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Tamer Elsayed (✉ tamer.el-sayed@abdn.ac.uk)

University of Aberdeen <https://orcid.org/0000-0002-1494-5334>

Hussein Sayed

Helwan University

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Bifurcation Analysis of Rotor/Bearing System using Third Order Journal Bearings Stiffness and Damping Coefficients

T.A. El-Sayed^{a,b,*}, Hussein Sayed^b

^a*Centre for Applied Dynamics Research, School of Engineering, University of Aberdeen, Aberdeen, AB24 3UE, UK*

^b*Department of Mechanical Design, Faculty of Engineering, Mataria, Helwan University, P.O. Box 11718, Helmeiat-Elzaton, Cairo, Egypt*

Abstract

Journal bearings have many applications in industry due to its high load carrying capacity. In addition proper design of journal bearings enables safe operation at very high speeds. However, they are susceptible to oil whirl instability which may cause bearing failure. The fluid film pressure distribution inside the journal bearing is described by Reynolds equation. Many studies had been done to approximate the bearing performance using first order bearing coefficients. Although this analysis is stable for evaluating the threshold speed but it is insensitive to limit cycles above the threshold speed. Much literature show that above the threshold speed, subcritical or supercritical bifurcations may be observed. Therefore, the aim of the present paper is to evaluate the third order bearing coefficients for a finite length journal bearing using finite perturbation method. The values of these coefficients are evaluated using infinitesimal perturbation analysis. These values are used to investigate the bifurcation stability of flexible Jeffcott rotor supported by two symmetric journal bearings. The effect of rotor stiffness ratio on the bifurcation stability of the system is investigated. The results of this work show that the third order parameters can be used to evaluate the type of bifurcation above the threshold speed.

Keywords: Journal bearing, bearing coefficient, infinitesimal perturbation, rotordynamics, Jeffcott rotor, numerical solution.

*Corresponding author

Email addresses: `tamer.elsayed@abdn.ac.uk` (T.A. El-Sayed),
`husseinsayed2008@m-eng.helwan.edu.eg` (Hussein Sayed)

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Nomenclature

c	Journal bearing radial clearance
c_{ij}	First order damping coefficient (N.s/m), $i, j = x, y$
c_{ijk}	Second order damping coefficient (N.s/m ²), $i, j, k, l = x, y$
c_{ijkl}	Third order damping coefficient (N.s/m ³), $i, j, k, l = x, y$
C_{ij}	Dimensionless linear damping coefficients $C_{ij} = \frac{c_{ij} c \Omega}{W}$
C_{ijk}	Dimensionless second order damping coefficients $C_{ijk} = \frac{c_{ijk} c^2 \Omega}{W}$
C_{ijkl}	Dimensionless third order damping coefficients $C_{ijkl} = \frac{c_{ijkl} c^3 \Omega}{W}$
e	Radial eccentricity
F_x, F_y	Bearing forces components in Cartesian coordinates N
$\bar{F}_X, \bar{F}_Y$	Dimensionless bearing forces components $\bar{F}_X = \frac{F_x}{W}, \bar{F}_Y = \frac{F_y}{W}$
h	Thickness of the fluid film (m)
H	Fluid film dimensionless thickness $H = \frac{h}{c}$
k_{ij}	1 st order stiffness coefficients(N/m), $i, j = x, y$
k_{ijk}	2 nd order Stiffness coefficients(N/m ²), $i, j, k = x, y$
k_{ijkl}	3 rd order stiffness coefficients(N/m ³), $i, j, k, l = x, y$
K_{ij}	1 st order dimensionless stiffness coefficients $K_{ij} = \frac{k_{ij} c}{W}$
K_{ijk}	2 nd order dimensionless stiffness coefficients $K_{ijk} = \frac{k_{ijk} c^2}{W}$
K_{ijkl}	3 rd order dimensionless stiffness coefficients $K_{ijkl} = \frac{k_{ijkl} c^3}{W}$
m	Total mass of the disc and shaft
m_d	Sum of disc and shaft around journal mass
m_j	Mass of shaft around the journal
$\bar{M}$	dimensionless mass $\bar{M} = \frac{m c \Omega^2}{W}$
$\bar{M}_{th}$	Threshold mass $\bar{M}_{th} = \frac{m_{th} \Omega_{th}^2}{W}$
p	Fluid film pressure (N/m ²)
P	Dimensionless pressure $P = \frac{p}{6\mu\Omega} \left(\frac{c}{R}\right)^2$
P_0	Dimensionless steady state pressure
P_ζ	1 st order pressure gradients $\zeta = x, y$
$P_{\zeta\eta}$	2 nd order pressure gradients $\zeta, \eta = x, y$
$P_{\zeta\eta\gamma}$	3 rd order pressure gradients $\zeta, \eta, \gamma = x, y$
R	Bearing Radius (m)
S	Sommerfeld number
T	periodic time
X, Y	Dimensionless displacements
X', Y', X'', Y''	Cartesian dimensionless velocities and accelerations.
z	Axial coordinate
ε	Eccentricity ratio (e/c)
θ	attitude angle
θ_1	small perturbation of attitude angle
μ	lubricant viscosity (N.s/m ²)
ρ	Lubricant density (kg/m ³)
τ	dimensionless time ($\tau = \Omega t$)
ϕ	the angular coordinate
Ω	the rotational speed of the journal rad/s

1. Introduction

Journal bearing is one of the crucial elements used in the engineering industry. It has many applications in heavy-duty ,moderate and high speed machinery such as reciprocating engines (Santos et al., Chun and Khonsari 2016), turbomachines (Salamone 1984, Ciulli et al. 2018, On et al. 2018), centrifugal pumps(Bai et al. 2019, Nishimura et al. 2018) and turbocharger (El-Sayed and Fatah 2016, Smolík et al. 2017, Novotný et al. 2018) ,etc. Many of these machines are required to spin at very high speeds to improve their efficiency. Therefore, it is important to investigate and evaluate the threshold whirling speed (Ehrich 1991, Vance et al. 2010). After this threshold oil whirl may be occurred. However, some applications are reported to operate at speeds higher than this threshold value in stable condition(Lund 1974). In addition, the experimental work of Muszynska (1988), Deepak and Noah (1998) showed that a stable whirling is occurred after threshold speed. Therefore, it is important to investigate the nonlinear behavior of journal bearings to improve the future design of rotating machinery.

