is n-random) $\equiv$ QM is ω-random.

Our purpose now is to find a bridge between n-randomness of QM and Sg-randomness (arithmetic Solovay generic randomness). The notion which does the job is the *weak* n-randomness (Definition 18), since

Theorem 5 ([31] Theorem 7.2.7, p.288) *Every n-random set is weakly n-random and every weakly $(n+1)$-random set is n-random*

and

Theorem 6 ([31] Theorem 7.2.28) *A set is Solovay n-generic iff it is weakly n-random.*

Finally

Corollary 2 *QM is Solovay ω-generic.*

This is because for any $n \geq 1$ there exists $n+1$ such that QM is $(n+1)$-random and from Theorems 5 and 6 QM is n-weakly random for any $n \geq 1$. But this is equivalent to n-Solovay genericity for any $n > 1$. □

3.3 QM and Solovay generic randomness — the infinite-dimensional case

We have just established the result that QM inherently deals with arithmetic random sequences, which means they avoid n-arithmetic measure zero sets. If such an infinite random sequence is represented by $r \in 2^{\omega}$, we can apply *miniaturisation* of Solovay forcing to the arithmetic realm [31] and consider such ω-Sg-random r as $r \notin L_{\omega}$ and $r \in L_{\omega}[r]$, where this last extension is the arithmetic Solovay forcing. L_{ω} represents Peano arithmetics and thus the forcing in this model is in fact realizable in any ZFC model M. In this section we will make one step further and show that turning to the infinite-dimensional Hilbert spaces, the suitable notion of randomness in this case 'enforces' the appearance of the set-theoretic Solovay forcing in a model of set theory expressing QM.

Before we proceed, it is important to see more clearly the relation of arithmetic Solovay forcing and set-theory random forcing. Both forcings are related to their languages, PA and ZF and give rise to results independent of the corresponding axiomatizations. However, in the realm of PA there exist independent statements which are also ZF independent (a good explanation of this issue is given in ref. [42]). Certainly there are ZF independent results which are not PA-generated (i.e. generated by arithmetic forcing in PA), since models of PA can be built within set theory and not conversely. This is why, while knowing that QM is Solovay arithmetic random, we still are searching for direct mechanism in QM generating and distinguishing these hyper random sequences from say 1-random. One way is to show that the set theory Solovay generic sequences are assigned to QM, the scenario followed by us in this section.

We know that there exists a model of ZFC, e.g. L_α, in which the mathematics of QM is expressible. In particular, given the Hilbert space $\mathcal{H}$ in L_α, we can define the lattice of projections $\mathbb{L}$ in L_α. There are three important observations for an infinite-dimensional $\mathcal{H}$.

1. One can not take L_ω instead of L_α, since e.g. we do not have the axiom of infinity in L_ω.
2. The local maximal complete Boolean algebras B's chosen from $\mathbb{L}$ (see Lemma 20 in the Methods section) are now atomless measure algebras in L_α.
3. A way to the classical 2-valued realm goes through the homomorphisms $h : B \to 2$.

The last point will be applied frequently below and requires more detailed understanding. Given a model where QM is described (the mathematics of QM is definable) like L_α and $\dim \mathcal{H} = \infty$, then B's are atomless measure algebras in L_α (see 2. above). Following [86], let M be a transitive model of ZFC and B the atomless Boolean measure algebra in M; then a homomorphism $h : B \to 2$ is M-completely additive whenever for any family $S \subset B$ for which there exists $\sup S$ in B (certainly in M) it holds

$$h(\sup S) = \sup \{h(s) \mid s \in S\} \text{ in } M\,.$$

The following result is crucial now (see [86, p.35] and ref. [60])

Lemma 10 *$h : B \to 2$ is M-completely additive $\iff$ $\mathcal{U} = h^{-1}(1)$ is a generic ultrafilter over M in B.*

However, the fundamental result from forcing extensions for countable transitive models M of ZFC (or for certain uncountable generalisation of them [86]) states that [48, p.7]

Lemma 11 *B is atomless in M $\iff$ a generic ultrafilter $\mathcal{U}$ of B is not in M.*

Taking $M = L_\alpha$ from Lemmas 10, 11 and Theorem 2 it follows that

1. $h : B \to 2$ is not in L_α even though $B \in L_\alpha$.
2. $\mathcal{U} = h^{-1}(1)$ is not in L_α.
3. $\mathcal{U} = r \in L_\alpha[r]$ where $L_\alpha[r]$ is the random Solovay extension of L_α.

The entire forcing constructions can be alternatively formulated via Boolean-valued models of ZFC as e.g. in Lemma 5. In the QM lattice of projections case it follows that to reach the classical 2-valued realm, i.e. to pass from the local Boolean $B \subset \mathbb{L}$ window to the 2-valued one, we have to go through the random forcing extension. Thus working in L_α and agreeing that the local Boolean contexts are the maximal complete Boolean algebras of projections chosen from $\mathbb{L}$ and that they lead to the 2-valued realms, we have to accept the random forcing extensions of L_α i.e. $L_\alpha[r]$. This last becomes the necessary component of the QM formalism on infinite-dimensional Hilbert space.

The Boolean-valued model L_α^B is realised internally in L_α [82], thus L_α^B are now indexed by local B's from $\mathbb{L}$. This point of view has been explored more extensively in [60].

Now we want to distinguish between the two cases: one that QM is formulated in a ZFC model M, like L_α, and another that QM is formulated in the entire universe V of sets or, even broader, in the informal environment of everyday physical reasoning. Let us quote again the Benioff's words in this respect [9]

> There is no suggestion in the literature that the real universe, V, of sets, if such exists, is as small and simple as L_α [originally M_0]. However, the point that can be made is that if there exists a universe V of sets as a ZFC model, which is more real than the others, then why V is one ZFC model and not another needs explaining. [...] physics may have something to say about this problem.

One of the results of our work is to present arguments in favour (almost 45 years after the original work by Benioff) that indeed QM needs a model of ZFC which is different than V. However, this is not to say that such a model would be an absolute, unchanged environment for physics (or just for QM). Rather, the change of models (random forcing extension) becomes a valid ingredient of the QM formalism, being a reason that it is more likely to consider the model and its extensions as a proper formal environment for QM, distinguished also by reality. The arguments do not serve as any formal proof that QM has to be enclosed by certain ZFC model, but we think that this option will be verified affirmatively in real experiments (see subsection 3.5 below).

Recall that for infinite-dimensional Hilbert spaces, one has to deal with random forcing extensions within QM (see points 1. - 3. after lemma 11 above). We have three possible (and so far formal) scenarios: 1. Random forcing is realised in the universe of sets V, 2. Random forcing is realised in L_α (or, perhaps, in another transitive set-model of ZFC), 3. Random forcing is realised in certain class-model of ZFC (e.g. inner models L, $L(\mathbb{R})$ etc.). We will not discuss the last option here — this will be a topic of the forthcoming work. We also do not discuss here some other categorical models for set theory like toposes or the like. We are down-to-earth in this respect and try to not introduce formal constructions without clear reasons for this, coming from QM itself. Thus our concern here is QM in V versus QM in another set-model of ZFC (represented by L_α) and this translates to 'random forcing in V' versus 'random forcing in a set-model'.

If QM is Solovay generic random, then there should exist a random sequence $r \in 2^\omega$ (realised formally by QM) which omits all measure zero sets in the considered model, i.e. V to begin with. Thus given the universe V of sets, we should have its extension by random forcing adding r to V. As is well-known, this kind of random forcing extensions of models of set theory which are not countable is problematic since e.g. for the uncountable models there frequently may not exist generic filters for the posets of forcing conditions (e.g. [7, p. 98]). For the universe V there is also the related difficulty, that there are already 'all' real numbers in V, i.e. entire Cantor set 2^ω, and none such r in V can omit all zero sets. Thus one can follow the constructions of Boolean-valued models and build V^B and recover its construction in a standard transitive set-model like L_α. Then the random forcing extension is also performed in L_α and the formal conclusions are valid for 'real' V^B/Ult, where Ult is some generic ultrafilter in B in L_α. This V^B/Ult is considered as a forcing extension $V[r]$ (see [7, p. 99-101]). However, this kind of forcing extension of V reduces in fact to the extension of L_α.

Another possibility is to consider the forcing over V as a way to prove theorems without even mentioning models of ZFC [65, p.281]. To this end one just needs to define the forcing relation $\Vdash$, ordering the formulas. In this case, however, one can not build random sequences extending V. So again true random formal environment for QM seems to be given by models of ZFC like L_α rather than V, since the proofs are less important than the model extension by adding a real. Thus having established that QM is Solovay generic random on infinite-dimensional Hilbert spaces and that there are infinite random sequences generated by the structure of QM or by outcomes of experiments (where Born rule for a single trial already determines randomness of the entire sequence), we arrive at the heuristic conclusion that (for infinite-dimensional Hilbert spaces)

QM in reality favours the situation where QM is realised in certain standard transitive models of ZFC different from V. (21)

Such possibility was so far formally applied by us in analysis of complexity of black holes in [62] and with drawing conclusions about the cosmological constant from the nontrivial model-theoretic vacuum of spacetime in [60]. Whether such an option for QM is indeed chosen by Nature should be decided experimentally (see sections 3.5 and 5).

Let us close this section by showing that none of the Cohen forcing extensions $M[r_C]$ contains any Sg random sequence, as claimed in the section 3.1. This result is a consequence of the following lemma. Let $R(M)$, $C(M)$ be sets of all generic reals, random and Cohen respectively, over a transitive ZFC model M. Let $\mathcal{N}$ and $\mathcal{M}$ be ideals of null and meager sets respectively in the space of Borel sets. The set of all Borel sets of measure zero coded in M is then $\mathcal{N} \cap M$ and the set of all meager sets coded in M is $\mathcal{M} \cap M$. The set of all real numbers in M is $2^\omega \cap M$ and in $M[r]$ it is $2^\omega \cap M[r]$, similarly for $M[c]$. Then it holds [64]

Lemma 12 *Let $r \in R(M)$ and $c \in C(M)$.*

i) It is provable in $M[r]$ that $2^\omega \cap M \in \mathcal{M}$ and $2^\omega \cap M \notin \mathcal{N}$.
ii) It is provable in $M[c]$ that $2^\omega \cap M \in \mathcal{N}$ and $2^\omega \cap M \notin \mathcal{M}$.

