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Path Integral Implementation of Relational Quantum Mechanics

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Relational formulation of quantum mechanics is based on the idea that relational properties among quantum systems, instead of the independent properties of a quantum system, are the most fundamental elements to construct quantum mechanics. In the recent works (J. M. Yang, Sci. Rep. 8:13305, 2018), basic relational quantum mechanics framework is formulated to derive quantum probability, Born's Rule, Schrödinger Equations, and measurement theory. This paper gives a concrete implementation of the relational probability amplitude by extending the path integral formulation. The implementation not only clarifies the physical meaning of the relational probability amplitude, but also gives several important applications. For instance, the double slit experiment can be elegantly explained. A path integral representation of the reduced density matrix of the observed system can be derived. Such representation is shown valuable to describe the interaction history of the measured system and a series of measuring systems. More interestingly, it allows us to develop a method to calculate entanglement entropy based on path integral and influence functional. Criteria of entanglement is proposed based on the properties of influence functional, which may be used to determine entanglement due to interaction between a quantum system and a classical field.

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I. INTRODUCTION

Quantum mechanics was originally developed as a physical theory to explain the experimental observations of a quantum system in a measurement. In the early days of quantum mechanics, Bohr had emphasized that the description of a quantum system depends on the measuring apparatus [1–3]. In more recent development of quantum interpretations, the dependency of a quantum state on a reference system was further recognized. The relative state formulation of quantum mechanics [4–6] asserts that a quantum state of a subsystem is only meaningful relative to a given state of the rest of the system. Similarly, in developing the theory of decoherence induced by environment [7–9], it is concluded that correlation information between two quantum systems is more basic than the properties of the quantum systems themselves. Relational Quantum Mechanics (RQM) further suggests that a quantum system should be described relative to another system, there is no absolute state for a quantum system [10, 11]. Quantum theory does not describe the independent properties of a quantum system. Instead, it describes the relation among quantum systems, and how correlation is established through physical interaction during measurement. The reality of a quantum system is only meaningful in the context of measurement by another system.

The idea that relational properties are more basic than the independent properties of a quantum system is profound. It should be considered a starting point for constructing the formulation of quantum mechanics. However, traditional quantum mechanics always starts with an observer-independent quantum state. It is of interest

to see if a quantum theory constructed based on relational properties can address some of the unanswered fundamental questions mentioned earlier. Such reconstruction program was initiated [10] and had some successes, for example, in deriving the Schrödinger Equation.

Recently, a similar reformulation of quantum mechanics was proposed [12, 13]. The reformulation is based on two basic ideas. 1.) Relational properties between the two quantum systems are the most fundamental elements to formulate quantum mechanics. 2.) A physical measurement of a quantum system is a probe-response interaction process. Thus, the framework to calculate the probability of an outcome when measuring a quantum system should model this bidirectional process. This implies the probability can be derived from product of two quantities with each quantity associated with a unidirectional process. Such quantity is defined as relational probability amplitude. Specifically, the probability of a measurement outcome is proportional to the summation of probability amplitude product from all alternative measurement configurations. The properties of quantum systems, such as superposition and entanglement, are manifested through the rules of counting the alternatives. As results, traditional quantum mechanics formulations can be rediscovered but with new insights on the origin of quantum probability. Schrödinger Equation is recovered when there is no entanglement in the relational probability amplitude matrix [12]. On the other hand, when there is change in the entanglement measure, the quantum measurement theory is obtained [13]. In essence, quantum mechanics demands a new set of rules to calculate measurement probability from an interaction process.

Although the concept of relational probability amplitude is useful to derive the quantum probability, its physical meaning is not obvious to understand. It is desirable to find an explicit calculation of the relational probability amplitude. It turns out that the path integral method

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can be used to achieve this goal and is briefly described in Ref. [12]. In this paper, the significance of the path integral implementation of relational quantum mechanics is fully developed. Besides providing the physical meaning of relation probability amplitude, the path integral formulation also has interesting applications. For instance, it can describe the history of a quantum system that has interacted with a series of measuring systems in sequence. As a result, the double slit experiment can be elegantly explained from the formulation developed here. More significantly, the coordinator representation of the reduced density matrix derived from this implementation allows us to develop a method to calculate entanglement entropy using path integral approach. We propose a criterion on whether there is entanglement between the system and external environment based on the influence functional. This enables us to calculate entanglement entropy of a physical system that interacts with classical fields, such as an electron in an electromagnetic field. Since entanglement entropy is an important concept in quantum information theory, the method described here may lead to new insight on the information aspect of a quantum system that interacts with classical fields.

The paper is organized as following. We first briefly review the relational formulation of quantum mechanics in Section II. In Section III A the path integral implementation of the relational probability amplitude is presented. It is shown to be compatible with the traditional path integral quantum mechanics, particularly on the definition of the influence functional. Section III B generalizes the formulation to describe the history of quantum state for the observed system that has interacted with a series of measuring systems in sequence. The formulation is applied to explain the double slit experiment in Section III C. Section III D introduces a method to calculate entanglement entropy between the interacting systems based on the path integral implementation. A criterion to determine whether entanglement entropy vanishes based on properties of the influence functional is discussed in Section IV. Section IV also summarizes the conclusions.

II. RELATIONAL FORMULATION OF QUANTUM MECHANICS

A. Terminologies

A *Quantum System*, denoted by symbol S , is an object under study and follows the laws of quantum mechanics. An *Apparatus*, denoted as A , can refer to the measuring devices, the environment that S is interacting with, or the system from which S is created. All systems are quantum systems, including any apparatus. Depending on the selection of observer, the boundary between a system and an apparatus can change. For example, in a measurement setup, the measuring system is an apparatus A , the measured system is S . However, the composite system $S + A$ as a whole can be considered a single system, relative to

another apparatus A' . In an ideal measurement to measure an observable of S , the apparatus is designed in such a way that at the end of the measurement, the pointer state of A has a distinguishable, one to one correlation with the eigenvalue of the observable of S .

The definition of *Observer* is associated with an apparatus. An observer, denoted as $\mathcal{O}$, is a person who can operate and read the pointer variable of the apparatus. Whether or not this observer (a person) is a quantum system is irrelevant in our formulation. An observer is defined to be physically local to the apparatus he associates with. This prevents the situation that $\mathcal{O}$ can instantaneously read the pointer variable of the apparatus that is space-like separated from $\mathcal{O}$.

In the traditional quantum measurement theory proposed by von Neumann [14], both the quantum system and the measuring apparatus follow the same quantum mechanics laws. After the interaction during the measurement process, both systems encode information each other, allowing an observer to infer measurement results of S by reading pointer variable of A . Quantum measurement is a question-and-answer bidirectional process. The measuring system interacts (or, disturbs) the measured system. The interaction in turn alters the state of the measuring system. As a result, a correlation is established, allowing the measurement result for S to be inferred from the pointer variable of A .

A *Quantum State* of S describes the complete information an observer $\mathcal{O}$ can know about S . From the examination on the measurement process and the interaction history of a quantum system, we consider a quantum state encodes the information relative to the measuring system or the environment that the system previously interacted with. In this sense, the quantum state of S is described relative to A . It is equivalent to say that the quantum state is relative to an observer $\mathcal{O}$ because there is no space-like separation between $\mathcal{O}$ and A . $\mathcal{O}$ operates A , reads the measurement outcomes of A , and has the complete control of A . The idea that a quantum state encodes information from previous interactions is also proposed in Ref [11]. The information encoded in the quantum state is the complete knowledge an observer can say about S , as it determines the possible outcomes of next measurement. When next measurement with another apparatus A' is completed, the description of quantum state is updated to be relative to A' .

