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A Fuzzy Linear Regression Model With Functional Predictors And Fuzzy Responses

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Abstract

A novel functional regression model was introduced, where the predictor was a curve linked to a scalar fuzzy response variable. An absolute error-based penalized method with SCAD loss function was proposed to evaluate the unknown components of the model. For this purpose, a concept of fuzzy-valued function was developed and discussed. Then, a fuzzy large number notion was proposed to estimate the fuzzy-valued function. Some common goodness-of-fit criteria were also used to examine the performance of the proposed method. Efficiency of the proposed method was then evaluated through two numerical examples, including a simulation study and an applied example in the scope of watershed management. The proposed method was also compared with several common fuzzy regression models in cases where the functional data was converted to scalar ones.

Keywords: Goodness-of-fit measure, functional fuzzy number, SCAD penalty, functional regression model.

1. Introduction

As the most basic and commonly used statistical technique, multiple regression analysis is utilized to estimate the relationships between one or more predictors (independent variables) and a response (dependent variable). Recently, many techniques have been proposed by different authors to combine the conventional statistical regression models with the concept of fuzzy set theory. In this regard, Chukhrova and Johannssen [1] provided a comprehensive systematic review of then-available methodologies and applications focused on fuzzy regression analysis as of 2019. Such studies can be classified as (1) possibilistic approaches, where linear and non-linear programming methods are minimized by minimizing the total spread of their fuzzy parameters, subject to the support observations at some specific levels (see for example Refs. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]), (2) fuzzy least squares and fuzzy least absolutes parametric/non-parametric methods, where the gap between the predicted fuzzy values and available fuzzy data is minimized with regard to various distance measures between two fuzzy numbers, covering the most commonly used linear and non-linear models (see for instance Refs. [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]), and (3) machine learning techniques, like evolutionary algorithms [30, 31, 32, 33, 34], support vector machines [35, 36, 37, 38], and neural networks embedded in fuzzy regression analysis [39, 40, 41, 42, 43], where the ideas and terminology relevant to biological evolution are used, such as mutation, recombination, reproduction and selection. Here the candidate solutions of the optimization problem represent individuals in a population. Accordingly, a fitness function is used to determine the quality of some solutions of the optimization problem with individuals in the underlying population. Cheng and Lee [44] investigated the two most basic non-parametric regression techniques, namely k-nearest neighbor smoothing and kernel smoothing, for a problem with crisp input and fuzzy output. They further formulated an algorithm to select the best smoothing parameters based on minimization of cross-validation criteria. Wang et al. [45] proposed a fuzzy non-parametric model with crisp input and LR-fuzzy output based on the local linear smoothing technique with a cross-validation procedure to select the optimal value of the smoothing parameter to fit the model. Additionally, Hesamian and Akbari [47] and Yang and Yin [47] proposed some fuzzy multiple regression model with fuzzy varying coefficients based on exact predictors and fuzzy responses.

All of the above mentioned fuzzy regression models relied on non-functional data. However, functional regression analysis [48] has received considerable attentions in various fields of application [49, 50, 51]. The basic idea behind functional regression analysis is to express each predictor in a repeatedly measured set of data as a smooth function and then draw information from the collection of the functional data. The term "functional" data traditionally refers to the data measured over an interval or a higher dimensional domain. Such data is recorded at discrete times to form a continuous function in order to (1) allow record evaluation at any point in time, (2) evaluate rates of change, (3) reduce noise, and (4) allow registration onto a common time-scale. From another pont of view, regression models with functional data can be classified into three classes: those with (1) functional predictor(s) and scalar response [52, 53], (2) scalar predictor(s) and functional response [54, 55, 48, 49, 56, 57, 58, 59, 60, 61] and (3) functional predictor(s) and functional response [62, 63, 64]. Many of these methods are direct extensions of the classical least squares, principal component, and partial least-squares procedures.

Previous studies on fuzzy regression analysis have been conducted on the basis of non-functional scalar/fuzzy quantities with exact/fuzzy or exact/fuzzy-valued (varying) coefficients. In this paper, however, a fuzzy functional linear regression modeling strategy is proposed based on functional predictors and a LR-fuzzy response and fuzzy varying coefficients. For many experts, a simple way to capture imprecision in a vague process is to express that as an LR-fuzzy number. Therefore, the LR-fuzzy numbers play an important role in many real-life applications of fuzzy inferences. To evaluate the unknown components of the proposed fuzzy functional coefficients, a criteria selection model is herein proposed via absolute error regularization and SCAD penalty.

The rest of this paper is organized as follows: Section 2 reviews some general concepts relevant to the fuzzy numbers. In Section 3, a methodology is proposed to estimate the fuzzy coefficients of a fuzzy regression model with functional predictors and fuzzy responses. A hybrid algorithm is also represented to evaluate the components of the proposed fuzzy functional regression model. Section 4 presents three numerical examples to evaluate the performance of the proposed method compared to other fuzzy multiple/non-linear/non-parametric regression methods in terms of some common performance measures. Finally, the main contributions of this paper are summarized in Section 5.

2. Fuzzy numbers

This section reviews some basic definitions of fuzzy numbers based on [65, 66]. A fuzzy set $\widetilde{A}$ of $\mathbb{R}$ (the real line) is defined by its membership function $\mu_{\widetilde{A}} : \mathbb{R} \to [0, 1]$. In addition, a fuzzy set $\widetilde{A}$ of $\mathbb{R}$ is called a fuzzy number (**FN**) if it is normal, i.e. there is a unique $x^*_{\widetilde{A}} \in \mathbb{R}$ so that $\mu_{\widetilde{A}}(x^*_{\widetilde{A}}) = 1$; and for every $\alpha \in [0, 1]$, the set $\widetilde{A}[\alpha] = \{x \in \mathbb{R} : \mu_{\widetilde{A}}(x) \geq \alpha\}$ is a nonempty compact interval in $\mathbb{R}$. This interval is denoted by $\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$, where $\widetilde{A}^L_\alpha = \inf\{x : x \in \widetilde{A}[\alpha]\}$ and $\widetilde{A}^U_\alpha = \sup\{x : x \in \widetilde{A}[\alpha]\}$. The set of all fuzzy numbers is denoted by $\mathbb{F}(\mathbb{R})$. It is worth noting that fuzzy numbers are approximate assessments, given by experts and accepted by decision-makers when access to more accurate values is either impossible or unnecessary. To simplify the fuzzy numbers representation and handling, several authors have captured the information contained in a (unimodal) fuzzy number using a functional parametric form known as LR-fuzzy number $\widetilde{A} = (a; l_a, r_a)_{LR}$. The membership function of a LR-fuzzy number $\widetilde{A}$ is defined by:

$$\mu_{\widetilde{A}}(x) = \begin{cases} L(\dfrac{a-x}{l_a}), & x \leq a, \\ R(\dfrac{x-a}{r_a}), & x > a, \end{cases} \tag{1}$$

where $a \in \mathbb{R}$, $l_a > 0$ and $r_a > 0$ are called the mean value, left and right spreads of $\widetilde{A}$, respectively. The shape function L (or R) is a decreasing function from $\mathbb{R}^+ \to [0,1]$ such that

1. $L(0) = 1$,
2. $L(x) < 1$ for every $x > 0$,
3. $L(x) > 0$ for every $x < 1$,
4. $L(1) = 0$ (or $L(x) > 0$ for any $x \in \mathbb{R}$ and $L(+\infty) = 0$).

An LR-number has been applied in various problems as a general model function of imprecision. In this paper, we employed the most commonly used LR-fuzzy numbers (with $L(x) = R(x) = \max\{0, 1-x\}$) so-called triangular fuzzy numbers (**TFN**s), to handle the imprecision in data set during numerical evaluations. The membership function of a triangular fuzzy number, denoted by $\widetilde{A} = (a; l_a, r_a)_T$, is given by:

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x-(a-l_a)}{l_a}, & a - l_a \leq x \leq a, \\ \frac{a+r_a-x}{r_a}, & a < x \leq a + r_a, \\ 0, & x \in \mathbb{R} - (a - l_a, a + r_a). \end{cases} \tag{2}$$

Definition 2.1. *[65] For a given $\widetilde{A} \in \mathbb{F}(\mathbb{R})$, the mapping $\widetilde{A}_\alpha : [0,1] \to \mathbb{R}$ is called α-values of $\widetilde{A}$ defined as follows:*

$$\widetilde{A}_\alpha = \begin{cases} \widetilde{A}^L[2\alpha] & \alpha \in [0, 0.5], \\ \widetilde{A}^U[2(1-\alpha)] & \alpha \in (0.5, 1], \end{cases} \tag{3}$$

where $\widetilde{A}^L[\alpha]$ and $\widetilde{A}^U[\alpha]$ denote the lower and upper limits of α-cuts of $\widetilde{A}$.

Example 2.1. *Let $\widetilde{A} = (a; l_a, r_a)_{LR}$ be an LR-fuzzy number. From Definition 2.1, one finds*

$$\widetilde{A}_\alpha = \begin{cases} a - l_a L^{-1}(2\alpha), & 0.0 \leq \alpha \leq 0.5, \\ a + r_a R^{-1}(2(1-\alpha)), & 0.5 \leq \alpha \leq 1.0. \end{cases}$$

For instance,

1. *If $\widetilde{A} = (a; l_a, r_a)_T$ is a triangular fuzzy number, then:*

$$\widetilde{A}_\alpha = \begin{cases} (a - l_a) + 2l_a\alpha, & 0.0 \le \alpha \le 0.50, \\ a + r_a - 2r_a(1-\alpha), & 0.50 < \alpha \le 1.0. \end{cases}$$

2. *Let $\widetilde{A} = (\mu, \sigma)_G$ be a Gaussian **FN** with the membership function of $\mu_{\widetilde{A}}(x) = \exp(-0.5[(x-\mu)/\sigma]^2)$, then:*

$$\widetilde{A}_\alpha = \begin{cases} \mu - \sigma\sqrt{-2\ln(2\alpha)}, & 0.0 < \alpha \le 0.50, \\ \mu + \sigma\sqrt{-2\ln(2(1-\alpha))}, & 0.50 < \alpha < 1.0. \end{cases}$$

Remark 2.1. *Since $\widetilde{A}_\alpha$ is a decreasing function of α, the relationship between α-values and α-cuts can be expressed as:*

$$\widetilde{A}[\alpha] = [\widetilde{A}^L[\alpha], \widetilde{A}^U[\alpha]] = [\widetilde{A}_{\alpha/2}, \widetilde{A}_{1-\alpha/2}]. \tag{4}$$

Therefore, having a sequence of α-values $\{\widetilde{A}_\alpha\}_{\alpha\in[0,1]}$, the membership function of $\widetilde{A}$ can be evaluated as follows:

$$\mu_{\widetilde{A}}(x) = \sup\{\alpha \in [0,1] : x \in [\widetilde{A}_{\alpha/2}, \widetilde{A}_{1-\alpha/2}]\},\ x \in \mathbb{R}.$$

In addition, for all $\widetilde{A}, \widetilde{B} \in \mathbb{F}(\mathbb{R})$, $\lambda \in \mathbb{R}$ and $\alpha \in [0,1]$, the addition and scalar multiplication operations between $\widetilde{A}$ and $\widetilde{B}$ can be evaluated as follows:

$$(\widetilde{A} \oplus \widetilde{B})_\alpha = \widetilde{A}_\alpha + \widetilde{B}_\alpha,$$
$$(\lambda \otimes \widetilde{A})_\alpha = \begin{cases} \lambda\widetilde{A}_\alpha & \textit{if} \quad \lambda > 0, \\ \lambda\widetilde{A}_{1-\alpha} & \textit{if} \quad \lambda < 0. \end{cases}$$

Such arithmetic operations will be applied to suggest a fuzzy multivariate regression model in next section.