The dynamic modelling of journal bearings requires the evaluation of the bearing forces. Since the fluid film pressure distribution inside the journal bearing can be described by the Reynolds equation, the bearing forces can be obtained from the integration of the pressure distribution inside the bearing. In case of short and long bearings, Reynolds equation can be simplified so that the bearing forces can be obtained analytically, see for example (Castro et al., Huang et al. 2017, Nishimura et al. 2018). Nishimura et al. (2018) investigated the nonlinear dynamics and stability of a vertical flexible rotor supported by short journal bearing. The bearing forces are evaluated and approximated based on the short bearing formula presented by Mori (1968). Castro et al. investigated the linear dynamics of rotors supported by short journal bearings. They investigated the instability ‘oil whirl’ and ‘oil whip’ during run-up and run-down. In case of finite bearing length where the analytical solution for Reynolds equation is not available, the bearing forces can be obtained by direct

integration of Reynolds equation. In dynamic analysis, this integration is required to be done each time step which is computationally expensive especially if very fine mesh is considered (Akers et al., Schweizer, Shin and Palazzolo 2020).

Alternatively, and more commonly is to represent the bearing using stiffness and damping coefficients. These coefficients are used to evaluate the bearing forces and to investigate the system dynamics. These coefficients are simply based on the Taylor expansion of the bearing forces. Much research considers only the linear first order approximations while others consider higher order approximation (Lund 1965, Lund and Thomsen 1978, Qiu and Tieu 1996, Miura et al. 2017). There are different methods to obtain the bearing coefficients, such as, experimental (Tiwari et al. 2004, Mao et al. 2016) analytical (Chasalevris and Sfyris, Wang and Khonsari 2006, Miraskari et al. 2018a), finite difference method (FDM) (Gero and McC. Ettles 1986, Qiu and Tieu 1995, Goodwin et al. 1997, Ebrat et al.), finite element method (FEM) (Wada et al. 1971, Mourelatos 1985, Arregui and Vázquez 2001, Nair and Nair 2004, Awasthi et al. 2006, Hili et al. 2010), mesh less method with radial basis (MMRB) (Nicoletti 2013) and computational fluid dynamics (CFD) (Gao et al. 2014, Zhang et al. 2015, Gao et al. 2016, Zhang et al. 2016).

Investigating the Hopf bifurcation stability is important because it represents the transformation from spin motion to periodic whirl motion. Although the bearing force linear first order approximation coefficients are sufficient to obtain the threshold speed, but they are not sufficient to judge the stability of periodic solution and whether it is subcritical or super critical. This motivates the research is to either use analytical solution for bearing forces where possible or using higher order bearing coefficients.

Dakel et al. (2014) investigated the nonlinear dynamics and stability of a rotor mounted hydrodynamic journal bearing. They investigated the stability based on both linear bearing coefficients and on bearing force from the analytical solution. Smolík et al. (2019) investigated the stability of rotor bearing system based on the threshold stability curves. They

used different approaches to calculate the bearing forces, infinitely short, infinitely short approximation with correction factor for finite length bearing, finite difference, and finite element. They also investigated the run-up time and acceleration.

Yu et al. (2005) introduced a time perturbation method to evaluate the bearing first order coefficient for aerodynamic compliant foil bearings. Weimin et al. (2012) extended a time perturbation method to evaluate the second order nonlinear coefficient for journal bearings. Miraskari et al. (2018b) investigated the nonlinear dynamics of flexible rotor supported by journal bearings. the second order nonlinear bearing coefficients are obtained using perturbation method. Then they used a shooting method to find the stability of the periodic solution. Chasalevris (2020) investigated the Hopf bifurcation of flexible rotor supported by journal bearings. He investigated the lemon bore and partial arc types of bearings. Chasalevris used Hopf bifurcation theory introduced by Hassard et al. (1981) to investigate the stability of periodic solution.

The previous literature shows that many studies for the nonlinear dynamics of journal bearing are based on the analytical solution which is only available in case of short or long bearings. Little studies are done based on the second order linear coefficient. Moreover, there is very little studies have done on higher order bearing coefficients. The main purpose of this study is to evaluate the third order bearing coefficient using finite perturbation method. The obtained coefficients are used to investigate the stability of a flexible rotor/disc supported on two symmetric hydrodynamic journal bearings.

After this introduction, the steps of how to obtain the journal bearing third order parameters are presented on the mathematical model section. Then the dynamic model of rotor supported by journal bearings is presented. The values of the obtained journal bearing coefficients are presented on the results and discussion section. The stability of the dynamical model is discussed at the dynamical results subsection. Finally, the sound results and conclusions are presented in conclusion section.

2. Analytical Model

The present model is for full circular journal bearing o_b is the center of the bearing, o_{js} is the center of the journal or shaft at the equilibrium position and o_j is the perturbed journal center. R is the radius of the journal bearing and R_j is the radius of the journal. The radial clearance is $c = R - R_j$. The height of the oil film between the journal and the bearing at any angle is h . e_0, θ_0 are the eccentricity and the attitude angle at the steady state equilibrium. O_{js} is the center of the journal or shaft at the equilibrium position in terms of dimensionless coordinates and O_j is the perturbed journal center in terms of dimensionless coordinates, see Figure 1-c.