It follows that even adding a single Cohen real c to M by Cohen forcing leads to $M[c]$ where $R(M)$ is empty. This is the reason that none Cohen forcing extension of L_α contains any random Solovay real — similar result has been obtained originally by Benioff by different means [10].

3.4 Forcing and randomness in QM

The purpose of this section is to demonstrate that the presence of the Solovay random sequences (random reals) is a generic feature of QM formalism (arithmetic Sg for finitely many dimensions and set theory Sg for infinitely many dimensions of $\mathcal{H}$). We focus here on the measurement process and problem of hidden variables. Thus we will obtain the result that there always exists a random sequence $r \in 2^{\omega}$, which is added in the generic measurement process. In the subsequent subsection we will show how the mechanism of adding random reals to models of set theory is related to hidden variables in QM. Both results require $\dim \mathcal{H} = \infty$. In the section 3.5 we will analyse the question about eventual practical use of that strong result.

3.4.1 Measurement and Randomness

This subsection can be seen as extending the ideas in [76], where the authors observed that in quantum measurements there are inherently associated problems of independence of logical formulas on certain sets of axioms coded by quantum states. They have analysed the logical independence from the point of view of information-theoretic perspective and found that there are finite sets of axioms and, independently of them, a formula can appear in experiments. This would correspond to arithmetic algorithmic randomness and the n-Solovay genericity in this context. The independence results are usually represented by forcing relations in rich theories like ZFC and we make the point that for infinite-dimensional Hilbert spaces, experiments in QM are carriers of set-theoretical random forcing.

Moreover, the authors of [76] also claim that

> [...] quantum probabilities can be seen as following from logical independence of mathematical propositions which are associated to the measurements without invoking quantum theory itself.

Here we find that indeed quantum experiments (in the systems on infinite-dimensional Hilbert spaces) typically go through the mixture of Boolean stages, which are not quantum but Boolean and despite this, they code the set-theoretical Solovay genericity and randomness. Hence, indeed, they do not need QM formalism in this sense. We also claim that this is a primary feature responsible for the quantum probabilities, although we do not develop this aspect here further, leaving that analysis to a separate publication.

A quantum measurement of an observable $\mathcal{O}$ in a state ψ will be called *generic* if the state ψ before measurement and the resulting state ϕ after the measurement, are different (nonparallel) vectors–states in infinite-dimensional Hilbert space, so ψ is not any eigenstate of the observable $\mathcal{O}$. We want to show that then a nontrivial random forcing is to be performed over a model of ZFC where QM is formulated in. First we prove

Lemma 13 *Any generic measurement in QM over a system formulated in infinite-dimensional Hilbert space of states, determines the pair (Boolean algebra B, ultrafilter $U \subset B$) where B is the measure algebra.*

From the point of view of the entire QM lattice $\mathbb{L}$, performance of the measurement m gives rise to the classical context, which is given for $\dim \mathcal{H} = \infty$ by B — the measure algebra. This is due to the fact that maximal atomless Boolean algebra of projections chosen from $\mathbb{L}$ for infinite-dimensional Hilbert spaces is the measure algebra B (see the 'Methods' section Lemma 19). Repeating the measurement in the same state, as the one fixed by just finished measurement, gives rise to the same value (with certainty) for the measured observable. Thus the procedure distinguishes all true elementary Yes/No questions as answered affirmatively within entire context of compatible projections (questions) in the maximal algebra B. This is determined by the spectral measure assigned to the observable $\mathcal{O}$, which in fact defines $\mathcal{O}$. In this way we have the valuation $h_m : B \to 2$ that is completely additive. The set of all 'true' projections (those with eigenvalue 1), compatible with the outcome of $\mathcal{O}$, defines the ultrafilter in B:

$$h_m^{-1}(1) = U_m \subset B\,.$$

The measurement m determines the local context B and the ultrafilter U as stated in the Lemma. □

Lemma 14 *Relative to a CTM M of ZFC, such that L_α is a submodel of M, any generic measurement in QM determines the pair*

$$(\textit{the Boolean algebra } B,\ \textit{generic ultrafilter } U \subset B)$$

in M.

This is just the relativisation of the Lemma 13 to any CTM M of ZFC. The condition $L_\alpha \subset M$ ensures that QM is properly represented in M. □

Let M be a carrier for mathematics of QM (QM be formulated in M). Then for observables acting on infinite-dimensional Hilbert space of states it holds true

Theorem 7 *Every generic measurement in QM (on systems with infinite-dimensional Hilbert spaces) gives rise to the classical realm by adding a random infinite sequence to the minimal CTM of ZFC.*

If a classical realm just after the measurement is attained, then the consistent valuation h_m over the maximal Boolean context is well-defined too. This fact can be understood in terms of a pseudoclassical state ψ_{psc} for certain (not necessarily commutative) propositional systems considered in QM (see the next subsection and [92]). In any classical realm such a state has to exist. For finite commutative propositional systems in QM [92] this state is always one of the states of the system. For propositional systems being atomic Boolean algebras (aBA), ψ_{psc} is one of the states on which (on 1-dimensional subspace of $\mathcal{H}$ containing ψ_{psc}) certain $p \in$ aBA projects. The true interest here is the case of atomless Boolean algebras of propositions (projections) and $\dim \mathcal{H} = \infty$. In this case, ψ_{psc} is precisely given by the ultrafilter on the measure algebra B [92,19], thus attaining the classical realm, which means having the ultrafilter ψ_{psc} defining valuation $h_{\psi_{psc}}$ on B such that $h^{-1}_{\psi_{pcl.}}(1) = \psi_{psc}$. Given a model M to which we are relativising the above construction, it holds $\psi_{psc} \notin M$ and $\psi_{psc} \in M[\psi_{psc}]$. Such a generic ultrafilter corresponds, however, to a generic random real [49] $r \in M[r] = M[\psi_{psc}]$. This r is represented as infinite binary random sequence with respect to $M = L_\alpha$. $\square$

Given the result established in the previous section, that QM is being realised in a standard model M of ZFC, in reality the random sequences within such 'localised' QM are represented by the random reals over M. Many interesting things can arise due to such 'world in a world' perspective which can be grasped by model theoretic means (e.g. [60]). We will comment in the next subsection that the classical realm as in Theorem 7 is described by localised HV (to a Boolean context) with respect to the quantum system under which the measurement has been performed.

Let us turn to the Boolean mixture of states which we have postulated in the 'Measurement' subsection of section 2 and which should be assigned to measurement process before the complete classical reduction to a 2-valued mixture of quantum states. The entire idea is based on the following observations.

i) Given a model M (e.g. CTM) and the complete measure algebra B in it, we are building in M the Boolean-valued model of ZFC, M^B (note that M^B is defined in M, cf. [82, p. 50]). This process is similar to building the Boolean universe of sets V^B internally to a model M [7].
ii) Let M be a model comprising the mathematics of QM. B is also present in the QM lattice of projections in M.
iii) QM in M involves inherently also Solovay forcing extensions $M[r]$ such that r are random reals not in M.
iv) $M[r]$ are canonically defined by quotient operation of the Boolean-valued model M^B by the ultrafilter U_r corresponding to r, i.e. $M[r] = M^B/U_r$ (see Lemma 5).

The above points indicate that QM and M^B are tied together quite strongly. We claim that such a statement is more than a formal coincidence and in fact M^B is a part of the structure of the measurement process in QM. This is where the Boolean mixture of states occurs. Note that in addition to i) - iv) above, taking $M = L_\alpha$ we have

v) L_α encloses QM and all random forcing extensions in an interfered mixture L^B_α. The 2-valued random reals emerging from the mixture L^B_α transcend the model L_α and constitute $\{L_\alpha[r] \,|\, r \text{ is a Solovay generic over } L_\alpha\}$.

That is why, given QM realised in L_α, we take L^B_α as the Boolean stage in the real measurement process: *Any measurement in QM factors through Boolean L^B_α before reaching the 2-valued classical stage.* In fact we easily show

Lemma 15 *Let QM be in L_α; then, the existence of the Boolean stage L_α^B in a generic measurement is equivalent to the existence of random forcing extensions for QM.*

The Boolean stage L_α^B determines every random extension $L_\alpha[r]$ by L_α^B/U_r and conversely, every random extension $L_\alpha[r]$ is isomorphic to L_α^B/U_r, which means that $L_\alpha[r]$ factors through L_α^B. $\square$

Theorem 8 *A generic measurement process in QM assigns the intermediate Boolean stage.*

From Theorem 7 we infer that random forcing extensions are assigned to measurement processes so the Boolean stages either. $\square$

The Boolean stage can be approached further by being represented by the mixture of states which are not derived from any pure state in QM. Thus it extends essentially the purification principle in QM (see e.g. ref. [39]). Moreover, the Boolean mixture of states arises precisely where classical spacetime overlaps the quantum realm and this overlapping is not reduced completely neither to pure quantum nor pure classical realms. Both these aspects will be addressed in a separate publication.

3.4.2 Hidden variables and randomness

As we discussed already in the 2 section, the very notion of hidden variables has several formulations; the strongest one that belongs to von Neumann [73] is the local, context independent LHV. Such LHV predicts that all potential measurements could lead to deterministic outcomes, irrespectively of setups of measuring devices and QM contexts. This was already refuted by von Neumann and also observed to be in contradiction with Kochen-Specker theorem for $\dim \mathcal{H} \geq 3$. On the level of quantum propositional systems (PS) (see [92]), the existence of HV entails the existence of the pseudoclassical state ψ_{psc} such that any proposition $P \in$ PS emerging in the measuring process in QM can be valuated uniquely as 'true' or 'false' in this state ψ_{psc}. If the semiclassical state exists independently on the contexts and interactions (measurements), then the emerging HV are in fact LHV. It is a well-known fact that there are no LHV for $\dim \mathcal{H} \geq 3$ and that there are possible (weaker) LHV for $\dim \mathcal{H} = 2$ (i.e. for certain subclasses of nonabelian PS's the semiclassical states exist) [92].

It seems that there is a natural and intuitive relation between randomness of experimental outcomes and the existence of hidden variables in QM. Namely, the existence of both should be inversely correlated, in a sense that the more random a phenomenon is, the less deterministic it appears and thus the hidden variables allowing for strict determination of the outcome are less probable to exist. In other words, if there were hidden variables for QM, then the description of a quantum system S by the use of such strong HV might be deterministic, hence an intrinsic randomness would be redundant.