B. Basic Formulation

The relational formulation of quantum mechanics [12] is based on a detailed analysis of the interaction process during measurement of a quantum system. First, from experimental observations, a measurement of a variable on a quantum system yields multiple possible outcomes randomly. Each potential outcome is obtained with a certain probability. We call each measurement with a

distinct outcome a quantum measurement event¹. Denote these alternative events with a set of kets $\{|s_i\rangle\}$ for S , where $(i = 0, \dots, N - 1)$, and a set of kets $\{|a_j\rangle\}$ for A , where $(j = 0, \dots, M - 1)$. A potential measurement outcome is represented by a pair of kets $(|s_i\rangle, |a_j\rangle)$. Second, a physical measurement is a bidirectional process, the measuring system and the measured system interact and modify the state of each other. The probability of finding a potential measurement outcome represented by a pair of kets $(|s_i\rangle, |a_j\rangle)$, p_{ij} , should be calculated by modeling such bidirectional process. This implies p_{ij} can be expressed as product of two numbers,

$$p_{ij} \propto Q_{ji}^{AS} R_{ij}^{SA}. \quad (1)$$

Q_{ji}^{AS} and R_{ij}^{SA} are not necessarily real non-negative number since each number alone only models a unidirectional process which is not a complete measurement process. On the other hand, p_{ij} is a real non-negative number since it models an actual measurement process. To satisfy such requirement, we further assume

$$Q_{ji}^{AS} = (R_{ij}^{SA})^*. \quad (2)$$

Written in a different format, $Q_{ji}^{AS} = (R_{ji}^{SA})^\dagger$. This means $Q^{AS} = (R^{SA})^\dagger$. Eq.(1) then becomes

$$p_{ij} = |R_{ij}^{SA}|^2 / \Omega \quad (3)$$

where Ω is a real number normalization factor. Q_{ji}^{AS} and R_{ij}^{SA} are called *relational probability amplitudes*. Given the relation in Eq.(2), we will not distinguish the notation R versus Q , and only use R .

The relational matrix R^{SA} gives the complete description of S . It provides a framework to derive the probability of future measurement outcome. Although R_{ij}^{SA} is a probability amplitude, not a probability real number, we assume it follows certain rules in the classical probability theory, such as multiplication rule, and sum of alternatives in the intermediate steps.

The set of kets $\{|s_i\rangle\}$, representing distinct measurement events for S , can be considered as *eigenbasis* of Hilbert space $\mathcal{H}_S$ with dimension N , and $|s_i\rangle$ is an eigenvector. Since each measurement outcome is distinguishable, $\langle s_i | s_j \rangle = \delta_{ij}$. Similarly, the set of kets $\{|a_j\rangle\}$ is eigenbasis of Hilbert space $\mathcal{H}_A$ with dimension N for the apparatus system A . The bidirectional process $|a_j\rangle \rightleftharpoons |s_i\rangle$ is called a *potential measurement configuration* in the joint Hilbert space $\mathcal{H}_S \oplus \mathcal{H}_A$.

To derive the properties of S based on the relational R , we examine how the probability of measuring S with

a particular outcome of variable q is calculated. It turns out such probability is proportional to the sum of weights from all applicable measurement configurations, where the weight is defined as the product of two relational probability amplitudes corresponding to the applicable measurement configuration. Identifying the applicable measurement configuration manifests the properties of a quantum system. For instance, before measurement is actually performed, we do not know that which event will occur to the quantum system since it is completely probabilistic. It is legitimate to generalize the potential measurement configuration as $|a_j\rangle \rightarrow |s_i\rangle \rightarrow |a_k\rangle$. In other words, the measurement configuration in the joint Hilbert space starts from $|a_j\rangle$, but can end at $|a_j\rangle$, or any other event, $|a_k\rangle$. Indeed, the most general form of measurement configuration in a bipartite system can be $|a_j\rangle \rightarrow |s_m\rangle \rightarrow |s_n\rangle \rightarrow |a_k\rangle$. Correspondingly, we generalize Eq.(1) by introducing a quantity for such configuration,

$$W_{jmnk}^{ASSA} = Q_{jm}^{AS} R_{nk}^{SA} = (R_{mj}^{SA})^* R_{nk}^{SA}. \quad (4)$$

The second step utilizes Eq.(2). This quantity is interpreted as a weight associated with the potential measurement configuration $|a_j\rangle \rightarrow |s_m\rangle \rightarrow |s_n\rangle \rightarrow |a_k\rangle$. Suppose we do not perform actual measurement and inference information is not available, the probability of finding S in a future measurement outcome can be calculated by summing W_{jmnk}^{ASSA} from all applicable alternatives of measurement configurations.

With this framework, the remaining task to calculate the probability is to correctly count the *applicable* alternatives of measurement configuration. This task depends on the expected measurement outcome. For instance, suppose the expected outcome of an ideal measurement is event $|s_i\rangle$, i.e., measuring variable q gives eigenvalue q_i . The probability of event $|s_i\rangle$ occurs, p_i , is proportional to the summation of W_{jmnk}^{ASSA} from all the possible configurations related to $|s_i\rangle$. Mathematically, we select all W_{jmnk}^{ASSA} with $m = n = i$, sum over index j and k , and obtain the probability p_i .

$$p_i \propto \sum_{j,k=0}^M (R_{ij}^{SA})^* R_{ik}^{SA} = \left| \sum_j R_{ij}^{SA} \right|^2. \quad (5)$$

This leads to the definition of wave function $\varphi_i = \sum_j R_{ij}$, so that $p_i = |\varphi_i|^2$. The quantum state can be described either by the relational matrix R , or by a set of variables $\{\varphi_i\}$. The vector state of S relative to A , is $|\psi\rangle_S^A = (\varphi_0, \varphi_1, \dots, \varphi_N)^T$ where superscript T is the transposition symbol. More specifically,

$$|\psi\rangle_S^A = \sum_i \varphi_i |s_i\rangle \quad \text{where } \varphi_i = \sum_j R_{ij}. \quad (6)$$

The justification for the above definition is that the probability of finding S in eigenvector $|s_i\rangle$ in future measurement can be calculated from it by defining a projection

¹ In the basic formulation described here, we assume finite number of quantum measurement events for the convenience of notation. It is possible to extend to infinite number of events. In fact, the path integral implementation, which is the main result of this paper, assume infinite number of positions that a system can be located. See Section III A.

operator $\hat{P}_i = |s_i\rangle\langle s_i|$. Noted that $\{|s_i\rangle\}$ are orthogonal eigenbasis, the probability is rewritten as:

$$p_i = \langle\psi|\hat{P}_i|\psi\rangle = |\varphi_i|^2 \quad (7)$$

Eqs.(5) and (6) are introduced on the condition that there is no entanglement (See Section II C for the definition of entanglement). between quantum system S and A . If there is entanglement between them, the summation in Eq.(5) over-counts the applicable alternatives of measurement configurations and should be modified accordingly. A more generic approach to describe the quantum state of S is the reduced density matrix formulation, which is defined as

$$\rho_S = RR^\dagger \quad (8)$$

The probability p_i is calculated using the projection operator $\hat{P}_i = |s_i\rangle\langle s_i|$

$$p_i = \text{Tr}_S(\hat{P}_i \hat{\rho}_S) = \sum_j |R_{ij}|^2. \quad (9)$$

The effect of a quantum operation on the relational probability amplitude matrix can be expressed through an operator. Defined an operator $\hat{M}$ in Hilbert space $\mathcal{H}_S$ as $M_{ij} = \langle s_i|\hat{M}|s_j\rangle$, The new relational probability amplitude matrix is obtained by

$$(R_{new}^{SA})_{ij} = \sum_k M_{ik}(R_{init}^{SA})_{kj}, \quad \text{or} \quad (10)$$

