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An analytical investigation for optimizing the support stiffness and positions of the bearings of a flexible rotor system

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ORIGINAL ARTICLE

An analytical investigation for optimizing the support stiffness and positions of the bearings of a flexible rotor system

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Received June xx, 201x; revised February xx, 201x; accepted March xx, 201x

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Abstract: The support stiffness and positions of the bearings can greatly affect the vibrations of flexible rotor systems. However, most previous works only focused on the effect of the support stiffness of the bearings on the critical speeds of the rigid rotor systems or modal characteristics including natural frequencies and mode shapes, which missed the combine effects of the support stiffness and positions of the bearings. To overcome this issue, an analytical dynamic model of a flexible rotor system based on the finite element (FE) method is proposed. The model considers the support stiffness of the bearings and rotational inertia of the rotor system. The frequency equation of the rotor system is established for solving the critical speeds. The critical speeds and modal deformations of the system from the presented model and the numerical model based on a commercial software are compared to verify the effectiveness of the proposed FE model. The effects of the support stiffness and positions of the bearings on the critical speeds of the flexible rotor system are analyzed. The results show that the critical speeds are positively correlated with the support stiffness. The critical speeds of the flexible rotor system are also greatly affected by the support positions of the bearing. This study can provide some guidance for the optimization design method of the support stiffness and positions of the bearings in the flexible rotor systems.

Keywords: Bearing stiffness optimization • bearing positions optimization • analytical model • vibrations • flexible rotor system

1 Introduction

Rotor systems in turbines, compressors, and turbojet engines are designed to be lighter and more flexible. Thus,

they can cause more difficult to control the system vibrations during the design processing of the rotor systems. As key parameters for the rotor systems, the unreasonable support stiffness and positions of the bearings may produce unacceptable subcritical superharmonic responses when the rotor speed is a fraction of the system natural frequency. Thus, an in-depth understanding of the vibrations of the rotor systems with different support stiffness and position cases is helpful for their optimal design.

Numerous previous works focused on the vibrations of the rotor systems. For instant, Chen and Wang [1] conducted a design optimization method for a rotor system based on the eigenvalues. Barrett and Flack [2] proposed an experimental investigation to analyze the effect of the support stiffness of the bearings on the stability and unbalance vibrations of a rotor system. Sinou et al. [3] presented the finite element (FE) and experimental methods to investigate the effect of the support stiffness of the bearings on the first forward and backward critical speeds. Sinou et al. [4] developed an experimental analysis to study the modal characteristics including modal frequencies and shapes of a flexible rotor system for different speed cases. Nagasaka et al. [5] analyzed the effects of the damping ratio, lateral force, and unbalance on the major and secondary critical speeds of a uniform rotor system. Dikmen et al. [6] presented the FE and experimental methods to the effect of the support stiffness of the bearings on the modal characteristics of a flexible rotor system. Jalali et al. [7] proposed a FE model based on a commercial software to study the critical speeds, unbalance response, and