Definition 2.2. *An L_p distance measure between two fuzzy numbers **FN**s $\widetilde{A}$ and $\widetilde{B}$ is defined as*

$$d_p(\widetilde{A}, \widetilde{B}) = \begin{cases} (\int_0^1 g(\alpha)|\widetilde{A}_\alpha - \widetilde{B}_\alpha|^p d\alpha)^{1/p}, & p \ge 1, \\ \sup_{\alpha\in[0,1]} |\widetilde{A}_\alpha - \widetilde{B}_\alpha|, & p = \infty, \end{cases}$$

where

$$g(\alpha) = \begin{cases} 4\alpha & 0 \le \alpha \le 0.5, \\ 4(1-\alpha) & 0.5 \le \alpha \le 1. \end{cases}$$

Any three **FN**s $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ satisfy the following conditions:

- $d_p(\widetilde{A}, \widetilde{B}) = 0$ if and only if $\widetilde{A} = \widetilde{B}$,
- $d_p(\widetilde{A}, \widetilde{B}) = d_p(\widetilde{B}, \widetilde{A})$,
- $d_p(\widetilde{A}, \widetilde{C}) \leq \big(d_p(\widetilde{A}, \widetilde{B}) + d_p(\widetilde{B}, \widetilde{C})\big)$.

It should be noted that $g(\alpha)$ also modifies the square error distance between the two **FN**s $\widetilde{A}$ and $\widetilde{B}$ since it focuses on the values near the centers rather than tails. This distance measure is used to evaluate the unknown components of the proposed fuzzy functional regression model and performances of the proposed fuzzy regression model compared to other fuzzy regression models.

Definition 2.3. *$\widetilde{A}(t)$ is said to be a (continuous) fuzzy-valued function on $[a, b]$ if $\widetilde{A}(t)$ is a **FN** for any $t \in [a, b]$.*

Definition 2.4. *Let $\widetilde{A}(t)$ be a fuzzy-functional-valued on $[a, b]$. The integration of $\widetilde{A}(.)$ over $[a, b]$ is defined to be a fuzzy number $\widetilde{\mathfrak{A}}$ with the following α-values:*

$$(\widetilde{\mathfrak{A}})_\alpha = \int_a^b (\widetilde{A}(t))_\alpha dt.$$

Example 2.2. *If $\widetilde{A}(t) = (0.5\sin(\pi t); 0.1t)_{LR}$ with $L(x) = \exp(-x/2)$ and $R(x) = \max\{0, 1 - x\}$ and $\widetilde{B}(t) = (t; 1/(1 + t))_G$ then, the plots of $\widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}}$ on $[0, 1]$ are shown in Fig. 1. In addition,*

$$d_2(\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{B}}) = (\int_0^1 g(\alpha)|\widetilde{\mathfrak{A}}_\alpha - \widetilde{\mathfrak{B}}_\alpha|^2 d\alpha)^{1/2} = 0.6342, and$$

$$d_1(\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{B}}) = \int_0^1 g(\alpha)|\widetilde{\mathfrak{A}}_\alpha - \widetilde{\mathfrak{B}}_\alpha| d\alpha = 0.6195.$$

Lemma 2.1. *Let $c, d \in \mathbb{R}$ and $\widetilde{A}(.)$ and $\widetilde{B}(.)$ be two fuzzy-valued functions. Then,*

$$\int \big((c \otimes \widetilde{A}(t)) \oplus (d \otimes \widetilde{B}(t))\big)dt = (c \otimes \widetilde{\mathfrak{A}}) \oplus (d \otimes \widetilde{\mathfrak{B}}),$$

where $\widetilde{\mathfrak{A}} = \int_0^1 (\widetilde{A}(t)dt$ and $\widetilde{\mathfrak{B}} = \int_0^1 (\widetilde{B}(t)dt$.

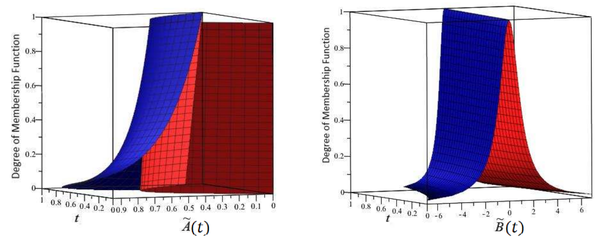

Figure 1: 3D-plot of $\widetilde{A}(t)$ and $\widetilde{B}(t)$ in Example 2.2.

Proof 2.1. *According to Definition 2.4 and arithmetic operations of α-values fuzzy numbers, it is easy to see that*

$$(\int \big((c \otimes \widetilde{A}(t)) \oplus (d \otimes \widetilde{B}(t))\big)dt)_\alpha =$$

$$\int \big((c \otimes \widetilde{A}(t)) \oplus (d \otimes \widetilde{B}(t))_\alpha\big)dt$$

$$= \int (c \otimes \widetilde{A}(t))_\alpha dt + \int (d \otimes \widetilde{B}(t))_\alpha dt.$$

Now, note that

$$\int (c \otimes \widetilde{A}(t))_\alpha dt = \begin{cases} c \int \widetilde{A}(t)_\alpha dt & c \geq 0, \\ c \int \widetilde{A}(t)_{1-\alpha} dt & c < 0, \end{cases}$$

and

$$\int (d \otimes \widetilde{B}(t))_\alpha dt = \begin{cases} d \int \widetilde{B}(t)_\alpha dt & d \geq 0, \\ d \int \widetilde{B}(t)_{1-\alpha} dt & d < 0. \end{cases}$$

Therefore, $\int (c \otimes \widetilde{A}(t))dt = c \otimes \widetilde{\mathfrak{A}}$ *and* $\int (d \otimes \widetilde{B}(t))dt = d \otimes \widetilde{\mathfrak{B}}$ *which completes the proof.*

Here a notion of large numbers is developed for a fuzzy-valued function of $\widetilde{A}(t)$.

Lemma 2.2. *Let $\widetilde{A}(t)$ be a fuzzy-valued function on $[0, 1]$. If $U_1, U_2, \ldots, U_N$ are independent random variables uniformly distributed over the interval $[a, b]$, then*

$$\lim_{N \to \infty} P\big(d_\infty(\int_a^b \widetilde{A}(t)dt, (\frac{b-a}{N}) \otimes \oplus_{i=1}^{N} \widetilde{A}(U_i)) = 0\big) = 1.$$

Proof 2.2. *According to Definition 2.2, first note that:*

$$d_\infty(\int_a^b \widetilde{A}(t)dt, (\frac{b-a}{N}) \otimes \bigoplus_{i=1}^{N} \widetilde{A}(U_i)) =$$

$$\sup_{\alpha\in[0,1]} |\int_a^b (\widetilde{A}(t))_\alpha dt - (\frac{b-a}{N}) \sum_{i=1}^{N} (\widetilde{A}(U_i))_\alpha|.$$

By strong law of large numbers [67], we know that

$$\lim_{N\to\infty} P(|\int_a^b (\widetilde{A}(t))_\alpha dt - (\frac{b-a}{N}) \sum_{i=1}^{N} (\widetilde{A}(U_i))_\alpha| = 0) = 1,$$

for any $\alpha \in [0,1]$. This simply concludes that $\lim_{N\to\infty} P(\sup_{\alpha\in[0,1]} |\int_a^b (\widetilde{A}(t))_\alpha dt - (\frac{b-a}{N}) \sum_{i=1}^{N} (\widetilde{A}(U_i))_\alpha| = 0) = 1$. This completes the proof.

Remark 2.2. *According to Lemma 2.2, the integration of $\widetilde{A}(.)$ over $[a,b]$ can be approximated by a Mont Carlo simulation as $(\frac{b-a}{N}) \otimes \bigoplus_{i=1}^{N} \widetilde{A}(U_i)$, that is:*

$$\int_a^b \widetilde{A}(t)dt \simeq (\frac{b-a}{N}) \otimes \bigoplus_{i=1}^{N} \widetilde{A}(U_i),$$

for a large value of N.

3. Fuzzy functional linear regression model

Functional data analysis is a fast evolving branch of applied statistics, with the functional regression becoming popular in recent years. In this section, a functional linear regression model with fuzzy functional predictors, fuzzy responses and fuzzy functional coefficients was developed. Denoting the observed data on n statistical units by $(\widetilde{y}_i, \boldsymbol{x}_i(.) = (x_{i1}(.), x_{i2}(.), \ldots, x_{ip}(.))^\top)$, consider the following fuzzy functional linear regression model:

$$\widetilde{y}_i = \widetilde{\alpha} \oplus \bigoplus_{j=1}^{d} \int_a^b (\widetilde{\beta}_j(t) \otimes x_{ij}(t))dt \oplus \widetilde{\epsilon}_i, \;\; i = 1, 2, \ldots, n, \tag{5}$$

where

1. $\widetilde{y}_i = (y_i; l_{y_i}, r_{y_i})_{L_i R_i}$ represent fuzzy responses,

2. $\widetilde{\alpha}$ denotes the unknown fuzzy intercept,

3. $\widetilde{\beta}_j(t) = (\beta_j(t); l_{\beta_j(t)}, r_{\beta_j(t)})_{L_j R_j}$ are the coefficients of the true fuzzy-valued function , and

4. $\widetilde{\epsilon}_i$ indicates a fuzzy error term.

Based on the fuzzy law of large numbers in fuzzy domain (Lemma 2.1), the fuzzy functional linear regression model (5) can be converted to a conventional fuzzy linear regression model:

$$\widetilde{y}_i = \widetilde{\alpha} \oplus (\frac{(b-a)}{N}) \bigoplus_{k=1}^{N} \bigoplus_{j=1}^{d} \left(\widetilde{\beta}_j(U_k) \otimes x_{ij}(U_k)\right) \oplus \widetilde{\epsilon}_i, \;\; i = 1, 2, \ldots, n, \tag{6}$$

where $U_1, U_2, \ldots, U_N$ are independent random variables uniformly distributed over the interval $[a, b]$ and $N \in \mathbb{N}$ is a large number.

Remark 3.1. *Note that the unknown fuzzy varying coefficients of model* (6) *can be presented as* $\boldsymbol{B} = (\boldsymbol{L}_B, B, \boldsymbol{R}_B)^\top$, *where:*

$$B = \begin{pmatrix} \beta_{11} & \beta_{21} & \ldots & \beta_{d1} \\ \vdots & \vdots & \ldots & \vdots \\ \beta_{N1} & \beta_{N2} & \ldots & \beta_{Nd} \end{pmatrix}, \tag{7}$$

$$\boldsymbol{L}_B = \begin{pmatrix} l_{\beta_{11}} & l_{\beta_{12}} & \ldots & l_{\beta_{1d}} \\ \vdots & \vdots & \ldots & \vdots \\ l_{\beta_{N1}} & l_{2\beta_{N2}} & \ldots & l_{\beta_{Nd}} \end{pmatrix}, \textit{and} \tag{8}$$

$$\boldsymbol{R}_B = \begin{pmatrix} r_{\beta_{11}} & r_{\beta_{12}} & \ldots & r_{\beta_{1d}} \\ \vdots & \vdots & \ldots & \vdots \\ r_{\beta_{N1}} & r_{2\beta_{N2}} & \ldots & r_{\beta_{Nd}} \end{pmatrix}, \tag{9}$$

where $\beta_{kj} = \beta_j(U_k)$, $l_{\beta_{kj}} = l_{\beta_j(U_k)}$ *and* $r_{\beta_{kj}} = r_{\beta_j(U_k)}$.