$$R_{new} = MR_{init}.$$

Consequently, the reduced density becomes,

$$\rho_{new} = R_{new}(R_{new})^\dagger = M\rho_{init}M^\dagger. \quad (11)$$

C. Entanglement Measure

The description of S using the reduced density matrix ρ_S is valid regardless there is entanglement between S and A . To determine whether there is entanglement between S and A , a parameter to characterize the entanglement measure should be introduced. There are many forms of entanglement measure [15, 16], the simplest one is the von Neumann entropy, which is defined as

$$H(\rho_S) = -\text{Tr}(\rho_S \ln(\rho_S)). \quad (12)$$

Denote the eigenvalues of the reduced density matrix ρ_S as $\{\lambda_i\}$, $i = 0, \dots, N$, the von Neumann entropy is calculated as

$$H(\rho_S) = -\sum_i \lambda_i \ln \lambda_i. \quad (13)$$

A change in $H(\rho_S)$ implies there is change of entanglement between S and A . Unless explicitly pointed out, we only consider the situation that S is described by a

single relational matrix R . In this case, the entanglement measure $E = H(\rho_S)$.

$H(\rho_S)$ enables us to distinguish different quantum dynamics. Given a quantum system S and its referencing apparatus A , there are two types of the dynamics between them. In the first type of dynamics, there is no physical interaction and no change in the entanglement measure between S and A . S is not necessarily isolated in the sense that it can still be entangled with A , but the entanglement measure remains unchanged. This type of dynamics is defined as *time evolution*. In the second type of dynamics, there is a physical interaction and correlation information exchange between S and A , i.e., the von Neumann entropy $H(\rho_S)$ changes. This type of dynamics is defined as *quantum operation*. *Quantum measurement* is a special type of quantum operation with a particular outcome. Whether the entanglement measure changes distinguishes a dynamic as either a time evolution or a quantum operation [12, 13].

As shown in this introduction section, the relational probability amplitude R_{ij} provides a complete description of the quantum system relative to a reference system. It is natural to ask what is the physical meaning of this quantity and how to mathematically calculate it. A concrete implementation of the relational quantum mechanics depends on how R_{ij} is calculated. This is main question we intend to answer in this paper.

III. RESULTS

A. Path Integral Implementation

This section shows that the relational probability amplitude can be explicitly calculated using the Path Integral formulation. Without loss of generality, the following discussion just focuses on one dimensional space-time quantum system. In the Path Integral formulation, the probability to find a quantum system moving from a point x_a at time t_a to a point x_b at time t_b is the absolute square of a probability amplitude, i.e., $P(b, a) = |K(b, a)|^2$. The probability amplitude is postulated as the sum of the contribution of phase from each path [18]:

$$K(b, a) = \int_a^b e^{(i/\hbar)S_p(x(t))} \mathcal{D}x(t) \quad (14)$$

where $\mathcal{D}x(t)$ denotes integral over all possible paths from point x_a to point x_b . It is the wave function for S moving from x_a to x_b [18]. The wave function of finding the particle in a region $\mathcal{R}_b$ previous to t_b can be expressed as

$$\varphi(x_b) = \int_{\mathcal{R}_b} e^{(i/\hbar)S_p(x(t))} \mathcal{D}x(t) \quad (15)$$

where x_b is the position of particle at time t_b . The integral over region $\mathcal{R}_b$ can be interpreted as integral of all paths ending at position x_b , with the condition that

each path lies in region $\mathcal{R}_b$ which is previous to time t_b . The rational of such definition can be found in Feynman's original paper [17].

Now let's consider how the relational matrix element can be formulated using similar formulation. Define $\mathcal{R}_b^S$ is the region of finding system S previous to time t_b , and $\mathcal{R}_b^A$ is the region of finding measuring system A previous to time t_b . We denote the matrix element as $R(x_b; y_b)$. Here the coordinates x_b and y_b act as indices to the system S and apparatus A , respectively. Borrowing the ideas described in Eq.(14), we propose that the relational matrix element is calculated as

$$R(x_b, y_b) = \int_{\mathcal{R}_b^S} \int_{\mathcal{R}_b^A} e^{(i/\hbar)S_p^{SA}(x(t), y(t))} \mathcal{D}x(t) \mathcal{D}y(t) \quad (16)$$

where the action $S_p^{SA}(x(t), y(t))$ consists three terms

$$S_p^{SA}(x(t), y(t)) = S_p^S(x(t)) + S_p^A(y(t)) + S_{int}^{SA}(x(t), y(t)). \quad (17)$$

The last term is the action due to the interaction between S and A when each system moves along its particular path. Eq.(16) is considered an extension of Postulate 1. We can validate Eq.(16) by deriving formulation that is consistent with traditional path integral. Suppose there is no interaction between S and A . The third term in Eq.(17) vanishes. Eq.(16) is decomposed to product of two independent terms,

$$R(x_b, y_b) = \int_{\mathcal{R}_b^S} e^{(i/\hbar)S_p^S(x(t))} \mathcal{D}x(t) \times \int_{\mathcal{R}_b^A} e^{(i/\hbar)S_p^A(y(t))} \mathcal{D}y(t) \quad (18)$$

Noticed that the coordinates y_a and y_b are equivalent of the index j in Eq.(6), the wave function of S can be obtained by integrating y_b over Eq.(18)

$$\begin{aligned} \varphi(x_b) &= \int_{-\infty}^{\infty} R(x_b, y_b) dy_b \\ &= \int_{\mathcal{R}_b^S} e^{(i/\hbar)S_p^S(x(t))} \mathcal{D}x(t) \\ &\quad \times \int_{-\infty}^{\infty} \int_{\mathcal{R}_b^A} e^{(i/\hbar)S_p^A(y(t))} \mathcal{D}y(t) dy_b \\ &= c \int_{\mathcal{R}_b^S} e^{(i/\hbar)S_p^S(x(t))} \mathcal{D}x(t) \end{aligned} \quad (19)$$

where constant c is the integration result of the second term in step two. The result is the same as Eq.(15) except an unimportant constant.

Next, we consider the situation that there is entanglement between S and A as a result of interaction. The third term in Eq.(17) does not vanish. We can no longer define a wave function for S . Instead, a reduced density matrix should be used to describe the state of the particle, $\rho = RR^\dagger$. Similar to Eq.(16), the element of the

reduced density matrix is

$$\begin{aligned} \rho(x_b, x'_b) &= \int dy_b \int_{\mathcal{R}_b^S} \int_{\mathcal{R}_{b'}^S} \int_{\mathcal{R}_b^A} \int_{\mathcal{R}_{b'}^A} e^{(i/\hbar)\Delta S} \\ &\quad \times \mathcal{D}x(t) \mathcal{D}x'(t) \mathcal{D}y(t) \mathcal{D}y'(t) \end{aligned} \quad (20)$$

where $\Delta S = S_p^S(x(t)) - S_p^S(x'(t)) + S_p^A(y(t)) - S_p^A(y'(t)) + S_{int}^{SA}(x(t), y(t)) - S_{int}^{SA}(x'(t), y'(t))$.

