Bilinear Form, Soliton, Breather, Hybrid and Periodic-Wave Solutions for a (3+1)-Dimensional Korteweg-De Vries Equation in a Fluid

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Bilinear form, soliton, breather, hybrid and periodic-wave solutions for a (3+1)-dimensional Korteweg-de Vries equation in a fluid

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Abstract

Fluids are studied in such disciplines as atmospheric science, oceanography and astrophysics. In this paper, we investigate a (3+1)-dimensional Korteweg-de Vries equation in a fluid. Bilinear form and N -soliton solutions are obtained, where N is a positive integer. Via the N -soliton solutions, we derive the higher-order breather solutions. We observe that the interaction between two perpendicular first-order breathers on the $x - y$ and $x - z$ planes and the periodic line wave interacts with the first-order breather on the $y - z$ plane, where x , y and z are the independent variables in the equation. Furthermore, we discuss the effects of α , β , γ and δ on the amplitudes of the second-order breathers, where α , β , γ and δ are the constant coefficients in the equation: Amplitude of the second-order breather decreases as α increases; Amplitude of the second-order breather increases as β increases; Amplitude of the second-order breather keeps invariant as γ and δ increase. Via the N -soliton solutions, hybrid solutions comprising the breathers and solitons are derived. Based on the Riemann theta function, we obtain the periodic-wave solutions. Furthermore, we find that the periodic-wave solutions approach to the one-soliton solutions under certain limiting condition.

Keywords: Fluid; (3+1)-dimensional Korteweg-de Vries equation; Bilinear form; Soliton solutions; Breather solutions; Hybrid solutions; Periodic-wave solutions

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1. Introduction

Fluids have been studied in such disciplines as atmospheric science, oceanography and astrophysics [1–4]. Analytic solutions for the nonlinear evolution equations (NLEEs) such as the soliton, breather and periodic-wave solutions have been applied in nonlinear optics, fluid mechanics and plasma physics [5–10]. Methods have been proposed to construct the analytic solutions, such as the bilinear method, Bäcklund transformation and Darboux transformation [11–13]. For instance, the multiple soliton solutions have been derived for a (3+1)-dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equation via the bilinear method [14]. Breathers have been investigated in a higher-order nonlinear Schrödinger equation based on the Darboux transformation [15]. Periodic-wave solutions have been studied in a (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation via the Riemann theta function [16].

Ref. [17] has considered a (3+1)-dimensional Korteweg-de Vries (KdV) equation in a fluid,

$$u_{ty} + u_{xxxy} + \alpha(u_y u_x)_x + \beta u_{xx} + \gamma u_{yy} + \delta u_{zy} = 0, \quad (1)$$

where u is a real differential function depending on the independent variables x, y, z and t , α, β, γ and δ are the real constants and the subscripts indicate the partial derivatives. Eq. (1) has some special cases in the fluids as follows:

- When $\alpha = 3$ and $\beta = \gamma = \delta = 0$, through the reduction $\partial_x = \partial_y$, Eq. (1) has been simplified as the (1+1)-dimensional KdV equation [18],

$$U_t + 6UU_x + U_{xxx} = 0, \quad (2)$$

which can describe certain phenomena in plasmas and fluids, such as the ion-acoustic solitons in a plasma and the shallow water waves under gravity [18–20], where $U(x, t) = u(x, t)_x$ is the wave function depending on the scaled space variable x and time variable t ;

- When $\alpha = -3$ and $\beta = \gamma = \delta = 0$, Eq. (1) has been degenerated to the (2+1)-dimensional BLMP equation [17, 21],

$$u_{yt} + u_{xxxy} - 3(u_y u_x)_x = 0. \quad (3)$$

Eq. (3) has been applied as a (2+1)-dimensional model for the interaction of the Riemann wave along the y axis and a long wave along the x axis [21];

- When $\alpha = 3, \beta = \gamma = 1$ and $\delta = 0$, Eq. (1) has been reduced into a (2+1)-dimensional extended BLMP equation for an incompressible fluid [22],

$$u_{ty} + u_{xxxy} + 3(u_y u_x)_x + \beta u_{xx} + \gamma u_{yy} = 0. \quad (4)$$

Eq. (1) has passed the Painlevé test, and one- and two-soliton solutions have been derived [17].

However, to our knowledge, bilinear form, N -soliton, breather, hybrid and periodic-wave solutions for Eq. (1) have not been considered, where N is a positive integer. In Section 2, bilinear form and N -soliton solutions for Eq. (1) will be studied. In Section 3, higher-order breather solutions for Eq. (1) will be obtained. In Section 4, hybrid solutions comprising the solitons and breathers for Eq. (1) will be derived. In Section 5, periodic-wave solutions and their asymptotic behaviors for Eq. (1) will be investigated. Our conclusions will be given in Section 6.

2. Bilinear form and N -soliton solutions for Eq. (1)

Via the variable transformation [17]

$$u = \frac{6}{\alpha}(\ln f)_x, \quad (5)$$

we derive the bilinear form for Eq. (1) as

$$(D_t D_y + D_x^3 D_y + \beta D_x^2 + \gamma D_y^2 + \delta D_z D_y) f \cdot f = 0, \quad (6)$$

where f is a real differentiable function of x , y , z and t , and D_x, D_y, D_z and D_t are the bilinear differential operators defined by [11]

$$D_x^l D_y^m D_z^n D_t^p F \cdot G = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^m \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'} \right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^p F(x, y, z, t) G(x', y', z', t')|_{x'=x, y'=y, z'=z, t'=t},$$

with F being a differentiable function of x , y , z and t , G being a differentiable function of the formal variable x' , y' , z' and t' , while l , m , n and p being the non-negative integers.