3.1. Estimation of unknown fuzzy coefficients and tuning constant

In order to estimate the fuzzy coefficients of the proposed fuzzy functional regression model (6), a regularization criterion that was originally presnted based on SCAD penalty was extended for the reduced fuzzy multivariate regression model (6), as follows:

$$(\widetilde{\widehat{\alpha}}, \widehat{\boldsymbol{B}}) = arg \min_{(\widetilde{\alpha}, \boldsymbol{B})} \Big\{ \sum_{i=1}^{n} d_1(\widetilde{y}_i, \widetilde{y}_i^{*}) + \sum_{j=1}^{d} \sum_{k=1}^{N} \rho_\lambda(|M_{\widetilde{\beta}_j(U_k)}|) \Big\}, \tag{10}$$

where $M_{\widetilde{\beta}_{jk}} = \max\{\beta_{jk}, l_{\beta_{jk}}, r_{\beta_{jk}}\}$,

$$\widetilde{y}_i^{*} = ((b-a)/N) \bigoplus_{k=1}^{N} \bigoplus_{j=1}^{d} \big(\widetilde{\beta}_j(U_k) \otimes x_{ij}(U_k)\big),$$

and

$$\rho_\lambda(|\theta|) = \begin{cases} \lambda|\theta| & |\theta| < \lambda, \\ -(\theta^2 - 2\lambda\theta|\theta| + \lambda^2)^2/5.4 & \lambda < |\theta| < 3.7\lambda, \\ 2.35\lambda^2 & |\theta| > 3.7\lambda. \end{cases} \tag{11}$$

According to the proposed fuzzy regression model, the unknown regression coefficients $(\widetilde{\alpha}, \boldsymbol{B})_\lambda$ and constant tuning parameter λ should be simultaneously estimated based on a set of observed values $(\widetilde{y}_1, \mathbf{x}_1^\top(.))$, ..., $(\widetilde{y}_n, \mathbf{x}_n^\top(.))$. Since $(\widetilde{\alpha}, \boldsymbol{B})_\lambda$ and λ were relaited on one another, beside the optimization problem given in Eq. (10)), the constant tuning parameter can be also evaluated by minimizing the cross validation criterion [68], i.e. $(\lambda_{(\widetilde{\alpha}, \boldsymbol{B})})_{opt} = arg \min_{\lambda > 0} CV(\lambda)$ where:

$$CV(\lambda) = \frac{1}{n} \sum_{j=1}^{n} d_2^2\big(\tilde{y}_j, \widetilde{\hat{y}}_{(\widetilde{\widehat{\alpha}}, \widehat{\boldsymbol{B}})_\lambda)^{(j)}}\big), \tag{12}$$

in which

$$((\widetilde{\widehat{\alpha}}, \widehat{\boldsymbol{B}})_\lambda)^{(j)} =$$

$$\arg \min_{(\widetilde{\alpha}, \boldsymbol{B})} \Big\{ \sum_{i(\neq j)=1}^{n} d_1(\widetilde{y}_i, \widetilde{y}_i^{*}) + \sum_{j=1}^{d} \sum_{k=1}^{N} \rho_\lambda(|M_{\widetilde{\beta}_j(U_k)}|) \Big\}, \tag{13}$$

For this purpose, the values $(\widetilde{\widehat{\alpha}}, \widehat{\boldsymbol{B}})_\lambda$ was computed for many values of λ, looking for an optimal λ_{opt} value that minimizes the one-out cross validation error (CV). Once found, the optimal value of $(\widetilde{\alpha}, \boldsymbol{B})$ was presented by $(\widetilde{\widehat{\alpha}}_{opt}, \widehat{\boldsymbol{B}}_{opt})$. To this end, the Mathematica software [69] was employed.

Remark 3.2. *To conduct a comparative study with other fuzzy regression models, four widely-used performance criteria for evaluation of fuzzy regression models were used [15]. These included:*

1. *Root mean square error (RMSE):*

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n} d_2^2(\widetilde{y}_i, \widehat{\widetilde{y}}_i)}{n}}. \tag{14}$$

2. *Mean absolute relative error (MARE):*

$$MARE = \frac{1}{n}\sum_{i=1}^{n} \frac{\int_0^1 |\widetilde{y}_i(x) - \widehat{\widetilde{y}}_i(x)|dx}{\int_0^1 \widetilde{y}_i(x)dx}. \tag{15}$$

3. *Mean similarity measure (MSM):*

$$MSM = \frac{1}{n}\sum_{j=1}^{n} S_{UI}(\widehat{\widetilde{y}}_j, \tilde{y}_j), \tag{16}$$

where

$$S_{UI}(\widehat{\widetilde{y}}_j, \tilde{y}_j) = \frac{Card(\widehat{\widetilde{y}}_j \cap \tilde{y}_j)}{Card(\widehat{\widetilde{y}}_j \cup \widehat{y}_j)},$$

4. *Area Under the Receiver Operating Characteristic Curve (AUROCC):*

$$AUROCC_m =$$

$$\frac{1}{2}\sum_{i=m}^{n}(RMSE_i + RMSE_{i-1})(coverage_i - coverage_{i-1}), \tag{17}$$

where

$$RMSE_i = \frac{\sum_{j=1}^{i} d_2^2(\widetilde{y}_j, \widehat{\widetilde{y}}_j)}{n},\ coverage_i = \frac{i}{n}.$$

In addition, to examine the relationship between $\widetilde{y}$ and $\widehat{\widetilde{y}}$ based on their scatter plots, the fuzzy response ($\widetilde{y}$) and the corresponding estimated value ($\widehat{\widetilde{y}}$) were converted to defuzzified to $M_{\widetilde{y}}$ and $M_{\widehat{\widetilde{y}}}$, respectively, according to the sugeno criteria [70]:

$$M_{\widetilde{y}} = \frac{\int x\mu_{\widetilde{y}}(x)dx}{\int \mu_{\widetilde{y}}(x)dx},\ M_{\widehat{\widetilde{y}}} = \frac{\int x\mu_{\widehat{\widetilde{y}}}(x)dx}{\int \mu_{\widehat{\widetilde{y}}}(x)dx}.$$

Remark 3.3. *It should be pointed out that the classical functional regression methods rely on some regularization basis functions such as B-spline. However, applying such methods in fuzzy domain, need more parameters in the proposed estimation procedure. However, by introducing the concept of fuzzy large number, such procedure was reduced to minimum parameters as much as possible.*

4. Application examples

The feasibility and effectiveness of the proposed fuzzy functional regression model were examined based on the performance measures explained in Remark 3.2.

Example 4.1. *(**A simulation study**) A set of $m = 10$ simulated data set with size of $n = 300$ were generated according to the following fuzzy functional regression model:*

$$\widetilde{y}_i = \widetilde{\alpha} \oplus \bigoplus_{j=1}^{3} \int_0^1 (\widetilde{\beta}_j(t) \otimes x_{ij}(t))dt \oplus \widetilde{\epsilon}_i, \ \ i = 1, 2, \ldots, n, \tag{18}$$

where

1. $\widetilde{\alpha} = (1; 0.2, 0.3)_T$.
2. $\widetilde{\beta}_1(t) = (\sin(\pi t); 0.1t, 0.2t)_T$, $\widetilde{\beta}_2(t) = (\exp(-2t); 0.2t, 0.4t)_{L_1 R_1}$, *and* $\widetilde{\beta}_3(t) = (-1 + 2t - 3t^2; 0.3t, 0.6t)_{L_2}$ *where* $L_1(x) = \exp(-x^2/2)$, $R_1(x) = 1/(1 + x^2)$ *and* $L_2(x) = \sqrt{1 - x^2}$.
3. $x_{i1}(t) = t(1-t)^{1.5} z_{i1} + w_{i1}$ *where* $z_{i1} \sim N(0, 0.1)$ *and* $w_{i1} \sim N(0, 0.9)$,
4. $x_{i2}(t) = 10t(t - 0.6)^2 z_{i2} + w_{i2}$ *where* $z_{i2} \sim N(0, 0.2)$ *and* $w_{i2} \sim N(0, 0.8)$,
5. $x_{i3}(t) = \exp(-t)\cos(4\pi t + 0.5) z_{i3} + w_{i3}$ *where* $z_{i3} \sim N(0, 0.3)$ *and* $w_{i3} \sim N(0, 0.7)$,
6. $\widetilde{\epsilon}_i = (\epsilon_i; |\epsilon_i|/(1 + |\epsilon_i|))_T$ *where* $\epsilon_i \sim N(0, 1)$.

3D-plots of $\widetilde{\beta}_1(t) - \widetilde{\beta}_3(t)$ are drawn in Fig. 2. The mean values of the performance measures of the proposed method are reported in Table 1. In particular, consider the performance of the proposed fuzzy functional regression model for the 5^{th} simulated data set (as shown in Table 2 and Fig. 3). The results indicate that $\widetilde{\widehat{\alpha}} = (0.91; 0.15, 0.36)_T$, with the performance of the proposed method also examined by comparing the defuzzified values of $M_{\widehat{\widetilde{y}}}$ and $M_{\widetilde{y}}$.

Table 1: Mean values of performance measures corresponding to the proposed method and some fuzzy multiple regression techniques for Example 4.1.

Performance	$\overline{MSM}$	$\overline{MARE}$	$\overline{RMSE}$	$\overline{AUCR}_{150}$	$\overline{AUCR}_{180}$
Result	0.810	14.738	10.820	31.527	23.785

Table 2: Performance measures corresponding to the 5^{th} simulated data set in Example 4.1.

Performance	MSM	$MARE$	$RMSE$	$AUCR_{150}$	$AUCR_{180}$
Result	0.825	12.829	9.028	25.728	19.882

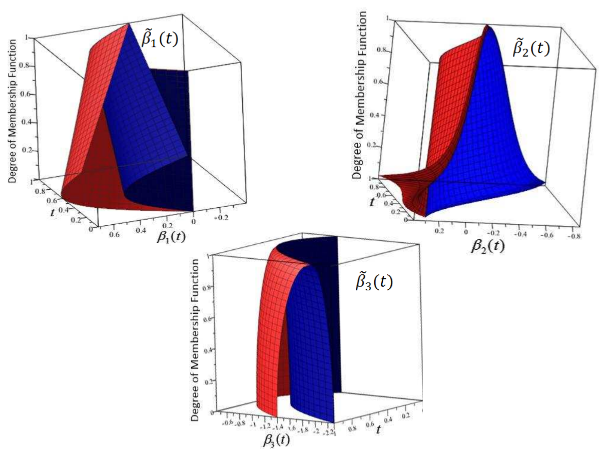

Figure 2: 3D-plot of $\widetilde{\beta}_1(t) - \widetilde{\beta}_3(t)$ in Example 4.1.

Example 4.2. *Prediction of suspended sediment load in a catchments area is very important as it can be used to evaluate the extent of the damage occurred in the catchment, the erosion hazard, and water management. In this example, prediction of the (annual) suspended sediment discharge (ton) based on stream water discharge (m^3 per day) of the Beheshtabad River (Chaharmahal and Bakhtiari Province, Iran) using the proposed fuzzy functional regression model is expected. Cutting through Beheshtabad Village, the river covers an area of 3866 m^2 (located between 31°28′N and 32°56′N latitude and 50°36′E and 51°45′E longitudes). This is an important stream as it supplies water for agricultural activities, fish farms, hydroelectric power plants, and drinking uses, making it important to monitor the suspended sediment load of this river. We hereby assume that the suspended sediment load is an imprecise quantity that can be expressed as symmetric* ***TFN****s. The values of suspended*

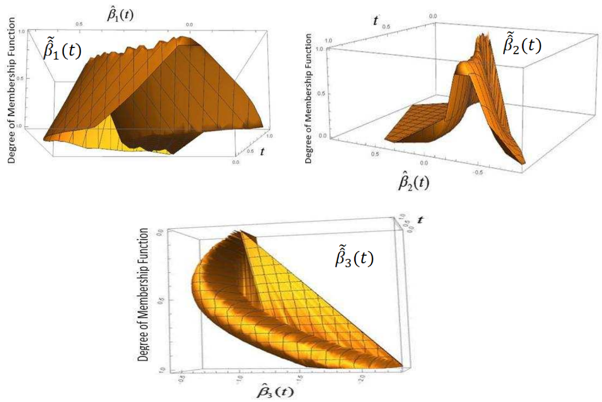

Figure 3: 3D-plot of $\widetilde{\widehat{\beta}}_1(t) - \widetilde{\widehat{\beta}}_3(t)$ in Example 4.1.