Here $x_b = x(t_b)$ and $x'_b = x'(t_b)$. The path integral over $\mathcal{D}y'(t)$ takes the same region (therefore same end point y_b) as the path integral over $\mathcal{D}y(t)$. After the path integral, an integral over y_b is performed. Eq.(20) is equivalent to the J function introduced in Ref [18]. We can rewrite the expression of ρ using the *influence functional*, $F(x(t), x'(t))$,

$$\begin{aligned} \rho(x_b, x'_b) &= \frac{1}{Z} \int_{\mathcal{R}_b^S} \int_{\mathcal{R}_{b'}^S} e^{(i/\hbar)[S_p^S(x(t)) - S_p^S(x'(t))]} \\ &\quad \times F(x(t), x'(t)) \mathcal{D}x(t) \mathcal{D}x'(t) \\ F(x(t), x'(t)) &= \int dy_b \int_{\mathcal{R}_b^A} e^{(i/\hbar)\Delta S'} \mathcal{D}y(t) \mathcal{D}y'(t) \end{aligned} \quad (21)$$

where $\Delta S' = S_p^A(y(t)) - S_p^A(y'(t)) + S_{int}^{SA}(x(t), y(t)) - S_{int}^{SA}(x'(t), y'(t))$.

where $Z = \text{Tr}(\rho)$ is a normalization factor to ensure $\text{Tr}(\rho) = 1$.

In summary, we show that the relational probability amplitude can be explicitly calculated through Eq.(16). R_{ij}^{SA} is defined as the sum of quantity $e^{iS_p/\hbar}$, where S_p is the action of the composite system $S + A$ along a path. Physical interaction between S and A may cause change of S_p , which is the phase of the probability amplitude. But $e^{iS_p/\hbar}$ itself is a probabilistic quantity, instead of a quantity associated with a physical property. With this definition and the results in Section II, we obtain the formulations for wave function in Eq. (19) and probability in Eq.(29) that are the consistent with those in traditional path integral formulation. The reduced density expression in Eq.(20), although equivalent to the J function in Ref [18], has richer physical meaning. For instance, we can calculate the entanglement entropy from the reduced density matrix. This will be discussed further in Section III D.

B. Interaction History of a Quantum System

One of the benefits of implementing the relational probability amplitude using path integral approach is that it is rather straightforward to describe the interaction history of a quantum system. Let's start with a

simple use case and later extending the formulation to a general use case.

Suppose up to time t_a , a quantum system S only interacts with a measuring system A . The detail of interaction is not important in this discussion. S may interact with A for a short period of time or may interact with A for the whole time up to t_a . Assume that after t_a , there is no further interaction between S and A . Instead S starts to interact with another measuring system A' , up to time t_b . Denote the trajectories of S, A, A' as $x(t), y(t), z(t)$, respectively. Up to time t_a , the relational matrix element is given by Eq.(16),

$$R(x_a, y_a) = \int_{\mathcal{R}_a^S} \int_{\mathcal{R}_a^A} e^{\frac{i}{\hbar} S_p^{SA}(x(t), y(t))} \mathcal{D}x(t) \mathcal{D}y(t). \quad (22)$$

Up to time t_b , the relational matrix element becomes

$$R(x_b, y_a, z_b) = \int_{\mathcal{R}_b^S} \int_{\mathcal{R}_a^A} \int_{\mathcal{R}_b^{A'}} \mathcal{D}x(t) \mathcal{D}y(t) \mathcal{D}z(t) \times \exp\left\{\frac{i}{\hbar} [S_p^{SA}(x(t), y(t)) + S_p^{SA'}(x(t), z(t))]\right\} \quad (23)$$

We can split region $\mathcal{R}_b^S$ into two regions, $\mathcal{R}_a^S$ and $\mathcal{R}_{ab}^S$, where $\mathcal{R}_{ab}^S$ is a region between time t_a and time t_b . This split allows us to express $R(x_b, y_a, z_b)$ in terms of $R(x_a, y_a)$,

$$R(x_b, y_a, z_b) = \int R(x_a, y_a) K(x_a, x_b, z_b) dx_a, \quad (24)$$

where

$$K(x_a, x_b, z_b) = \int_{\mathcal{R}_{ab}^S} \int_{\mathcal{R}_b^{A'}} \mathcal{D}x(t) \mathcal{D}z(t) \exp\left\{\frac{i}{\hbar} S_p^{SA'}(x(t), z(t))\right\} \quad (25)$$

From Eq.(25), one can derive the reduced density matrix element for S ,

$$\begin{aligned} \rho(x_b, x'_b) &= \int \int R(x_b, y_a, z_b) R^*(x'_b, y_a, z_b) dy_a dz_b \\ &= \int \int \rho(x_a, x'_a) G(x_a, x'_a; x_b, x'_b) dx_a dx'_a \end{aligned} \quad (26)$$

where

$$G(x_a, x_b; x'_a, x'_b) = \int K(x_a, x_b, z_b) K^*(x'_a, x'_b, z_b) dz_b. \quad (27)$$

Normalizing the reduced density matrix element, we have

$$\varrho((x_b, x'_b)) = \frac{1}{Z} \int \int \rho(x_a, x'_a) G(x_a, x'_a; x_b, x'_b) dx_a dx'_a \quad (28)$$

where the normalization factor $Z = \text{Tr}(\rho)$.

The reduced density matrix allows us to calculate the probability of finding the system in a particular state. For instance, the probability of finding the system initially in a state $\psi(x_b)$. Defining a project operator

$\hat{P} = |\psi(x_b)\rangle\langle\psi(x_b)|$, the probability is calculated, similar to Eq.(9), as

$$\begin{aligned} p(\psi) &= \text{Tr}(\rho \hat{P}) = \langle\psi(x_b)|\varrho|\psi(x_b)\rangle \\ &= \int \int \psi^*(x'_b) \psi(x_b) \varrho(x_b, x'_b) dx_b dx'_b \end{aligned} \quad (29)$$

To find the particle moving from a particular position $\bar{x}_b$ at time t_b , we substitute $\psi(x_b) = \delta(x_b - \bar{x}_b)$ into Eq.(29),

$$\begin{aligned} p(\bar{x}_b) &= \int \int \varrho(x_b, x'_b) \delta(x_b - \bar{x}_b) \delta(x'_b - \bar{x}_b) dx_b dx'_b \\ &= \varrho(\bar{x}_b, \bar{x}_b). \end{aligned} \quad (30)$$

Suppose there is no interaction between S and A' after time t_a , the action $S_p^{SA'}(x(t), z(t))$ consists only two independent terms, $S_p^{SA'}(x(t), z(t)) = S_p^S(x(t)) + S_p^{A'}(z(t))$. This allows us to rewrite Eq.(25) as a product of two terms:

$$\begin{aligned} K(x_a, x_b, z_b) &= \int_{\mathcal{R}_{ab}^S} \mathcal{D}x(t) \exp\left\{\frac{i}{\hbar} S_p^S(x(t))\right\} \\ &\times \int_{\mathcal{R}_b^{A'}} \mathcal{D}z(t) \exp\left\{\frac{i}{\hbar} S_p^{A'}(z(t))\right\} \\ &= K(x_a, x_b) \int_{\mathcal{R}_b^{A'}} \mathcal{D}z(t) \exp\left\{\frac{i}{\hbar} S_p^{A'}(z(t))\right\} \end{aligned} \quad (31)$$

Consequently, function $G(x_a, x'_a; x_b, x'_b)$ is rewritten as

$$\begin{aligned} G(x_a, x'_a; x_b, x'_b) &= K(x_a, x_b) K^*(x'_a, x'_b) \\ &\times \int dz_b \int \int_{\mathcal{R}_b^{A'}} \mathcal{D}z(t) \mathcal{D}z'(t) \exp\left\{\frac{i}{\hbar} [S_p^{A'}(z(t)) - S_p^{A'}(z'(t))]\right\}. \end{aligned} \quad (32)$$

The integral in the above equation is simply a constant, denoted as C . Thus, $G(x_a, x'_a; x_b, x'_b) = CK(x_a, x_b) K^*(x'_a, x'_b)$. Substituting this into Eq.(28), one obtains

$$\varrho(x_b, x'_b) = \frac{1}{Z} \int \int K(x_a, x_b) \rho(x_a, x_b) K^*(x'_a, x'_b) dx_a dx'_a. \quad (33)$$

The constant C is absorbed into the normalization factor. Eqs.(33) and (30) together can explain the double slit experiment, as will be shown in next Section.