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operational deflection shapes of a flexible rotor system. Birchfield et al. [8] used the transfer function method to study the eigenvalues of a rotor system with the flexible foundations. Nagesh et al. [9] developed the FE and experimental methods to study the modal characteristics of a flexible rotor system. Sinou and Thouverez [10] presented an experimental method to study the effect of the bearing temperature on the critical speeds and unbalance response of a flexible rotor system. Lazarus et al. [11] proposed a FE model based on modal analysis to study the unbalance response of a flexible rotor system. Sapanen et al. [12] introduced a numerical approach based on the multibody and FE methods to analyze the superharmonic responses of a flexible rotor system. Han and Chu [13] established a Jeffcott rotor model considering a transverse crack and asymmetric inertia to study the effect of the crack on the system vibrations. Wang et al. [14] proposed a FE model for a flexible rotor system to study the effect of the shaft anisotropy on the whirling and forced response. Zou et al. [15] developed a vibration model to study the forward and backward frequencies of a flexible rotor system. Zhou et al. [16] established a nonlinear rotor-bearing model to investigate the nonlinear characteristics. Hu and Palazzolo [17] introduced a FE model including the gyroscopic and support stiffness of the bearings to study the modal characteristics of a flexible rotor system. Jin et al. [18] proposed an analytical model to study the bearing varying compliance on the nonlinear dynamic of a rotor system. Heidari and Safarpour [19] proposed H_∞ and H_2 methods to obtain the optimum support stiffness and damping ratio of a flexible rotor system. Li et al. [20] presented a general vibration model to study the vibrations of a flexible rotor system. AL-Shudeifat [21] studied the new backward whirl response of a cracked rotor system. Zheng et al. [22] developed a FE model to study the effects of the support stiffness of the bearings and material properties of the rotor on the double frequency vibrations of a flexible rotor system. As the above listed descriptions, most previous works only focused on the effect of the support stiffness of the bearings on the critical speeds of the rigid rotor systems or modal characteristics including natural frequencies and mode shapes, few works focused on the combine effects of the support stiffness and positions of the bearings on both the critical speeds and modal characteristics of the flexible rotor system.

This work proposes an analytical dynamic model of a flexible rotor system based on the FE method. The model considers the support stiffness of the bearings and rotational inertia of the rotor system. The rotor is modelled as Timoshenko beams. The contact stiffness in the bearings is

obtained by using Hertzian contact method. The frequency equation of the rotor system is established for solving the critical speeds. The critical speeds and modal deformations of the system from the presented model and the FE model based on a commercial software are compared to verify the effectiveness of the presented dynamic model. The effects of the support stiffness and positions of the bearings on the critical speeds of the flexible rotor system are analyzed.

2 A proposed FE Model and Frequency Equation of the Flexible Rotor System

A FE model of the flexible rotor system is shown in Figure 1. There are three rigid disks on the shaft. The three rigid disks have different masses and moments of inertia. The shaft and disks are rigidly connected. There are three support positions on the shaft, and the three positions are located at the middle position and two ends of the shaft. Each support position only considers the same stiffness of bearings in X and Y directions.

Figure 1 A FE model of the flexible rotor system

2.1 A FE model of the flexible rotor system

According to the FE method in Ref. [23], the equations of motion for the proposed FE model of the rotor system are given by

$$\begin{bmatrix} \mathbf{M}_1 & \\ & \mathbf{M}_1 \end{bmatrix} \{\ddot{\mathbf{u}}\} + \begin{bmatrix} \Omega \mathbf{J}_1 & \\ & -\Omega \mathbf{J}_1 \end{bmatrix} \{\dot{\mathbf{u}}\} + \begin{bmatrix} \mathbf{K}_1 & \\ & \mathbf{K}_1 \end{bmatrix} \{\mathbf{u}\} = \{\mathbf{0}\} \quad (1)$$

where $\mathbf{u}$, $\dot{\mathbf{u}}$, and $\ddot{\mathbf{u}}$ are the displacement, velocity and acceleration vectors of each node, respectively, and Ω is the rotational angular velocity; In Eq. (1), $\mathbf{M}_1$ is the assembled mass matrix of the rotor system, which is composed by the mass matrices of all nodes $\mathbf{M}_s^{(i)}$ ($i=1, 2, \dots, 11$) and the mass matrix of the disk $\mathbf{M}_d$; $\mathbf{J}_1$ is the assembled gyroscopic matrix of the rotor system, which is composed by the

gyroscopic matrices of all nodes $\mathbf{G}_s^{(i)}$ and the gyroscopic matrix of the disk $\mathbf{J}$; $\mathbf{K}_1$ is the assembled stiffness matrix of the rotor system, which is composed by the stiffness matrices of all nodes $\mathbf{K}_s^{(i)}$; and $\{\mathbf{0}\}$ is the null vector.

