

# On induced map $f^\#$ in fuzzy set theory and its applications to fuzzy continuity

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**Abstract** In this paper, we introduce  $\#$  image of a fuzzy set which gives an induced map  $f^\#$  corresponding to any function  $f : X \rightarrow Y$ , where  $X$  and  $Y$  are crisp sets. With this, we present a new vision of studying fuzzy continuous mappings in fuzzy topological spaces where fuzzy continuity explains the term of closeness in the mathematical models. We also define the concept of fuzzy saturated sets which helps us to prove some new characterizations of fuzzy continuous mappings in terms of interior operator rather than closure operator.

**Keywords** fuzzy set · fuzzy topology · fuzzy saturated · fuzzy continuous · interior · closure

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## 1 Introduction

The theory of fuzzy sets, which proposes to deal with uncertainty, vagueness and unclear boundaries was first introduced by L.A. Zadeh in [7]. The basic idea of a fuzzy set concerns the flexibility over the concept of belongingness. Since its advent, research in soft set theory and its applications in several directions has been growing exponentially resulting in the great success from the theoretical and technological point of view. Therefore, fuzzification of various classical concepts related to topology has been done by various authors in [1], [6], [4] etc. Chang introduced the concept of fuzzy topological spaces in [3]

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and generalized the notion of continuity to what he named fuzzy continuous (F-continuous) functions and studied some of its characterizations. To introduce fuzzy continuity, he introduced image and inverse image of a fuzzy set corresponding to any function  $f : X \rightarrow Y$  and established some properties of fuzzy sets induced by mappings.

In [4], Ming et. al. introduced the concept of closure and interior of a fuzzy set in a fuzzy topological space and studied some of its properties which are similar to that in general topological space. They also studied some more characterizations of fuzzy continuous functions in their other paper [5]. In [2], Arkhangel'skii introduced a new map  $f^\# : \wp(X) \rightarrow \wp(Y)$  given by  $f^\#(E) = \{y \in Y \mid f^{-1}(y) \subset E\}$  for any subset  $E$  of  $X$ , where  $\wp(X)$  is a collection of all subsets of  $X$ . This gives us the idea about the possibility of introducing such type of induced map using fuzzy subsets and use it to find possible characterizations of fuzzy continuity.

In this paper, apart from image and inverse image of a fuzzy set we introduce  $\#$  image of a fuzzy set. Equivalently, given any function  $f : X \rightarrow Y$ , where  $X$  and  $Y$  are crisp sets, it gives rise to a induced map  $f^\# : FZ(X) \rightarrow FZ(Y)$ , where by  $FZ(X)$ , we mean collection of all fuzzy subsets of  $X$  (Definition 7). We prove the monotonicity of map  $f^\#$  along with its various properties (Lemma 1). Relationship between  $\#$  image of a fuzzy set and image of a fuzzy set is also given (Lemma 2). Further, we define fuzzy saturated sets and obtain its characterizations. It is also shown that surjective maps can be characterized in terms of  $\#$  image of a fuzzy set (Theorem 4). Finally, using  $f^\#$  map, we give characterizations of fuzzy continuous maps in terms of interior operator rather than closure operator (Theorem 5) and using saturated sets (Theorem 6).

## 2 Preliminaries

In this section we will give some basic definitions and results that we need in our further sections.

Let  $X = \{x\}$  be a space of points then a *fuzzy set*  $E$  in  $X$  is a function  $\mu_E$  from  $X$  to closed interval  $[0, 1]$  i.e.  $\mu_E : X \rightarrow [0, 1]$ . In other words, a fuzzy set is characterized by a membership function which associates with each  $x$  in  $X$  its "grade of membership"  $\mu_E(x)$ , in  $E$ . An empty fuzzy set, denoted by  $\phi$  is defined as  $\mu_\phi(x) = 0$  for all  $x$  in  $X$  and fuzzy set  $X$  is defined as  $\mu_X(x) = 1$  for all  $x$  in  $X$ . For a fuzzy set  $E$ ,

$$Hgt(E) = \sup\{\mu_E(x) \mid x \in X\}$$

is known as *height* of  $E$  and a fuzzy set  $E$  is said to be *normalized* if and only if  $Hgt(E) = 1$ . Also, we have

**Definition 1** [3] "Let  $A$  and  $B$  be fuzzy sets in a space  $X = \{x\}$  with membership functions  $\mu_A(x)$  and  $\mu_B(x)$  respectively. Then

1.  $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$  for all  $x$  in  $X$ .

2.  $A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$  for all  $x$  in  $X$ .
3.  $C = A \cup B \Leftrightarrow \mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$  for all  $x$  in  $X$ .
4.  $C = A \cap B \Leftrightarrow \mu_C(x) = \min\{\mu_A(x), \mu_B(x)\}$  for all  $x$  in  $X$ .
5.  $E = A^c \Leftrightarrow \mu_E(x) = 1 - \mu_A(x)$  for all  $x$  in  $X$ ."