Figure 1: Full circular journal bearing (a) schematic of the journal bearing coordinates (b) journal bearing mesh (c) perturbed journal bearing in dimensionless coordinates X and Y

The pressure distribution inside the fluid film can be described by Reynolds equation as follows:

$$\frac{1}{R^2} \frac{\partial}{\partial \phi} \left(\frac{\rho h^3}{12\mu} \frac{\partial p}{\partial \phi} \right) + \frac{\partial}{\partial z} \left(\frac{\rho h^3}{12\mu} \frac{\partial p}{\partial z} \right) = \frac{\Omega}{2} \frac{\partial (\rho h)}{\partial \phi} + \rho \frac{\partial h}{\partial t} \quad (1)$$

where h and p are the oil film thickness and pressure respectively. ϕ is the angular coordinate with respect to vertical axis, t is the time in sec., Ω is the rotational speed of the shaft in rad/sec.. x and y are the sectional bearing coordinates as shown in Figure 1-a. z is the axial coordinate as shown in Figure 1-b. The fluid film viscosity and density are μ in N.s/m² and ρ in kg/m³ respectively.

Reformulating the Reynolds equation Eq. (1) for the dimensionless representation, it can be written as:

$$\frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P}{\partial Z} \right) = \frac{\partial H}{\partial \phi} + 2 \frac{\partial H}{\partial \tau} \quad (2)$$

H is the oil film dimensionless thickness, $P = \frac{p}{6\mu\Omega} \left(\frac{c}{R} \right)^2$ is the dimensionless pressure, $Z = \frac{z}{R}$ is dimensionless location in axial direction. The dimensionless time is $\tau = \Omega t$. To evaluate the bearing coefficients a small perturbation is introduced from the steady state point as shown in Figure 1-c. The height in dimensionless can be presented as in Eq.(3). After perturbation the eccentricity ratio, attitude angle and dimensionless height can be obtained from Eqs.(4), (5) and (6) respectively. Since the analysis in the present article is based on the third order approximation, All the differences with order higher than three is omitted.

$$H = 1 + \varepsilon \cos(\phi - \theta) = 1 + \varepsilon \cos(\varphi) \quad (3)$$

For which ε is the eccentricity ratio and θ is the attitude angle, for a small perturbation

$$\varepsilon = \varepsilon_0 + \delta\varepsilon \quad (4)$$

$$\theta = \theta_0 + \delta\theta \quad (5)$$

$$H = 1 + (\varepsilon_0 + \delta\varepsilon) \cos(\varphi_0 - \delta\theta) = 1 + (\varepsilon_0 + \delta\varepsilon) [\cos(\varphi_0) \cos(\delta\theta) + \sin(\varphi_0) \sin(\delta\theta)] \quad (6)$$

For small $\delta\theta$, substitute $\cos(\delta\theta) = 1 - \frac{\delta\theta^2}{2}$ and $\sin(\delta\theta) = \delta\theta - \frac{\delta\theta^3}{6}$, then equation (6) can be written as

$$H = 1 + (\varepsilon_0 + \delta\varepsilon) \left[\cos(\varphi_0) \left(1 - \frac{\delta\theta^2}{2} \right) + \sin(\varphi_0) \left(\delta\theta - \frac{\delta\theta^3}{6} \right) \right] \quad (7)$$

Equation (7) can be expanded as follows

$$\begin{aligned} H &= 1 + \varepsilon_0 \cos(\varphi_0) + \varepsilon_0 \delta\theta \sin(\varphi_0) + \delta\varepsilon \cos(\varphi_0) + \delta\varepsilon \delta\theta \sin(\varphi_0) - \\ &\quad \frac{1}{2} \varepsilon_0 \delta\theta^2 \cos(\varphi_0) - \frac{\delta\theta^2}{2} \delta\varepsilon \cos(\varphi_0) - \varepsilon_0 \frac{\delta\theta^3}{6} \sin(\varphi_0) - \frac{\delta\varepsilon \delta\theta^3}{6} \sin(\varphi_0) \\ &= H_0 + H_1 + H_2 + H_3 + O(\delta\theta^4) \end{aligned} \quad (8)$$

for which

$$H_0 = 1 + \varepsilon_0 \cos(\varphi_0) \quad (9)$$

$$H_1 = \varepsilon_0 \delta\theta \sin(\varphi_0) + \delta\varepsilon \cos(\varphi_0) \quad (10)$$

$$H_2 = \delta\varepsilon \delta\theta \sin(\varphi_0) - \frac{1}{2} \varepsilon_0 \delta\theta^2 \cos(\varphi_0) \quad (11)$$

$$H_3 = -\frac{\delta\theta^2}{2} \delta\varepsilon \cos(\varphi_0) - \varepsilon_0 \frac{\delta\theta^3}{6} \sin(\varphi_0) \quad (12)$$

Now it is important to transform the height equation from polar coordinates to Cartesian coordinates. The transformation relations from ε and θ coordinates to X and Y coordinates are as follows:

$$\delta\varepsilon = \delta X \sin(\theta_0) + \delta Y \cos(\theta_0) \quad (13)$$

$$\varepsilon_0 \delta\theta = \delta X \cos(\theta_0) - \delta Y \sin(\theta_0) \quad (14)$$

Substituting by the transformation relations in Eqs. (13) and (14) into Eqs. (9) to (12), results in

$$H_0 = 1 + \varepsilon_0 \cos(\varphi_0) \quad (15)$$

$$H_1 = \delta X \sin \phi + \delta Y \cos \phi \quad (16)$$

$$\begin{aligned} H_2 = & \frac{\delta X^2}{2 \varepsilon_0} (2 \sin \theta_0 \cos \theta_0 \sin \varphi_0 - \cos^2 \theta_0 \cos \varphi_0) \\ & + \frac{\delta X \delta Y}{\varepsilon_0} (\sin \theta_0 \cos \theta_0 \cos \varphi_0 + \cos^2 \theta_0 \sin \varphi_0 - \sin^2 \theta_0 \sin \varphi_0) \\ & + \frac{\delta Y^2}{2 \varepsilon_0} (-2 \sin \theta_0 \cos \theta_0 \sin \varphi_0 - \sin^2 \theta_0 \cos \varphi_0) \end{aligned} \quad (17)$$