2.2 Frequency Equation of the Flexible Rotor System

When the gyro torque is considered, the shaft will be bent due to the unbalanced mass excitation. Both the orbit and rotary motion of the rotor are formulated at the same time, where line #1 is the axis of orbit motion and line #2 is the axis of rotary motion as shown in Figure 2. In Figure 2, ω_F and ω_B are the orbit motion speeds in the forward and backward whirling directions, respectively; and Ω are the rotational speed of the rotor. When the directions of ω_F and Ω are same, it is the forward whirling (FW) motion, and when the directions of ω_B and Ω are different, it is the backward whirling (BW) motion.

Figure 2 FW and BW motions of the flexible rotor system

When the rotational speed is Ω , the frequency equation for the rotor system is formulated as

$$|-\mathbf{M}_1\omega^2 + \mathbf{J}_1\Omega\omega + \mathbf{K}_1| = 0 \quad (2)$$

where ω is the whirling angular velocity. By solving Eq. (2), the frequencies for the FW and BW motions can be obtained. These frequencies can reflect the variation of angular velocity of whirling motion during the changing processing of Ω . If $\Omega = \pm\omega$ is substituted into Eq. (2), the critical speeds and natural frequencies for the FW and BW motions can be solved, respectively.

2.3 Model Validation

In order to verify the accuracy of the proposed FE model of the flexible rotor system, the critical speeds and vibrations from the proposed FE model and numerical model from the commercial software are compared. The Campbell diagram from the numerical model from the commercial software is shown in Figure 3. The critical speeds of the flexible rotor system from the numerical model from the commercial software can be depicted in the Campbell diagram. The critical speeds and differences between the proposed FE model and numerical model from the commercial software are listed in Table 1. It can be seen that the differences between the proposed FE model and numerical model from the commercial software are less than 10%. The results can give some validation for the proposed FE model.

Figure 3 Campbell diagram of the commercial software

Table 1 Critical speeds and differences between the proposed FE model and numerical model from the commercial software

Methods	FW critical speed (r/min)			BW critical speed (r/min)		
	Mode 1	Mode 2	Mode 3	Mode 1	Mode 2	Mode 3
FE model	10029	17887	22873	8674	14272	21379
Numerical model	9652	17504	22562	8222	13507	19707
Differences	3.8%	2.1%	1.4%	5.2%	5.4%	7.8%

In Figures 4 to 9, the mode shapes of the flexible rotor system for the proposed FE model and numerical model from the commercial software at different critical speeds are shown. The first three natural frequencies of the proposed FE model for the FW motion are 167 Hz, 298 Hz, and 381 Hz, respectively; and those for the BW motion are 145 Hz, 238 Hz, and 356 Hz, respectively. The first three natural frequencies of numerical model from the commercial software for the FW motion are 161 Hz, 292 Hz, and 376 Hz, respectively; and those for the BW motion are 137 Hz, 225 Hz, and 328 Hz, respectively. For the FW motion, the differences of the natural frequencies between the proposed FE model and numerical model are 3.6%, 2.0%, and 1.3%, respectively. For the BW motion, the differences of the natural frequencies between the proposed FE model and

numerical model are 5.5%, 5.5%, and 7.9%, respectively. It can be clearly seen that the vibrations for the proposed FE model and numerical model from the commercial software are similar and their shapes match perfectly. As a consequence, the proposed FE model is an effective one for solving the critical speeds.