In order to derive the N -soliton solutions for Eq. (1), we assume the following expression:

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots + \varepsilon^N f_N, \quad (7)$$

where ε is a real constants and f_ϵ 's ($\epsilon = 1, 2, 3, \dots, N$) are the real functions of x , y , z and t . Substituting Expansion (7) into Bilinear Form (6) and making the coefficients on ε^ϵ vanish, the N -soliton solutions for Eq. (1) can be derived as

$$u = \frac{6}{\alpha}(\ln f)_x, \quad (8a)$$

$$f = \sum_{\mu=0,1} \exp \left[\sum_{j=1}^N \mu_j \xi_j + \sum_{1 \leq i < j}^N \mu_i \mu_j \ln(A_{ij}) \right], \quad (8b)$$

where

$$\begin{aligned}\xi_i &= a_i(x + b_i y + c_i z + d_i t) + \xi_i^0, \quad d_i = -a_i^2 - b_i^{-1}\beta - b_i\gamma - c_i\delta, \quad A_{ij} = \frac{C_{ij}}{M_{ij}}, \\ C_{ij} &= 3a_i^2 b_i^2 b_j + 3a_j b_i^2 b_j^2 - 3a_i a_j b_i b_j (b_i + b_j) - (b_i - b_j)^2 \beta, \\ M_{ij} &= 3a_i^2 b_i^2 b_j + 3a_j b_i^2 b_j^2 + 3a_i a_j b_i b_j (b_i + b_j) - (b_i - b_j)^2 \beta,\end{aligned}$$

$A_{ij} > 0$, b_i 's $\neq 0$ and $M_{ij} \neq 0$, $i, j = 1, 2, \dots, N$, a_i 's, b_i 's, c_i 's and ξ_i^0 's are the complex constants, $\sum_{\mu=0,1}$ denotes a summation over all the possible combinations of $\mu_1 = 0, 1$, $\mu_2 = 0, 1, \dots$ and $\mu_N = 0, 1$, and $\sum_{1 \leq i \leq j}^N$ indicates a summation over all possible pairs (i, j) under the condition $1 \leq i \leq j \leq N$.

3. higher-order breather solutions for Eq. (1)

In this section, motivated by Ref. [23], we construct the higher-order breather solutions for Eq. (1) with certain values of the parameters a_i 's, b_i 's, c_i 's, and ξ_i^0 's in N -Soliton Solutions (8) with N being an even integer. We assume that

$$\begin{aligned}N &= 2T, \quad a_r = a_{T+r}^* = a_{r1} + i a_{r2}, \quad b_r = b_{T+r}^* = b_{r1} + i b_{r2}, \\ c_r &= c_{T+r}^* = c_{r1} + i c_{r2}, \quad \xi_r^0 = \xi_{T+r}^{0*} = \xi_{r1}^0 + i \xi_{r2}^0,\end{aligned}\tag{9}$$

where a_{r1} 's, a_{r2} 's, b_{r1} 's, b_{r2} 's, c_{r1} 's, c_{r2} 's, ξ_{r1}^0 's and ξ_{r2}^0 's are all the real constants, T is a positive integer, $r = 1, 2, \dots, T$ and $i^2 = -1$. The T th-order breather solutions for Eq. (1) are given as

$$u = \frac{6}{\alpha} (\ln f)_x, \tag{10a}$$

$$f = \sum_{\mu=0,1} \exp \left[\sum_{j=1}^{2T} \mu_j \xi_j + \sum_{1 \leq i \leq j}^{2T} \mu_i \mu_j \ln (A_{ij}) \right], \tag{10b}$$

where

$$\begin{aligned}\xi_r &= \xi_{T+r}^* = \xi_{r1} + i \xi_{r2}, \\ \xi_{r1} &= a_{r1}x + (a_{r1}b_{r1} - a_{r2}b_{r2})y + (a_{r1}c_{r1} - a_{r2}c_{r2})z - [a_{r1}^3 - 3a_{r1}a_{r2}^2 - \\ &\quad (b_{r1}^2 + b_{r2}^2)^{-1} (a_{r1}b_{r1} + a_{r2}b_{r2})\beta - (a_{r1}b_{r1} - a_{r2}b_{r2})\gamma - (a_{r1}c_{r1} - a_{r2}c_{r2})\delta]t + \xi_{r1}^0, \\ \xi_{r2} &= a_{r2}x + (a_{r1}b_{r2} + a_{r2}b_{r1})y + (a_{r1}c_{r2} + a_{r2}c_{r1})z + [a_{r2}^3 - 3a_{r1}^2a_{r2} - \\ &\quad (b_{r1}^2 + b_{r2}^2)^{-1} (a_{r1}b_{r2} - a_{r2}b_{r1})\beta + (a_{r1}b_{r2} + a_{r2}b_{r1})\gamma + (a_{r1}c_{r2} + a_{r2}c_{r1})\delta]t + \xi_{r2}^0, \\ A_{ij} &= \frac{C_{ij}}{D_{ij}}, \\ C_{ij} &= 3a_i^2 b_i^2 b_j + 3a_j b_i^2 b_j^2 - 3a_i a_j b_i b_j (b_i + b_j) - (b_i - b_j)^2 \beta, \\ D_{ij} &= 3a_i^2 b_i^2 b_j + 3a_j b_i^2 b_j^2 + 3a_i a_j b_i b_j (b_i + b_j) - (b_i - b_j)^2 \beta,\end{aligned}$$

with $A_{r,T+r} > 1$.