**Definition 2** [3] "A family  $T$  of fuzzy sets in  $X$  is said to be a *fuzzy topology* if it satisfies following conditions:

- (a)  $\phi, X \in T$ .
- (b) If  $A, B \in T$  then  $A \cap B \in T$ .
- (c) If  $A_i \in T$  for each  $i \in I$  then  $\cup A_i \in T$ .

Every member of  $T$  is called a  $T$ -open fuzzy set and the pair  $(X, T)$  is known as fuzzy topological space or fts for short. Also a fuzzy set is  $T$ -closed if and only if its complement is  $T$ -open."

**Definition 3** [4] "Let  $(X, T)$  be fuzzy topological space and  $E$  be any fuzzy set in  $X$ . Then the union of all  $T$ -open fuzzy sets contained in  $E$  is called *interior* of  $E$ , denoted by  $E^o$ . Equivalently,  $E^o$  is the largest open set contained in  $E$  and  $(E^o)^o = E^o$ ."

**Definition 4** [4] "Let  $(X, T)$  be fuzzy topological space and  $E$  be any fuzzy set in  $X$ . Then the intersection of all  $T$ -closed fuzzy sets containing  $E$  is called the *closure* of  $E$ , denoted by  $\overline{E}$ . Clearly,  $\overline{E}$  is the smallest  $T$ -closed fuzzy set containing  $E$  and  $\overline{\overline{E}} = \overline{E}$ ."

**Theorem 1** [4] "In a fuzzy topological space  $(X, T)$ , for any fuzzy subset  $E$  of  $X$ ,  $(\overline{E})^c = (E^c)^o$  and so  $\overline{E^c} = (E^o)^c$ ."

**Definition 5** [3] "Let  $f : X \rightarrow Y$  be a function and  $E$  be a fuzzy set in  $X$  with membership function  $\mu_E(x)$  then the image of  $E$ , denoted by  $f(E)$  is a fuzzy set in  $Y$  whose membership function is defined by,

$$\begin{aligned} \mu_{f(E)}(y) &= \sup_{z \in f^{-1}(y)} \mu_E(z) && \text{if } f^{-1}(y) \neq \phi \\ &= 0, && \text{otherwise.} \end{aligned}$$

for all  $y \in Y$  where  $f^{-1}(y) = \{x \mid f(x) = y\}$ .

Also for a fuzzy set  $B$  in  $Y$  with membership function  $\mu_B(y)$ , inverse image of  $B$ , denoted by  $f^{-1}(B)$  is a fuzzy set in  $X$  whose membership function is defined by

$$\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$$

for all  $x$  in  $X$ ."

**Theorem 2** [3] "Let  $f : X \rightarrow Y$  be any function. Then for any fuzzy subsets  $E, F$  in  $X$  and  $B$  in  $Y$ , the following holds :

1.  $f^{-1}(B^c) = (f^{-1}(B))^c$ .

2.  $(f(E))^c \subset f(E^c)$ .
3. If  $B_1 \subset B_2$  then  $(f^{-1}(B_1)) \subset (f^{-1}(B_2))$  where  $B_1$  and  $B_2$  are fuzzy sets in  $Y$ .
4. If  $E \subset F$  then  $f(E) \subset f(F)$ .
5.  $f(f^{-1}(B)) \subset B$  and equality holds if  $f$  is surjective.
6.  $E \subset f^{-1}(f(E))$  and equality holds if  $f$  is injective.
7. Let  $g$  be a function from  $Y$  to  $Z$  then  $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$  for any fuzzy set  $C$  in  $Z$ , where  $g \circ f$  is the composition of  $g$  and  $f$ .

**Definition 6** [4] “A function  $f : (X, T) \rightarrow (Y, U)$  is *fuzzy continuous* (or *F-continuous*) if and only if the inverse image of each  $U$ -open fuzzy set is  $T$ -open fuzzy set.”

**Theorem 3** [5] “A function  $f : (X, T) \rightarrow (Y, U)$  is *fuzzy continuous* (or *F-continuous*) if and only if  $f(\overline{E}) \subset \overline{f(E)}$ , for any fuzzy set  $E$  in  $X$ .”

### 3 Induced map $f^\#$

We begin by introducing  $\#$  image of a fuzzy set which gives rise to the induced map  $f^\# : FZ(X) \rightarrow FZ(Y)$  corresponding to any function  $f : X \rightarrow Y$ .

**Definition 7** Let  $X = \{x\}$  and  $Y = \{y\}$  be spaces of points. Let  $f$  be a function from  $X$  to  $Y$  and  $E$  be a fuzzy set in  $X$  with membership function  $\mu_E(x)$ . Then  $\#$  image of a fuzzy set  $E$ , written as  $f^\#(E)$ , is a fuzzy set in  $Y$  with membership function defined by

$$\begin{aligned} \mu_{f^\#(E)}(y) &= \inf_{z \in f^{-1}(y)} \mu_E(z) && \text{if } f^{-1}(y) \neq \phi \\ &= 1, && \text{otherwise.} \end{aligned}$$

for all  $y \in Y$  where  $f^{-1}(y) = \{x \mid f(x) = y\}$ .