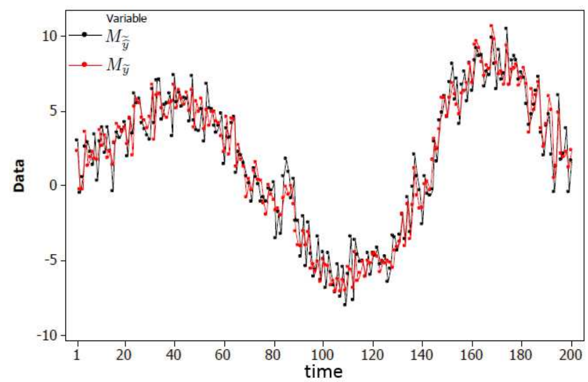

Figure 4: Comparison of $M_{\widetilde{\widetilde{y}}}$ versus $M_{\widetilde{y}}$ in Example 4.1.

sediment load ($\widetilde{y}_i = (y_i; l_{y_i})_T$) and stream water discharge $x_i(t)$, $t = 1, 2, \ldots, 93$ (in summer, 93 days, as per Solar Hijri calendar) were collected during 2000-2019. The time series plots of $x_i(t)$ were smoothed via non-parametric kernel fitting [77] by Minitab software. For each year, a nonlinear regression model was considered: $x_j^i = x_i(j) + \epsilon_j$, $j = 1, 2, \ldots, 93$, and $i = 2000, 2001, \ldots, 2019$. Using the (local) Nadariya-Watson estimator, $x_i(.)$ can be estimated as $\widehat{x}_i(j^*) = \sum_{j=1}^{93} w_h(j) x_j^i$ where $w_h(j) = K(\frac{j-j^*}{h}) / \sum_{j=1}^{93} K(\frac{j-j^*}{h})$ in which h is a bandwidth and K is a kernel. In this regard, the so-called triweight kernel was utilized. The cross-validation criterion was also employed to evaluate the optimal value of h. The plots of $x_i(t)$ and their smoothed functions (as well as $\widetilde{y}_i = (y_i; l_{y_i})_T$) are given in Figs. 5-8. Consider the *following univariate fuzzy functional regression model:*

$$\widetilde{y}_i = \widetilde{\alpha} \oplus \int_1^{93} (\widetilde{\beta}(t) \otimes x_i(t))dt \oplus \widetilde{\epsilon}_i,\ i = 2000, \ldots, 2019. \tag{19}$$

According to the proposed method, this gives the following fuzzy linear regression:

$$\widetilde{y}_i = \widetilde{\alpha} \oplus (\frac{92}{N}) \bigoplus_{k=1}^{93} \big(\widetilde{\beta}(U_k) \otimes x_i(U_k)\big) \oplus \widetilde{\epsilon}_i,\ i = 2000, \ldots, 2019, \tag{20}$$

*where $\widetilde{\beta}(U_k)$ generates **TFN**s, $U_1, U_2, \ldots, U_N$ are independent random variables uniformly distributed over the interval $[1, 93]$, and $N \in \mathbb{N}$ is a large number. Here, it was assumed that $N = 100$. The results of performance evaluations are summarized in Table 3. In addition, Fig. 9 presents the 3D-plot of $\widetilde{\widetilde{\beta}}(t)$. In order to evaluate the effect of functional predictors on the fuzzy response, the functional data was converted to scalar values (mean values over summer). Then, the proposed method was compared with some common fuzzy regression models (Hao and Chiang [35], Wang et al. [45]), Atalay et al. [3], Choi and Buckley [71], Choi and Yoon [72], Kula and Apaydin [73], Zeng et al. [74], D'Urso and Gastaldi [75] and Roldan Lopez de Hierro et al. [76])). For this purpose, let us consider the conventional fuzzy linear regression model in cases where $x_i(t)$s are converted to mean values of x_i:*

$$\widetilde{y}_i = \widetilde{\alpha} \oplus \int_1^{93} (\widetilde{\beta}(t) \otimes x_i)dt \oplus \widetilde{\epsilon}_i = \widetilde{\alpha} \oplus (\widetilde{\beta}' \otimes x_i) \oplus \widetilde{\epsilon}_i,$$

where $\widetilde{\beta}' = \int_0^{93}(\widetilde{\beta}(t))dt$. The results of some common fuzzy univariate linear/non-linear regression models are summarized in Table 3. A comparison among different methods indicates that the proposed method in this study led to more accurate results in terms of $MSM = 0.70, MARE = 11.682, RMSE = 9.22$ and $AUROCC_15 = 10.04$. The accuracy of the proposed method along with other ones were also examined by comparing the corresponding $M_{\widetilde{\widetilde{y}}}$ and $M_{\widetilde{y}}$ values, as shown in Figs. 10-11, further confirming the superiority of the hereby presented method over the other methods for this simulation example. Therefore, incorporation of functional data into a fuzzy regression model is expected to lead to more accurate performance measures compared to the conventional fuzzy regression models with scalar data.

5. Conclusion

Functional regression models are used to evaluate the complex relationship between repeatedly measured variables. In this paper, a regression model was built

Table 3: Coefficients of the model and performance measures corresponding to some fuzzy regression techniques in Example 4.2.

Method	Fuzzy coefficients	MSM	$MARE$	$RMSE$	$AUROCC_{15}$
Choi and Yoon	$\widehat{\widetilde{\beta}}_0 = (14.77; 0.010)_T$, $\widehat{\widetilde{\beta}}_1 = (-30.11; 0.15)_T$, $\widehat{\widetilde{\beta}}_2 = (13.77; 0.12)_T$	0.51	17.05	15.46	15.94
Zeng et al.	$\widehat{\widetilde{\beta}}_0 = (15.78; 0.06)_T$, $\widehat{\widetilde{\beta}}_1 = (-31.78; 0.11)_T$, $\widehat{\widetilde{\beta}}_2 = (15.88; 0.01)_T$	0.54	15.83	13.73	14.17
Kula and Apaydin	$\widehat{\widetilde{\beta}}_0 = (15.88; 0.10)_T$, $\widehat{\widetilde{\beta}}_1 = (-33.55; 0.15)_T$, $\widehat{\widetilde{\beta}}_2 = (13.77; 0.12)_T$	0.54	16.82	13.73	15.42
Choi and Buckley	$\widehat{\widetilde{\beta}}_0 = (12.01; 0.06)_T$, $\widehat{\widetilde{\beta}}_1 = (-27.77; 0.24)_T$, $\widehat{\widetilde{\beta}}_2 = (118.80; 0.09)_T$	0.52	18.32	15.79	14.73
Wang et al.	$\widehat{h} = 0.372$	0.53	15.93	14.88	16.01
Hao and Chiang	$\widehat{\widetilde{\beta}}_0 = (15.66; 0.03)_T$, $\widehat{\widetilde{\beta}}_1 = (-36.17; 0.08)_T$, $\widehat{\widetilde{\beta}}_2 = (16.49; 0.06)_T$, $H = 0.5$, $K = 150$, $P = 30$	0.59	11.98	11.21	11.39
Proposed	$\widehat{\widetilde{\alpha}} = (15.82; 0.12)_T$, $\widehat{\lambda} = 2.35$	0.70	10.68	9.22	10.04
Roldan Lopez de Hierro et al.	$\widehat{\widetilde{\beta}}_0 = (15.92; 0.05)_T$, $\widehat{\widetilde{\beta}}_1 = (-32.29; 0.07)_T$, $\widehat{\widetilde{\beta}}_2 = (17.92; 0.06)_T$	0.57	13.82	10.61	11.96
D'Urso and Gastaldi	$\widehat{\widetilde{y}}(x) = (f(x); l_{f(x)})_T$, $f(x) = 14.55 - 35.88x + 18.99x^2$, $l_{f(x)} = -5.407 + 8.602x$	0.51	17.05	15.46	15.94
Atalay et al.	$\widehat{h} = \widehat{k}_1 = \widehat{k}_2 = 0.5$	0.56	15.42	14.27	13.88

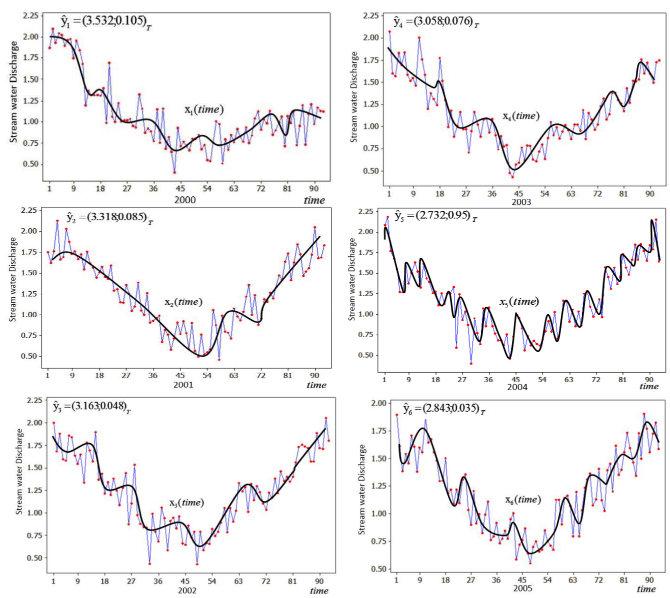

Figure 5: Time series plots of the stream water discharge and their smooth functions in Example 4.2 (1).

for a functional fuzzy response where the predictors were functions. To this end, the concept of fuzzy integral of a fuzzy-valued function was first defined. Then, a fuzzy estimated value of the fuzzy integral of the fuzzy-valued function was proposed using the large numbers theorem. Then a regularization technique was adopted with absolute error deviation, SCAD penalty, and cross-validation criteria to evaluate the

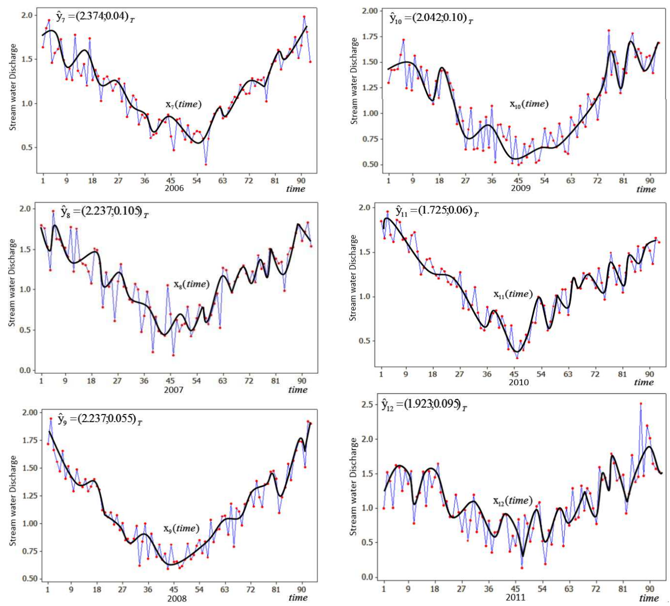

Figure 6: Time series plots of the stream water discharge and their smooth functions in Example 4.2 (2).

coefficients and tuning constant of the fuzzy-valued function. The proposed regression model was subsequently examined according to several goodness-of-fit criteria via an applied example and a simulation study. The results were compared to those of some common fuzzy linear regression models in cases where the functional data was reduced to exact values. The findings clearly indicated the higher efficiency of the proposed method in this research over other techniques. The proposed method can be applied for virtually any kind of LR-fuzzy response. Further research works may focus on extending the proposed model to the the cases where the predictors are also fuzzy-valued functions. A sensitivity analysis with respect to outliers can represent another potential topic for further studies.