In general one can not refute the existence of nonlocal hidden variables in QM (see also subsection 2.3). Based on findings regarding genericity in the previous sections and the results about HV and genericity discussed in [92, 19] we can work out certain important properties of HV and randomness via models of ZFC. We proceed as follows. QM can be realised in the standard model of ZFC like L_α. Then the Solovay forcing extensions of L_α correspond to local Boolean frames in QM and to adding a random sequence $r \in L_\alpha[r]$ as in the previous subsection. One of the main results of Van Wesep [92] was showing that

> [genHV] Any quasi-classical state in QM (i.e. that corresponding to HV) is realised by a generic ultrafilter G in an (not necessary Boolean) algebraic system of propositions. In particular it follows: For infinite-dimensional Hilbert space any realisation of HV factors through random forcing extensions of local Boolean frames in $\mathbb{L}$.

This is due to the observation that forcings (for infinite-dimensional Hilbert spaces) are determined by completely additive homomorphisms $h_r : B \to 2$, where B is the measure algebra in L_α and the generic fillters in the Wesep's algebraic systems of propositions (containing B) are also determined by true propositions about the state ψ_{scl}. It follows that the realistic LHV for QM require the existence (in infinite-dimensional case) the existence of random generic extensions for any local frame B in $\mathbb{L}$. One obvious situation guaranteeing this would be the existence of all possible random extensions in such LHV theory (say MWI). This is, however, easily refuted for CTM of ZFC due to the following, quite natural supposition

> [HV] If HV exist (in a possibly weaker form), then there exists a ZFC model N containing the realisation of this HV.

Lemma 16 *There does not exist any CTM N containing all random extensions of L_α.*

This is due to the existence of uncountably many of random forcing extensions over L_α outside (in the entire set universe V), while in a CTM N there are merely countably many such extensions (random real numbers $\subset R_N$). □

This is something to be expected from the point of view of realisation of different contexts in a single CTM model N — noncommuting observables cannot be measured simultaneously. However, such result seems to be highly excessive, since it should hold even for two different incompatible contexts for $\dim \mathcal{H} \geq 3$ and not only for as many as uncountably many of them.

Given two different MASA's (or equivalently two different maximal Boolean algebras of projections from $\mathbb{L}$, contexts) with respect to which two noncommuting observables are defined, there should not exist any single measurement (expressed by a third Boolean context — MASA) recapitulating possible values of these two observables. In forcing terms:

> [HV+forcing] There exist two different random forcing extensions, $M_1 = L_\alpha[U_1], M_2 = L_\alpha[U_2]$, such that there does not exist any other forcing extension $M_3 = L_\alpha[U_3]$ such that M_1, M_2 are submodels of M_3.

Recall that all $L_\alpha[U]$ preserves ordinal numbers of L_α (Theorem 2 (14)), thus the below result by Hamkins resolves in affirmative the above claim

Lemma 17 (Hamkins[44]) *For any CTM M and its random extension $M_1 = M[U_1]$, there always exists a CTM $M_2 = M[U_2]$ of ZFC such that there does not exist any forcing random extension $M[U]$ with the same ordinals and containing M_1, M_2 as its submodels.*

Thus interpreting incompatible MASA's (those corresponding to noncommuting observables) by non-extendible forcing extensions as above, we have a set-theoretical obstruction for the existence of HV. This procedure works even for the pair of noncommuting observables (one does not need to take infinitely many of them).

Note that all forcing extensions $L_\alpha[U]$ coexist virtually in the Boolean model L_α^B (Lemma 5 (16)) since

$$L_\alpha^B/U \simeq L_\alpha[U]\,.$$

However, taking quotient by U_1, i.e. L_α^B/U_1, means that incompatible frame $L_\alpha^B/U_2 = L_\alpha[U_2]$ is automatically given by nonextendible simultaneously ultrafilter (context), as in Lemma 17.

The interesting question to ask now is whether it is possible to overcome set-theoretical difficulties as in Lemma 17. Indeed, one attempt of this kind is to allow for models of ZFC which require additional ordinal numbers. In that case one can find a model of ZFC containing both $L_\alpha[U_1], L_\alpha[U_2]$ as submodels. We will address this in a separate forthcoming publication, noting here only that such strong models will be generic for the description of spacetime in the quantum regime and hence connected with a version of quantum gravity emerging in this context (see also [60]). From the point of view of QM, such super-models with additional strong set-theoretical suppositions will be applied to characterise the setup of no-signaling QM theories, which are able to produce correlations stronger than predicted by Bell's inequalities. These kinds of theories shed also new light on the realisation of LHV within this stronger setup (e.g. [81]).

3.5 Is QM random?

Can we decide by any experimental verification whether QM is really random or not? This seems to be a difficult problem. We found formally that any measurement in spacetime with a wavefunction reduction leads to a kind of HV (localised in spacetime) and adds a random sequence to the minimal model of ZFC. However, is it really possible that we can find any direct experimental verification for the fact that QM is actually encoded in L_α, or in any other model of ZFC? One indication toward such possibility is the formalization [60] which has led to the layered structure of spacetime, where different models of ZFC (corresponding to different maximal Boolean algebras in the lattice of projections) appear. One important ingredient of this construction was the formulation of QM in a certain base (possibly countable) model of ZFC. If we ensured ourselves that indeed Nature has chosen such solution, i.e. QM is in a model of ZFC rather than in a broader universe like V or eventually in partly informal physical environment, we would know that QM *has to* act through the models where it is formulated in. This is especially important when we investigate the interaction of quantum- and general relativity spacetime regimes. From the point of view of randomness, the claim that a single sequence of outcomes is random is out of reach. However, when QM is indeed formulated in certain model, say L_α, we would be sure that such sequences *have to* appear in real physical processes in the overlapping regions of QM and spacetime (e.g. [62]). Presumably such effects can be assigned to regime where quantum gravity (QG) description becomes valid. From that point of view, one procedure for detecting randomness in quantum mechanics in reality could be analysing the spectra and disturbances in the CMB data. The expected results of such analysis could be to find traces of set-theoretical forcing, considered now as physical process, which would indicate the existence of models of ZFC, changed by random forcing. Thus searching for a realisation QM in a CTM of ZFC in Nature could be in fact looking for primordial footprints of the countable models or/and forcing in CMB map. If confirmed, there have to exist random sequences relative to the models.

Another related possibility would be to draw and analyse physical conclusions from the existence of models of ZFC, in which QM is supposed to be formulated. This 'QM internal to models' would probably become a necessary ingredient of quantum systems, or even entire Universe or spacetime, on certain stages of their developments. If we would be able to collect representative collections of such conclusions then we could consider the use of ZFC models in the context of QM as a valid ingredient of our understanding of the world around us. This is the work under development, but we can mention that the value and origins of the cosmological constant has been obtained by us, based on the assumption of QM internal to models of ZFC (see [60]). Moreover, the first step toward understanding the structure of black holes (as well as quantum regime of spacetime) has been performed again on the supposition of 'QM internal to models' (see [62]) and we expect that similar technique will shed a light on the problem of information loss in black holes (work also in progress). This kind of efforts belong rather to indirect proofs of randomness of QM.

In spite of lack of defined procedures and sufficient experimental precision in analysing CMB data, one could instead perform more direct attempts to find strong randomness around us. This possibly can go through careful analysis of the experimental results of QM or designing new experiments. A very promising direction is the one coming from the loophole-free Bell-like experiments, where the violation of Bell inequalities is certified by impossibility of superluminal signals in spacetime [13, 78]. In the context of the analysis in this paper, the point would be to design and perform experiments, which not only show the algorithmic (arithmetical Solovay genericity) randomness of sequences, but also indicate their set-theoretic Solovay genericity. The loophole-free Bell tests as proposed and described in [13, 78] indeed lead to the generation of true random numbers which, however, belong to the arithmetic class. This is due to the finite number of axioms coded by the states and finite-dimensional Hilbert spaces underlying that. The flat Minkowski structure of spacetime guarantees the constant and finite value of the speed of light, hence forbids any superluminal signals, and thus protects arithmetic randomness of QM. Can we modify somehow the procedures toward detecting set-theoretical randomness by some new mechanism, protecting this randomness in spacetime? Let us add nonvanishing 4-curvature to this scenario. We consider the possibility that there exist certain curvature-driven tests, whose passing would indicate the set-theoretic Solovay randomness of QM in spacetime. The possibility that the curvature fluctuations in the CMB spectra would code (in principle) varying models of ZFC for QM, belongs to this class. On theoretical ground there is a kind of fundamental connection between curvature of certain open smooth 4-manifolds and topology. This opens the possibility, that the Solovay genericity in models of ZFC would be supported by topological invariants underlying 4-curvature. A particularly attractive direction in this context is an exotic smoothness in dimension 4 (exotic smooth open 4-spaces like exotic R^4), which has been recently shown to be responsible for the violation of the strong censorship conjecture (SCC) [34,36]. Such results indicate that

1. A degree of spacetime unpredictability in SCC is now translated to gravitational instantons, represented by exotic R^4 breaking SCC [36].
2. Exotic R^4's are Riemannian, non-flat (in any of its smooth metrics) smooth manifolds and their curvature is topologically protected. This last leads to observed physical effects as e.g. the cosmological constant [5].