We wish to generalize Eqs.(23) and (28) to describe a series of interaction history of the quantum system S . Suppose quantum system S has interacted with a series of measuring systems $A_1, A_2, \dots, A_n$ along the history but interacts with one measuring system at a time. Specifically, S interacts with A_1 up to time t_1 . From t_1 to t_2 , it only interacts with A_2 . From t_2 to t_3 , it only interacts with A_3 , and so on. Furthermore, we assume there is no interaction among these measuring systems. They are all independent. Denote the trajectories of these measuring systems as $y^{(1)}(t), y^{(2)}(t), \dots, y^{(n)}(t)$, and $y^{(1)}(t_1) = y_b^{(1)}, y^{(2)}(t_2) = y_b^{(2)}, \dots, y^{(n)}(t_n) = y_b^{(n)}$.

With this model, we can write down the relational matrix element for the whole history

$$\begin{aligned} R^{(n)}(x_b^{(n)}, y_b^{(1)}, y_b^{(2)}, \dots, y_b^{(n)}) \\ = \int_{\mathcal{R}_n^S} \mathcal{D}x(t) \prod_{i=1}^n \left\{ \int_{\mathcal{R}_i^A} \mathcal{D}y^{(i)}(t) \right\} \\ \times \exp\left\{ \frac{i}{\hbar} \sum_{j=1}^n [S_p^{(j)}(x(t), y^{(j)}(t))] \right\} \end{aligned} \quad (34)$$

where $\mathcal{R}_n^S$ is the region of finding the measured system S previous to time t_n , $\mathcal{R}_i^A$ is the region of finding measuring system A_i between time t_{i-1} to t_i . Action

$$S_p^{(j)}(x(t), y^{(j)}(t)) = \int_{t_{j-1}}^{t_j} L(x(t), \dot{x}(t), y^{(j)}(t), \dot{y}^{(j)}(t)) dt \quad (35)$$

is the integral of the Lagrangian over a particular path p lying in region $\mathcal{R}_n^S \cup \mathcal{R}_j^A$. The reason the total action among S and $A_1, A_2, \dots, A_n$ is written as a summation of individual action between S and A_i is due to the key assumption that S only interacts with one measuring system A_i at a time, and the interaction with each of measuring system A_i is independent from each other. This assumption further allows us to separate path integral over region $\mathcal{R}_n^S$ into two parts, one is integral over region previous to t_{n-1} , $\mathcal{R}_{n-1}^S$, and the other is the integral over region between t_{n-1} and t_n , $\mathcal{R}_{n-1,n}^S$. Thus,

$$\begin{aligned} R^{(n)}(x_b^{(n)}, y_b^{(1)}, y_b^{(2)}, \dots, y_b^{(n)}) \\ = \int R^{(n-1)}(x_b^{(n-1)}, y_b^{(1)}, y_b^{(2)}, \dots, y_b^{(n-1)}) \\ \times K(x_b^{(n-1)}, x_b^{(n)}, y_b^{(n)}) dx_b^{(n-1)} \end{aligned} \quad (36)$$

and

$$\begin{aligned} K(x_b^{(n-1)}, x_b^{(n)}, y_b^{(n)}) = \int_{\mathcal{R}_{n-1,n}^S} \mathcal{D}x(t) \int_{\mathcal{R}_n^A} \mathcal{D}y^{(n)} \\ \times \exp\left\{ \frac{i}{\hbar} S_p^{(n)}(x(t), y^{(n)}(t)) \right\} \end{aligned} \quad (37)$$

Similar to Eq.(18), the reduced density matrix element at t_n is

$$\begin{aligned} \rho(x_b^{(n)}, x_{b'}^{(n)}) &= \left[\prod_{i=1}^{i=n} \int dy_b^{(i)} \right] R^{(n)}(x_b^{(n)}, y_b^{(1)}, y_b^{(2)}, \dots, y_b^{(n)}) \\ &\times (R^{(n)}(x_b^{(n)}, y_b^{(1)}, y_b^{(2)}, \dots, y_b^{(n)}))^* \\ &= \int \int \rho(x_b^{(n-1)}, x_{b'}^{(n-1)}) \\ &\times G(x_b^{(n-1)}, x_b^{(n)}, x_{b'}^{(n-1)}, x_{b'}^{(n)}) dx_b^{(n-1)} dx_{b'}^{(n-1)} \end{aligned} \quad (38)$$

where

$$\begin{aligned} G(x_b^{(n-1)}, x_b^{(n)}, x_{b'}^{(n-1)}, x_{b'}^{(n)}) = \\ \int K(x_b^{(n-1)}, x_b^{(n)}, y_b^{(n)}) K^*(x_{b'}^{(n-1)}, x_{b'}^{(n)}, y_b^{(n)}) dy_b^{(n)}. \end{aligned} \quad (39)$$

The normalized version of the reduced density matrix element is given by

$$\varrho(x_b^{(n)}, x_{b'}^{(n)}) = \frac{\rho(x_b^{(n)}, x_{b'}^{(n)})}{\text{Tr}(\rho(t_n))} \quad (40)$$

and the probability of finding S in a position $\bar{x}_b^{(n)}$ at time t_n is

$$p(\bar{x}_b^{(n)}) = \varrho(\bar{x}_b^{(n)}, \bar{x}_b^{(n)}). \quad (41)$$

Recall that $x_b^{(n)}, x_{b'}^{(n)}$ are two different positions of finding S at time t_n . The probability of finding S at position $\bar{x}_b^{(n)}$ is simply a diagonal element of the reduced density matrix.

C. The Double Slit Experiment

In the double slit experiment, an electron passes through a slit configuration and is detected at position x of the destination screen. Denote the probability that the particle is detected at position x as $p_1 = |\varphi_1(x)|^2$ when only slit T_1 is opened, and as $p_2 = |\varphi_2(x)|^2$ when only slit T_2 is opened. If both slits are opened, the probability that the particle is detected at position x after passing through the double slit is given by $p(x) = |\varphi_1(x) + \varphi_2(x)|^2$, which is different from $p_1 + p_2 = |\varphi_1(x)|^2 + |\varphi_2(x)|^2$. Furthermore, when a detector is introduced to detect exactly which slit the particle goes through, the probability becomes $|\varphi_1(x)|^2 + |\varphi_2(x)|^2$. This observation was used by Feynman to introduce the concept of probability amplitude and the rule of calculating the measurement probability as the absolute square of the probability amplitude [18]. However, why the probability is the absolute square of the probability amplitude is not explained in Ref. [18]. By recognizing the measurement probability is a directional process, the measurement probability is shown to be the absolute square of probability amplitude [12]. In this section we will show that these observations can be readily explained by applying Eqs.(33) and (30).

Denote the location of T_1 is x_1 and location of T_2 is x_2 . Suppose both slits are opened. The electron passes through the double slit at t_a and detected at the destination screen at t_b . When the electron passes through the double slit, there is interaction between the electron and the slit. Denote the relational matrix after interaction as R . Suppose there is no entanglement after the electron passes through the slits, there is no inference information on exactly which slit the electron passes through. Further assuming equal probability for the electron passing either T_1 or T_2 , the state vector is represented by a superposition state $|\psi_a\rangle = (1/\sqrt{2})(|x_1\rangle + |x_2\rangle)$. The reduced density operator at t_a is $\varrho(t_a) = (1/2)(|x_1\rangle + |x_2\rangle)(\langle x_1| + \langle x_2|)$.