Taking $T = 1$ into Solutions (10), the first-order breather solutions can be derived as

$$u = \frac{6}{\alpha}(\ln f)_x, \quad (11a)$$

$$f = 1 + e^{\xi_1} + e^{\xi_2} + A_{12}e^{\xi_1+\xi_2}, \quad (11b)$$

where

$$\begin{aligned} \xi_1 &= \xi_2^* = \xi_{11} + i\xi_{12}, \\ \xi_{11} &= a_{11}x + (a_{11}b_{11} - a_{12}b_{12})y + (a_{11}c_{11} - a_{12}c_{12})z - [a_{11}^3 - 3a_{11}a_{12}^2 - \\ &\quad (b_{11}^2 + b_{12}^2)^{-1}(a_{11}b_{11} + a_{12}b_{12})\beta - (a_{11}b_{11} - a_{12}b_{12})\gamma - (a_{11}c_{11} - a_{12}c_{12})\delta]t + \xi_{11}^0, \\ \xi_{12} &= a_{12}x + (a_{11}b_{12} + a_{12}b_{11})y + (a_{11}c_{12} + a_{12}c_{11})z + [a_{12}^3 - 3a_{11}^2a_{12} - \\ &\quad (b_{11}^2 + b_{12}^2)^{-1}(a_{11}b_{12} - a_{12}b_{11})\beta + (a_{11}b_{12} + a_{12}b_{11})\gamma + (a_{11}c_{12} + a_{12}c_{11})\delta]t + \xi_{12}^0, \\ A_{12} &= 1 - \frac{6a_{12}^2b_{11}(b_{11}^2 + b_{12}^2)}{3a_{11}(a_{11}b_{11} - a_{12}b_{12})(b_{11}^2 + b_{12}^2) + b_{12}^2\beta}, \end{aligned}$$

with $A_{12} > 1$.

The periods of the first-order breathers are $\frac{2\pi}{a_{12}}$ along the x direction, $\frac{2\pi}{a_{11}b_{12}+a_{12}b_{11}}$ in the y direction and $\frac{2\pi}{a_{11}c_{12}+a_{12}c_{11}}$ in the z direction. The locations of the first-order breathers are related to ξ_{11} . When $a_{12} = b_{12} = c_{12} = \xi_{12}^0 = 0$ in Solutions (11), the first-order breather solutions can be degenerated to the one-soliton solutions for Eq. (1).

When supposing $T = 2$ in Solutions (10), we can obtain the second-order breather solutions for Eq. (1). The second-order breathers depict the interactions between the two first-order breathers. As discussed on the first-order breathers, we can construct the second-order breathers comprising different first-order breathers. For example, the second-order breathers consisting of two perpendicular first-order breathers can be constructed under the condition $a_{11}a_{21} + (a_{11}b_{11} - a_{12}b_{12})(a_{21}b_{21} - a_{22}b_{22}) + (a_{11}c_{11} - a_{12}c_{12})(a_{21}c_{21} - a_{22}c_{22}) = 0$. We observe that interaction between two perpendicular first-order breathers on the $x - y$ and $x - z$ planes, as presented in Figs. 1(a) and (b). In Figs. 1(c), since $a_{11} \neq 0$, $a_{11}b_{11} - a_{12}b_{12} = 0$ and $a_{11}c_{11} - a_{12}c_{12} = 0$, ξ_{11} is not related to y and z . One of the first-order breathers is reduced into the periodic line wave. When $t = -\frac{1}{3}$, we observe the first-order breather, as shown in Fig. 1(c₁). As t goes by, we can find that the periodic line wave appears and interacts with the first-order breather, as shown in Fig. 1(c₂). When $t = \frac{1}{3}$, we find that the amplitude of the periodic line wave decreases and the amplitude of the first-order breather keeps unchanged, as shown in Fig. 1(c₃).

Furthermore, we investigate the influences of the coefficients in Eq. (1) on the amplitudes of the second-order breathers. Comparing Figs. 2 with Figs. 1, when α increases, we find that the amplitude of the second-order breather decreases. Comparing Figs. 3 with Figs. 1, when β increases, we observe that the amplitude of the second-order breather increases. Comparing

Figs. 4 and 5 with Figs. 1, we find that when γ and δ increase, the amplitude of the second-order breather keeps invariant. Under the condition $\frac{a_{11}}{a_{21}} = \frac{a_{11}b_{11}-a_{12}b_{12}}{a_{21}b_{21}-a_{22}b_{22}} = \frac{a_{11}c_{11}-a_{12}c_{12}}{a_{21}c_{21}-a_{22}c_{22}}$, we can derive the second-order breather containing two parallel first-order breathers, as shown in Figs. 6.