It follows that topological invariants underlying exotic R^4 may serve as suitable spacetime certificate for the Sg randomness of QM. Interestingly, it was shown recently that quantum computations can be based on the topological constructions underlying exotic R^4's [79]. This indicates a rather deep theoretical connection on the level of topology between spacetime structure, QM and randomness. From a practical point of view, since gravity is very weak comparing to other forces we need extreme densities of energies or cosmological scales of our Universe to 'see' gravitational effects responsible for randomness of QM (see [62,35,87]). Even in the case of confirmation of the Solovay genericity in such extreme phenomena, their practical applicability would require much additional and conceptual work. Eventual useful working generators of such topologically certified Solovay generic random 'numbers' have to wait until the randomness is confirmed and this seems to be out of the scope of the current research. However, one promising direction, allowing even for laboratory investigations of set-theoretical Solovay genericity of QM, could follow from the recent work on searching for quantum gravity effects on interacting gravitationally micron size masses in spacetime [20,84]. The authors claim that there are different spacetime regions which undergo entanglement, which 'entangles' the spacetime structure into QM entanglement and could lead to possible measurable effects. There are arguments that gravitons have to mediate this interaction between the masses. Such nontrivial space-

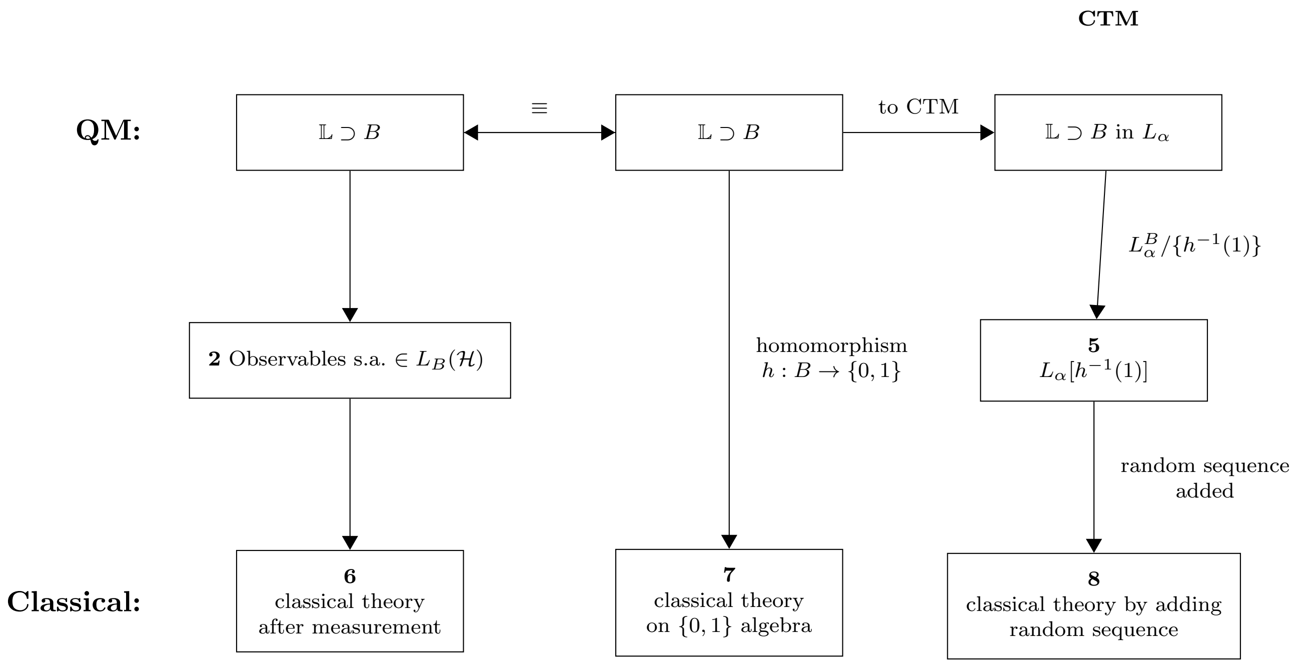

Fig. 1 Three different yet parallel ways to reach classical environment from QM, which are the basis for set-theoretical Solovay genericity of QM. The most right column results from embedding QM internally to L_α and extending by adjoining a random sequence.

time structure and gravity presence in QM entanglement could be a potential source for Solovay generic random numbers. Certainly, conceptual and experimental investigations in future will show whether this option can lead to such effects.

4 Discussion

Let us quote the words of Hans Freudenthal from 1969 [66, p. 8]

> 'It may be taken for granted that any attempt at defining disorder in a formal way will lead to a contradiction. This does not mean that the notion of disorder is contradictory. It is so, however, as soon as I try to formalize it.'

Randomness and total disorder are different names for the same phenomenon. By placing QM into a ZFC model M, we are close to grasping randomness as forcing properties. Thus we have an access to the external perspective (namely the universe V), where random forcing, extending M, is well-defined and leads to the formalization of randomness by varying models of ZFC. The method of formalization has been successfully employed, what led to the recovering

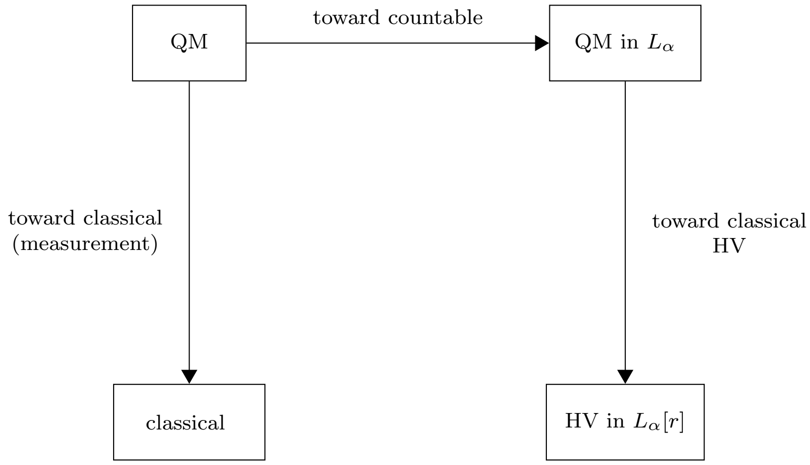

Fig. 2 We have two directions of transforming QM: one is the measurement process leading to classical realm, the other is the transformation from continuous to countable realm. QM, when formulated in a countable model like L_α, uncovers its strict random nature, which is another complementary side of a measurement. On the countable realm one can understand the appearance of the localised HV in the extended model $L_\alpha[r]$.

new aspects of QM, together with the new concept of quantum randomness. In particular we have demonstrated that QM is generic Solovay random, in a sense that for finite-dimensional Hilbert spaces it is arithmetic (algorithmic) ω-Solovay random, while for infinite-dimensional Hilbert spaces QM it is a set-theoretical Solovay generic random. We also found that there exists a standard transitive model of ZFC, in which presumably QM is formulated in reality. This last result was established heuristically by collecting formal arguments against opposite hypothesis that QM is in the universe V of set theory (including the informal version of QM). Another possibility would be to take nonstandard models of set theory for QM or more broadly models in categories. This kind of models indeed appeared already here, where we have discussed the Boolean-valued models of set theory, which are in fact Boolean toposes (sheaves of sets on atomless Boolean algebras) extending the 2-valued classical models to Boolean-valued ones. The next step would be to consider models in non-Boolean toposes, i.e. intuitionistic ones, or in other categories. These cases are not analysed here even though they could presumably appear in some regime of the final quantum theory of gravity, where logic could be seen as a physical variable.

The randomness *à la* Solovay appearing in QM could be considered as a purely mathematical fact, which is formally unavoidable. The extended set-theoretical randomness is a part of QM formalism, whenever a quantum system overlaps with the quantum regime of spacetime (infinite-dimensional Hilbert spaces). All of this is a strong formal indication supporting eventual efforts for the search of experimental verifications of set-theoretical Solovay randomness, which would extend the case of algorithmic Solovay genericity for finite-dimensional state spaces. Some of the issues like those are discussed in sections 3.5 and 5, where one can find indications eventually leading to direct and indirect tests of randomness in QM, with particular emphasis on extended randomness. Below we will concentrate on certain consequences or speculations about the latter, which have not been touched in the main text.

4.0.1 Forcing and many worlds interpretation of QM

The formalism of QM, which we have reformulated such that random forcing became a crucial part of the structure and the carrier of randomness, highlights also another feature important from the point of view of interpretations of QM. The idea is that different models of ZFC represent different 'worlds' as in many-worlds interpretation (MWI) of QM. So far we have been working in Hilbert space traditional formalism of QM, leaving aside deep interpretational questions. The thorough analysis of the role and explanation of the enhanced Solovay randomness from different interpretational points of view deserves further studies. In the original MWI formulation we have incompatible (independent) worlds external to the formalism of QM, where each of the worlds represents the possible outcome before measurement-driven collapse of the wave function, thus choosing a single one with known Born probability. The intrinsic randomness of QM in MWI seems to be absent if one takes the totality of possible worlds as real and any world does not interact with another [91,11].

In our approach, where alleged worlds would correspond to models of ZFC, another possibility appears. We have shown that such picture is already a part of QM formalism on infinite-dimensional Hilbert spaces at the price of certain modifications of MWI. Namely, in every measurement a random sequence is added by random forcing. However, this forcing extensions describe the births of new (set-theoretic) universes and this is unavoidable and ever-present in every (generic) measurement. Different and incompatible forcing extensions correspond to different incompatible local classical frames of QM observables. Indeed, set theory enables one to speak effectively about the process of birth and change of the (mathematical) worlds-universes. Thus we can summarise above as

> [ZFC and MWI] Within the Hilbert space formulation of QM, the set-theoretic universes correspond to physical universes in MWI scenario. In any generic measurement there are random (Solovay generic) sequences (reals) added by forcing and new worlds-models emerge. The worlds communicate by enhanced random channels which differ from the original MWI, where no communication is allowed. The new measurements in the extended models give rise to the births of new generic extensions which, again, can communicate the ground model with the extended ones by the enhanced random channels, and so on. From the point of view of the extended model $M[r]$ the ground model M is not

accessible randomly, but rather as a submodel (what is not provable in $M[r]$).

Moreover, Boolean intermediate stage of the measurement (see the subsection 2.1.2) allows one to speak about the state where all the (incompatible) QM windows create the Boolean mixture, where they interact in a specific and well-defined mathematically way. This is realised by the Boolean–valued model L_{α}^{B}. So our MWI scenario realised by models of ZFC (where QM is formulated in) predicts the enhanced (maximal) randomness in QM by which the model before and after the generic measurement can communicate. This may have an impact on the problems like no-signaling or causality in theories of gravity.

4.0.2 No-signaling and exceeding the Bell bounds

The suggestion that gravity, or rather spacetime, in quantum regime may be a source for enhanced randomness in QM requires some comments. Such scenario implies that QM may be somehow modified in the QG limit, where also spacetime undergoes changes and the amplified randomness results. In this paper's terminology, random forcing becomes the structural component of both QM and gravity with spacetime (see also [62,60]).