Figure 4 First mode shapes for the BW motion from (a) the proposed FE model (145 Hz) and (b) numerical model (137 Hz)

Figure 5 Second mode shapes for the BW motion from (a) the proposed FE model (238 Hz) and (b) numerical model (225 Hz)

Figure 6 Third mode shapes for the BW motion from (a) the

proposed FE model (356 Hz) and (b) numerical model (328 Hz)

Figure 7 First mode shapes for the FW motion from (a) the proposed FE model (167 Hz) and (b) numerical model (161 Hz)

Figure 8 Second mode shapes for the FW motion from (a) the proposed FE model (298 Hz) and (b) numerical model (292 Hz)

Figure 9 Third mode shapes for the FW motion from (a) the proposed FE model (381 Hz) and (b) numerical model (376 Hz)

3 Numerical Analyses

To analyze the effect of bearing stiffness on the critical speeds of the flexible rotor system, the first critical speeds

for the FW motion are calculated by Eq. (2) for different bearing stiffness cases. To analyze the effects of support positions A, B, and C on the critical speeds of the flexible rotor system, three support position cases are discussed as shown in Figure 10, where their variation ranges x_A , x_B , and x_C are defined be from 0 mm to 40 mm. Under the above conditions, the first critical speeds for the FW motion are calculated by Eq. (2) for different support position case.

3.1 Effect of the Support Stiffness on the Critical Speeds of the Flexible Rotor System

The effect of support stiffness on the first critical speed for the FW motion are shown in Figure 11. The support stiffness are from 1×10^7 N/m to 1×10^9 N/m. In Figure 11, the first critical speed for the FW motion increases with the increment of the support stiffness. When K_b is larger than 5×10^8 N/m, the increasing rate of the first critical speed will slow down. It seems that the critical speeds are positively correlated with the bearing support stiffness.

Figure 11 Effect of the support stiffness on the critical speeds of the flexible rotor system

3.2 Effect of the Support Positions on the Critical Speeds of the Flexible Rotor System

3.2.1 Case One

For case one, one support position is fixed and the other two support positions are the variable ones. The effect of this support position case on the first critical speed for the FW motion are depicted in Figure 12. In Figure 12(a), x_A is fixed

at 40 mm, the first critical speed of the rotor system increases with the decrement of the x_B ; and the first critical speed of the rotor system increases with the increment of x_C . In Figure 12(b), x_B is fixed at 40 mm, the first critical speed of the rotor system increases with the increment of the x_A ; and the first critical speed of the rotor system increases with the decrement of x_C . In Figure 12(c), x_C is fixed at 40 mm, the first critical speed of the rotor system increases with the increment of the x_A ; and the first critical speed of the rotor system increases with the increment of x_B . Figure 12 gives that the first critical speed of the rotor system is greatly affected by the bearing support positions. Thus, the support position optimization can be helpful for controlling the first critical speed and the relative vibrations.

Figure 12 First critical speeds for the FW motion of the flexible rotor system when (a) $x_A=40$ mm, (b) $x_B=40$ mm, and (c) $x_C=40$ mm

3.2.2 Case Two

For case two, the variations of three positions x_A , x_B , and x_C are same, whose values are defined as Δx . The effect of this support position case on the first critical speed for the FW motion is depicted in Figure 13. In Figure 13(a), $K_b = 1 \times 10^8$ N/m, the first critical speed for the FW motion increases with the increment of the variation Δx . In Figure 13(b), $K_b = 1 \times 10^9$ N/m, when Δx is less than 50 mm, the first critical speed for the FW motion increases with the increment of Δx ; when Δx is 50 mm, the first critical speed for the FW motion reaches the maximum one; when Δx is larger than 50 mm, the first critical speed for the FW motion decreases with the increment of Δx . Figure 13 also gives that the first critical speed of the rotor system is greatly affected by the bearing support positions. Similarly, the results show that the support position optimization can be helpful for controlling the first critical speed and the relative vibrations.