Figs. 1. The second-order breather via Solutions (10) with $T = 2$, $\alpha = \beta = \gamma = \delta = \epsilon = 1$, $a_1 = a_3^* = -1 - 2i$, $b_1 = b_3^* = c_1 = c_3^* = -\frac{1}{2} - \frac{1}{4}i$, $a_2 = a_4^* = 2i$, $b_2 = b_4^* = -1 + i$, $c_2 = c_4^* = -\frac{1}{2} + 2i$, $\xi_1^0 = \xi_2^0 = \xi_3^0 = \xi_4^0 = 0$ and (a) $z = 0$; (b) $y = 0$; (c) $x = 0$.

Figs. 2. The same as Figs. 1 expect that $\alpha = 3$.

Figs. 3. The same as Figs. 1 expect that $\beta = 5$.

Figs. 4. The same as Figs. 1 expect that $\gamma = 10$.

(a) $t = 0$

(b) $t = 0$

(c) $t = 0$

Figs. 5. The same as Figs. 1 expect that $\delta = 10$.

(a₁) $t = -3$

(a₂) $t = 0$

(a₃) $t = 3$

(b₁) $t = -3$

(b₂) $t = 0$

(b₃) $t = 3$

Figs. 6. The second-order breather via Solutions (10) with $T = 2$, $\alpha = \beta = \gamma = \delta = \epsilon = 1$, $a_1 = a_3^* = -\frac{1}{2} - \frac{1}{2}i$, $b_1 = b_3^* = -\frac{1}{4} - \frac{1}{2}i$, $c_1 = c_3^* = -\frac{1}{4} - \frac{1}{8}i$, $a_2 = a_4^* = \frac{1}{2} + \frac{3}{4}i$, $b_2 = b_4^* = -\frac{1}{2} - \frac{1}{2}i$, $c_2 = c_4^* = -\frac{1}{2} - \frac{1}{4}i$, $\xi_1^0 = \xi_2^0 = \xi_3^0 = \xi_4^0 = 0$ and (a) $z = 0$; (b) $y = 0$; (c) $x = 0$.

4. Hybrid solutions comprising solitons and breathers for Eq. (1)

Hybrid solutions consisting of Θ solitons and Λ breathers for Eq. (1) are derived with the following constraints in Solutions (8):

$$\begin{aligned} N &= 2\Lambda + \Theta, \quad a_\nu = a_{\Lambda+\nu}^* = a_{\nu 1} + ia_{\nu 2}, \quad b_\nu = b_{\Lambda+\nu}^* = b_{\nu 1} + ib_{\nu 2}, \\ c_\nu &= c_{\Lambda+\nu}^* = c_{\nu 1} + ic_{\nu 2}, \quad \xi_\nu^0 = \xi_{\Lambda+\nu}^{0*} = \xi_{\nu 1}^0 + i\xi_{\nu 2}^0, \quad a_s = a_{s1}, \quad b_s = b_{s1}, \\ c_s &= c_{s1}, \quad \xi_s^0 = \xi_{s1}^0, \quad (\nu = 1, 2, \dots, \Lambda, \quad s = 2\Lambda + 1, 2\Lambda + 2, \dots, N) \end{aligned} \quad (12)$$

where Θ and Λ are the positive integers, a_{s1} 's, b_{s1} 's, c_{s1} 's and ξ_{s1}^0 's are the real constants.

For example, the hybrid solutions comprising the first-order breather and one kink-type soliton are obtained from Solutions (8) with the parameters as

$$\begin{aligned} N &= 3, \quad \Lambda = 1, \quad \Theta = 1, \quad a_1 = a_2^* = a_{11} + ia_{12}, \quad b_1 = b_2^* = b_{11} + ib_{12}, \\ c_1 &= c_2^* = c_{11} + ic_{12}, \quad \xi_1^0 = \xi_2^{0*} = \xi_{11}^0 + i\xi_{12}^0, \quad a_3 = a_{31}, \quad b_3 = b_{31}, \\ c_3 &= c_{31}, \quad \xi_3^0 = \xi_{31}^0. \end{aligned} \quad (13)$$

Under the condition $\frac{a_{11}}{a_{31}} = \frac{a_{11}b_{11}-a_{12}b_{12}}{a_{31}b_{31}} = \frac{a_{11}c_{11}-a_{12}c_{12}}{a_{31}c_{31}}$, we observe that the first-order breather parallels with one kink-type soliton, as shown in Figs. 7. When $t = 0$, the first-order breather interacts with the kink-type soliton. After the interaction, the first-order breather and the kink-type soliton keep their velocities and shapes unchanged, meaning that the interaction is elastic. When $N = 4$, $\Lambda = 1$ and $\Theta = 2$ via Constraints (12), the interaction among the first-order breather and two kink-type solitons is shown in Figs. 8.

Figs. 7. Interaction between the first-order breather and one kink-type soliton via Solutions (8) with $\alpha = 3$, $\beta = \gamma = \delta = \epsilon = 1$, $a_1 = a_2^* = 1 + 2i$, $b_1 = b_2^* = 1 + i$, $c_1 = c_2^* = 1 - \frac{1}{2}i$, $a_3 = c_3 = 2$, $b_3 = -1$ and (a) $z = 0$; (b) $y = 0$; (c) $x = 0$.