The possibility widely discussed in recent years is that QM would be one of the limiting realisations of the theories which break the maximal Bell bound (i.e. the Tsirelson bound) for the quantum correlations, i.e. $> 2\sqrt{2}$. Namely, the nonlocality and relativistic causality imply QM, although there is some space for causal nonlocal theories breaking the maximal Bell bound which are not QM [80]. Not being QM might indicate that the theories are not physical. However, when one allows for modifications of QM, the theories might serve as a useful alternative. There are no-signaling theories (preserving relativistic causality) and nonlocal ones, which realise maximal violation of the Clauser, Horne, Shimony and Holt inequality and thus demonstrate that superquantum correlations call for a form of nonlocality stronger than in QM. Thus maybe any valid extension of QM could be toward allowing for such strong nonlocality? Is it possible at all that QM would undergo such amplification of nonlocality? One such natural possibility would be the regime where spacetime itself becomes highly nonlocal, hence the spacetime-settled observer could notice superquantum correlations for quantum systems in such spacetime. There might be randomness for such amplified quantum systems as well [47]. However, as shown in [30,21] there are trivial information complexities (almost) always assigned to the superquantum correlations, hence the Authors claimed that the correlations are (almost entirely) excluded from the physical world, leaving QM as the only possibility.

Still, there is a need for a physical principle which should be added to fundamental laws, prohibiting such superquantum correlations or rather theories. The principle called Information causality (IC) has been proposed and analysed in [77]. This principle is to some degree similar to the relativistic causality, which fixes the speed of light c (in vacuum) in spacetime. However, c applied to entire spacetime can be exceeded (e.g. cosmological inflation), although this fact cannot be used to transmit information by any superluminal channel. Similarly, IC can be applied to entire spacetime, this time in a quantum regime. Such a spacetime model has been proposed in [60] where the structural ingredient is the random forcing extending models of ZFC. The same structural ingredient of QM appears also here, what makes both QM and spacetime compatible structures in a quantum regime. Moreover, the forcing extensions extend also spacetime given locally as R^4 in M to R^4 in $M[r]$. Such extensions are not restricted by IC. Namely, from the point of view of provability in ZFC there is no distinction between R^4_M and $R^4_{M[r]}$ and r can appear as trivial. However, from the external point of view of models of ZFC, the properties that are independent of axioms of ZFC indicate that there is no limitation imposed on the randomness of r and relative information complexity in spacetime (between local regions of spacetime which could be entangled [84]) allowing for forcing extensions. Thus structural inherent random extensions of spacetime are well-defined even though they are trivial from the point of view of ZFC axiomatic system. Such effects in spacetime can lead to indirectly observed consequences, similarly as inflation in cosmology. Nevertheless, the relation of a spacetime in quantum regime and quantum systems is a vast, unexplored territory and one motivation for studies of randomness in QM is the hope to better understand some parts. This work is under current development.

4.0.3 Future perspectives and research.

Even without assuming any connection of an enhanced randomness and no-signaling theories, there are still interesting questions in physics and mathematics regarding this new kind of random behaviour. They may seem speculative at present, although they are natural and they show possible further research directions. First such domain can be the one about quantum computations. We think that the Solovay set-theoretical genericity might correspond to the randomness characterising future quantum computers, similarly as algorithmic randomness is challenging the current RNGs. One way to such an enhancement is to include gravity in the quantum regime eventually, which means that there are models of ZFC where QM is defined in and they become active elements of the quantum regime of spacetime and gravity. Even though this seems to require cosmological scales (or subplanckian energies) for the verification at present, the entire task may find its realisability at laboratory scales even in the not-so-distant future [84].

Further, in high energies of Planck scale this kind of randomized effects could dominate spacetime, such that the change of the models of ZFC by forcing rather than (what happens in the) models themselves become the true fabric of spacetime. This is more-or-less the extension of the idea of the structure of quantum spacetime discussed in the previous subsection and partly investigated in [60]. Additionally, the point of view that the Planck regime of spacetime should be characterised by high complexity classes has been considered in recent years [87]. In his work Susskind et al. considered black holes as objects which should be described by formal computational complexity classes. The classes have much to do with degrees of genericity which, however, come from Cohen extensions rather than random ones. We have also analysed black holes from the Solovay random and ZFC models point of view [62]. Thus our work shows a kind of complementarity of generic formal descriptions such that it follows the duality of category and measure known and widely studied in mathematics (e.g. [75]). In general, one sees a dependence of the enhanced randomness and the quantum regime of gravity. Considering the relation seriously and making it even stronger, we come to the following hypothesis

> [QG and Rand] Detecting the QG effects is as difficult as detecting the new random effects of QM in spacetime.

It follows that testing experimentally QG can be considered as a quest for Solovay genericity in a spacetime in quantum regime. Thus we find certain indications in QM formalism, which may shed light on the fundamental problem of formulating a successful theory of QG, that are new and promising. The analyses of this kind of expectations and eventual substantiating of them will be performed in the future work.

Additionally, there has appeared rather accidentally a method of forcing employment in analysing the mathematics of smoothness structures on 4-spacetime (e.g. [58,12,59,60]). We expect to develop systematically this approach, which again should have an impact (through forcing) on QM and gravity as phenomena reconciled in physical spacetime.

Final important remark relates the freedom with the choice of a (standard, transitive) model of ZFC, which would be the best for QM formulated in it. Here 'the best' means best-suited for the description of phenomena in the real world, but also there are formal arguments, which were not explicitely stated here, distinguishing certain models. The choice of L_α is rather dictated by historical reasons (Benioff's proof), although it is also constructible minimal and simple. In our future work we are going to find further mathematical and perhaps experimental reasons for taking this or another model as 'the best' for QM. In this respect we consider the fact that continuum hypothesis may fail in quantum spacetime limit and the models can be uncountable as an important indication. Mathematics distinguishes certain models like Solovay or the universal model by Woodin, where additionally the presence of large cardinals could play some role in our quest for successful formulation of quantum gravity. In particular one can find gravitons in such formal spacetimes without engaging the entire procedure of the quantization of gravity field. The issue related to the above is an extension of ZFC by certain additional axioms like large cardinals. We think this is a central problem for choosing a particular model of ZFC. The results will be published separately but let us comment here on certain physical aspects of the approach. The main idea is that extensions of ZFC become necessary in high Planckian energies in spacetime. In such regime the description of spacetime and gravity requires models of ZFC with certain classes of large cardinals. One can try to relate the extensions of models with extreme processes in spacetime accompanying approaching singularities, with gravitational collapse leading to formation of black holes in particular. So the physical gravitational collapse would be correlated with the formal Lévy collapse of e.g. $\aleph_1$ to $\aleph_0$. The process is the same as used by Solovay to introduce random forcing in [86]. In this publication the model created via Lévy collapse describes real line of numbers such that all its subsets are Lebesgue measurable, which comes at the cost of an inaccessible cardinal appearance. This model is well-suited to the extended set-theoretical randomness considered here as generated by QM (see also [33]). The Solovay model is an inner model of set theory, i.e. all ordinals are present, so the relation to physics may indicate that one should work rather in inner models than in the minimal countable one. This is at least for Sub-Planckian energies characterizing, in particular, the near singularity region of black holes. Theses observations focus our attention on the mathematical inner model program (IMP) of Woodin, aiming at finding the ultimate model of ZFC compatible with various large cardinals (thus excluding the Gödel constructible universe). Here we come across the problem of continuum hypothesis in the context of physics one more time. Is it possible that the problem of a proper description of spacetime and gravity in extreme (singular) regime is somehow correlated with LC hypothesis and IMP program? We consider such direction as very intriguing and will present first results in the forthcoming publication. All this formal considerations in the context of physics constitute a new perspective for proper defining the QG puzzle in a quite unexpected way. The work on these topics is currently under intense development. The formal results presented in this paper seem to be merely a tip of the iceberg of this new approach to quantum-mechanical properties of spacetime and gravity. The approach which is uncovering deep interplay between fundamental aspects of set theory, like continuum hypothesis or the HOD hypothesis of Woodin, with unknown properties of spacetime, gravity and QM. Thus the problem of choosing the proper extension of the ZFC model for QM is likely to be highly nontrivial and fruitful.

5 Methods

The methodology of this paper relies on formalization and general reasoning, that allows to prove the main results and making the logical order of the presentation. We choose the tools of 1st order theories (like ZFC or PA) and their models, since PA itself is a natural home for algorithmic randomness considered in the literature and ZFC serves as a minimal generalisation thereof. The structure of reasoning is schematically represented in Figs. 1 and 2 and reflects horizontal and vertical ways of our approach to randomness in QM. The vertical direction is distinguished by the inherent logic of measurements in QM, which leads from quantum to classical regime. It is further decomposed in the horizontal directions into various stages connected with depth of formal models of ZFC. In general, this is a way from continuous to countable universes of set theory. The measurement process in the standard Hilbert space approach to QM performed on a quantum system S leads to the state of S

such that quick subsequent repetition of the measurement leaves the system in the same state and gives the outcome with probability one. The last one is a fully deterministic process and can be embedded and described in the classical environment. The entire path from quantum S, through measurement to the classical environment can be represented as

$$\mathbb{L} \supset B \xrightarrow{\text{s.a. oper.}} \mathcal{A} \xrightarrow{\text{measurement in } \psi_0\text{, eigenvalue pr.}} \text{deterministic repetitions of measurements in } \psi_0$$

Here $\mathcal{A}$ is the space of self-adjoint operators on the infinite-dimensional Hilbert space $\mathcal{H}$. At the same time, more formally one obtains that reaching the classical realm can be performed by 2-valued valuations on the Boolean contexts in $\mathbb{L}$. The next and even more formal stage is to take this in a standard transitive model of ZFC and subsequent step is to consider this construction in a CTM M of ZFC. This shows the necessity of random forcing in the mathematical structure of QM. The forcing is by the same token the instrument allowing for detection of the n-randomness of QM.

5.1 Mathematical randomness of QM

The method of showing that QM is random in a strict mathematical sense relies on formalization as defined in [60] and follows the argumentation that QM is 1-random from [68].

5.1.1 QM is 1-random

Let A be a self-adjoint operator in the space $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. The extension of the Born measure μ_A over 2^ω follows the general procedure of extending a measure μ on a measurable space (X, μ) onto the measure μ^∞ on X^ω. At first one considers cylindrical σ-algebra S of subsets of X^ω generated by sets of the form

$$\mathbf{C} = \prod_{i=1}^{N} C_i \times \prod_{k=N+1}^{\infty} X_k, \tag{22}$$

where $X_k = X$ for all $k \geq N+1$ and $C_i \in Bor(X)$ for all $i = 1, 2, ..., N$. Then the measure on all sets $\mathbf{C} \in S$ is given by

$$\mu_S^\infty(\mathbf{C}) = \mu(C_1) \cdot ... \cdot \mu(C_N)\,. \tag{23}$$

The probability measure μ_S^∞ on S ($\mu(X) = 1$) extends uniquely to the probability measure μ^∞ on X^ω, which is essentially the $N \to \infty$ case of eq. (22). The uniqueness follows from [32, Th. 8.2.2].