The dimensionless pressure distribution $P(X, Y, X', Y')$ in the fluid film can be expanded using Taylor expansion as follows:

$$\begin{aligned} P = & P_0 + P_X \delta X + P_Y \delta Y + P_{X'} \delta X' + P_{Y'} \delta Y' + 0.5 P_{XX} \delta X^2 + 0.5 P_{YY} \delta Y^2 \\ & + P_{XY} \delta X \delta Y + P_{XX'} \delta X \delta X' + P_{XY'} \delta X \delta Y' + P_{YX'} \delta Y \delta X' + P_{YY'} \delta Y \delta Y' \\ & + \frac{1}{6} P_{XXX} \delta X^3 + \frac{1}{6} P_{YYY} \delta Y^3 + \frac{1}{2} P_{XXY} \delta X^2 \delta Y + \frac{1}{2} P_{XY Y} \delta X \delta Y^2 \\ & + \frac{1}{2} P_{XX X'} \delta X^2 \delta X' + \frac{1}{2} P_{XX Y'} \delta X^2 \delta Y' + \frac{1}{2} P_{Y Y X'} \delta Y^2 \delta X' \\ & + \frac{1}{2} P_{Y Y Y'} \delta Y^2 \delta Y' + P_{XY X'} \delta X \delta Y \delta X' + P_{XY Y'} \delta X \delta Y \delta Y' \end{aligned} \quad (18)$$

where,

$$X = \frac{x}{c}, Y = \frac{y}{c}, ' = \frac{d}{d\tau}, \tau = \Omega t, X' = \frac{x}{\Omega c}, Y' = \frac{y}{\Omega c}$$

$$P_X = \frac{\partial P}{\partial X}, P_{XX} = \frac{\partial^2 P}{\partial X^2}, P_{XXX} = \frac{\partial^3 P}{\partial X^3}, P_{XY Y'} = \frac{\partial^3 P}{\partial X \partial Y \partial Y'} \text{ and so on.}$$

To obtain the steady state pressure P_0 the Reynolds equation in Eq. (2) are solved using finite difference relations. Then to obtain the first order pressure gradient the Reynolds Eq. (2) is differentiated with respect to X, Y, X' and Y' and rearranged as shown in Eq.(19). Similarly, the second order pressure gradient can be obtained by differentiating

the Reynolds first derivative equations in Eq. (19) with respect to X , Y , X' and Y' as shown in Eq.(20). Likewise, the third order pressure gradient in Eq. (18) can be obtained by differentiating the second order derivative of Eq.(20) with respect to X , Y , X' and Y' as shown in Eqs.(21) to (30). Solving Eqs. (19) to (30) using finite difference the first, second and third order pressure gradient can be obtained. It is worth noting that during the solution of these equations the first, second and third dimensionless height are required which are listed in Appendix A.

$$\begin{aligned}
& \frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P^*}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P^*}{\partial Z} \right) = \\
& \frac{\partial H_0}{\partial \phi} & P^* = P_0 \\
& \frac{\partial H_X}{\partial \phi} - \frac{\partial}{\partial \phi} \left(3H_0^2 H_X \frac{\partial P_0}{\partial \phi} \right) - \frac{\partial}{\partial Z} \left(3H_0^2 H_X \frac{\partial P_0}{\partial Z} \right) & P^* = P_X \\
& \frac{\partial H_Y}{\partial \phi} - \frac{\partial}{\partial \phi} \left(3H_0^2 H_Y \frac{\partial P_0}{\partial \phi} \right) - \frac{\partial}{\partial Z} \left(3H_0^2 H_Y \frac{\partial P_0}{\partial Z} \right) & P^* = P_Y \\
& 2H_X & P^* = P_{X'} \\
& 2H_Y & P^* = P_{Y'}
\end{aligned} \tag{19}$$

$$\begin{aligned}
& \frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P^*}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P^*}{\partial Z} \right) = \\
& \quad \frac{\partial H_{XX}}{\partial \phi} - \frac{\partial}{\partial \phi} \left[6H_0^2 H_X \frac{\partial P_X}{\partial \phi} + (6H_0 H_X^2 + 3H_0^2 H_{XX}) \frac{\partial P_0}{\partial \phi} \right] & P^* = P_{XX} \\
& \quad - \frac{\partial}{\partial Z} \left[6H_0^2 H_X \frac{\partial P_X}{\partial Z} + (6H_0 H_X^2 + 3H_0^2 H_{XX}) \frac{\partial P_0}{\partial Z} \right] \\
& \quad \frac{\partial H_{YY}}{\partial \phi} - \frac{\partial}{\partial \phi} \left[6H_0^2 H_Y \frac{\partial P_Y}{\partial \phi} + (6H_0 H_Y^2 + 3H_0^2 H_{YY}) \frac{\partial P_0}{\partial \phi} \right] & P^* = P_{YY} \\
& \quad - \frac{\partial}{\partial Z} \left[6H_0^2 H_Y \frac{\partial P_Y}{\partial Z} + (6H_0 H_Y^2 + 3H_0^2 H_{YY}) \frac{\partial P_0}{\partial Z} \right] \\
& \quad \frac{\partial H_{XY}}{\partial \phi} - \frac{\partial}{\partial \phi} \left[3H_0^2 H_Y \frac{\partial P_X}{\partial \phi} + 3H_0^2 H_X \frac{\partial P_Y}{\partial \phi} + (6H_0 H_X H_Y + 3H_0^2 H_{XY}) \frac{\partial P_0}{\partial \phi} \right] & P^* = P_{XY} \\
& \quad - \frac{\partial}{\partial Z} \left[3H_0^2 H_Y \frac{\partial P_X}{\partial Z} + 3H_0^2 H_X \frac{\partial P_Y}{\partial Z} + (6H_0 H_X H_Y + 3H_0^2 H_{XY}) \frac{\partial P_0}{\partial Z} \right] \\
& \quad 2H_{XX} - \frac{\partial}{\partial \phi} \left(3H_0^2 H_X \frac{\partial P_{X'}}{\partial \phi} \right) - \frac{\partial}{\partial Z} \left(3H_0^2 H_X \frac{\partial P_{X'}}{\partial Z} \right) & P^* = P_{XX'} \\
& \quad 2H_{XY} - \frac{\partial}{\partial \phi} \left(3H_0^2 H_X \frac{\partial P_{Y'}}{\partial \phi} \right) - \frac{\partial}{\partial Z} \left(3H_0^2 H_X \frac{\partial P_{Y'}}{\partial Z} \right) & P^* = P_{XY'} \\
& \quad 2H_{XY} - \frac{\partial}{\partial \phi} \left(3H_0^2 H_Y \frac{\partial P_{X'}}{\partial \phi} \right) - \frac{\partial}{\partial Z} \left(3H_0^2 H_Y \frac{\partial P_{X'}}{\partial Z} \right) & P^* = P_{YX'} \\
& \quad 2H_{YY} - \frac{\partial}{\partial \phi} \left(3H_0^2 H_Y \frac{\partial P_{Y'}}{\partial \phi} \right) - \frac{\partial}{\partial Z} \left(3H_0^2 H_Y \frac{\partial P_{Y'}}{\partial Z} \right) & P^* = P_{YY'} \\
\end{aligned} \tag{20}$$