Figs. 8. Interaction among the first-order breather and two kink-type solitons via Solutions (8) with $\alpha = c_4 = 3$, $\beta = \gamma = \delta = \epsilon = b_3 = a_4 = 1$, $a_1 = a_2^* = 1 + 2i$, $b_1 = b_2^* = 1 + i$, $c_1 = c_2^* = 1 + \frac{1}{2}i$, $a_3 = 2$, $c_3 = -1$, $b_4 = -2$ and (a) $z = 0$; (b) $y = 0$; (c) $x = 0$.

5. Periodic-wave solutions and asymptotic behaviors for Eq. (1)

5.1 Periodic-wave solutions for Eq. (1)

In this section, motivated by Ref. [16], the Riemann theta function will be applied to construct the periodic-wave solutions for Eq. (1). We derive the one-Riemann theta function as

$$f = \psi(\zeta_1, \mu_1) = \sum_{\rho=-\infty}^{+\infty} e^{\pi i \rho^2 \mu_1 + 2\pi i \rho \zeta_1}, \quad (14)$$

where $\zeta_1 = Q_1 x + B_1 y + W_1 z + R_1 t + \epsilon_1$, μ_1 is a pure imaginary number and $\text{Im}(\mu_1) > 0$, ρ is an integer, Q_1, B_1, W_1, R_1 and ϵ_1 are the real constants. Substituting Transformation (5) and Expression (14) into Eq. (1). We obtain

$$\begin{aligned} \vartheta(D_x, D_y, D_z, D_t) \psi(\zeta_1, \mu_1) \cdot \psi(\zeta_1, \mu_1) = \\ (D_t D_y + D_x^3 D_y + \beta D_x^2 + \gamma D_y^2 + \delta D_z D_y + c) \psi(\zeta_1, \mu_1) \cdot \psi(\zeta_1, \mu_1) = 0, \end{aligned} \quad (15)$$

where $\vartheta(D_x, D_y, D_z, D_t)$ is a polynomial about D_x, D_y, D_z and D_t , and c is an integral constant.

Bilinear derivative operators possess the following properties [16]:

$$D_x^l D_y^m D_z^n D_t^p e^{v_1} \cdot e^{v_2} = (\varpi_1 - \varpi_2)^l (o_1 - o_2)^m (\chi_1 - \chi_2)^n (\varsigma_1 - \varsigma_2)^p e^{v_1 + v_2}, \quad (16)$$

where $v_{i_1} = \varpi_{i_1} x + o_{i_1} y + \chi_{i_1} z + \varsigma_{i_1} t + h_{i_1}$ ($i_1 = 1, 2$), and $\varpi_{i_1}, o_{i_1}, \chi_{i_1}, \varsigma_{i_1}$ and h_{i_1} being the complex numbers. Applying Eqs. (14) and (16), we can rewrite Expression (15) as

$$\begin{aligned} & \vartheta(D_x, D_y, D_z, D_t) \psi(\zeta_1, \mu_1) \cdot \psi(\zeta_1, \mu_1) \\ &= \sum_{\varrho=-\infty}^{+\infty} \sum_{\rho=-\infty}^{+\infty} \vartheta(D_x, D_y, D_z, D_t) e^{\pi i \rho^2 \mu_1 + 2\pi i \rho \zeta_1} \cdot e^{\pi i \varrho^2 \mu_1 + 2\pi i \varrho \zeta_1} \\ &= \sum_{\varrho=-\infty}^{+\infty} \sum_{\rho=-\infty}^{+\infty} \vartheta [2i\pi(\rho - \varrho)Q_1, 2i\pi(\rho - \varrho)B_1, 2i\pi(\rho - \varrho)W_1, 2i\pi(\rho - \varrho)R_1] \\ & \quad \times e^{\pi i(\rho^2 + \varrho^2)\mu_1 + 2\pi i(\rho + \varrho)\zeta_1} \\ &= \sum_{\varrho'=-\infty}^{+\infty} \tilde{\vartheta}(\varrho') e^{2\pi i \varrho' \zeta_1}, \quad \varrho' = \varrho + \rho, \end{aligned} \quad (17)$$

where

$$\begin{aligned} & \tilde{\vartheta}(\varrho') \\ &= \sum_{\rho=-\infty}^{+\infty} \vartheta [2i\pi(2\rho - \varrho')Q_1, 2i\pi(2\rho - \varrho')B_1, 2i\pi(2\rho - \varrho')W_1, 2i\pi(2\rho - \varrho')R_1] e^{\pi i[\rho^2 + (\rho - \varrho')^2]\mu_1} \\ & \stackrel{\rho=\varrho'+1}{=} \sum_{\rho'=-\infty}^{+\infty} \vartheta \{2i\pi[2\rho' - (\varrho' - 2)]Q_1, 2i\pi[2\rho' - (\varrho' - 2)]B_1, 2i\pi[2\rho' - (\varrho' - 2)]W_1, \\ & \quad 2i\pi[2\rho' - (\varrho' - 2)]R_1\} \times e^{\pi i\{\rho'^2 + [\rho' - (\varrho' - 2)]^2\}\mu_1} \cdot e^{2\pi i(\varrho' - 1)\mu_1} \\ &= \tilde{\vartheta}(\varrho' - 2) e^{2\pi i(\varrho' - 1)\mu_1} = \dots = \begin{cases} \tilde{\vartheta}(0) e^{i\pi \varrho' \mu_1}, & \varrho' \text{ is even,} \\ \tilde{\vartheta}(1) e^{i\pi(\varrho' + 1)\mu_1}, & \varrho' \text{ is odd,} \end{cases} \quad \varrho', \rho' \in \mathbb{Z}. \end{aligned} \quad (18)$$