Thus given the Born measure μ_A on the spectrum σ_A of a self-adjoint operator A in the state ψ, we are looking for the induced measure on σ_A^ω where infinite sequences of the outcomes of repeating measurements of A in the state ψ live. This is essentially $\mu^\infty = \mu_A^\infty$ on $X = \sigma_A$ generated by $N \to \infty$ limit of eqs. (23) and (22) as discussed above.

Let us consider now $X = 2 = \{0, 1\}$ and take the product measure μ^∞ on 2^ω. The infinite binary sequence of QM outcomes can be obtained by repeating measurements of an elementary projection P in a state ψ, where the process is suitably prepared by making reset on a system every time each measurement has been performed (e.g. [9]). Note that $\sigma_P = \{0, 1\}$ for any projection P. Also in this case the Born measure μ_P determines uniquely the measure μ_P^∞ on 2^ω and this elementary case is sufficient to capture randomness of binary sequences in QM. Namely it holds [68, Th. 4.1, Cor. 5.2]

Lemma 18 *According to the measure μ_P^∞, the set of 1-random sequences in 2^ω has measure 1. The measure of the set of nonrandom binary infinite sequences generated by the Born measure μ_P in QM vanishes.*

It follows that the Born's probability of occurrence of infinite binary sequence of outcomes in QM which is nonrandom is zero. Given the definition of n-randomness, [n-RAND], of QM* and hence QM in section 3.2, we conclude that

Corollary 3 *QM is 1-random.*

This is extended to n-randomness via Solovay generic forcing extensions assigned inherently to QM in the main body of the paper (see section 3.2).

5.2 Measure algebra as maximal Boolean subalgebra of projections in $\mathbb{L}(\mathcal{H})$

Theorem 9 ([52], Th. 9.4.1) *Let $\dim \mathcal{H} = \infty$, let μ be the Lebesgue measure on the σ-algebra of Borel subsets on $[0, 1]$ and let $\mathcal{L}(\mathcal{H})$ be the extension of the space of bounded linear operators on $\mathcal{H}$ containing also unbounded operators. Maximal abelian von Neumann subalgebras (MASA's) of $\mathcal{L}(\mathcal{H})$ are unitarily equivalent to*

$N_a \oplus N_c$ *where* N_a *is the atomic part and* N_c *the continuous part*;

N_a *is generated by the projections on the vectors of the base in* $\mathcal{H}$;

N_c *is the algebra of (essentially bounded) measurable functions* $L^\infty([0, 1], \mu)$

Thus maximal complete Boolean subalgebras in $\mathbb{L}$ (for $\dim \mathcal{H} = \infty$) have the general form of

Lemma 19 *$B = B_a \oplus B_c$, where B_a is the atomic Boolean algebra and B_c is the measure algebra B .*

B is the algebra of Borel subsets of $[0, 1]$ modulo the ideal $\mathcal{N}$ of μ-measure zero Borel subsets

$$B = Bor([0, 1])/_{\mathcal{N}} \,. \tag{24}$$

Lemma 20 ([40, 52]) *B is atomless Boolean algebra.*

The above property really distinguishes infinite-dimensional case, since

Corollary 4 ([40, 52]) *If* $\dim \mathcal{H} < \infty$ *then maximal complete Boolean algebras of projections chosen from the lattice* $\mathbb{L}$ *are atomic.*

Due to the atomless property of B the nontrivial random forcing is an inherent part of the QM formalism (e.g. [48, Proposition 2.1]). This is also responsible for some random Solovay generic phenomena in QM.

5.3 Experimental tests for quantum-mechanical randomness

As already stated in section 2.2.1, the notion of arithmetic randomness is typically described by one of the following: patternlessness, incompressibility and unpredictability. Although often understood as roughly equivalent, they are verified quite independently though indeed are equivalent as 1-random phenomenon. Nevertheless for higher $n > 1$ they may diverge. In this section we comment on the possible tests for quantum-mechanical arithmetic randomness from the experimental point of view. The issue of eventual verification of the set-theoretic Solovay randomness of QM has been discussed in section 3.5.

Let us start with a brief discussion on a general procedure of random number generation. Recall that, being incomputable, it is not a feasible task to decide the randomness of a given binary sequence, generated by some (possibly hidden) algorithm. One can easily propose sequences that look much like fully random, such as binary expansions of π or Champernowne's number,[1] in spite of them acquiring values through well-defined, deterministic algorithms, hence being computable. It was von Neumann, who observed that "there is no such thing as a random number — there are only methods to produce random numbers".[72] The case for π can be taken to the limit by considering hypothetically perfectly random sequence, restored subsequently from the memory of some device.[11] The sequence itself clearly loses then its unpredictability. This simple example points at the fact that every practical randomness test (e.g. relative frequency, Borel normality, a battery of statistical tests etc.) provides only a necessary condition for the sequence to be random. In other words, such a test shows to what extent a sequence can be considered random, and which sequences are to be suspected as not random with a high probability. After all, Ramsey's theory suggests that even an infinite random sequence eventually would contain some patterns or correlations, hence could fail a particular statistical test in principle.[25]

Ultimately, the only way to evade above issues and to hope for the sufficient condition for randomness is to take the generation process into account. Therefore, to guarantee the randomness of a numeric sequence, an emphasis has been put on physical processes generating sequences and at least "certifying" some level of randomness. From the point of view of physics, the most convincing choices seem to be classical stochastic, classical chaotic and quantum-mechanical systems.

For the classical case, both stochastic and chaotic systems are deterministic in principle, although not entirely predictable. Indeed, it is an up-to-date and widespread technique to use the so-called pseudorandom number generators (PRNGs) for practical purposes.[71] These rely on the mixture of two components: first, a stochastic numeric component (a seed) is chosen, usually in the sense of a hardware-generated signal; second, a highly nonlinear function is applied recursively, starting with the seed value. The particular seed comes through variety of methods, from the computer's Unix time in microseconds to user's mouse movements. Although the "random" process ends immediately after the seed generation and the rest is completely deterministic and predictable, the methods of PRNGs are fast and sufficiently reliable for most applications. After all, cryptographic PRNGs are designed primarily to hide the deterministic algorithm; still, one should keep in mind that "anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin".[72] An exemplification of the quote could be the ongoing process of preventing cryptographic attacks by continuous PRNGs development.[55, 69] Still, the quality of randomness of such generated sequence depends somehow on the computer performance and a state-of-the-art cryptanalysis.

Therefore, if one is interested in random numbers coming from formally indeterministic processes, independent of a third-party resources, quantum mechanics seems to be the way to go. Here, the principles of a quantum-mechanical measurement (at least in the widely accepted Copenhagen interpretation with the Born rule, discussed in the earlier part) are believed to ensure that the outcomes of a particularly devised measurement cannot be predicted in principle. For example, a careful experiment conducted on the photons going through a polarization beam splitter leads to the sequence of allegedly unpredictable, random outcomes.[50] Such setups usually employ two-dimensional state spaces (as a space for a photon with two polarizations), due to their immediate interpretation in terms of binary sequences; note, however, that the only formal basis for such a sequence to claim its randomness is again the Born rule. Therefore, to stand on firmer theoretical ground, one turns to QM constructions that certify unpredictability more rigorously. In particular, we focus here on two milestones of QM, namely the Kochen–Specker theorem and Bell inequalities.

As already stated, the Kochen–Specker theorem claims that, for systems of Hilbert space $\mathcal{H}$ dimensionality greater than 2, it is impossible to assign in advance the outcomes of measurement in a noncontextual (i.e. independent of the measurement method) way. As a consequence, careful setup of such quantum system can give a theoretically certified method for random sequence generation. Since the theorem demands at least 3 dimensions to prove noncontextuality, the fundamental quantum-mechanical system is based on a qutrit (spin-1) system,[63] which is a three-dimensional analogy to the more familiar qubit (spin-$\frac{1}{2}$) unit. Moreover, a particular, constructive variant of the Kochen–Specker theorem has been proved: to find specific observables of indefinite values, one needs to recognize one-dimensional projections of which the system is not an eigenstate.[2] Recently, it has been demonstrated rigorously that, based on the above indefinitess criterion and the qutrit experimental setup, a measurement yields an indefinite result, when performed on the system in a certain, prepared eigenstate.[3] Moreover, these results can be shown to be unpredictable (i.e. there is no computable function returning those outcomes).

Concerning experimental realisation of Bell's inequalities to random number generation, it has been shown[78] that a quantum-mechanical systems composed of two sufficiently separated subsystems becomes a basis for another certification of randomness. In more detail, the subsystems are ions trapped in vacuum chambers, emitting photons; entangling the photons in a beamsplitter results then in the entanglement of the ions, and the detection of photons in appropriate basis yields the measurement of two-state (qubit) ion systems. Then the observed violation of Bell's inequalities corresponds exactly to the fact, that there is no deterministic process describing the series of measurements (in other words, the outcomes could not be predetermined in advance).

Let us now discuss the so-called a posteriori tests, which constitute a more standard approach to randomness testing. Namely, the process of sequence generation is irrelevant here and the sequence itself is the main subject. In particular, we refer to the work[1], where an emphasis has been put on the algorithmic randomness testing. Again, the first remark is that an algorithmic randomness of a sequence, as understood in terms of incompressibility, is not effectively decidable. One way out is to design carefully the test that explores an algorithmic randomness as much as possible, and compare the result with the case of chosen PRNGs. In the work cited, the authors studied the qutrit measurement results described above, against five deterministic, pseudo-random sequences. The first test for Borel normality revealed the substantial bias in the case of a quantum sequence; this, although expected, can be post-processed in principle e.g. by the so-called von Neumann's trick.[63] Then, four tests aimed at an inspection of algorithmic randomness (so-called Chaitin–Schwartz–Solovay–Strassen tests) have been applied;[1] while the hope was to provide the major distinction in computability between truly random and pseudo-random sequences, the results were not staggering: the tests either proved no significant difference, or, in case of differences, they are likely due to the mentioned earlier bias. Recently, slightly more optimistic, but similar results have been demonstrated.[54] This also raises a potential area to investigate other tests for algorithmic randomness in future.