$\langle x_2 |$). Its matrix element is

$$\begin{aligned} \varrho(x_a, x'_a) &= \langle x_a | \hat{\rho}_a | x'_a \rangle \\ &= \frac{1}{2} (\delta(x_a - x_1) + \delta(x_a - x_2)) \\ &\quad \times (\delta(x'_a - x_1) + \delta(x'_a - x_2)) \end{aligned} \quad (42)$$

where the property $\langle x_a | x_i \rangle = \delta(x_a - x_i)$ is used in the last step. Later at time t_b , according to Eq.(33), the density matrix element for the electron at a position x_b in the detector screen is

$$\begin{aligned} \varrho(x_b, x'_b) &= \\ \frac{1}{2} \int \int_{-\infty}^{\infty} K(x_b, x_a) \varrho(x_a, x'_a) K^*(x'_b, x'_a) dx_a dx'_a \\ &= \frac{1}{2} [K(x_b, x_1) K^*(x'_b, x_1) + K(x_b, x_2) K^*(x'_b, x_2) \\ &\quad + K(x_b, x_1) K^*(x'_b, x_2) + K(x_b, x_2) K^*(x'_b, x_1)]. \end{aligned} \quad (43)$$

According to Eq. (30), the probability of finding the electron at a position x_b is

$$\begin{aligned} p(x_b) &= \varrho(x_b, x_b) \\ &= \frac{1}{2} [K(x_b, x_1) K^*(x_b, x_1) + K(x_b, x_2) K^*(x_b, x_2) \\ &\quad + K(x_b, x_1) K^*(x_b, x_2) + K(x_b, x_2) K^*(x_b, x_1)] \\ &= \frac{1}{2} |K(x_b, x_1) + K(x_b, x_2)|^2 \\ &= \frac{1}{2} |\varphi_1(x_b) + \varphi_2(x_b)|^2 \end{aligned} \quad (44)$$

where $\varphi_1(x_b) = K(x_b, x_1)$ and $\varphi_2(x_b) = K(x_b, x_2)$. Therefore, the probability to find the electron showing up at position x_b of the detector D is $p(x_b, t_b) = (1/2)|\varphi_1(x_b, t_b) + \varphi_2(x_b, t_b)|^2$.

Now suppose there is a weak measurement such that another apparatus, A' , is added to detect whether the electron passes through T_1 or T_2 . A' must interact with the electron in order to detect whether electron passes through T_1 or T_2 . Denote the pointer states of A' that are corresponding to the electron passing through T_1 and T_2 as $|T_1\rangle$ and $|T_2\rangle$, respectively. After the interaction, there is an entanglement between the electron and A' . As a weak measurement (or, non-selective measurement), the quantum state of S is not yet projected to either eigenstate of $|x_1\rangle$ or $|x_2\rangle$. The state vector of the composite system of electron and A' is $|\Psi\rangle = 1/\sqrt{2}(|x_1\rangle|T_1\rangle + |x_2\rangle|T_2\rangle)$, thus the reduced density operator for the electron after passing the slit configuration is $\hat{\rho}_a = Tr_T(|\Psi\rangle\langle\Psi|) = \frac{1}{2}(|x_1\rangle\langle x_1| + |x_2\rangle\langle x_2|)$. Its matrix element is

$$\begin{aligned} \varrho(x_a, x'_a, t_a) &= \langle x_a | \hat{\rho}_a | x'_a \rangle \\ &= \frac{1}{2} (\delta(x_a - x_1) \delta(x'_a - x_1) \\ &\quad + \delta(x_a - x_2) \delta(x'_a - x_2)). \end{aligned} \quad (45)$$

Substituted this into Eq.(33), the reduced density matrix element at t_b is

$$\begin{aligned} \varrho(x_b, x'_b) &= \\ \frac{1}{2} \int \int_{-\infty}^{\infty} K(x_b, x_a) \varrho(x_a, x'_a) K^*(x'_b, x'_a) dx_a dx'_a \\ &= \frac{1}{2} [K(x_b, x_1) K^*(x'_b, x_1) + K(x_b, x_2) K^*(x'_b, x_2)]. \end{aligned} \quad (46)$$

From this, the probability of finding the electron at position x_b is

$$\begin{aligned} p(x_b) &= \varrho(x_b, x_b) \\ &= \frac{1}{2} [K(x_b, x_1) K^*(x_b, x_1) + K(x_b, x_2) K^*(x_b, x_2)] \\ &= \frac{1}{2} (|K(x_b, x_1)|^2 + |K(x_b, x_2)|^2) \\ &= \frac{1}{2} (|\varphi_1(x_b)|^2 + |\varphi_2(x_b)|^2) \end{aligned} \quad (47)$$

There is no interference term $\varphi_1\varphi_2$ in Eq.(47). This result can be further understood as following. The interaction between S and A' results in the entanglement between these two systems. This is equivalent to say that A' has “measured” S . Therefore, A' has extract information from S . The indeterminacy of which eigenstate S is in disappears since such information can be inferred if we know which eigenstate A' is in. Thus, the interference terms should be excluded to avoid over-counting alternatives that contribute to the measurement probability (see more detailed explanation of this probability counting rule in Ref. [12]), resulting in Eq.(47).

D. The Von Neumann Entropy

The definition of Von Neumann entropy in Eq.(12) calls for taking the logarithm of the density matrix. This is a daunting computation task when the reduced density matrix element is defined using path integral formulation as Eq.(21). Brute force calculation of the eigenvalues of the reduced density matrix may be possible if one approximates the continuum of the position with a lattice model with spacing ϵ , and then takes the lattice spacing $\epsilon \rightarrow 0$ to find the limit. On the other hand, in quantum field theory, there is a different approach to calculate entropy based on the “replica trick” [19–21]. This approach allows one to calculate the von Neumann entropy without taking the logarithm. We will briefly describe this approach and apply it here. In the case of finite degree of freedom, the eigenvalues of the reduced density matrix λ_i must lie in the interval $[0, 1]$ such that $\sum_i \lambda_i = 1$. This means the sum $Tr(\rho^n) = \sum_i \lambda_i^n$ is convergent. This statement is true for any $n > 1$ even n is not an integer. Thus, the derivative of $Tr(\rho^n)$ with respect to n exists. It can be shown that

$$H(\rho_S) = - \lim_{n \rightarrow 1} \frac{\partial}{\partial n} Tr(\rho_S^n) = \lim_{n \rightarrow 1} H_S^{(n)} \quad (48)$$

where the Rényi entropy is defined as

$$H_S^{(n)} = \frac{1}{1-n} \ln \text{Tr}(\rho_S^n). \quad (49)$$

The replica trick calls for calculating ρ_S^n for integers n first and then analytically continuing to real number n . In this way, calculation of the von Neumann entropy is turned into the calculation of $\text{Tr}(\rho_S^n)$. But first, we have to construct ρ_S^n from the path integral version of reduced density matrix element in Eq.(21).