$$\begin{aligned}
& \frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P_{XXX}}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P_{XXX}}{\partial Z} \right) = \\
& \quad \frac{\partial H_{XXX}}{\partial \phi} - \frac{\partial}{\partial \phi} \left[9H_0^2 H_X \frac{\partial P_{XX}}{\partial \phi} + (18H_0 H_X^2 + 9H_0^2 H_{XX}) \frac{\partial P_X}{\partial \phi} \right. \\
& \quad \left. + (6H_X^3 + 18H_0 H_X H_{XX} + 3H_0^2 H_{XXX}) \frac{\partial P_0}{\partial \phi} \right] \\
& \quad - \frac{\partial}{\partial Z} \left[9H_0^2 H_X \frac{\partial P_{XX}}{\partial Z} + (18H_0 H_X^2 + 9H_0^2 H_{XX}) \frac{\partial P_X}{\partial Z} + (6H_X^3 + 18H_0 H_X H_{XX} + 3H_0^2 H_{XXX}) \frac{\partial P_0}{\partial Z} \right] \\
\end{aligned} \tag{21}$$

$$\begin{aligned}
& \frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P_{XXY}}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P_{XXY}}{\partial Z} \right) = \\
& \frac{\partial H_{XXY}}{\partial \phi} - \frac{\partial}{\partial \phi} \left[3H_0^2 H_Y \frac{\partial P_{XX}}{\partial \phi} + 6H_0^2 H_X \frac{\partial P_{XY}}{\partial \phi} + (12H_0 H_X H_Y + 6H_0^2 H_{XY}) \frac{\partial P_X}{\partial \phi} \right. \\
& \left. + (6H_0 H_X^2 + 3H_0^2 H_{XX}) \frac{\partial P_Y}{\partial \phi} + (6H_Y H_X^2 + 12H_0 H_X H_{XY} + 6H_0 H_Y H_{XX} + 3H_0^2 H_{XXY}) \frac{\partial P_0}{\partial \phi} \right] \\
& - \frac{\partial}{\partial Z} \left[3H_0^2 H_Y \frac{\partial P_{XX}}{\partial Z} + 6H_0^2 H_X \frac{\partial P_{XY}}{\partial Z} + (12H_0 H_X H_Y + 6H_0^2 H_{XY}) \frac{\partial P_X}{\partial Z} \right. \\
& \left. + (6H_0 H_X^2 + 3H_0^2 H_{XX}) \frac{\partial P_Y}{\partial Z} + (6H_Y H_X^2 + 12H_0 H_X H_{XY} + 6H_0 H_Y H_{XX} + 3H_0^2 H_{XXY}) \frac{\partial P_0}{\partial Z} \right] \\
& \quad (22)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P_{XYX}}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P_{XYX}}{\partial Z} \right) = \\
& \frac{\partial H_{XYX}}{\partial \phi} - \frac{\partial}{\partial \phi} \left[3H_0^2 H_X \frac{\partial P_{YY}}{\partial \phi} + 6H_0^2 H_Y \frac{\partial P_{XY}}{\partial \phi} + (12H_0 H_X H_Y + 6H_0^2 H_{XY}) \frac{\partial P_Y}{\partial \phi} \right. \\
& \left. + (6H_0 H_Y^2 + 3H_0^2 H_{YY}) \frac{\partial P_X}{\partial \phi} + (6H_X H_Y^2 + 12H_0 H_Y H_{XY} + 6H_0 H_X H_{YY} + 3H_0^2 H_{XYX}) \frac{\partial P_0}{\partial \phi} \right] \\
& - \frac{\partial}{\partial Z} \left[3H_0^2 H_X \frac{\partial P_{YY}}{\partial Z} + 6H_0^2 H_Y \frac{\partial P_{XY}}{\partial Z} + (12H_0 H_X H_Y + 6H_0^2 H_{XY}) \frac{\partial P_Y}{\partial Z} \right. \\
& \left. + (6H_0 H_Y^2 + 3H_0^2 H_{YY}) \frac{\partial P_X}{\partial Z} + (6H_X H_Y^2 + 12H_0 H_Y H_{XY} + 6H_0 H_X H_{YY} + 3H_0^2 H_{XYX}) \frac{\partial P_0}{\partial Z} \right] \\
& \quad (23)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P_{YY}}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P_{YY}}{\partial Z} \right) = \\
& \frac{\partial H_{YY}}{\partial \phi} - \frac{\partial}{\partial \phi} \left[9H_0^2 H_Y \frac{\partial P_{YY}}{\partial \phi} + (18H_0 H_Y^2 + 9H_0^2 H_{YY}) \frac{\partial P_Y}{\partial \phi} \right. \\
& \left. + (6H_Y^3 + 18H_0 H_Y H_{YY} + 3H_0^2 H_{YY}) \frac{\partial P_0}{\partial \phi} \right] \\
& - \frac{\partial}{\partial Z} \left[9H_0^2 H_Y \frac{\partial P_{YY}}{\partial Z} + (18H_0 H_Y^2 + 9H_0^2 H_{YY}) \frac{\partial P_Y}{\partial Z} + (6H_Y^3 + 18H_0 H_Y H_{YY} + 3H_0^2 H_{YY}) \frac{\partial P_0}{\partial Z} \right]
\end{aligned} \tag{24}$$