- Remark 1* (1) If  $f : X \rightarrow Y$  is surjective map then  $f^\#(E) \subset f(E)$  for every fuzzy subset  $E$  in  $X$ , since  $f^{-1}(y) \neq \phi$  for all  $y \in Y$ .  
 (2) If  $f : X \rightarrow Y$  is injective map then  $f(E) \subset f^\#(E)$  for every fuzzy subset  $E$  in  $X$ , since

$$\mu_{f^\#(E)}(y) = \inf_{x \in f^{-1}(y)} \mu_E(x) = \mu_E(x) = \sup_{x \in f^{-1}(y)} \mu_E(x),$$

where  $f^{-1}(y) = x$  for some  $y \in Y$ .

**Note :** From the definition of  $f^\#$  map, it can be seen easily that  $\#$  image of any fuzzy set is normalized fuzzy set for any map  $f : X \rightarrow Y$ , which is not surjective. But if  $f$  is surjective then  $\#$  image of a fuzzy set need not be normalized fuzzy set.

**Definition 8** Let  $f : X \rightarrow Y$  be a function. Define a fuzzy set  $E^\# = f^{-1}(f^\#(E))$  in  $X$  with membership function as

$$\mu_{E^\#}(x) = \mu_{f^{-1}(f^\#(E))}(x) = \mu_{f^\#(E)}(f(x))$$

The following Lemma gives some properties of  $f^\#$  map.

**Lemma 1** Let  $f : X \rightarrow Y$  be any function and  $E, F$  be fuzzy subsets of  $X$  and  $B$  be fuzzy subset of  $Y$ . Then

- (a) If  $E \subset F$  then  $(f^\#(E)) \subset (f^\#(F))$ .
- (b)  $f^{-1}(f^\#(E)) \subset E$  i.e.  $E^\# \subset E$  and equality holds if grade of membership  $\mu_E(z)$  is constant on fibers.
- (c)  $B \subset f^\#(f^{-1}(B))$  and equality holds if  $f$  is surjective.
- (d)  $f(E^\#) = f^\#(E) \cap f(X)$ .
- (e)  $f^\#(E \cap F) = f^\#(E) \cap f^\#(F)$
- (f)  $f^\#(\phi) = (f(X))^c$ .
- (g)  $f^\#(X) = Y$ .
- (h)  $X^\# = X$  and  $\phi^\# = \phi$ .
- (i)  $f(E^\#) = f^\#(E) \cap f(E)$ .
- (j)  $f^\#(f^{-1}(f^\#(E))) = f^\#(E^\#) = f^\#(E)$
- (k)  $f^{-1}(f^\#(f^{-1}(B))) = f^{-1}(B)$ .

*Proof* (a) Let  $E \subset F$  then  $\mu_E(z) \leq \mu_F(z)$  for all  $z \in X$ . Therefore, for any  $y \in Y$  such that  $f^{-1}(y) \neq \phi$

$$\begin{aligned} \mu_{f^\#(E)}(y) &= \inf_{z \in f^{-1}(y)} \mu_E(z) \\ &\leq \inf_{z \in f^{-1}(y)} \mu_F(z) \\ &= \mu_{f^\#(F)}(y) \end{aligned}$$

and if  $f^{-1}(y) = \phi$  then obviously  $\mu_{f^\#(E)}(y) = \mu_{f^\#(F)}(y)$  and so  $f^\#(E) \subset f^\#(F)$ .

(b) For all  $x \in X$ ,

$$\begin{aligned} \mu_{f^{-1}(f^\#(E))}(x) &= \mu_{f^\#(E)}(f(x)) \\ &= \inf_{z \in f^{-1}(f(x))} \mu_E(z) \\ &\leq \mu_E(x) \end{aligned}$$

Therefore,  $f^{-1}(f^\#(E)) = E^\# \subset E$ .

Further, if grade of membership  $\mu_E(z)$  is constant on fibers i.e. for all  $z$  in  $f^{-1}(f(x))$ ,  $\mu_E(z)$  remains same then

$$\inf_{z \in f^{-1}(f(x))} \mu_E(z) = \mu_E(x)$$

Therefore,  $\mu_{f^{-1}(f^\#(E))}(x) = \mu_E(x)$  for each  $x$  in  $X$  and so  $f^{-1}(f^\#(E)) = E$ . Hence, in particular, if  $f$  is injective then  $f^{-1}(f^\#(E)) = E$ .

(c) For each  $y \in Y$ , such that  $f^{-1}(y) \neq \phi$ , then

$$\begin{aligned}\mu_{f^\#(f^{-1}(B))}(y) &= \inf_{z \in f^{-1}(y)} \mu_{f^{-1}(B)}(z) \\ &= \inf_{z \in f^{-1}(y)} \mu_B f(z) \\ &= \inf_{z \in f^{-1}(y)} \mu_B(y) \\ &= \mu_B(y)\end{aligned}$$

and if  $f^{-1}(y) = \phi$  then

$$\mu_B(y) \leq 1 = \mu_{f^\#(f^{-1}(B))}(y)$$

Therefore,  $B \subset f^\#(f^{-1}(B))$ . In particular, it can be seen easily that if  $f$  **is surjective map then**  $B = f^\#(f^{-1}(B))$ .