Compliance with ethical standards

Conflict of interest: The authors declare that they have no conflict of interest.

Ethical standard: This article does not contain any studies with human participants or animals performed by the authors.

CRediT authorship contribution statement:

Gholamreza Hesamian: Conceptualization, Methodology, Writing-review & edit-

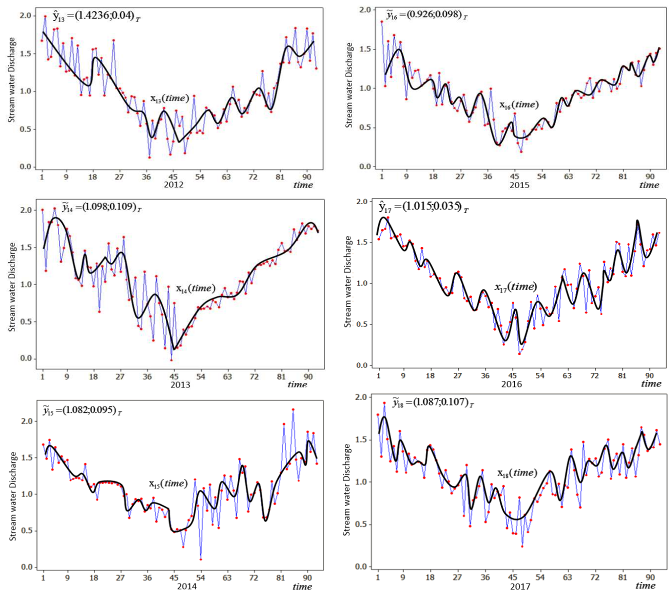

Figure 7: Time series plots of the stream water discharge and their smooth functions in Example 4.2 (3).

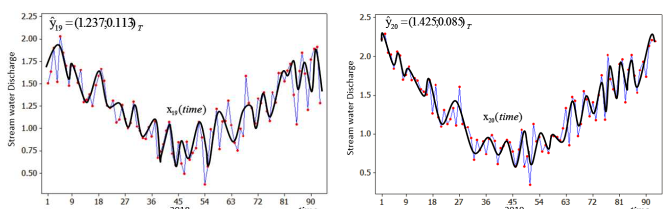

Figure 8: Time series plots of the stream water discharge and their smooth functions in Example 4.2 (4).

ing. **Mohammad Ghasem Akbari**: Software, Validation, Investigation.

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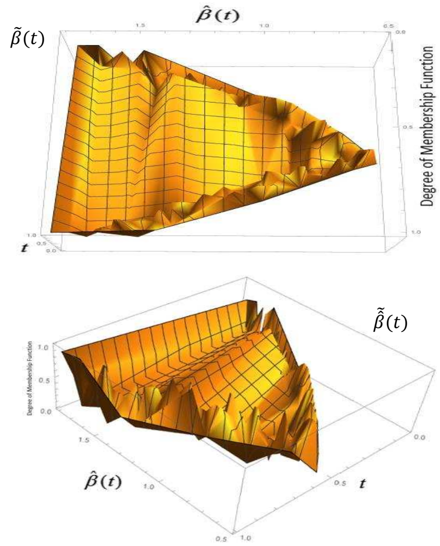

Figure 9: 3D-plot of $\widetilde{\beta}(t)$ and $\widetilde{\widetilde{\beta}}(t)$ in Example 4.2.

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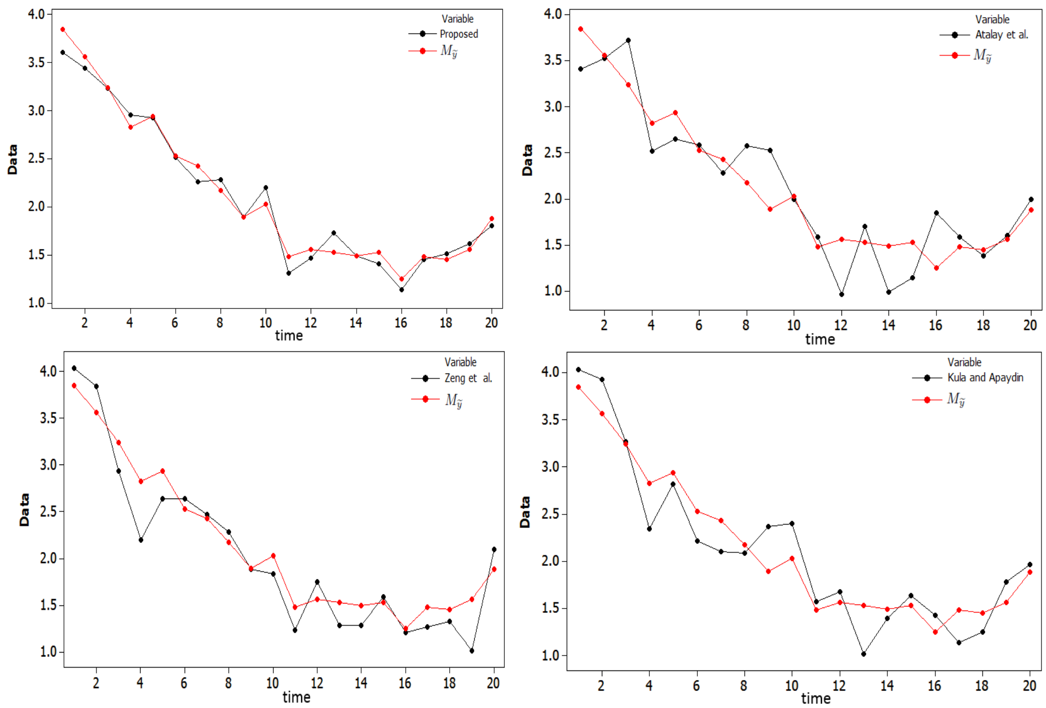

Figure 10: Comparison of $M_{\widetilde{\widetilde{y}}}$ versus $M_{\widetilde{y}}$ in Example 4.2 (1).

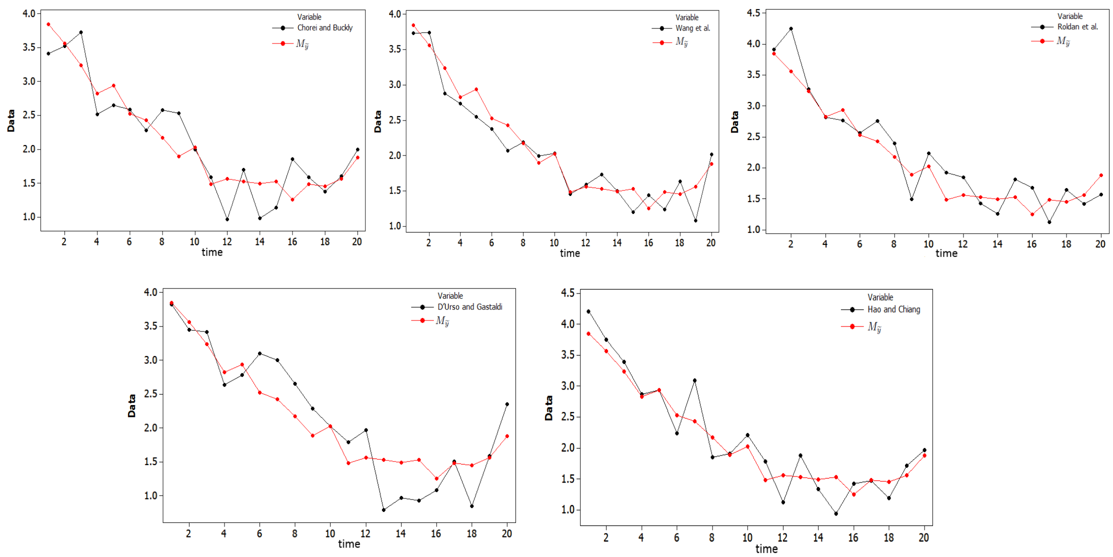

Figure 11: Comparison of $M_{\widetilde{\widehat{y}}}$ versus $M_{\widetilde{y}}$ in Example 4.2 (2).