To summarize, an algorithmic randomness of QM is currently verified within two main steps: first, we certify the randomness on theoretical, formal grounds (as the Kochen–Specker theorem or Bell-type inequalities); second, we verify the outcoming sequences to pass through the tests all patternless sequences should pass. While the former does not raise serious doubts and gets different formulation under our work (namely the randomness is to be certified by passing through ZFC models, that occurs during the measurement process), the latter gives usually ambigous results, that should be also adressed thoroughly in future from the point of view of algorithmic randomness described in the present work.

References

1. Abbott, A.A., Calude, C.S., Dinneen, M.J., Huang, N.: Experimentally probing the algorithmic randomness and incomputability of quantum randomness. Physica Scripta **94**, 045103 (2019). DOI https://doi.org/10.1088/1402-4896/aaf36a
2. Abbott, A.A., Calude, C.S., Svozil, K.: A variant of the kochen-specker theorem localising value indefiniteness. Journal of Mathematical Physics **56**(10), 102201 (2015). DOI 10.1063/1.4931658. URL http://dx.doi.org/10.1063/1.4931658
3. Agüero Trejo, J.M., Calude, C.S.: A new quantum random number generator certified by value indefiniteness. Theoretical Computer Science (2020). DOI https://doi.org/10.1016/j.tcs.2020.08.014. URL http://www.sciencedirect.com/science/article/pii/S0304397520304679
4. Anastopoulos, C., Hu, B.L.: A master equation for gravitational decoherence: Probing the textures of spacetime. Classical and Quantum Gravity **30**, 165007 (2013). DOI https://doi.org/10.1088/0264-9381/30/16/165007
5. Asselmeyer-Maluga, T., Król, J.: How to obtain a cosmological constant from small exotic R^4. Physics of the Dark Universe **19**, 66–77 (2018). DOI https://doi.org/10.1016/j.dark.2017.12.002
6. Bartoszyński, T., Judah, H.: Set Theory. On the Structure of the Real Line. A K Peters, Wellesley, Massachusetts, USA (1995)
7. Bell, J.L.: Set Theory. Boolean-valued Models and Independence Proofs. Clarendon Press, Oxford, UK (2005)
8. Beltrametti, E., Cassinelli, G.: The Logic of Quantum Mechanics. Addison-Wesley, Massachusetts, USA (1981)
9. Benioff, P.A.: Models of Zermelo Frankel set theory as carriers for the mathematics of physics. I. Journal of Mathematical Physics **17(5)**, 618 (1976). DOI https://doi.org/10.1063/1.522953
10. Benioff, P.A.: Models of Zermelo Frankel set theory as carriers for the mathematics of physics. II. Journal of Mathematical Physics **17(5)**, 629 (1976). DOI https://doi.org/10.1063/1.522954
11. Bera, N.N., Acin, A., Kuś, M., Mitchell, M., Lewenstein, M.: Randomness in quantum mechanics: Philosophy, physics and technology. Reports on Progress in Physics **80(12)**, 124001 (2017). DOI https://doi.org/10.1088/1361-6633/aa8731
12. Bielas, K., Klimasara, P., Król, J.: The structure of the real line in quantum mechanics and cosmology. Acta Physica Polonica B **46**, 2375 (2015). DOI https://doi.org/10.5506/APhysPolB.46.2375
13. Bierhorst, P., Knill, E., Glancy, S., Zhang, Y., Mink, A., Jordan, S., Rommal, A., Liu, Y., Christensen, B., Nam, S.W., Stevens, M.J., Shalm, L.K.: Experimentally generated randomness certified by the impossibility of superluminal signals. Nature **556**, 223–226 (2018). DOI https://doi.org/10.1038/s41586-018-0019-0
14. Blencowe, M.P.: Effective field theory approach to gravitationally induced decoherence. Phys. Rev. Lett. **111**, 021302 (2013). DOI https://doi.org/10.1103/PhysRevLett.111.021302
15. Bohm, D.: Quantum Theory. Prentice-Hall, New York, USA (1951)
16. Bohm, D.: A suggested interpretation of the quantum theory in terms of "hidden" variables. I. Physical Review **85**, 166–179 (1952). DOI https://doi.org/10.1103/PhysRev.85.166
17. Bohm, D.: A suggested interpretation of the quantum theory in terms of "hidden" variables. II. Physical Review **85(2)**, 180–193 (1952). DOI https://doi.org/10.1103/physrev.85.180
18. Bongaarts, P.: Quantum Theory: A Mathematical Approach. Springer International Publishing, Switzerland (2015)

19. Boos, W.: Mathematical quantum theory I: Random ultrafilters as hidden variables. Synthese **107**, 83–143 (1996). DOI https://doi.org/10.1007/BF00413903
20. Bose, S., Mazumdar, A., Morley, G.W., Ulbricht, H., Toros, M., Paternostro, M., Geraci, A., Barker, P., Kim, M.S., Milburn, G.: A spin entanglement witness for quantum gravity. Physical Review Letters **119**, 240401 (2017). DOI https://doi.org/10.1103/PhysRevLett.119.240401
21. Brassard, G., Buhrman, H., Linden, N., Méthot, A.A., Tapp, A., Unger, F.: Limit on nonlocality in any world in which communication complexity is not trivial. Physical Review Letters **96**, 250401 (2006). DOI https://doi.org/10.1103/PhysRevLett.96.250401
22. Brendle, J.: Mutual generics and perfect free subsets. Acta Mathematica Hungarica **82**, 143–161 (1999). DOI https://doi.org/10.1023/A:1026458210592
23. Brunner, N., Pironio, S., Acin, A., Gisin, N., Méthot, A.A., Scarani, V.: Testing the dimension of hilbert spaces. Physical Review Letters **100**(21) (2008). DOI 10.1103/physrevlett.100.210503. URL http://dx.doi.org/10.1103/PhysRevLett.100.210503
24. Calude, C.S.: Algorithmic randomness, quantum physics, and incompleteness. In: M. Margenstern (ed.) Machines, Computations, and Universality, *MCU 2004. Lecture Notes in Computer Science*, vol. 3354. Springer, Berlin, Heidelberg (2005). DOI https://doi.org/10.1007/978-3-540-31834-7-1
25. Calude, C.S.: Quantum Randomness: From Practice to Theory and Back, pp. 169–181. Springer International Publishing, Cham (2017)
26. Calude, C.S., Hertling, P.H., Svozil, K.: Embedding quantum universes into classical ones. Foundations of Physics **29(3)**, 349–390 (1999)
27. Calude, C.S., Jürgensen, H.: Is complexity a source of incompleteness? Centre for Discrete Mathematics and Theoretical Computer Science, CDMTCS Research Report 241, (2004). Available online: http://arxiv.org/abs/math.LO/0408144
28. Coladangelo, A., Stark, J.: An inherently infinite-dimensional quantum correlation. Nature Communications **11**, 3335 (2020). DOI https://doi.org/10.1038/s41467-020-17077-9
29. Dalla Chiara, M.L., Giuntini, R., Greechie, R.: Reasoning in Quantum Theory: Sharp and Unsharp Quantum Logics, *Trends in Logic*, vol. 22. Springer, Netherlands (2015)
30. van Dam, W.: Nonlocality and communication complexity. Ph.D. thesis, University of Oxford (2000). See also http://arxiv.org/quant-ph/0501159
31. Downey, R.G., Hirschfeldt, D.R.: Algorithmic Randomness and Complexity. Springer Science+Business Media, New York, USA (2010)
32. Dudley, R.D.: Real Analysis and Probability. Cambridge Studies in Advanced Mathematics Vol. 74. Cambridge University Press, Cambridge, UK (2002)
33. Duranda, B., Kanovei, V., Uspensky, V.A., Vereshchagin, N.: Do stronger definitions of randomness exist? Theoretical Computer Science **290**, 1987–1996 (2003). DOI https://doi.org/10.1016/S0304-3975(02)00040-3
34. Etesi, G.: Exotica and the status of the strong cosmic censor conjecture in four dimensions. Classical and Quantum Gravity **34**, 245010 (2017). DOI https://doi.org/10.1088/1361-6382/aa945b
35. Etesi, G.: On a possibly pure set-theoretic contribution to black hole entropy. Foundations of Science **25(2)**, 327–340 (2020). DOI https://doi.org/10.1007/s10699-019-09625-4
36. Etesi, G.: Global solvability of the vacuum Einstein equation and the strong cosmic censor conjecture in four dimensions. Journal of Geometry and Physics **164**, 104164 (2021). DOI 10.1016/j.geomphys.2021.104164. Available online: arXiv:1903.02855v4
37. Farah, I., Wofsey, E.: Set theory and operator algebras. In: Appalachian Set Theory: 2006–2012, pp. 63–120. Cambridge University Press, Cambridge, UK (2012)
38. Farhi, E., Goldstone, J., Gutmann, S.: How probability arises in quantum mechanics. Annals of Physics (New York) **192**, 368–382 (1989). DOI https://doi.org/10.1016/0003-4916(89)90141-3
39. Galley, T.D., Masanes, L.: Any modification of the Born rule leads to a violation of the purification and local tomography principles. Quantum **2**, 104 (2018). ArXiv:1801.06414v4 [quant-ph]
40. de Groote, H.F.: Stone spectra. Observables I (2005). Available online: arXiv:math-ph/0509020