(d) For each  $y \in Y$ , if  $f^{-1}(y) \neq \phi$ , then

$$\begin{aligned}\mu_{f(X)}(y) &= \sup_{z \in f^{-1}(y)} \mu_X(z) \\ &= \sup_{z \in f^{-1}(y)} 1 \\ &= 1\end{aligned}$$

and if  $f^{-1}(y) = \phi$ , then

$$\begin{aligned}\mu_{f^\#(E)}(y) &= 1 \\ \mu_{f(X)}(y) &= 0\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mu_{f^\#(E) \cap f(X)}(y) &= \min.\{\mu_{f^\#(E)}(y), \mu_{f(X)}(y)\} \\ &= \begin{cases} \mu_{f^\#(E)}(y) & f^{-1}(y) \neq \phi \\ 0 & f^{-1}(y) = \phi \end{cases}\end{aligned}$$

and

$$\begin{aligned}\mu_{f(E^\#)}(y) &= \begin{cases} \sup_{z \in f^{-1}(y)} \mu_{E^\#}(z) & f^{-1}(y) \neq \phi \\ 0 & f^{-1}(y) = \phi \end{cases} \\ &= \begin{cases} \sup_{z \in f^{-1}(y)} \mu_{f^\#(E)}(y) & f^{-1}(y) \neq \phi \\ 0 & f^{-1}(y) = \phi \end{cases} \\ &= \begin{cases} \mu_{f^\#(E)}(y) & f^{-1}(y) \neq \phi \\ 0 & f^{-1}(y) = \phi \end{cases}\end{aligned}$$

Hence  $f(E^\#) = f^\#(E) \cap f(X)$ .

(e) For each  $y \in Y$ , if  $f^{-1}(y) \neq \phi$ , then

$$\begin{aligned}
 \mu_{f^\#(E \cap F)}(y) &= \inf_{z \in f^{-1}(y)} \mu_{E \cap F}(z) \\
 &= \inf_{z \in f^{-1}(y)} \inf\{\mu_E(z), \mu_F(z)\} \\
 &= \inf\left\{ \inf_{z \in f^{-1}(y)} \mu_E(z), \inf_{z \in f^{-1}(y)} \mu_F(z) \right\} \\
 &= \inf\{\mu_{f^\#(E)}(y), \mu_{f^\#(F)}(y)\} \\
 &= \mu_{f^\#(E) \cap f^\#(F)}(y)
 \end{aligned}$$

and if  $f^{-1}(y) = \phi$  then obviously  $\mu_{f^\#(E \cap F)}(y) = \mu_{f^\#(E) \cap f^\#(F)}(y)$ . Hence  $f^\#(E \cap F) = f^\#(E) \cap f^\#(F)$ .

(f) For each  $y \in Y$ , if  $f^{-1}(y)$  is not empty, then

$$\begin{aligned}
 \mu_{f^\#(\phi)}(y) &= \inf_{z \in f^{-1}(y)} \mu_\phi(z) \\
 &= \inf_{z \in f^{-1}(y)} 0 \\
 &= 0
 \end{aligned}$$

and

$$\begin{aligned}
 \mu_{f(X)^c}(y) &= 1 - \mu_{f(X)}(y) \\
 &= 1 - \sup_{z \in f^{-1}(y)} \mu_X(z) \\
 &= 1 - 1 \\
 &= 0
 \end{aligned}$$

If  $f^{-1}(y) = \phi$  then  $\mu_{f^\#(\phi)}(y) = 1$   
and

$$\begin{aligned}
 \mu_{f(X)^c}(y) &= 1 - \mu_{f(X)}(y) \\
 &= 1 - 0 \\
 &= 1
 \end{aligned}$$

Hence  $f^\#(\phi) = f(X)^c$ .

(g) For each  $y \in Y$ , if  $f^{-1}(y)$  is not empty, then

$$\begin{aligned}
 \mu_{f^\#(X)}(y) &= \inf_{z \in f^{-1}(y)} \mu_X(z) \\
 &= 1 \\
 &= \mu_Y(y)
 \end{aligned}$$

Also if  $f^{-1}(y) = \phi$  then

$$\mu_{f^\#(X)}(y) = 1 = \mu_Y(y)$$

(h) For each  $x \in X$ ,

$$\begin{aligned}
 \mu_{X^\#}(x) &= \mu_{f^{-1}(f^\#(X))}(x) \\
 &= \mu_{f^\#(X)}(f(x)) \\
 &= \inf_{z \in f^{-1}(f(x))} \mu_X(z) \\
 &= \inf_{z \in f^{-1}(f(x))} 1 \\
 &= 1 \\
 &= \mu_X(x)
 \end{aligned}$$

Similarly,  $\phi^\# = \phi$ .