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Figures

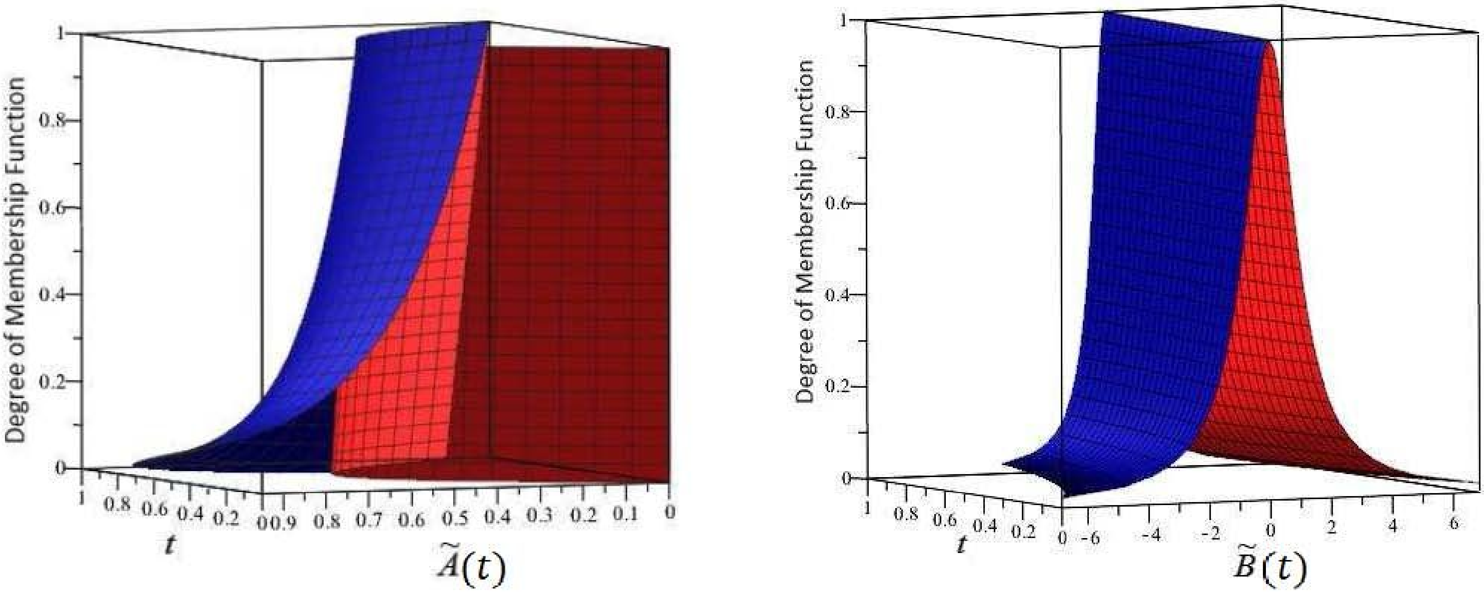

Figure 1

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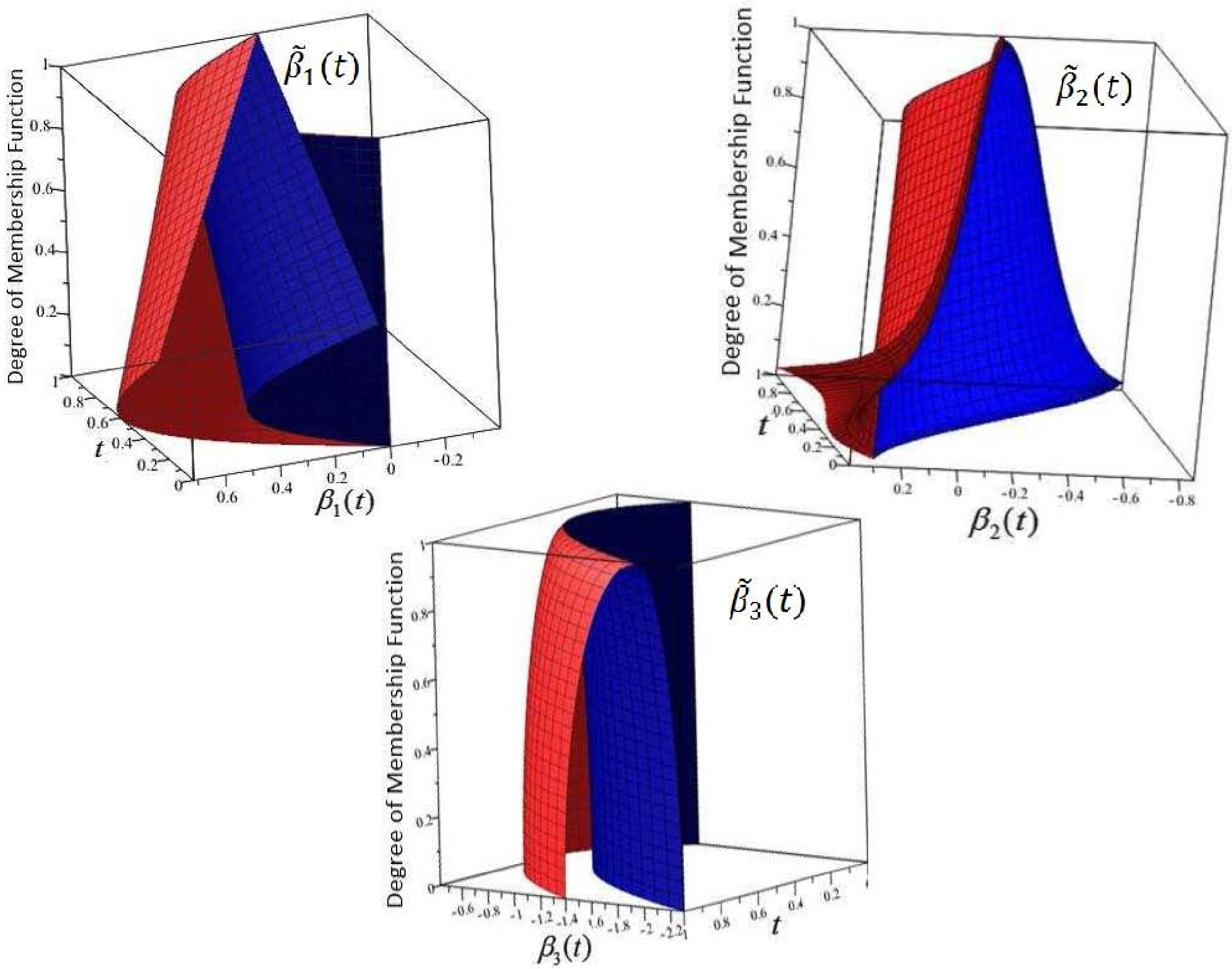

Figure 2

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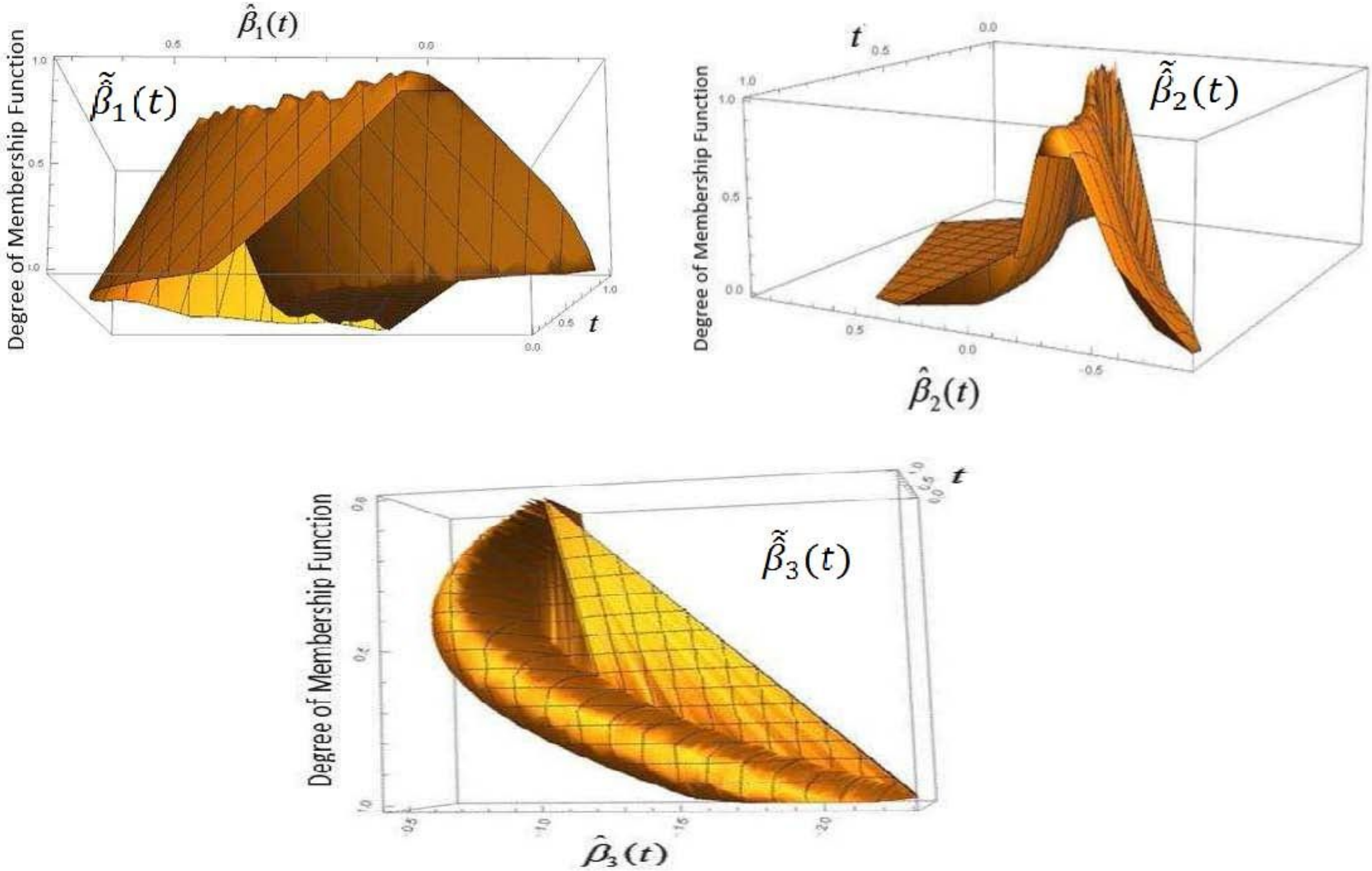

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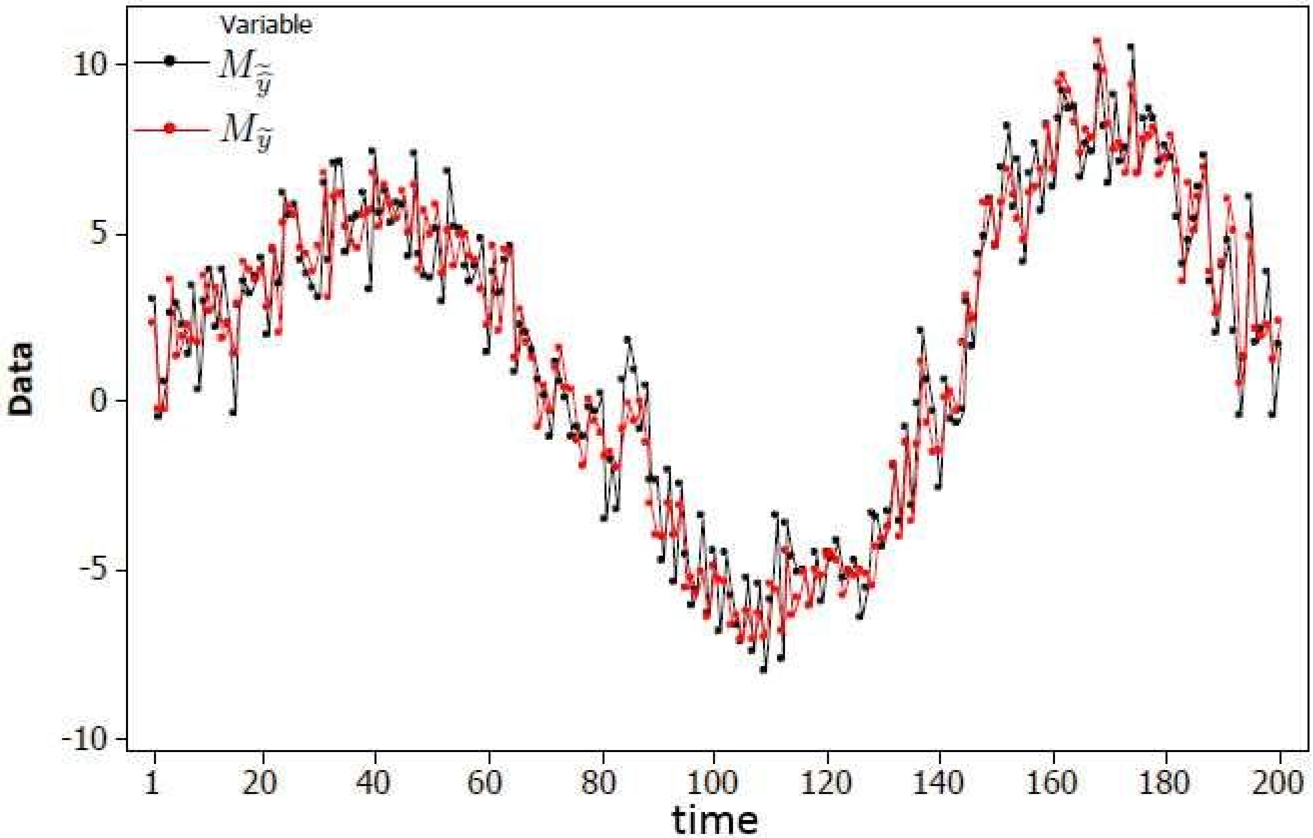