$$\begin{aligned}
& \frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P_{XX}}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P_{XX}}{\partial Z} \right) = \\
& 2H_{XXX} - \frac{\partial}{\partial \phi} \left[6H_0^2 H_X \frac{\partial P_{XX}}{\partial \phi} + (6H_0 H_X^2 + 3H_0^2 H_{XX}) \frac{\partial P_{X'}}{\partial \phi} \right] \\
& - \frac{\partial}{\partial Z} \left[6H_0^2 H_X \frac{\partial P_{XX}}{\partial Z} + (6H_0 H_X^2 + 3H_0^2 H_{XX}) \frac{\partial P_{X'}}{\partial Z} \right]
\end{aligned} \tag{25}$$

$$\begin{aligned}
& \frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P_{XY}}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P_{XY}}{\partial Z} \right) = \\
& 2H_{XYY} - \frac{\partial}{\partial \phi} \left[6H_0^2 H_X \frac{\partial P_{XY}}{\partial \phi} + (6H_0 H_X^2 + 3H_0^2 H_{XX}) \frac{\partial P_{Y'}}{\partial \phi} \right] \\
& - \frac{\partial}{\partial Z} \left[6H_0^2 H_X \frac{\partial P_{XY}}{\partial Z} + (6H_0 H_X^2 + 3H_0^2 H_{XX}) \frac{\partial P_{Y'}}{\partial Z} \right]
\end{aligned} \tag{26}$$

$$\begin{aligned}
& \frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P_{YY}}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P_{YY}}{\partial Z} \right) = \\
& 2H_{XYY} - \frac{\partial}{\partial \phi} \left[6H_0^2 H_Y \frac{\partial P_{YX'}}{\partial \phi} + (6H_0 H_Y^2 + 3H_0^2 H_{YY}) \frac{\partial P_{X'}}{\partial \phi} \right] \\
& - \frac{\partial}{\partial Z} \left[6H_0^2 H_Y \frac{\partial P_{YX'}}{\partial Z} + (6H_0 H_Y^2 + 3H_0^2 H_{YY}) \frac{\partial P_{X'}}{\partial Z} \right]
\end{aligned} \tag{27}$$

$$\begin{aligned}
& \frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P_{YY'Y'}}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P_{YY'Y'}}{\partial Z} \right) = \\
& 2H_{YY'Y} - \frac{\partial}{\partial \phi} \left[6H_0^2 H_Y \frac{\partial P_{YY'}}{\partial \phi} + (6H_0 H_Y^2 + 3H_0^2 H_{YY'}) \frac{\partial P_{Y'}}{\partial \phi} \right] \\
& - \frac{\partial}{\partial Z} \left[6H_0^2 H_Y \frac{\partial P_{YY'}}{\partial Z} + (6H_0 H_Y^2 + 3H_0^2 H_{YY'}) \frac{\partial P_{Y'}}{\partial Z} \right]
\end{aligned} \tag{28}$$

$$\begin{aligned}
& \frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P_{XX'X'}}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P_{XX'X'}}{\partial Z} \right) = \\
& 2H_{XX'Y} - \frac{\partial}{\partial \phi} \left[3H_0^2 H_Y \frac{\partial P_{XX'}}{\partial \phi} + 3H_0^2 H_X \frac{\partial P_{YX'}}{\partial \phi} + (6H_0 H_X H_Y + 3H_0^2 H_{XY}) \frac{\partial P_{X'}}{\partial \phi} \right] \\
& - \frac{\partial}{\partial Z} \left[3H_0^2 H_Y \frac{\partial P_{XX'}}{\partial Z} + 3H_0^2 H_X \frac{\partial P_{YX'}}{\partial Z} + (6H_0 H_X H_Y + 3H_0^2 H_{XY}) \frac{\partial P_{X'}}{\partial Z} \right]
\end{aligned} \tag{29}$$

$$\begin{aligned}
& \frac{\partial}{\partial \phi} \left(H^3 \frac{\partial P_{XY'Y'}}{\partial \phi} \right) + \frac{\partial}{\partial Z} \left(H^3 \frac{\partial P_{XY'Y'}}{\partial Z} \right) = \\
& 2H_{XY'Y} - \frac{\partial}{\partial \phi} \left[3H_0^2 H_Y \frac{\partial P_{XY'}}{\partial \phi} + 3H_0^2 H_X \frac{\partial P_{YY'}}{\partial \phi} + (6H_0 H_X H_Y + 3H_0^2 H_{XY}) \frac{\partial P_{Y'}}{\partial \phi} \right] \\
& - \frac{\partial}{\partial Z} \left[3H_0^2 H_Y \frac{\partial P_{XY'}}{\partial Z} + 3H_0^2 H_X \frac{\partial P_{YY'}}{\partial Z} + (6H_0 H_X H_Y + 3H_0^2 H_{XY}) \frac{\partial P_{Y'}}{\partial Z} \right]
\end{aligned} \tag{30}$$

2.1. Nonlinear bearing forces

Expanding the dimensionless bearing forces $\bar{F}_X$ and $\bar{F}_Y$ to the third order using Taylor series, for which $\bar{F}_X = \frac{F_x}{W}$ and $\bar{F}_Y = \frac{F_y}{W}$, where F_x is the bearing force in x direction, F_y is

the bearing force in y direction and W is the bearing load.