(i) For each  $y \in Y$ , if  $f^{-1}(y) \neq \phi$ , then

$$\begin{aligned}
 \mu_{f^\#(E) \cap f(E)}(y) &= \inf\{\mu_{f^\#(E)}(y), \mu_{f(E)}(y)\} \\
 &= \inf\left\{\inf_{z \in f^{-1}(y)} \mu_E(z), \sup_{z \in f^{-1}(y)} \mu_E(z)\right\} \\
 &= \inf_{z \in f^{-1}(y)} \mu_E(z)
 \end{aligned}$$

and if  $f^{-1}(y) = \phi$  then

$$\begin{aligned}
 \mu_{f^\#(E)}(y) &= 1 \\
 \mu_{f(E)}(y) &= 0
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 \mu_{f^\#(E) \cap f(E)}(y) &= \begin{cases} \mu_{f^\#(E)}(y) & f^{-1}(y) \neq \phi \\ 0 & f^{-1}(y) = \phi \end{cases} \\
 &= \mu_{f(E^\#)}(y)
 \end{aligned}$$

Hence  $f(E^\#) = f^\#(E) \cap f(E)$ .

(j) For each  $y \in Y$ , if  $f^{-1}(y) \neq \phi$ , then

$$\begin{aligned}
 \mu_{f^\#(f^{-1}(f^\#(E)))}(y) &= \mu_{f^\#(E^\#)}(y) \\
 &= \inf_{z \in f^{-1}(y)} \mu_{E^\#}(z) \\
 &= \inf_{z \in f^{-1}(y)} \mu_{f^\#(E)}(f(z)) \\
 &= \mu_{f^\#(E)}(y)
 \end{aligned}$$

and if  $f^{-1}(y) = \phi$  then

$$\mu_{f^\#(f^{-1}(f^\#(E)))}(y) = \mu_{f^\#(E^\#)}(y) = 1 = \mu_{f^\#(E)}(y)$$

Hence  $f^\#(f^{-1}(f^\#(E))) = f^\#(E^\#) = f^\#(E)$ .



(k) For each  $x \in X$ ,

$$\begin{aligned}
 \mu_{f^{-1}(f^\#(f^{-1}(B)))}(x) &= \mu_{f^\#(f^{-1}(B))}f(x) \\
 &= \inf_{z \in f^{-1}(f(x))} \mu_{f^{-1}(B)}(z) \\
 &= \inf_{z \in f^{-1}(f(x))} \mu_B f(z) \\
 &= \mu_B f(x) \\
 &= \mu_{f^{-1}(B)}(x)
 \end{aligned}$$

Hence  $f^{-1}(f^\#(f^{-1}(B))) = f^{-1}(B)$ .

As we have seen in Lemma 1(e) that intersection is preserved by  $f^\#$  map. But the following Example shows that union need not be preserved by  $f^\#$  map.

*Example 1* Let  $X = \{x_1, x_2, x_3, x_4\}$  and  $Y = \{y_1, y_2, y_3, y_4\}$  be spaces of points. Consider a function  $f : X \rightarrow Y$  defined by  $f(x_1) = y_1$ ,  $f(x_2) = y_2$ ,  $f(x_3) = y_2$  and  $f(x_4) = y_3$  and the fuzzy subsets  $E = \{(x_1, 0.1), (x_2, 0.4), (x_3, 0.5), (x_4, 0.3)\}$  and  $F = \{(x_1, 0.5), (x_2, 0.6), (x_3, 0.3), (x_4, 0.8)\}$  of  $X$ . Then  $E \cup F = \{(x_1, 0.5), (x_2, 0.6), (x_3, 0.5), (x_4, 0.8)\}$ . Now,

$$\begin{aligned}
 \mu_{f^\#(E \cup F)}(y_2) &= \inf_{x \in f^{-1}(y_2)} \mu_{E \cup F}(x) \\
 &= \inf\{\mu_{E \cup F}(x_2), \mu_{E \cup F}(x_3)\} \\
 &= \inf\{0.6, 0.5\} \\
 &= 0.5
 \end{aligned}$$

and

$$\begin{aligned}
 \mu_{f^\#(E) \cup f^\#(F)}(y_2) &= \sup\{\mu_{f^\#(E)}(y_2), \mu_{f^\#(F)}(y_2)\} \\
 &= \sup\{0.4, 0.3\} \\
 &= 0.4
 \end{aligned}$$

Hence  $f^\#(E \cup F) \neq f^\#(E) \cup f^\#(F)$ .

The following Lemma gives some more properties of  $f^\#$  map which we shall need to characterize fuzzy continuity.

**Lemma 2** Let  $f : X \rightarrow Y$  be any function and  $E$  be any fuzzy subset of  $X$  and  $B$  be any fuzzy subset of  $Y$ . Then

- (a)  $f^{-1}(B) \subset E$  if and only if  $B \subset f^\#(E)$ .
- (b)  $(g \circ f)^\# = g^\# \circ f^\#$ .
- (c)  $f^\#(E^c) = (f(E))^c$  and so  $f^\#(E) = (f(E^c))^c$  and  $f(E) = (f^\#(E^c))^c$ .