Eq. (18) implies that $\tilde{\vartheta}(\varrho')$ for $\varrho' \in \mathbb{Z}$ are completely dominated by $\tilde{\vartheta}(0)$ and $\tilde{\vartheta}(1)$, where $\mathbb{Z}$ denotes an integer set. When $\tilde{\vartheta}(0) = \tilde{\vartheta}(1) = 0$, we have $\vartheta(D_x, D_y, D_z, D_t)\psi(\zeta_1, \mu_1) \cdot \psi(\zeta_1, \mu_1) = 0$. Through Expression (18), $\tilde{\vartheta}(0)$ and $\tilde{\vartheta}(1)$ can be derived as

$$\begin{aligned}\tilde{\vartheta}(0) &= \sum_{\rho=-\infty}^{+\infty} \vartheta[4\rho\pi i Q_1, 4\rho\pi i B_1, 4\rho\pi i W_1, 4\rho\pi i R_1] e^{2\rho^2\pi i \mu_1} \\ &= \sum_{\rho=-\infty}^{+\infty} (-16\rho^2\pi^2 B_1 R_1 + 256\rho^4\pi^4 Q_1^3 B_1 - 16\beta\rho^2\pi^2 Q_1^2 - 16\gamma\rho^2\pi^2 B_1^2 - \\ &\quad 16\delta\rho^2\pi^2 B_1 W_1 + c) e^{2\rho^2\pi i \mu_1} = 0,\end{aligned}\tag{19a}$$

$$\begin{aligned}\tilde{\vartheta}(1) &= \sum_{\rho=-\infty}^{+\infty} \vartheta[2i\pi(2\rho-1)Q_1, 2i\pi(2\rho-1)B_1, 2i\pi(2\rho-1)W_1, 2i\pi(2\rho-1)R_1] e^{\pi i(2\rho^2-2\rho+1)\mu_1} \\ &= \sum_{\rho=-\infty}^{+\infty} [-4(2\rho-1)^2\pi^2 B_1 R_1 + 16(2\rho-1)^4\pi^4 Q_1^3 B_1 - 4\beta(2\rho-1)^2\pi^2 Q_1^2 - \\ &\quad 4\gamma(2\rho-1)^2\pi^2 B_1^2 - 4\delta(2\rho-1)^2\pi^2 B_1 W_1 + c] e^{\pi i(2\rho^2-2\rho+1)\mu_1} = 0.\end{aligned}\tag{19b}$$

Through the notations

$$\begin{aligned}\Delta &= e^{\pi i \mu_1}, \quad p_{11} = - \sum_{\rho=-\infty}^{+\infty} 16\rho^2\pi^2 B_1 \Delta^{2\rho^2}, \quad p_{12} = \sum_{\rho=-\infty}^{+\infty} \Delta^{2\rho^2}, \\ p_{21} &= - \sum_{\rho=-\infty}^{+\infty} 4(2\rho-1)^2\pi^2 B_1 \Delta^{2\rho^2-2\rho+1}, \quad p_{22} = \sum_{\rho=-\infty}^{+\infty} \Delta^{2\rho^2-2\rho+1}, \\ q_1 &= \sum_{\rho=-\infty}^{+\infty} (-256\rho^4\pi^4 Q_1^3 B_1 + 16\beta\rho^2\pi^2 Q_1^2 + 16\gamma\rho^2\pi^2 B_1^2 + 16\delta\rho^2\pi^2 B_1 W_1 + c) \Delta^{2\rho^2}, \\ q_2 &= \sum_{\rho=-\infty}^{+\infty} [-16(2\rho-1)^4\pi^4 Q_1^3 B_1 + 4\beta(2\rho-1)^2\pi^2 Q_1^2 + 4\gamma(2\rho-1)^2\pi^2 B_1^2 + \\ &\quad 4\delta(2\rho-1)^2\pi^2 B_1 W_1 + c] \Delta^{2\rho^2-2\rho+1},\end{aligned}\tag{20}$$

Eqs. (19) can be converted into the following linear system about R_1 and c , i.e.,

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} R_1 \\ c \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.\tag{21}$$

Solving System (21), the periodic-wave solutions for Eq. (1) can be derived as

$$u = \frac{6}{\alpha} [\ln \psi(\zeta_1, \mu_1)]_x, \quad R_1 = \frac{p_{22}q_1 - p_{21}q_2}{p_{11}p_{22} - p_{12}p_{21}}, \quad c = \frac{p_{21}q_1 - p_{11}q_2}{p_{12}p_{21} - p_{11}p_{22}}.\tag{22}$$