ρ_S^n is basically the multiplication of n copies of the same density matrix, i.e., $\rho_S^n = \rho_S^{(1)} \rho_S^{(2)} \dots \rho_S^{(n)}$, at time t_b . To simplify the notation, we will drop the subscript S . Since all the calculations are at time t_b , we also drop subscript b in Eq.(21). Denote the element of i th copy of reduced density matrix as $\rho(x_-^{(i)}, x_+^{(i)})$, where $x_-^{(i)} = x_L^{(i)}(t_b)$ and $x_+^{(i)} = x_R^{(i)}(t_b)$ are two different positions at time t_b , and $x_L^{(i)}(t)$ and $x_R^{(i)}(t)$ are two different trajectories used to perform the path integral for i th copy of reduced density matrix element as defined in Eq.(21). With these notation, the matrix element of ρ_S^n is

$$\begin{aligned} \rho^n(x_-^{(1)}, x_+^{(n)}) &= \int dx_+^{(1)} dx_-^{(2)} \dots dx_+^{(n-1)} dx_-^{(n)} \rho(x_+^{(1)}, x_-^{(1)}) \dots \\ &\times \rho(x_+^{(n)}, x_-^{(n)}) \delta(x_+^{(1)} - x_-^{(2)}) \dots \delta(x_+^{(n-1)} - x_-^{(n)}) \quad (50) \\ &= \int \prod_{i=1}^{(n-1)} [dx_+^{(i)} dx_-^{(i+1)} \rho(x_-^{(i)}, x_+^{(i)})] \\ &\times \delta(x_+^{(i)} - x_-^{(i+1)}) \rho(x_-^{(n)}, x_+^{(n)}). \end{aligned}$$

The delta function is introduced in the calculation above to ensure that when multiplying two matrices, the second index of the first matrix element and first index of the second matrix element are identical in the integral. From Eq.(50), we find the trace of ρ^n is

$$\begin{aligned} \text{Tr}(\rho^n) &= \int \rho^n(x_-^{(1)}, x_+^{(n)}) \delta(x_+^{(n)} - x_-^{(1)}) dx_+^{(n)} dx_-^{(1)} \\ &= \int \prod_{i=1}^n \{ \rho(x_-^{(i)}, x_+^{(i)}) \delta(x_+^{(i)} - x_-^{(mod(i,n)+1)}) \\ &\times dx_+^{(i)} dx_-^{(mod(i,n)+1)} \} \quad (51) \end{aligned}$$

where $\text{mod}(i, n) = i$ for $i < n$ and $\text{mod}(i, n) = 0$ for $i = n$. To further simplifying the notation, denote $\Delta_{+-}^{(i)} = \delta(x_+^{(i)} - x_-^{(mod(i,n)+1)})$, $dx_{+-}^{(i)} = dx_+^{(i)} dx_-^{(mod(i,n)+1)}$ and $Z(n) = \text{Tr}(\rho^n)$, the above equation is rewritten in a more compact form

$$Z(n) = \int \prod_{i=1}^n \{ \rho(x_-^{(i)}, x_+^{(i)}) \Delta_{+-}^{(i)} dx_{+-}^{(i)} \} \quad (52)$$

Substituting Eq.(21) into (52), we get

$$\begin{aligned} Z(n) &= \int \prod_{i=1}^n \{ \Delta_{+-}^{(i)} dx_{+-}^{(i)} \\ &\times \int_{\mathcal{R}_L^{(i)}} \int_{\mathcal{R}_R^{(i)}} e^{(i/\hbar)[S(x_L^{(i)}(t) - S(x_R^{(i)}(t))]} \quad (53) \\ &\times F(x_L^{(i)}(t), x_R^{(i)}(t)) \mathcal{D}x_L^{(i)}(t) \mathcal{D}x_R^{(i)}(t) \}. \end{aligned}$$

We have omitted the normalization so far. To remedy this, the normalized $Z(n) = \text{Tr}(\rho^n)$ should be rewritten as

$$\mathcal{Z}(n) = \text{Tr}(\rho^n) = \frac{Z(n)}{Z(1)^n}. \quad (54)$$

where $Z(1) = Z = \text{Tr}(\rho)$ as defined in Eq.(21). Once $\mathcal{Z}(n)$ is calculated, the von Neumann entropy is obtained through

$$H(\rho) = - \lim_{n \rightarrow 1} \frac{\partial}{\partial n} \text{Tr}(\rho_S^n) = - \lim_{n \rightarrow 1} \frac{\partial}{\partial n} \mathcal{Z}(n) \quad (55)$$

Eq.(53) appears very complicated. Let's validated in the case that there is no interaction between S and A . One would expect there is no entanglement between S and A . Thus, the entropy should be zero. We can check whether this is indeed the case based on Eq.(53). In this case, the influence functional is simply a constant. Eq.(21) is simplified into

$$\rho(x_b, x'_b) = \frac{1}{Z} \varphi(x_b) \varphi^*(x'_b) = \frac{1}{Z} \rho(x_b, x'_b) \quad (56)$$

where $\varphi(x_b)$ has been given in Eq.(19). Taking trace of the above equation,

$$1 = \text{Tr}(\rho) = \frac{1}{Z} \int \varphi(x_b) \varphi^*(x_b) dx_b \quad (57)$$

gives

$$Z = \int \varphi(x_b) \varphi^*(x_b) dx_b \quad (58)$$

Eq.(56) implies that S is in a pure state, since by definition, $\varphi(x_b) = \langle x_b | \varphi \rangle = \int \delta(x - x_b) \varphi(x) dx$, so that

$$\rho(x_b, x'_b) = \frac{1}{Z} \langle x_b | \varphi \rangle \langle \varphi | x'_b \rangle. \quad (59)$$

Multiplication of density matrix that represents a pure state gives the same density matrix itself. Using the same notation as in Eq.(50), we obtain

$$\begin{aligned} \rho^2(x_-^{(1)}, x_+^{(2)}) &= \int \rho(x_-^{(1)}, x_+^{(1)}) \rho(x_-^{(2)}, x_+^{(2)}) \delta(x_+^{(1)} - x_-^{(2)}) dx_+^{(1)} dx_-^{(2)} \\ &= \int \rho(x_-^{(1)}, x_+^{(1)}) \rho(x_+^{(1)}, x_+^{(2)}) dx_+^{(1)} \\ &= \int \varphi(x_-^{(1)}) \varphi^*(x_+^{(1)}) \varphi(x_+^{(1)}) \varphi^*(x_+^{(2)}) dx_+^{(1)} \\ &= Z \rho(x_-^{(1)}, x_+^{(2)}). \quad (60) \end{aligned}$$

From this we can deduce that $\rho^n = Z^{n-1}\rho$. This gives $Z(n) = \text{Tr}(\rho^n) = Z^n$, and

$$\mathcal{Z}(n) = \frac{Z(n)}{Z(1)^n} = 1. \quad (61)$$

It is independent of n , thus

$$H(\varrho) = -\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \mathcal{Z}(n) = 0, \quad (62)$$

as expected. The von Neumann entropy is only non-zero when there is an interaction between S and A . The effect of the interaction is captured in the influence functional. Concrete form of the influence functional should be constructed in order to find examples where the von Neumann entropy is non-zero.

IV. DISCUSSION AND CONCLUSION

A. The G Function

The G function introduced in Eq.(27) can be rewritten in terms of the influence functional. To do this, we first rewrite Eq.(25) as

$$\begin{aligned} K(x_a, x_b, z_b) &= \int_{\mathcal{R}_{ab}^S} \mathcal{D}x(t) \exp\left\{\frac{i}{\hbar} S_p^S(x(t))\right\} \\ &\times \int_{\mathcal{R}_b^{A'}} \mathcal{D}z(t) \exp\left\{\frac{i}{\hbar} [S_p^{A'}(z(t)) + S_{int}^{SA'}(x(t), z(t))]\right\} \end{aligned} \quad (63)$$

Substituting this into Eq.(27), we have

$$\begin{aligned} G(x_a, x_b; x'_a, x'_b) &= \int_{\mathcal{R}_{ab}^S} \int_{\mathcal{R}_{a'b'}^{S'}} e^{(i/\hbar)[S_p^S(x(t)) - S_p^S(x'(t))]} \\ &\times F(x(t), x'(t)) \mathcal{D}x(t) \mathcal{D}x'(t) \end{aligned} \quad (64)$$