*Proof*(a) Firstly, let  $f^{-1}(B) \subset E$  then using Lemma 1(a), it follows that  $f^\#(f^{-1}(B)) \subset f^\#(E)$  and so  $B \subset f^\#(f^{-1}(B)) \subset f^\#(E)$  using Lemma 1(c). Conversely, let  $B \subset f^\#(E)$  then  $f^{-1}(B) \subset f^{-1}(f^\#(E))$ . Therefore, by Lemma 1(b), it follows that  $f^{-1}(B) \subset f^{-1}(f^\#(E)) \subset E$ .

- (b) Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be any two maps then  $g \circ f : X \rightarrow Z$ . Therefore, for each  $z \in Z$ , we have

$$\begin{aligned} \mu_{g \circ f \#(E)}(z) &= \mu_{g \#(f \#(E))}(z) \\ &= \begin{cases} \inf_{y \in g^{-1}(z)} \mu_{f \#(E)}(y) & g^{-1}(z) \neq \phi \\ 1 & g^{-1}(z) = \phi \end{cases} \\ &= \begin{cases} \inf_{y \in g^{-1}(z)} \left\{ \begin{cases} \inf_{x \in f^{-1}(y)} \mu_E(x) & f^{-1}(y) \neq \phi \\ 1 & f^{-1}(y) = \phi \end{cases} \right\}, & g^{-1}(z) \neq \phi \\ 1 & g^{-1}(z) = \phi \end{cases} \end{aligned}$$

Now, three cases arise :

(i) If  $g^{-1}(z) \neq \phi$  and  $f^{-1}(y) \neq \phi$  then  $(g \circ f)^{-1}(z) \neq \phi$ .

(ii) If  $g^{-1}(z) \neq \phi$  and  $f^{-1}(y) = \phi$  then  $(g \circ f)^{-1}(z) = \phi$ .

(iii) If  $g^{-1}(z) = \phi$  then  $(g \circ f)^{-1}(z) = \phi$ .

It follows that

$$\begin{aligned} \mu_{g \circ f \#(E)}(z) &= \begin{cases} \inf_{y \in g^{-1}(z)} \inf_{x \in f^{-1}(y)} \mu_E(x) & g^{-1}(z) \neq \phi \text{ and } f^{-1}(y) \neq \phi \\ 1 & g^{-1}(z) \neq \phi \text{ and } f^{-1}(y) = \phi \\ 1 & g^{-1}(z) = \phi \end{cases} \\ &= \begin{cases} \inf_{x \in (g \circ f)^{-1}(z)} \mu_E(x) & (g \circ f)^{-1}(z) \neq \phi \\ 1 & (g \circ f)^{-1}(z) = \phi \end{cases} \\ &= \mu_{(g \circ f) \#(E)}(z) \end{aligned}$$

Hence  $(g \circ f)^\# = g^\# \circ f^\#$ .

- (c) For each  $y \in Y$ , if  $f^{-1}(y) \neq \phi$ , then

$$\begin{aligned} \mu_{f \#(E^c)}(y) &= \inf_{z \in f^{-1}(y)} \mu_{E^c}(z) \\ &= \inf_{z \in f^{-1}(y)} \{1 - \mu_E(z)\} \\ &= 1 - \sup_{z \in f^{-1}(y)} \mu_E(z) \\ &= 1 - \mu_{f(E)}(y) \\ &= \mu_{(f(E))^c}(y) \end{aligned}$$

and if  $f^{-1}(y) = \phi$  then

$$\begin{aligned} \mu_{f \#(E^c)}(y) &= 1 \\ \mu_{(f(E))^c}(y) &= 1 - \mu_{f(E)}(y) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Hence  $f^\#(E^c) = (f(E))^c$  and so  $f(E) = (f^\#(E^c))^c$ . Replacing  $E$  by  $E^c$  we get,  $f^\#(E) = (f(E^c))^c$ .

**Definition 9** A fuzzy subset  $E$  of  $X$  is called fuzzy saturated subset of  $X$  if  $E = f^{-1}(B)$  for some fuzzy subset  $B$  of  $Y$  i.e.  $\mu_E(x) = \mu_{f^{-1}(B)}(x)$  for each  $x \in X$ .

*Remark 2* For any map  $f : X \rightarrow Y$ ,  $E^\#$  is fuzzy saturated for each fuzzy subset  $E$  of  $X$ , since,  $f^{-1}(f(E^\#)) = f^{-1}(f(f^{-1}(f^\#(E)))) = f^{-1}(f^\#(E)) = E^\#$ .

The following two Lemmas characterize fuzzy saturated sets in terms of image and  $\#$  image of a fuzzy set.

**Lemma 3** A subset  $E$  of  $X$  is fuzzy saturated if and only if  $E = f^{-1}(f(E))$ .

*Proof* Let  $E$  be fuzzy saturated subset of  $X$  then  $E = f^{-1}(B)$  for some fuzzy subset  $B$  of  $Y$ . Therefore,  $\mu_E(x) = \mu_{f^{-1}(B)}(x)$  for each  $x \in X$  and so  $\mu_E(x_1) = \mu_E(x_2)$  whenever  $f(x_1) = f(x_2)$ . Now, for each  $x \in X$

$$\begin{aligned} \mu_{f^{-1}(f(E))}(x) &= \mu_{f(E)}(f(x)) \\ &= \sup_{z \in f^{-1}(f(x))} \mu_E(z) \\ &= \mu_E(x), \end{aligned}$$

since for all  $z \in f^{-1}(f(x))$ ,  $\mu_E(z) = \mu_E(x)$ . Hence  $f^{-1}(f(E)) = E$ .