$$\begin{aligned}
\bar{F}_\zeta = & \bar{F}_{\zeta_0} + \frac{\partial \bar{F}_\zeta}{\partial X} \delta X + \frac{\partial \bar{F}_\zeta}{\partial Y} \delta Y + \frac{\partial \bar{F}_\zeta}{\partial X'} \delta X' + \frac{\partial \bar{F}_\zeta}{\partial Y'} \delta Y' + 0.5 \frac{\partial^2 \bar{F}_\zeta}{\partial X^2} \delta X^2 + \\
& 0.5 \frac{\partial^2 \bar{F}_\zeta}{\partial Y^2} \delta Y^2 + \frac{\partial^2 \bar{F}_\zeta}{\partial X \partial Y} \delta X \delta Y + \frac{\partial^2 \bar{F}_\zeta}{\partial X \partial X'} \delta X \delta X' + \frac{\partial^2 \bar{F}_\zeta}{\partial X \partial Y'} \delta X \delta Y' + \frac{\partial^2 \bar{F}_\zeta}{\partial Y \partial X'} \delta Y \delta X' + \\
& \frac{\partial^2 \bar{F}_\zeta}{\partial Y \partial Y'} \delta Y \delta Y' + \frac{1}{6} \frac{\partial^3 \bar{F}_\zeta}{\partial X^3} \delta X^3 + \frac{1}{6} \frac{\partial^3 \bar{F}_\zeta}{\partial Y^3} \delta Y^3 + 0.5 \frac{\partial^3 \bar{F}_\zeta}{\partial X^2 \partial Y} \delta X^2 \delta Y + \\
& 0.5 \frac{\partial^3 \bar{F}_\zeta}{\partial X \partial Y^2} \delta X \delta Y^2 + 0.5 \frac{\partial^3 \bar{F}_\zeta}{\partial X^2 \partial X'} \delta X^2 \delta X' + 0.5 \frac{\partial^3 \bar{F}_\zeta}{\partial X^2 \partial Y'} \delta X^2 \delta Y' + \\
& 0.5 \frac{\partial^3 \bar{F}_\zeta}{\partial Y^2 \partial X'} \delta Y^2 \delta X' + 0.5 \frac{\partial^3 \bar{F}_\zeta}{\partial Y^2 \partial Y'} \delta Y^2 \delta Y' + \frac{\partial^3 \bar{F}_\zeta}{\partial X \partial Y \partial X'} \delta X \delta Y \delta X' + \frac{\partial^3 \bar{F}_\zeta}{\partial X \partial Y \partial Y'} \delta X \delta Y \delta Y'
\end{aligned} \tag{31}$$

where $\zeta = X, Y$.

The bearing forces in X, Y directions in Eq. (31) can be written using the bearing first, second and third order coefficients as shown below:

$$\begin{aligned}
\bar{F}_\zeta = & \bar{F}_{\zeta_0} + K_{\zeta X} \delta X + K_{\zeta Y} \delta Y + C_{\zeta X} \delta X' + C_{\zeta Y} \delta Y' + 0.5 K_{\zeta XX} \delta X^2 + \\
& 0.5 K_{\zeta YY} \delta Y^2 + K_{\zeta XY} \delta X \delta Y + C_{\zeta XX} \delta X \delta X' + C_{\zeta XY} \delta X \delta Y' + C_{\zeta YX} \delta Y \delta X' + \\
& C_{\zeta YY} \delta Y \delta Y' + \frac{1}{6} K_{\zeta XXX} \delta X^3 + \frac{1}{6} K_{\zeta YYY} \delta Y^3 + 0.5 K_{\zeta XXY} \delta X^2 \delta Y + \\
& 0.5 K_{\zeta XYY} \delta X \delta Y^2 + 0.5 C_{\zeta XXX} \delta X^2 \delta X' + 0.5 C_{\zeta XXY} \delta X^2 \delta Y' + \\
& 0.5 C_{\zeta YYX} \delta Y^2 \delta X' + 0.5 C_{\zeta YYY} \delta Y^2 \delta Y' + C_{\zeta XYX} \delta X \delta Y \delta X' + C_{\zeta XYY} \delta X \delta Y \delta Y'
\end{aligned} \tag{32}$$

for which $X = \frac{x}{c}$ and $Y = \frac{y}{c}$ are the dimensionless coordinates, c is the radial clearance.

$K_{ij} = \frac{k_{ij} c}{W}$ and $C_{ij} = \frac{c_{ij} c \Omega}{W}$ are the dimensionless linear stiffness and damping coefficients respectively, where $(i, j = X, Y)$. k_{ij} and c_{ij} are stiffness and damping dimensional coefficients respectively.

$K_{ijk} = \frac{k_{ijk} c^2}{W}$ and $C_{ijk} = \frac{c_{ijk} c^2 \Omega}{W}$ are the second order dimensionless stiffness and damping coefficients respectively where $i, j, k = X, Y$. k_{ijk} and c_{ijk} are the second order stiffness

and damping dimensional coefficients respectively.

$K_{ijkl} = \frac{k_{ijkl} c^3}{W}$ and $C_{ijkl} = \frac{c_{ijkl} c^3 \Omega}{W}$ are the third order dimensionless stiffness and damping coefficients where $i, j, k, l = X, Y$. k_{ijkl} and c_{ijkl} are the third order stiffness and damping dimensional coefficients.

The bearing forces can be calculated by integrating the pressure over the area as shown below

$$\bar{F}_X = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P \sin \phi d\phi dZ \quad (33)$$

$$\bar{F}_Y = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P \cos \phi d\phi dZ \quad (34)$$

Comparing Eqs. (18),(32) to (34) the steady state force can be obtained as:

$$\bar{F}_{X_0} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_0 \sin \phi d\phi dZ \quad (35)$$

$$\bar{F}_{Y_0} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_0 \cos \phi d\phi dZ \quad (36)$$

It is worth noting that through the present analysis the only steady state load in y direction is the weight and no static load in x direction. Therefore, $F_{x0} = 0$ and $F_{y0} = W$ which also means that

$$\bar{F}_{X_0} = 0, \bar{F}_{Y_0} = 1 \quad (37)$$

Using this condition and by an iterative method the steady state attitude angle can be obtained. Also, the eight first order coefficients can be obtained from the following equations.