Figure 4

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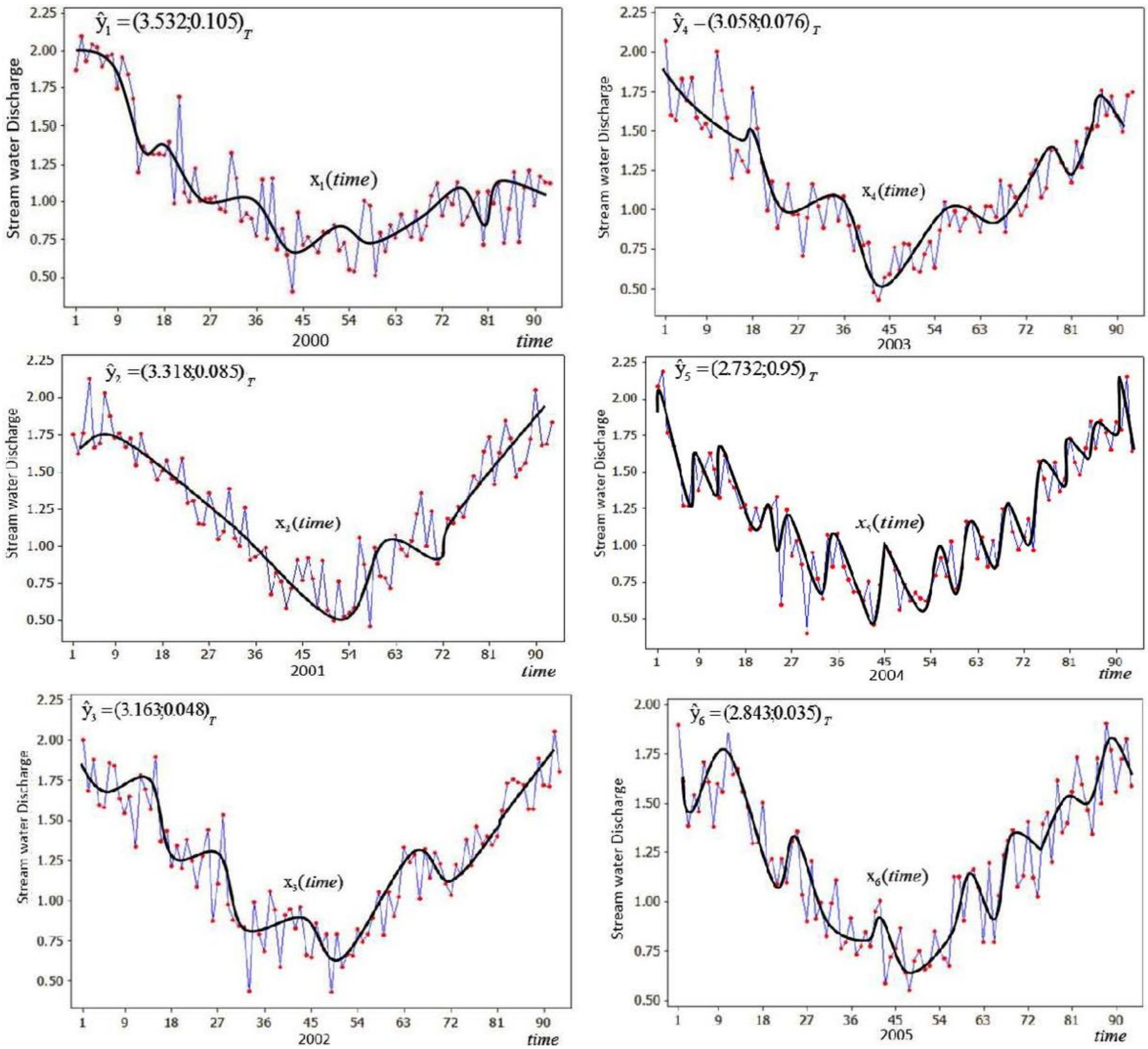

Figure 5

Time series plots of the stream water discharge and their smooth functions in Example 4.2(1).

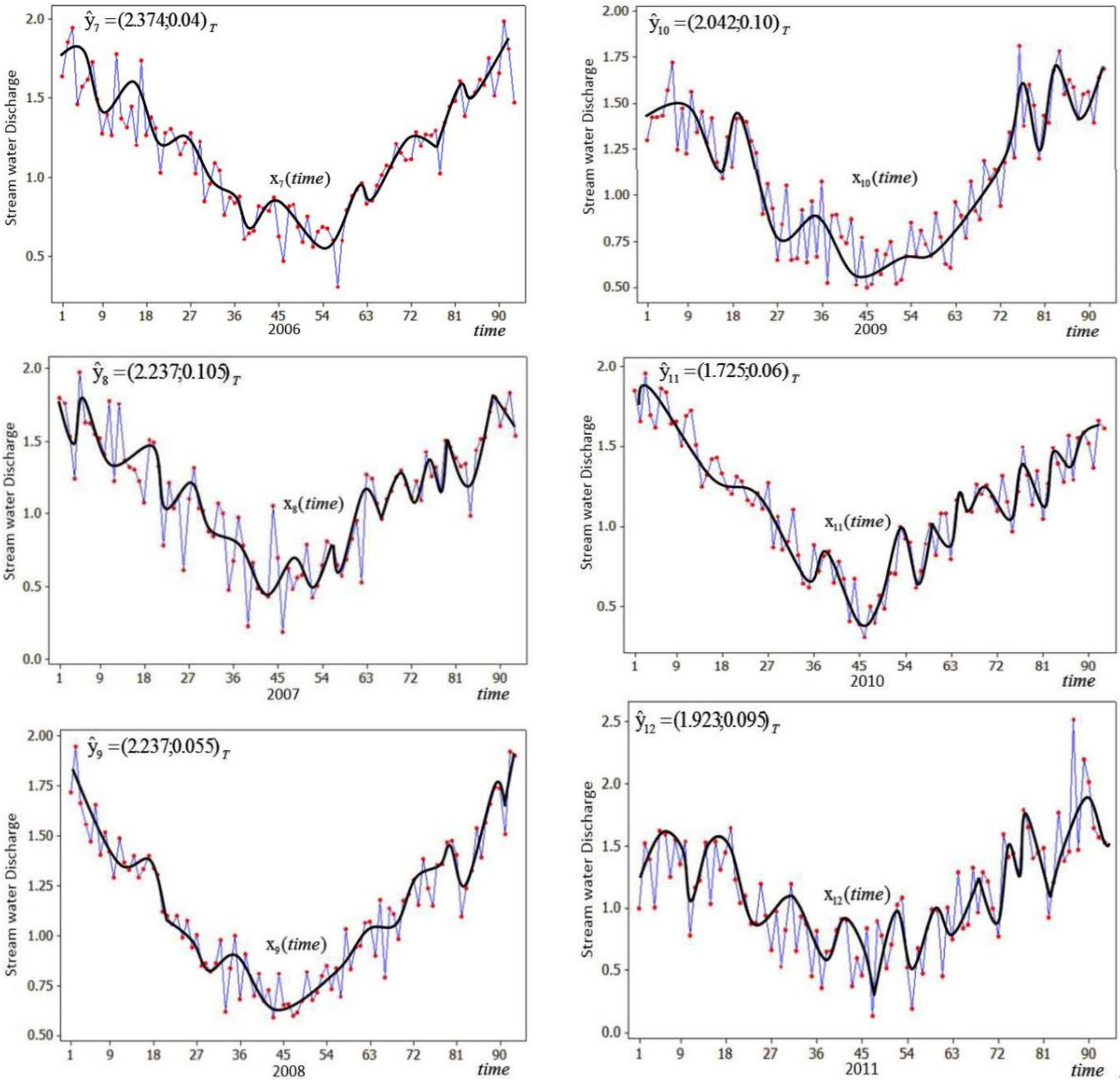

Figure 6

Time series plots of the stream water discharge and their smooth functions in Example 4.2(2).

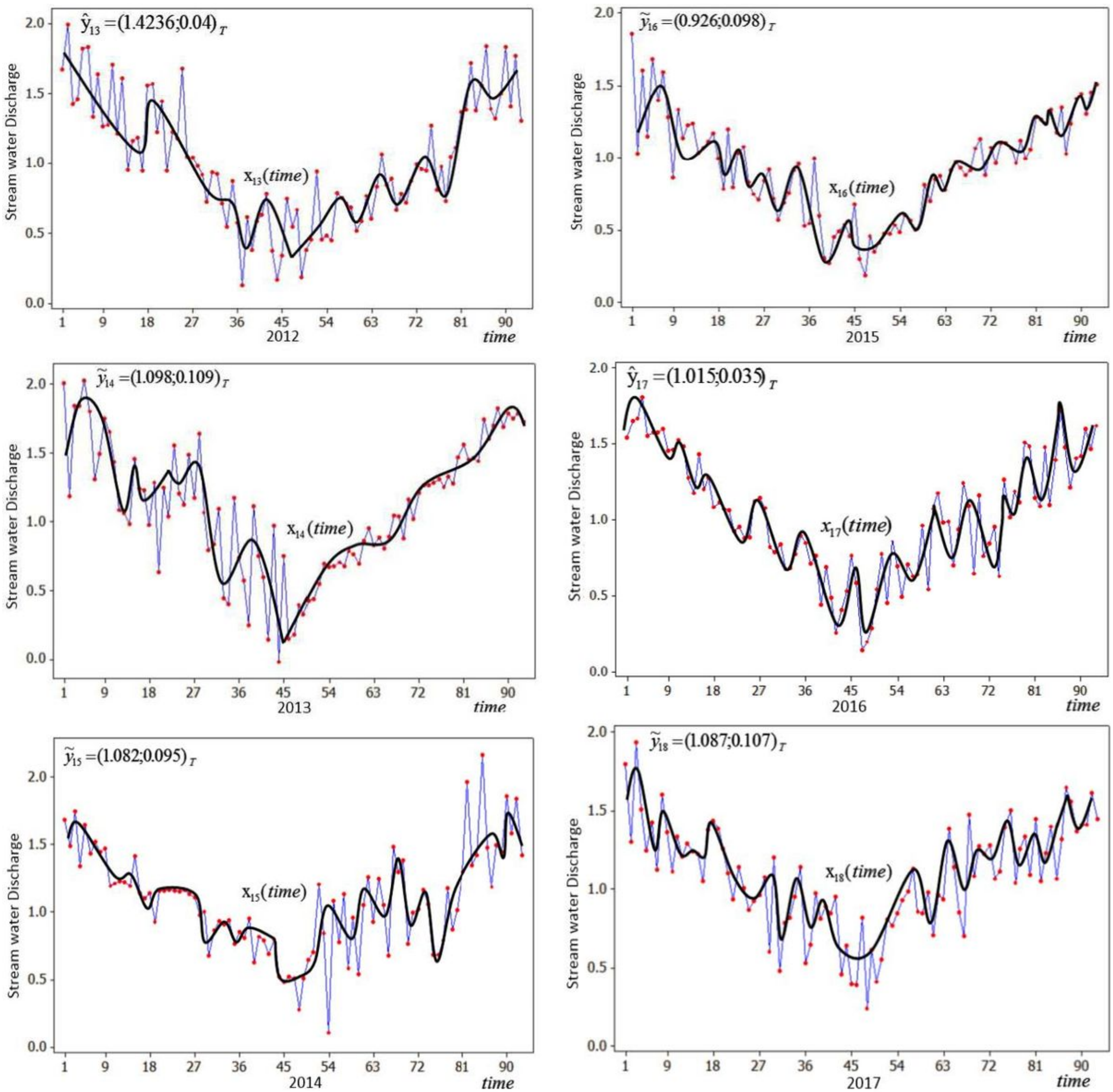

Figure 7

Time series plots of the stream water discharge and their smooth functions in Example 4.2(3).

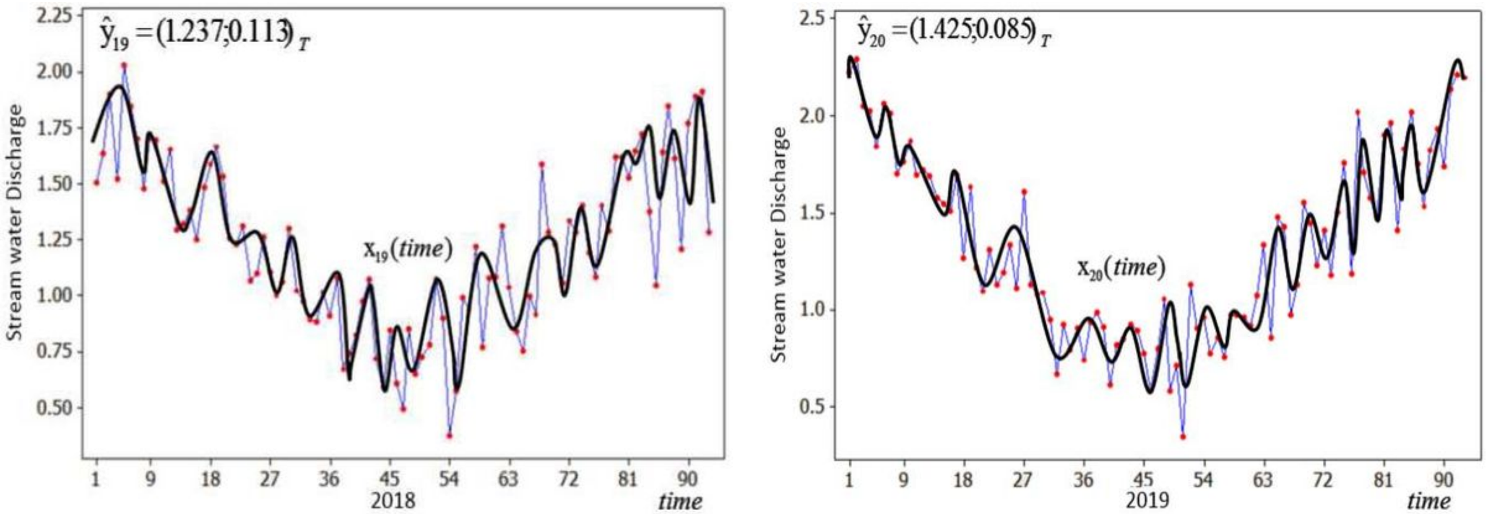

Figure 8

Time series plots of the stream water discharge and their smooth functions in Example 4.2(4).