**Lemma 4** For any map  $f : X \rightarrow Y$  and any fuzzy subset  $E$  in  $X$ ,  $E$  is fuzzy saturated if and only if  $E = E^\#$ .

*Proof* Let  $E$  be any fuzzy saturated subset in  $X$  then

$$\begin{aligned} \mu_{E^\#}(x) &= \mu_{f^{-1}(f^\#(E))}(x) \\ &= \mu_{f^\#(E)}(f(x)) \\ &= \inf_{z \in f^{-1}(f(x))} \mu_E(z) \\ &= \mu_E(x) \end{aligned}$$

Hence  $E = E^\#$ .

Conversely, let  $E = E^\#$  then by Remark 2, it follows that  $E$  is fuzzy saturated subset of  $X$ .

The following result gives a new characterization of surjective maps using  $\#$  image of a fuzzy set.

**Theorem 4** Let  $f$  be a function from  $X$  to  $Y$  where  $X = \{x\}$  and  $Y = \{y\}$  be spaces of points then  $f$  is surjective if and only if  $f(E^\#) = f^\#(E)$ .

*Proof* For each  $y \in Y$ ,  $f^{-1}(y) \neq \phi$ , since  $f$  is surjective. Therefore,

$$\begin{aligned}\mu_{f(E^\#)}(y) &= \sup_{z \in f^{-1}(y)} \mu_{E^\#}(z) \\ &= \sup_{z \in f^{-1}(y)} \mu_{f^{-1}(f^\#(E))}(z) \\ &= \sup_{z \in f^{-1}(y)} \mu_{f^\#(E)}(f(z)) \\ &= \mu_{f^\#(E)}(y)\end{aligned}$$

Conversely, let  $f(E^\#) = f^\#(E)$  then for each  $y \in Y$

$$\mu_{f(E^\#)}(y) = \mu_{f^\#(E)}(y)$$

which is possible if and only if  $f^{-1}(y) \neq \phi$  for each  $y \in Y$  i.e. if and only if  $f$  is surjective, since by definition

$\mu_{f(E^\#)}(y) = 0$  and  $\mu_{f^\#(E)}(y) = 1$ , if  $f^{-1}(y) = \phi$ .

Equivalently, we can say that  $f$  is surjective if and only if  $f^\#(\phi) = \phi$ .

#### 4 Applications of $f^\#$ to fuzzy continuity

The following theorem gives the characterization of fuzzy continuous maps in terms of interior in fuzzy topology.

**Theorem 5** *Let  $f$  be a function from  $X$  to  $Y$  where  $X = \{x\}$  and  $Y = \{y\}$  be spaces of points then  $f$  is fuzzy continuous if and only if for each fuzzy subset  $E$  of  $X$ ,  $(f^\#(E))^o \subset f^\#(E^o)$ .*

*Proof* By Theorem 3, fuzzy continuity of  $f$  is equivalent to  $f(\overline{E}) \subset \overline{f(E)}$ , for each fuzzy subset  $E$  of  $X$  which is again equivalent to  $[f^\#((\overline{E})^c)]^c \subset \overline{(f^\#(E^c))^c}$  by part (c) of Lemma 2. But  $[(f^\#(E^c))^c]^c = [(f^\#(E^c))^o]^c$ , since  $(\overline{E^c}) = (E^o)^c$  and  $[f^\#((\overline{E})^c)]^c = [(f^\#((E^c)^o))]^c$  since  $(\overline{E})^c = (E^c)^o$  by Theorem 1. Therefore,  $f(\overline{E}) \subset \overline{f(E)}$  is equivalent to  $(f^\#(E^c))^o \subset f^\#((E^c)^o)$  which holds for arbitrary fuzzy subsets  $E$  of  $X$ . Hence  $f$  is fuzzy continuous if and only if for each fuzzy subset  $E$  of  $X$ ,  $(f^\#(E))^o \subset f^\#(E^o)$ .

The following corollary characterizes fuzzy continuity for surjective maps.

**Corollary 1** *Let  $f : X \rightarrow Y$  be any surjective map. Then  $f$  is fuzzy continuous if and only if for each fuzzy subset  $E$  of  $X$ ,  $(f(E^\#))^o \subset f((E^o)^\#)$ .*

*Proof* Proof follows from Theorems 5 and 4.

The following theorem gives equivalent conditions for a surjective fuzzy continuous map.