Figure 13 First critical speeds for the FW motion of the flexible rotor system when (a) $K_b = 1 \times 10^8$ N/m, and (b) $K_b = 1 \times 10^9$ N/m

3.2.3 Case Three

Figure 14 The first critical speeds for the FW motion of the flexible rotor system when (a) x_B and x_C , (b) x_A and x_C , and (c) x_A and x_B are same

For case three, the variations of two positions are same and the other one position is different (such as $x_A = x_B \neq x_C$). The effect of this support position case on the first critical speed for the FW motion is depicted in Figure 14. In Figure 14(a), x_B and x_C are same, the first critical speed for the FW motion increases when x_B and x_C are close to 42 mm; and the first critical speed for the FW motion increases with the increment of x_A . In Figure 14(b), x_A and x_C are same, the first critical speed for the FW motion increases with the decrements of x_A and x_C ; and the first critical speed for the FW motion increases with the increment of x_B . In Figure 14(c), x_A and x_B are same, the first critical speed for the FW motion increases with the increments of x_A and x_B ; and the

first critical speed for the FW motion increases with the decrement of x_c . Figure 14 gives that the first critical speed of the rotor system is greatly affected by the bearing support positions too. Moreover, the results also depict that the support position optimization can be helpful for controlling the first critical speed and the relative vibrations.

4 Conclusions

This work proposes an analytical FE model of a flexible rotor system based on the FE method in the literature. The proposed model considers the support stiffness of the bearings and rotational inertia of the rotor system. The frequency equation of the rotor system is established for solving the critical speeds. The critical speeds, mode shapes, and natural frequencies of the system from the proposed FE model and numerical model based on a commercial software are compared to verify the effectiveness of the proposed FE model. The effects of the support stiffness and positions of the support bearings on the critical speeds of the flexible rotor system are analyzed. The results show that the critical speeds are positively correlated with support stiffness. The critical speeds of the flexible rotor system are greatly affected by the support positions of the bearing too. Thus, the support position optimization can be helpful for controlling the first critical speed and the relative vibrations. This study can provide some guidance for the optimization design method of the support stiffness and positions of the bearings in the flexible rotor systems.

5 Declaration

Acknowledgements

Not applicable

Funding

Supported by National Natural Science Foundation of China (No. 51605051 and 51975068).

Availability of data and materials

The datasets supporting the conclusions of this article are included within the article.

Authors' contributions

The author's contributions are as follows: Jing Liu was in charge of the whole trial; Jing Liu and Changke Tang wrote the manuscript; Jing Liu and Changke Tang assisted with sampling and laboratory analyses.

Competing interests

The authors declared that they have no conflicts of interest to this work.

Consent for publication

Not applicable

Ethics approval and consent to participate

Not applicable

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Biographical notes

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Appendix

Based on the FE method in Ref. [Error! Bookmark not defined.], a shaft in a rotor system can be divided into i Timoshenko beam elements. The parameters of i th Timoshenko beam element are listed in Table 2.

Table 2 i th Timoshenko beam element parameters

Parameter	Value
length	l
radius	r
elastic modulus	E
Moment of inertia	I
Poisson's ratio	μ
Rotation speed	ω

Figure 15 i th Timoshenko beam element

The mass matrix $\mathbf{M}_s^{(i)}$, gyroscopic matrix $\mathbf{G}_s^{(i)}$ and stiffness matrix $\mathbf{K}_s^{(i)}$ of the beam element are as follows.

(1) The mass matrix of the beam element $\mathbf{M}_s^{(i)}$ is expressed as

$$\mathbf{M}_s^{(i)} = \mathbf{M}_{sT}^{(i)} + \mathbf{M}_{sR}^{(i)} \quad (A1)$$

where $\mathbf{M}_{sT}^{(i)}$ and $\mathbf{M}_{sR}^{(i)}$ are

$$\mathbf{M}_{sT}^{(i)} = \frac{\mu l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ \text{sym} & & & 4l^2 \end{bmatrix} \quad (A2)$$

$$\mathbf{M}_{sR}^{(i)} = \frac{\mu r^2}{120l} \begin{bmatrix} 36 & 3l & -36 & 3l \\ & 4l^2 & -3l & -l^2 \\ & & 36 & -3l \\ \text{sym} & & & 4l^2 \end{bmatrix} \quad (A3)$$