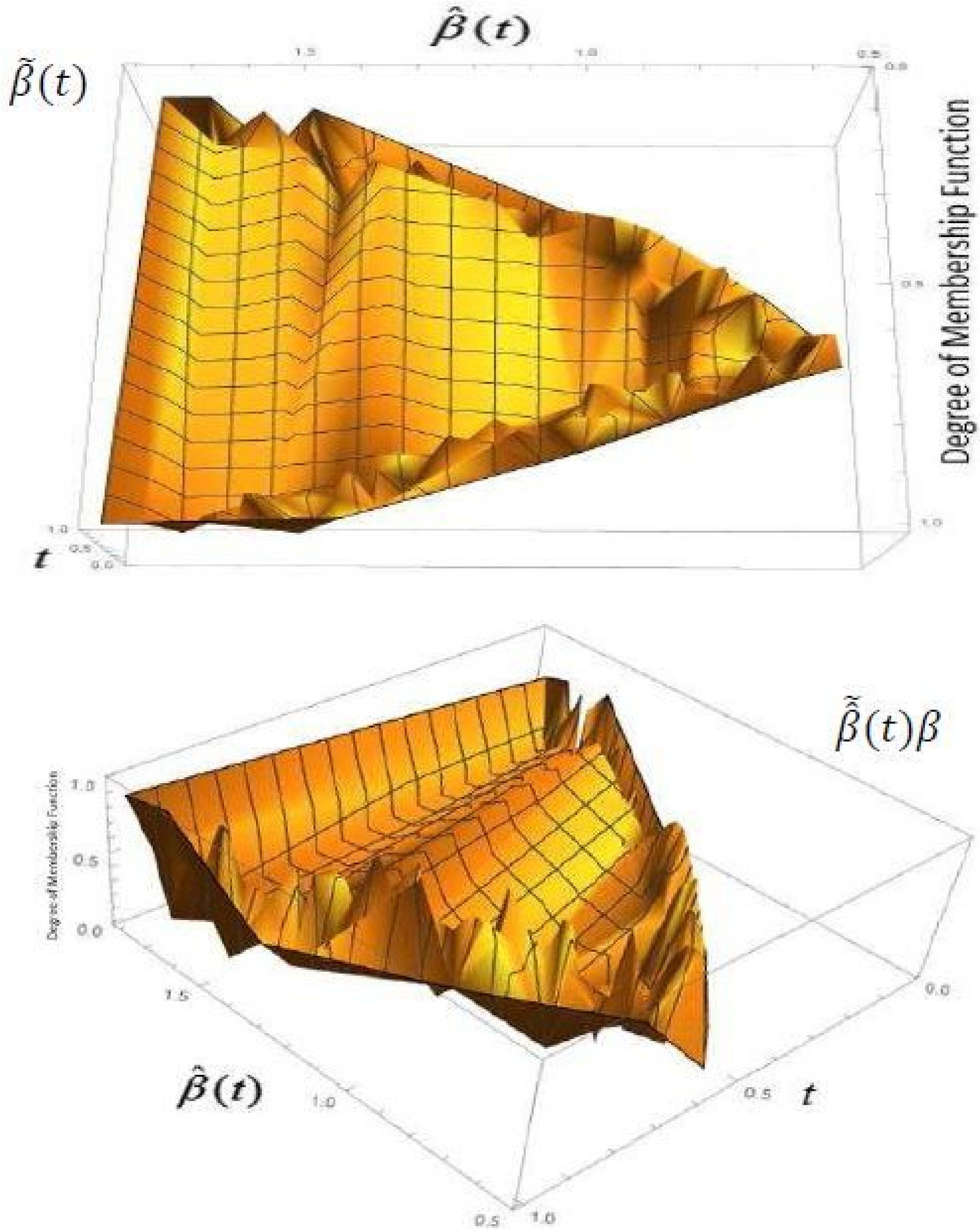

Figure 9

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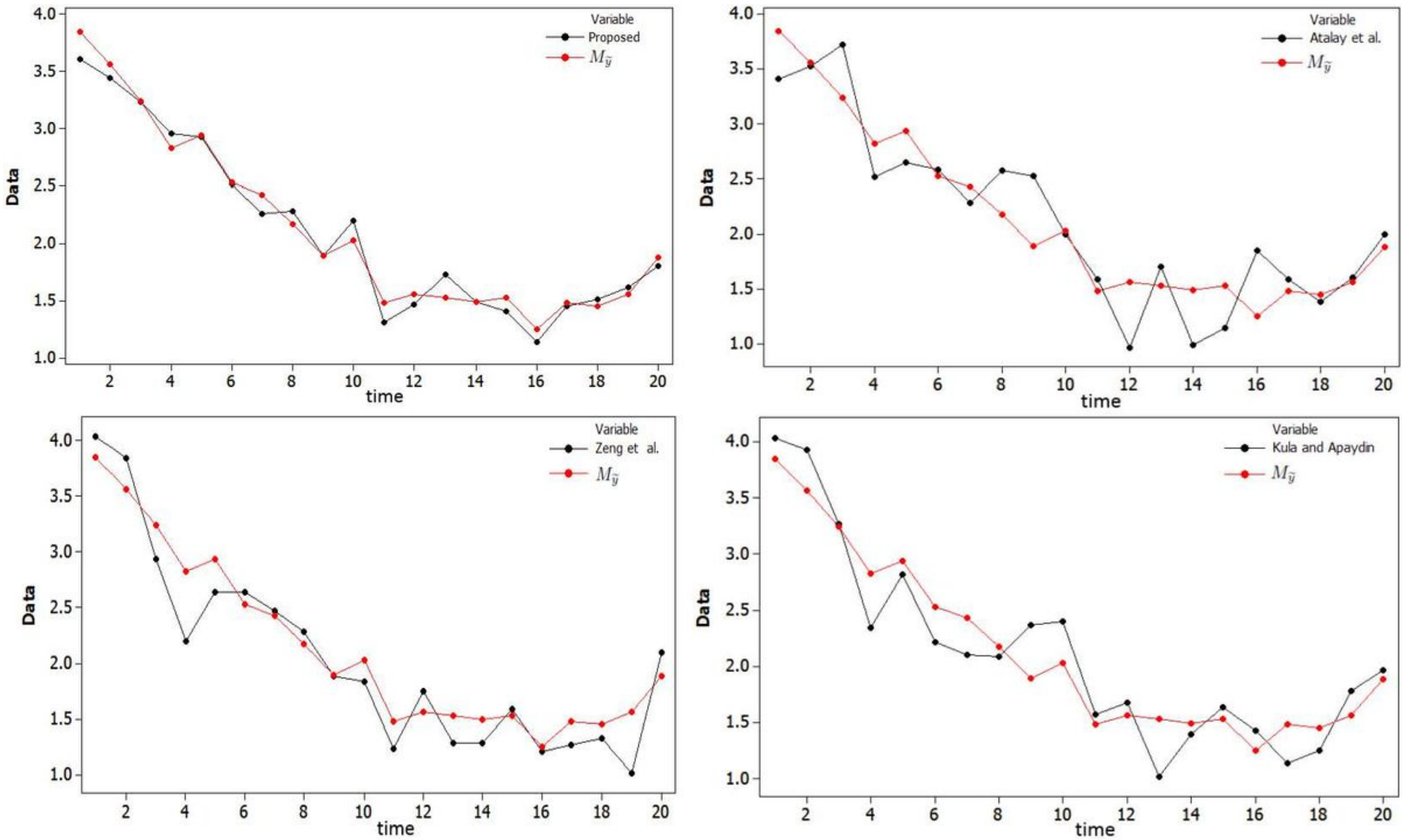

Figure 10

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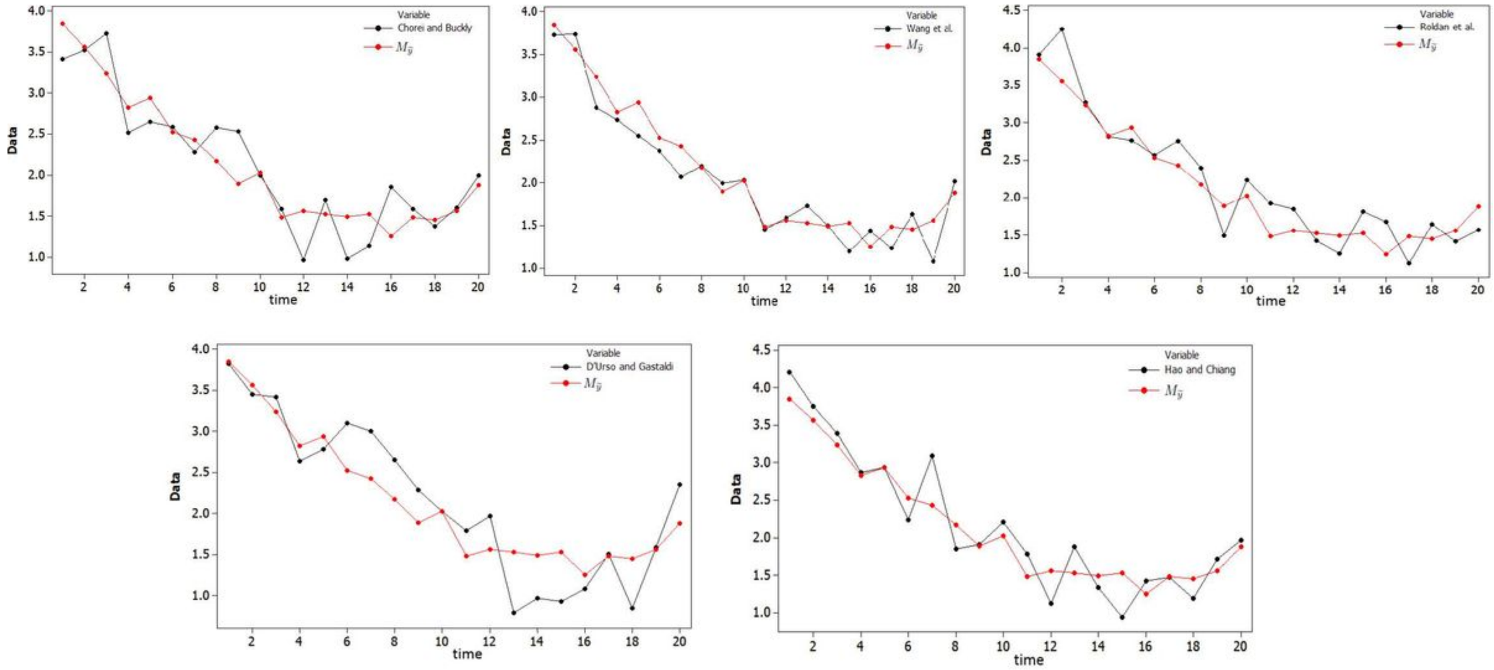

Figure 11

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