**Theorem 6** *Let  $f : X \rightarrow Y$  be any surjective map. Then the following are equivalent.*

- (a)  $f$  is fuzzy continuous.
- (b)  $E^\#$  is fuzzy open in  $X$  whenever  $f(E^\#)$  is fuzzy open in  $Y$ .
- (c)  $E^\#$  is fuzzy open in  $X$  whenever  $f^\#(E)$  is fuzzy open in  $Y$ .
- (d) for any fuzzy saturated set  $E$  in  $X$ ,  $E$  is fuzzy closed in  $X$  whenever  $f(E)$  is fuzzy closed in  $Y$ .
- (e) for any fuzzy saturated set  $E$  in  $X$ ,  $E$  is fuzzy open in  $X$  whenever  $f^\#(E)$  is fuzzy open in  $Y$ .

*Proof* (a)  $\Rightarrow$  (b) : Let  $f(E^\#)$  be fuzzy open in  $Y$  then  $f$  is fuzzy continuous implies that  $f^{-1}f(E^\#)$  is fuzzy open in  $X$  and so  $f^{-1}(f(f^{-1}(f^\#(E))))$  is fuzzy open in  $X$ . But  $f^{-1} \circ f \circ f^{-1} = f^{-1}$  Therefore,  $f^{-1}(f^\#(E))$  is fuzzy open in  $X$  and so  $E^\#$  is fuzzy open in  $X$ .

(b)  $\Rightarrow$  (c) : Let  $f^\#(E)$  be fuzzy open in  $Y$  then  $f$  is surjective implies  $f(E^\#)$  is fuzzy open in  $Y$  and so by (b),  $E^\#$  is fuzzy open in  $X$ .

(c)  $\Rightarrow$  (d) : Let  $f(E)$  be fuzzy closed in  $Y$ , where  $E$  is fuzzy saturated set in  $X$  then  $(f(E))^c$  is fuzzy open in  $Y$  and so by Lemma 2(c),  $f^\#(E^c)$  is fuzzy open in  $Y$ . Therefore by (c),  $(E^c)^\#$  is fuzzy open in  $X$  and so  $E^c$  is fuzzy open in  $X$ , since  $E$  is fuzzy saturated implies  $E^c$  is fuzzy saturated. Hence  $E$  is fuzzy closed in  $X$  and so (d) holds.

(d)  $\Rightarrow$  (e) : Let  $f^\#(E)$  be fuzzy open in  $Y$  for fuzzy saturated subset  $E$  of  $X$ . Then  $(f^\#(E))^c$  is fuzzy closed in  $Y$  and so  $f(E^c)$  is fuzzy closed in  $Y$  by Lemma 2(c). Therefore (d) implies that  $E^c$  is fuzzy closed in  $X$  and so  $E$  is fuzzy open in  $X$  and so (e) holds.

(e)  $\Rightarrow$  (a) : Let  $B$  be any fuzzy open subset of  $Y$  and let  $f^{-1}(B) = E$ . Then  $B = f^\#(f^{-1}(B)) = f^\#(E)$  by Lemma 1(c). Therefore  $f^\#(E)$  is fuzzy open in  $Y$  and so by (e),  $E$  is fuzzy open in  $X$ , since  $E$  is fuzzy saturated subset of  $X$ . Hence  $f^{-1}(B)$  is fuzzy open in  $X$  and so  $f$  is fuzzy continuous.

The following theorem gives the characterization of fuzzy continuity using fuzzy saturated sets.

**Theorem 7** *Let  $f : X \rightarrow Y$  be any surjective map. Then  $f$  is fuzzy continuous if and only if for any fuzzy saturated subset  $E$  of  $X$ ,  $E$  is fuzzy open in  $X$  whenever  $f(E)$  is fuzzy open in  $Y$ .*

*Proof* Let  $f$  be fuzzy continuous and  $f(E)$  be fuzzy open in  $Y$  for fuzzy saturated subset  $E$  of  $X$ . Then  $f^{-1}(f(E))$  is fuzzy open in  $X$  and so  $E$  is fuzzy open in  $X$ .

Conversely, let  $B$  be fuzzy open subset of  $Y$  and  $f^{-1}(B) = E$ . Then  $f^{-1}(B)$  is fuzzy saturated subset of  $X$  and  $f(f^{-1}(B)) = B$  is fuzzy open in  $Y$ , since  $f$  is surjective. Therefore,  $f^{-1}(B)$  is fuzzy open in  $X$  and so  $f$  is fuzzy continuous.

## 5 Conclusion

Topological spaces not only have applications in mathematics but in various other fields like physics, geographical systems, computer, chemistry etc. But because of uncertainties and incomplete information of an element, it is difficult to apply the concept of topology in all real life problems. To overcome this difficulty, fuzzy sets and hence fuzzy topological spaces were introduced which deals with uncertainty of an object. In various real life applications of fuzzy set theory, fuzzy functions play an important role especially in fuzzy control and approximate reasoning. Motivated by the applications of fuzzy functions, we have introduced a new type of fuzzy function denoted by  $f^\#$  and have discussed their properties. Further, it motivates to contribute in the theoretical study on fuzzy continuous functions. We can also study fuzzy open and fuzzy closed functions using  $f^\#$  which will be our next target and to study fuzzy topological properties preserved by these maps.

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