(2) The gyroscopic matrix of the beam element $\mathbf{G}_s^{(i)}$ is given as

$$G_s^{(i)} = \omega \mathbf{J}_s^{(i)} = \frac{\mu \omega r^2}{60l} \begin{bmatrix} 60 & 3l & -36 & 3l \\ & 4l^2 & -3l & -l^2 \\ & & 36 & -3l \\ \text{sym} & & & 4l^2 \end{bmatrix} \quad (\text{A4})$$

(3) The stiffness matrix of the beam element $\mathbf{K}_s^{(i)}$ is given by

$$\mathbf{K}_s^{(i)} = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{sym} & & & 4l^2 \end{bmatrix} \quad (\text{A5})$$

Moreover, $\mathbf{M}_s^{(i)}$, $\mathbf{G}_s^{(i)}$ and $\mathbf{K}_s^{(i)}$ are divided into four matrices as follows

$$\mathbf{M}_s^{(i)} = \begin{bmatrix} \mathbf{m}_{11}^{(i)} & \mathbf{m}_{12}^{(i)} \\ \mathbf{m}_{21}^{(i)} & \mathbf{m}_{22}^{(i)} \end{bmatrix}, \mathbf{J}_s^{(i)} = \begin{bmatrix} \mathbf{J}_{11}^{(i)} & \mathbf{J}_{12}^{(i)} \\ \mathbf{J}_{21}^{(i)} & \mathbf{J}_{22}^{(i)} \end{bmatrix}, \mathbf{K}_s^{(i)} = \begin{bmatrix} \mathbf{K}_{11}^{(i)} & \mathbf{K}_{12}^{(i)} \\ \mathbf{K}_{21}^{(i)} & \mathbf{K}_{22}^{(i)} \end{bmatrix} \quad (\text{A6})$$

For a rotor system with 11 nodes connected by 7 beam elements, mass matrix $\mathbf{M}_s^{(i)}$, gyroscopic matrix $\mathbf{G}_s^{(i)}$ and stiffness matrix $\mathbf{K}_s^{(i)}$ of each element can be integrated to the assembled mass matrix $\mathbf{M}_1$, the assembled gyroscopic matrix $\mathbf{G}_1$ and the assembled stiffness matrix $\mathbf{K}_1$. The specific compositions are

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{m}_{11}^{(1)} & \mathbf{m}_{12}^{(1)} & & & & & \\ \mathbf{m}_{21}^{(1)} & \mathbf{m}_{22}^{(1)} + \mathbf{m}_{11}^{(2)} & \mathbf{m}_{12}^{(2)} & & & & \\ & \mathbf{m}_{21}^{(2)} & \mathbf{m}_{22}^{(2)} + \mathbf{m}_{11}^{(3)} + \mathbf{M}_{d1} & \mathbf{m}_{12}^{(3)} & & & \\ & & \mathbf{m}_{21}^{(3)} & \mathbf{m}_{22}^{(3)} + \mathbf{m}_{11}^{(4)} & \mathbf{m}_{12}^{(4)} & & \\ & & & \mathbf{m}_{21}^{(4)} & \mathbf{m}_{22}^{(4)} + \mathbf{m}_{11}^{(5)} + \mathbf{M}_{d2} & \mathbf{m}_{12}^{(5)} & \\ & & & & \mathbf{m}_{21}^{(5)} & \mathbf{m}_{22}^{(5)} + \mathbf{m}_{11}^{(6)} + \mathbf{M}_{d3} & \mathbf{m}_{12}^{(6)} \\ & & & & & \mathbf{m}_{21}^{(6)} & \mathbf{m}_{22}^{(6)} + \mathbf{m}_{11}^{(7)} & \mathbf{m}_{12}^{(7)} \\ & & & & & & \mathbf{m}_{21}^{(7)} & \mathbf{m}_{22}^{(7)} \end{bmatrix} \quad (\text{A7})$$