41. Gudder, S.P.: On hidden variable theories. Journal of Mathematical Physics **11**, 431 (1970). DOI https://doi.org/10.1063/1.1665156
42. Hajek, P.: Arithmetic hierarchy and complexity of computation. Theoretical Computer Science **8**, 227–237 (1979)
43. Halvorson, H., Clifton, R.: Maximal beable subalgebras of quantum mechanical observables. International Journal of Theoretical Physics **38**, 2441–2484 (1999). DOI 10.1023/A:1026628407645
44. Hamkins, J.D.: Upward closure and amalgamation in the generic multiverse of a countable model of set theory. Presented at: Recent developments in axiomatic set theory. Research Institute for Mathematical Sciences (RIMS), Kyoto University, Japan (2015). Available online: http://arXiv:1511.01074
45. Harding, J., Navara, M.: Subalgebras of orthomodular lattices. Order **28**, 549–563 (2011). DOI 10.1007/s11083-010-9191-z
46. Heunen, C., Landsman, N.P., Spitters, B.: Bohrification of operator algebras and quantum logic. Synthese **186**, 719–752 (2012). DOI 10.1007/s11229-011-9918-4
47. Jaeger, G.S.: Quantum and super–quantum correlations. In: Beyond the Quantum, pp. 146–151. World Scientific, Singapore (2007)
48. Jech, T.: Multiple forcing. Cambridge Tracts in Mathematics Vol. 88. Cambridge University Press, Cambridge, USA (1986)
49. Jech, T.: Set Theory. Springer Verlag, Berlin (2003, Third millennium edition)
50. Jennewein, T., Achleitner, U., Weihs, G., Weinfurter, H., of Vienna, A.Z.U., of Innsbruck, U., of Munich, U.: A fast and compact quantum random number generator. Review of Scientific Instruments **71**, 1675–1680 (2000)
51. Judah, H., Repický, M.: Amoeba reals. Journal of Symbolic Logic **60**, 1168–1185 (1999). DOI https://doi.org/10.2307/2275880
52. Kadison, R.V., Ringrose, J.R.: Fundamentals of the Theory of Operator Algebras II. AMS, Providence, Rhode Island, USA (1997)
53. Kautz, S.M.: Degrees of Random Sets. Ph.D. thesis, Cornell University (1991)
54. Kavulich, J.T., Van Deren, B.P., Schlosshauer, M.: Searching for evidence of algorithmic randomness and incomputability in the output of quantum random number generators. Physics Letters A **388**, 127032 (2021). DOI 10.1016/j.physleta.2020.127032. URL http://dx.doi.org/10.1016/j.physleta.2020.127032
55. Kelsey, J., Schneier, B., Wagner, D., Hall, C.: Cryptanalytic attacks on pseudorandom number generators. In: S. Vaudenay (ed.) Fast Software Encryption, pp. 168–188. Springer Berlin Heidelberg, Berlin, Heidelberg (1998)
56. Kochen, S., Specker, E.P.: The problem of hidden variables in quantum mechanics. Journal of Mathematics and Mechanics **17**, 59–87 (1967)
57. Kovalsky, M.G., Hnilo, A.A., Agüero, M.B.: Kolmogorov complexity of sequences of random numbers generated in Bell's experiments. Physical Review A **98**, 042131 (2018). DOI https://doi.org/10.1103/PhysRevA.98.042131
58. Król, J.: Background independence in quantum gravity and forcing constructions. Foundations of Physics **34**, 361–403 (2004). DOI https://doi.org/10.1023/B:FOOP.0000019620.04821.a2
59. Król, J.: Model and set-theoretic aspects of exotic smoothness structures on $\mathbb{R}^4$. Fundam. Theor. Phys. **183**, 217–240 (2016). DOI https://doi.org/10.1007/978-3-319-31299-6 13
60. Król, J., Asselmeyer-Maluga, T.: Quantum mechanics, formalization and the cosmological constant problem. Foundations of Science **25**, 879–904 (2020). DOI https://doi.org/10.1007/s10699-019-09646-z
61. Król, J., Asselmeyer-Maluga, T., Bielas, K., Klimasara, P.: From quantum to cosmological regime. The role of forcing and exotic 4-smoothness. Universe **3(2), 31**, 1–11 (2017). DOI https://doi.org/10.3390/universe3020031
62. Król, J., Klimasara, P.: Black holes and complexity via constructible universe. Universe **6(11)**, 198 (2020). DOI https://doi.org/10.3390/universe6110198
63. Kulikov, A., Jerger, M., Potočnik, A., Wallraff, A., Fedorov, A.: Realization of a quantum random generator certified with the kochen-specker theorem. Physical Review Letters **119**(24) (2017). DOI 10.1103/physrevlett.119.240501. URL http://dx.doi.org/10.1103/PhysRevLett.119.240501

64. Kunen, K.: Random and Cohen reals. In: Handbook of set-theoretic topology, p. 887–911. North-Holland, Amsterdam, The Netherlands (1984)
65. Kunen, K.: Set Theory. Studies in Logic Mathematical Logic and Foundations Volume 34. College Publications, King's College, London, UK (2013)
66. van Lambalgen, M.: Random Sequences. Ph.D. thesis, Universiteit van Amsterdam (1987)
67. Landsman, K.: Foundations of Quantum Theory. From Classical Concepts to Operator Algebras. Springer Open, Cham, Switzerland (2017)
68. Landsman, K.: Randomness? What Randomness? Foundations of Physics **50**, 61–104 (2020). DOI https://doi.org/10.1007/s10701-020-00318-8
69. Lenstra, A.K., Hughes, J.P., Augier, M., Bos, J.W., Kleinjung, T., Wachter, C.: Ron was wrong, whit is right. IACR Cryptol. ePrint Arch. **2012**, 64 (2012)
70. Marangon, D.G., Vallone, G., Villoresi, P.: Source-device-independent ultrafast quantum random number generation. Physical Review Letters **118**(6) (2017). DOI 10.1103/physrevlett.118.060503. URL http://dx.doi.org/10.1103/PhysRevLett.118.060503
71. Menezes, A.J., Vanstone, S.A., Oorschot, P.C.V.: Handbook of Applied Cryptography, 1st edn. CRC Press, Inc., USA (1996)
72. von Neumann, J.: Various techniques used in connection with random digits. In: A.S. Householder, G.E. Forsythe, H.H. Germond (eds.) Monte Carlo Method, *National Bureau of Standards Applied Mathematics Series*, vol. 12, chap. 13, pp. 36–38. US Government Printing Office, Washington, DC (1951)
73. von Neumann, J.: Mathematical Foundations of Quantum Mechanics. Princeton University Press, Princeton, USA (1955)
74. Nies, A.: Computability and Randomness. Oxford Logic Guides 51. Oxford University Press, New York, USA (2009). DOI https://doi.org/10.1142/1524
75. Oxtoby, J.C.: Measure and Category. A Survey of the Analogies between Topological and Measure Spaces. Graduate Texts in Mathematics. Springer Verlag, New York (1980)
76. Paterek, T., Kofler, J., Prevedel, R., Klimek, P., Aspelmeyer, M., Zeilinger, A., Brukner, C.: Logical independence and quantum randomness. New Journal of Physics **12**, 013019 (2010). DOI https://doi.org/10.1088/1367-2630/12/1/013019
77. Pawlowski, M., Paterek, T., Kaszlikowski, D., Scarani, V., Winter, A., Zukowski, M.: Information causality as a physical principle. Nature **461**, 1101–1104 (2010). DOI https://doi.org/10.1038/nature08400
78. Pironio, S., Acín, A., Massar, S., Boyer de la Giroday, A., Matsukevich, D.N., Maunz, P., Olmschenk, S., Hayes, D., Luo, L., Manning, T.A., Monroe, C.: Random numbers certified by Bell's theorem. Nature **464**, 1021–1024 (2010). DOI https://doi.org/10.1038/nature09008
79. Planat, M., Aschheim, R., Amaral, M.M., Irwin, K.: Quantum computation and measurements from an exotic space-time R^4. Universe **12**, 736 (2020). DOI https://doi.org/10.3390/sym12050736
80. Popescu, S., Rohrlich, D.: Nonlocality as an axiom for quantum theory. Foundations of Physics **24 (3)**, 379 (1994). DOI https://doi.org/10.1007/BF02058098
81. Ramanathan, R., Banacki, M., Rodriguez, R.R., Horodecki, P.: Single trusted qubit is necessary and sufficient for quantum realisation of extremal no-signaling correlations (2020). Available online: arXiv:2004.14782
82. Roitman, J.: Adding random or a Cohen real: topological consequences and the effect on Martin's axiom. Fundamenta Mathematicae **103**, 47–60 (1979). DOI https://doi.org/10.4064/fm-103-1-47-60
83. Sanz, A.S.: Bohm's approach to quantum mechanics: Alternative theory or practical picture? Frontiers in Physics **14**, 11301 (2019). DOI https://doi.org/10.1007/s11467-018-0853-4
84. Schmöle, J., Dragosits, M., Hepach, H., Aspelmeyer, M.: A micromechanical proof-of-principle experiment for measuring the gravitational force of milligram masses. Classical and Quantum Gravity **33**, 125031 (2016). DOI https://doi.org/10.1088/0264-9381/33/12/125031
85. Singh, A., Carroll, S.M.: Modeling Position and Momentum in Finite-Dimensional Hilbert Spaces via Generalized Pauli Operators. arXiv e-prints arXiv:1806.10134 (2018)

86. Solovay, R.M.: A model of set theory in which every set of reals is Lebesgue measurable. Annales of Mathematics **92(1)**, 1–56 (1970). DOI https://doi.org/10.2307/1970696
87. Susskind, L.: Three Lectures on Complexity and Black Holes. Springer, New York, USA (2020)
88. Svozil, K.: Randomness & Undecidability in Physics. World Scientific, Singapore (1993). DOI https://doi.org/10.1142/1524
89. Svozil, K.: Quantum Logic. Springer, Singapore (1998)
90. Takeuti, G.: Two Applications of Logic to Mathematics. Princeton University Press, Princeton, USA (1978)
91. Vaidman, L.: Quantum theory and determinism. Quantum Studies: Mathematics and Foundations **1**, 5–38 (2014). DOI https://doi.org/10.1007/s40509-014-0008-4
92. Van Wesep, R.A.: Hidden variables in quantum mechanics: Generic models, set-theoretic forcing, and the emergence of probability. Annals of Physics **321**, 2453–2490 (2006)
93. Viale, M., Audrito, G., Steila, S.: Iterated forcing, category forcings, generic ultrapowers, generic absoluteness (2019). Available online (access on 16.06.2021): http://www.logicatorino.altervista.org/matteo viale/book.pdf
94. Yang, J.M.: A relational formulation of quantum mechanics. Scientific Reports **8**, 13305 (2018). DOI https://doi.org/10.1038/s41598-018-31481-8
95. Yurtsever, U.: Quantum mechanics and algorithmic randomness. Complexity **6(1)**, 27–34 (2000). DOI https://doi.org/10.1002/1099-0526(200009/10)6:1¡27::AID-CPLX1004¿3.0.CO;2-R
96. Żurek, W.H.: Probabilities from entanglement, Born's rule $p_k = |\psi_k|^2$ from envariance. Physical Review A **71**, 052105 (2005). DOI https://doi.org/10.1103/PhysRevA.71.052105

Conflict of interest

The authors declare that they have no conflict of interest.