Consequently, the normalized reduced density matrix element in Eq.(28) becomes

$$\begin{aligned} \varrho((x_b, x'_b)) &= \frac{1}{Z} \int \int_{\mathcal{R}_{ab}^S} \int_{\mathcal{R}_{a'b'}^{S'}} e^{(i/\hbar)[S_p^S(x(t)) - S_p^S(x'(t))]} \\ &\times \rho(x_a, x'_a) F(x(t), x'(t)) \mathcal{D}x(t) \mathcal{D}x'(t) dx_a dx'_a. \end{aligned} \quad (65)$$

This gives the same result as in Ref. [22]. However, there is advantage of using the G function instead of the influence functional F because the complexity of path integral is all captured inside the G function, making it mathematically more convenient. This can be shown in the following modified double slit experiment example. Suppose after the electron passing the double slit, there is no detector next to the slits to detect which split the electron passing through, but there is another system B that continues to interact with S till S reaches the destination screen detector. Alternatively, this can be an

electromagnetic field between the double slit and the destination screen detector. We will need to apply Eq.(28) instead of (33) to calculate the reduced density matrix element. In this case, we simply substitute Eq.(42) into Eq.(28) and obtain

$$\begin{aligned} \varrho(x_b, x'_b) &= \frac{1}{2} [G(x_1, x_b; x_1, x'_b) + G(x_1, x_b; x_2, x'_b) \\ &+ G(x_2, x_b; x_1, x'_b) + G(x_2, x_b; x_2, x'_b)]. \end{aligned} \quad (66)$$

The probability of finding the electron at position x_b is

$$\begin{aligned} p(x_b) &= \varrho(x_b, x_b) \\ &= \frac{1}{2} [G(x_1, x_b; x_1, x_b) + G(x_1, x_b; x_2, x_b) \\ &+ G(x_2, x_b; x_1, x_b) + G(x_2, x_b; x_2, x_b)]. \end{aligned} \quad (67)$$

From the definition of function $G(x_a, x_b; x'_a, x'_b)$ in Eq.(27), it is easy to derive the following property,

$$G(x_a, x_b; x'_a, x'_b) = G^*(x'_a, x'_b; x_a, x_b). \quad (68)$$

When $x_a = x'_a$ and $x_b = x'_b$, we have $G(x_a, x_b; x_a, x_b) = G^*(x_a, x_b; x_a, x_b)$. Thus, $G(x_a, x_b; x_a, x_b)$ must be a real function. We denote it as $G_R(x_a, x_b)$. With these properties, Eq.(67) can be further simplified as

$$\begin{aligned} p(x_b) &= \frac{1}{2} [G_R(x_1, x_b) + G_R(x_2, x_b)] \\ &+ \text{Re}[G(x_1, x_b; x_2, x_b)]. \end{aligned} \quad (69)$$

This is consistent with the requirement that $p(x_b)$ must be a real number. The second term $\text{Re}[G(x_1, x_b; x_2, x_b)]$ is an interference quantum effect due to the fact that the initial state after passing through the double slit is a pure state. This interference term also depends on the interaction between S and B . If the interaction is adjustable, the probability distribution will be adjusted accordingly. For instance, if there is an electromagnetic field between the double slit and the destination screen that interacts with the electron, tuning the electromagnetic field will cause the probability distribution $p(x_b)$ to change through the interference term in Eq.(69). Presumably, the Aharonov–Bohm effect [23] can be explained through Eq.(69) as well. The detailed calculation will be published in a later paper.

B. Influence Functional and Entanglement Entropy

In Section IIID, we show that in the condition that there is no interaction, the influence functional is a constant and therefore the entanglement entropy is zero. We can relax this condition to detect whether there is entanglement. Suppose the influence functional can be decomposed in the following production form,

$$F(x(t), x'(t)) = f(x(t)) f^*(x'(t)). \quad (70)$$

Such a form of influence functional satisfies the rule [18] $F(x(t), x'(t)) = F^*(x'(t), x(t))$. Eq.(70) implies that the

entanglement entropy is still zero even there is interaction between S and A . The reason for this is that the reduced density matrix element can be still written as the form of Eq.(56) but with

$$\varphi(x_b) = \int_{\mathcal{R}_b} e^{(i/\hbar)S(x(t))} f(x(t)) \mathcal{D}x(t). \quad (71)$$

Again Eq.(56) shows S is in a pure state therefore the reasoning process from Eqs.(56) to (62) is applicable here.

We now examine the entanglement entropy for some general forms of influence functional discussed in Ref. [18]. The most general exponential functional in linear form is

$$F(x(t), x'(t)) = \exp\{i \int x(t)g(t)dt - i \int x'(t)g(t)dt\} \quad (72)$$

where $g(t)$ is a real function. Clearly this form satisfies the condition specified in Eq.(70) since we can take $f(x(t)) = \exp[i \int x(t)g(t)dt]$ and $f(x'(t)) = \exp[i \int x'(t)g(t)dt]$. The entanglement entropy is zero with this form of influence functional.

On the other hand, the most general Gaussian influence functional [18]

$$F(x(t), x'(t)) = \exp\left\{\int \int [\alpha(t, t')x(t') - \alpha^*(t, t')x'(t')][x(t) - x'(t)]dt dt'\right\} \quad (73)$$

where $\alpha(t, t')$ is an arbitrary complex function, defined only for $t > t'$. This form of influence functional cannot be decomposed to satisfy the condition specified in Eq.(70). The entanglement entropy may not be zero in this case. It will be of interest to further study the influence functional of some actual physical setup and calculate the entanglement entropy explicitly.

C. Conclusions

We show that the relational probability amplitude can be implemented using path integral approach. The formulation is consistent with the results from the traditional path integral quantum mechanics. The significance of such implementation is two-folds. First, it gives a clearer meaning of the relational probability amplitude. R_{ij}^{SA} is defined as the sum of quantity $e^{iS_p/\hbar}$, where S_p is the action of the composite system $S + A$ along a path. Physical interaction between S and A may cause change

of S_p , which is the phase of a complex number. This complex number, $e^{iS_p/\hbar}$, is just a probabilistic quantity, instead of a quantity associated with a physical property. Second, it gives a natural derivation of the coordinator representation of the reduced density matrix of the observed system. Based on the coordinator representation of the reduced density matrix element, it is mathematically convenient to develop formulation for some of interesting physical processes and concepts. For instance, we can describe the interaction history of the measured system and a series of measuring systems or environment, and use this formulation to elegantly explain the double slit experiment. A more interesting application of the coordinator representation of the reduced density matrix is the method to calculate entanglement entropy using path integral approach. This will allow us to potentially calculate entanglement entropy of a physical system that interacts with classical fields, such as an electron in an electromagnetic field. Section IV B gives a criterion on whether there is entanglement between the system and external environment based on the influence functional. Since entanglement entropy is an important concept in quantum information theory, the method described here may lead to new insight on the information aspect of a physical system that interacts with classical fields. This is a topic for further research.

Ref. [12, 13] and this paper together show that quantum mechanics can be constructed by shifting the starting point from the independent properties of a quantum system to the relational properties between a measured quantum system and a measuring quantum system. In essence, quantum mechanics demands a new set of rules to calculate probability of a potential outcome from the physical interaction in quantum measurement. The relational formulation of quantum mechanics is not only significant at the conceptual level, but also valuable at the application level. At the conceptual level, the difference between the relational formulation and traditional formulation results in fundamental consequence in some special scenario. This is demonstrated in resolving the EPR paradox [13]. At the application level, the path integral implementation of relational probability amplitude leads to new tool to calculate entanglement entropy based on the coordinator representation of the reduced density matrix. Ultimately, we hope the relational formulation presented in Ref. [12, 13] and this paper can be one step towards a better understanding of quantum mechanics.

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