$$\mathbf{J}_1 = \begin{bmatrix} \mathbf{J}_{11}^{(1)} & \mathbf{J}_{12}^{(1)} & & & & & \\ \mathbf{J}_{21}^{(1)} & \mathbf{J}_{22}^{(1)} + \mathbf{J}_{11}^{(2)} & \mathbf{J}_{12}^{(2)} & & & & \\ & \mathbf{J}_{21}^{(2)} & \mathbf{J}_{22}^{(2)} + \mathbf{J}_{11}^{(3)} + \mathbf{J} & \mathbf{J}_{12}^{(3)} & & & \\ & & \mathbf{J}_{21}^{(3)} & \mathbf{J}_{22}^{(3)} + \mathbf{J}_{11}^{(4)} & \mathbf{J}_{12}^{(4)} & & \\ & & & \mathbf{J}_{21}^{(4)} & \mathbf{J}_{22}^{(4)} + \mathbf{J}_{11}^{(5)} + \mathbf{J} & \mathbf{J}_{12}^{(5)} & \\ & & & & \mathbf{J}_{21}^{(5)} & \mathbf{J}_{22}^{(5)} + \mathbf{J}_{11}^{(6)} + \mathbf{J} & \mathbf{J}_{12}^{(6)} \\ & & & & & \mathbf{J}_{21}^{(6)} & \mathbf{J}_{22}^{(6)} + \mathbf{J}_{11}^{(7)} & \mathbf{J}_{12}^{(7)} \\ & & & & & & \mathbf{J}_{21}^{(7)} & \mathbf{J}_{22}^{(7)} \end{bmatrix}$$

(A8)

$$\mathbf{K}_1 = \begin{bmatrix} \mathbf{K}_{11}^{(1)} & \mathbf{K}_{12}^{(1)} & & & & & \\ \mathbf{K}_{21}^{(1)} & \mathbf{K}_{22}^{(1)} + \mathbf{K}_{11}^{(2)} + \mathbf{K}_x & \mathbf{K}_{12}^{(2)} & & & & \\ & \mathbf{K}_{21}^{(2)} & \mathbf{K}_{22}^{(2)} + \mathbf{K}_{11}^{(3)} & \mathbf{K}_{12}^{(3)} & & & \\ & & \mathbf{K}_{21}^{(3)} & \mathbf{K}_{22}^{(3)} + \mathbf{K}_{11}^{(4)} + \mathbf{K}_x & \mathbf{K}_{12}^{(4)} & & \\ & & & \mathbf{K}_{21}^{(4)} & \mathbf{K}_{22}^{(4)} + \mathbf{K}_{11}^{(5)} & \mathbf{K}_{12}^{(5)} & \\ & & & & \mathbf{K}_{21}^{(5)} & \mathbf{K}_{22}^{(5)} + \mathbf{K}_{11}^{(6)} & \mathbf{K}_{12}^{(6)} \\ & & & & & \mathbf{K}_{21}^{(6)} & \mathbf{K}_{22}^{(6)} + \mathbf{K}_{11}^{(7)} + \mathbf{K}_x & \mathbf{K}_{12}^{(7)} \\ & & & & & & \mathbf{K}_{21}^{(7)} & \mathbf{K}_{22}^{(7)} \end{bmatrix}$$

(A9)

where the mass matrix of the disk $\mathbf{M}_d$, the gyroscopic matrix of the disk $\mathbf{J}$ and the stiffness matrix of the bearing $\mathbf{K}_x$ are represented as

$$\mathbf{M}_d = \begin{bmatrix} m_d & 0 \\ 0 & J_d \end{bmatrix} \quad (\text{A10})$$

$$\mathbf{J} = \begin{bmatrix} 0 & 0 \\ 0 & J_p \end{bmatrix} \quad (\text{A11})$$

$$\mathbf{K}_x = \begin{bmatrix} K_b & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{A12})$$

where m_d , J_d , J_p , and K_b are the mass, the diametral moment of inertia, the polar moment of inertia of the disk and the support stiffness in the X direction, whose calculation methods are given in Refs. [24-26] for different bearing types.