5.2 The asymptotic behaviors of periodic-wave solutions for Eq. (1)

In this section, we will study the asymptotic behaviors of Periodic-Wave Solutions (22). Eqs. (20) can be expanded as

$$\begin{aligned}
p_{11} &= -32\pi^2 B_1 \left(\Delta^2 + 4\Delta^8 + \cdots + \rho^2 \Delta^{2\rho^2} + \cdots \right), \\
p_{12} &= 1 + 2 \left(\Delta^2 + \Delta^8 + \cdots + \Delta^{2\rho^2} + \cdots \right), \\
p_{21} &= -8\pi^2 B_1 \left(\Delta + 9\Delta^5 + \cdots + (2\rho - 1)^2 \Delta^{2\rho^2 - 2\rho + 1} + \cdots \right), \\
p_{22} &= 2 \left(\Delta^1 + \Delta^5 + \cdots + \Delta^{2\rho^2 - 2\rho + 1} + \cdots \right), \\
q_1 &= 2 \left(-256\pi^4 Q_1^3 B_1 + 16\beta\pi^2 Q_1^2 + 16\gamma\pi^2 B_1^2 + 16\delta\pi^2 B_1 W_1 + c \right) \Delta^2 + \\
&\quad 2 \left(-4096\pi^4 Q_1^3 B_1 + 64\beta\pi^2 Q_1^2 + 64\gamma\pi^2 B_1^2 + 64\delta\pi^2 B_1 W_1 + c \right) \Delta^8 + \cdots + \\
&\quad 2 \left(-256\rho^4 \pi^4 Q_1^3 B_1 + 16\beta\rho^2 \pi^2 Q_1^2 + 16\gamma\rho^2 \pi^2 B_1^2 + 16\delta\rho^2 \pi^2 B_1 W_1 + c \right) \Delta^{2\rho^2} + \cdots, \\
q_2 &= 2 \left(-16\pi^4 Q_1^3 B_1 + 4\beta\pi^2 Q_1^2 + 4\gamma\pi^2 B_1^2 + 4\delta\pi^2 B_1 W_1 + c \right) \Delta + \\
&\quad 2 \left(-1296\pi^4 Q_1^3 B_1 + 36\beta\pi^2 Q_1^2 + 36\gamma\pi^2 B_1^2 + 36\delta\pi^2 B_1 W_1 + c \right) \Delta^5 + \cdots + \\
&\quad 2 \left[-16(2\rho - 1)^4 \pi^4 Q_1^3 B_1 + 4\beta(2\rho - 1)^2 \pi^2 Q_1^2 + 4\gamma(2\rho - 1)^2 \pi^2 B_1^2 + \right. \\
&\quad \left. 4\delta(2\rho - 1)^2 \pi^2 B_1 W_1 + c \right] \Delta^{2\rho^2 - 2\rho + 1} + \cdots,
\end{aligned} \tag{23}$$

and substituting Expressions (23) into System (21), we obtain

$$\begin{aligned}
\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} &= \Gamma_0 + \Gamma_1 \Delta + \Gamma_2 \Delta^2 + \cdots, \\
\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \Upsilon_0 + \Upsilon_1 \Delta + \Upsilon_2 \Delta^2 + \cdots,
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
\Gamma_0 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & 0 \\ -8\pi^2 B_1 & 2 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} -32\pi^2 B_1 & 2 \\ 0 & 0 \end{pmatrix}, \\
\Gamma_3 = \Gamma_4 &= \mathbf{0}, \quad \Gamma_5 = \begin{pmatrix} 0 & 0 \\ -72\pi^2 B_1 & 2 \end{pmatrix}, \cdots, \quad \Upsilon_0 = \Upsilon_3 = \Upsilon_4 = \mathbf{0}, \\
\Upsilon_1 &= \begin{pmatrix} 0 \\ \kappa_1 \end{pmatrix}, \quad \Upsilon_2 = \begin{pmatrix} 0 \\ \kappa_2 \end{pmatrix}, \quad \Upsilon_5 = \begin{pmatrix} 0 \\ \kappa_5 \end{pmatrix}, \cdots,
\end{aligned} \tag{25}$$

with

$$\begin{aligned}
\kappa_1 &= 2 \left(-16\pi^4 Q_1^3 B_1 + 4\beta\pi^2 Q_1^2 + 4\gamma\pi^2 B_1^2 + 4\delta\pi^2 B_1 W_1 + c \right), \\
\kappa_2 &= 2 \left(-256\pi^4 Q_1^3 B_1 + 16\beta\pi^2 Q_1^2 + 16\gamma\pi^2 B_1^2 + 16\delta\pi^2 B_1 W_1 + c \right), \\
\kappa_5 &= 2 \left(-1296\pi^4 Q_1^3 B_1 + 36\beta\pi^2 Q_1^2 + 36\gamma\pi^2 B_1^2 + 36\delta\pi^2 B_1 W_1 + c \right),
\end{aligned}$$

Then, R_1 and c in System (21) can be rewritten as

$$\begin{pmatrix} R_1 \\ c \end{pmatrix} = \Psi_0 + \Psi_1 \Delta + \Psi_2 \Delta^2 + \dots, \quad (26)$$

$$\Psi_0 = \begin{pmatrix} \frac{2\Upsilon_0^{[1]} - \Upsilon_1^{[2]}}{8\pi^2 B_1} \\ \Upsilon_0^{[1]} \end{pmatrix}, \quad \Psi_1 = \begin{pmatrix} \frac{2\Upsilon_1^{[1]} - (\Upsilon_2 - \Gamma_2 \Psi_0)^{[2]}}{8\pi^2 B_1} \\ \Upsilon_1^{[1]} \end{pmatrix}, \dots,$$

where $\Upsilon^{[\iota]} (\iota = 1, 2)$ denotes the ι -th elements of the two-dimensional vector Υ . From Expressions (26), we obtain

$$\Psi_0 = \begin{pmatrix} \frac{-16\pi^4 Q_1^3 B_1 + 4\beta\pi^2 Q_1^2 + 4\gamma\pi^2 B_1^2 + 4\delta\pi^2 B_1 W_1}{4\pi^2 B_1} \\ 0 \end{pmatrix}, \quad \Psi_1 = \mathbf{0}, \dots \quad (27)$$

Substituting Expressions (27) into Expressions (24) and taking $\Delta \rightarrow 0$, we obtain

$$c \rightarrow 0, \quad R_1 \rightarrow \frac{-16\pi^4 Q_1^3 B_1 + 4\beta\pi^2 Q_1^2 + 4\gamma\pi^2 B_1^2 + 4\delta\pi^2 B_1 W_1}{4\pi^2 B_1}. \quad (28)$$

Assuming

$$Q_1 = \frac{a_1}{2i\pi}, \quad B_1 = \frac{a_1 b_1}{2i\pi}, \quad W_1 = \frac{a_1 c_1}{2i\pi}, \quad \epsilon_1 = \frac{\xi_1^0 - i\pi\mu_1}{2i\pi}, \quad (29)$$

we have

$$\begin{aligned} 2i\pi\zeta_1 &= 2i\pi \left(Q_1 x + B_1 y + W_1 z + \frac{-16\pi^4 Q_1^3 B_1 + 4\beta\pi^2 Q_1^2 + 4\gamma\pi^2 B_1^2 + 4\delta\pi^2 B_1 W_1}{4\pi^2 B_1} t + \epsilon_1 \right) \\ &= a_1(x + b_1 y + c_1 z + d_1 t) + \xi_1^0 - i\pi\mu_1 \\ &= \xi_1 - i\pi\mu_1. \end{aligned} \quad (30)$$

Substituting Expression (30) into Eq. (14), we further derive that

$$\begin{aligned} \psi(\zeta_1, \mu_1) &= \sum_{\rho=-\infty}^{+\infty} e^{\pi i \rho^2 \mu_1 + 2\pi i \rho \zeta_1} = 1 + (e^{2\pi i \zeta_1} + e^{-2\pi i \zeta_1}) \Delta + \dots = 1 + e^{\xi_1} + e^{-\xi_1} \Delta^2 + \dots \\ &\stackrel{\Delta \rightarrow 0}{=} 1 + e^{\xi_1}. \end{aligned} \quad (31)$$

According to the above discussion, we find that Periodic-Wave Solutions (22) approach to one-soliton solutions with the limiting condition $\Delta \rightarrow 0$.

6. Conclusions

Fluids have been studied in such disciplines as atmospheric science, oceanography and astrophysics. In this paper, we have studied a (3+1)-dimensional KdV equation in a fluid, i.e, Eq. (1). Via Transformation (5) and the bilinear method, Bilinear Form (6) and N -Soliton Solutions (8) for Eq. (1) have been obtained, where N is a positive integer. Based

on N -Soliton Solutions (8), we have obtained the T th-order breather solutions for Eq. (1), i.e, Solutions (10), where T is a positive integer. Hybrid solutions comprising the solitons and breathers for Eq. (1) have been constructed via N -Soliton Solutions (8) under Constraints (12). Via the Riemann theta function, Periodic-Wave Solutions (22) for Eq. (1) have been obtained.

Via Solutions (10), we have observed that the interaction between the two perpendicular first-order breathers on the $x - y$ and $x - z$ planes and the periodic line wave interacts with the first-order breather on the $y - z$ plane, as shown in Figs. 1. Furthermore, we have discussed the effects of α , β , γ and δ on the amplitudes of the second-order breathers, where α , β , γ and δ are the constant coefficients in Eq. (1): Amplitude of the second-order breather decreases as α increases, as shown in Figs. 2; Amplitude of the second-order breather increases as β increases, as shown in Figs. 3; Amplitude of the second-order breather keeps invariant as γ and δ increase, as shown in Figs. 4 and 5. We have observed the interaction between the two parallel first-order breathers, as shown in Figs. 6. We have seen that the first-order breather interacts with one kink-type soliton and the interaction is elastic, as shown in Figs. 7. Interaction among the first-order breather and two kink-type solitons has been shown in Figs. 8. With the asymptotic behaviors of Periodic-Wave Solutions (22), we have found that Periodic-Wave Solutions (22) approach to the one-soliton solutions under the limit process.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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Figures

Figure 1

"Please see the Manuscript PDF file for the complete figure caption".

(a) $t = 0$

(b) $t = 0$

(c) $t = 0$

Figure 2

The same as Figs. 1 expect that $\alpha = 3$.

(a) $t = 0$

(b) $t = 0$

(c) $t = 0$

Figure 3

The same as Figs. 1 expect that $\beta = 5$.

(a) $t = 0$

(b) $t = 0$

(c) $t = 0$

Figure 4

The same as Figs. 1 expect that $\gamma = 10$.

(a) $t = 0$

(b) $t = 0$

(c) $t = 0$

Figure 5

The same as Figs. 1 expect that $\delta = 10$.

Figure 6

"Please see the Manuscript PDF file for the complete figure caption".

Figure 7

"Please see the Manuscript PDF file for the complete figure caption".

Figure 8

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