$$K_{X\zeta} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_{\zeta} \sin \phi d\phi dZ \quad (38)$$

$$K_{Y\zeta} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_{\zeta} \cos \phi d\phi dZ \quad (39)$$

$$C_{X\zeta} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_{\zeta'} \sin \phi d\phi dZ \quad (40)$$

$$C_{Y\zeta} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_{\zeta'} \cos \phi d\phi dZ \quad (41)$$

where $\zeta = X, Y$ which means that by replacing ζ by either X or Y the eight equation for the eight coefficients can be obtained from Eqs.(38) to (41) In addition, the fourteen second order coefficients can be calculated from

$$K_{X\zeta\eta} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_{\zeta\eta} \sin \phi d\phi dZ \quad (42)$$

$$K_{Y\zeta\eta} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_{\zeta\eta} \cos \phi d\phi dZ \quad (43)$$

$$C_{X\zeta\eta} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_{\zeta\eta'} \sin \phi d\phi dZ \quad (44)$$

$$C_{Y\zeta\eta} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_{\zeta\eta'} \cos \phi d\phi dZ \quad (45)$$

where each of ζ and η can be either X, Y . This means that Eqs.(42) to (45) represent sixteen equations. These equations are reduced to fourteen equations because $K_{XYX} = K_{XXY}$ and $K_{YXY} = K_{YYX}$. Similarly, the twenty Third order coefficients can be evaluated from the following equations

$$K_{X\zeta\eta\gamma} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_{\zeta\eta\gamma} \sin \phi d\phi dZ \quad (46)$$

$$K_{Y\zeta\eta\gamma} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_{\zeta\eta\gamma} \cos \phi d\phi dZ \quad (47)$$

$$C_{X\zeta\eta\gamma} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_{\zeta\eta\gamma'} \sin \phi d\phi dZ \quad (48)$$

$$C_{Y\zeta\eta\gamma} = \int_0^{\frac{L}{R}} \int_0^{2\pi} -P_{\zeta\eta\gamma'} \cos \phi d\phi dZ \quad (49)$$

where each of ζ , η and γ can be either X, Y . This means that thirty-two equation can be obtained from Eqs. (46) to(49) but these equations are reduced to only twenty because of similarity of several coefficients such as $K_{XYXX} = K_{XXYX} = K_{XXXY}$.

2.2. Rotor-bearing model

In this section, the analytical model for elastic rotor supported on two symmetric journal bearings is investigated. The model is four degree of freedom in x and y directions. In this model both the mass of rotor inside the journal m_j and disk mass m_d are considered, as shown in Figure 2. A static load $2W$ is applied on the disc.

Figure 2: (a) Elastic rotor supported on two symmetric journal bearings. (b) Bearing coordinates.

The equations of motion for the presented system can be presented as

$$\begin{aligned}
 2m_j\ddot{x}_j + k_s(x_j - x_d) &= -2F_x \\
 2m_j\ddot{y}_j + k_s(y_j - y_d) &= -2F_y \\
 m_d\ddot{x}_d + k_s(x_d - x_j) &= 0 \\
 m_d\ddot{y}_d + k_s(y_d - y_j) &= 2W
 \end{aligned} \tag{50}$$

For which m_j is the mass of shaft inside and around the journal and m_d is the sum of the disc and remaining part of the shaft mass, k_s is the shaft stiffness, F_x and F_y are the bearing forces in x and y directions respectively, W is the bearing load and x_j, y_j, x_d, y_d are journal mass geometrical center and disc geometrical center current positions with respect to the journal steady state position. $m = m_d + 2m_j$ is the total mass of the rotor.

Transforming the equations of motion Eq.(50) into dimensionless form we get the follow-

ing equation

$$\begin{aligned}
2M_J X_J'' + K_S (X_J - X_D) &= -2\bar{F}_x \\
2M_J Y_J'' + K_S (Y_J - Y_D) &= -2\bar{F}_y \\
M_D X_D'' + K_S (X_D - X_J) &= 0 \\
M_D Y_D'' + K_S (Y_D - Y_J) &= 2
\end{aligned} \tag{51}$$

where

$$\begin{aligned}
X_J &= \frac{x_j}{c}, Y_J = \frac{y_j}{c}, X_D = \frac{x_D}{c}, Y_D = \frac{y_D}{c}, ' = \frac{d}{d\tau}, '' = \frac{d^2}{d\tau^2}, \tau = \Omega t \\
X_J'' &= \frac{\ddot{x}_j}{\Omega^2 c}, Y_J = \frac{\ddot{y}_j}{\Omega^2 c}, X_D'' = \frac{\ddot{x}_d}{\Omega^2 c}, Y_D'' = \frac{\ddot{y}_d}{\Omega^2 c}, K_S = \frac{k_s c}{W}, M_J = \frac{m_j \Omega^2 c}{W} \text{ and } M_D = \frac{m_d \Omega^2 c}{W}.
\end{aligned}$$

M_J and M_D are the dimensionless mass of the journal and disc respectively. Through the current paper the following mass ratio are considered $2M_J = 0.1M$ and $M_D = 0.9M$. The dimensionless shaft stiffness is K_S . The rotational speed is Ω and the dimensionless time is τ . The dimensionless static equilibrium forces are $\bar{F}_{x_0} = 0$ and $\bar{F}_{y_0} = 1$. Substituting Eqs.(32) and(37) in Eq.(51), and converting the resulting equations into in the state space

form, the final resulting equations can be written as:

$$\begin{aligned}
x'_1 &= x_2 \\
x'_2 &= \frac{K_s(x_5-x_1)}{2M_J} - \frac{1}{M_J} \left(\begin{aligned} &K_{XX}x_1 + C_{XX}x_2 + K_{XY}x_3 + C_{XY}x_4 + 0.5 K_{XXX}x_1^2 \\ &+ 0.5 K_{XYY}x_3^2 + K_{XXY}x_1x_3 + C_{XXX}x_1x_2 + C_{XXY}x_1x_4 \\ &+ C_{XYX}x_3x_2 + C_{XYX}x_3x_4 + \frac{1}{6}K_{XXXX}x_1^3 + \frac{1}{6}K_{XYYY}x_3^3 \\ &+ 0.5K_{XXXY}x_1^2x_3 + 0.5K_{XXYY}x_1x_3^2 + 0.5C_{XXXX}x_1^2x_2 + \\ &0.5C_{XXXY}x_1^2x_4 + 0.5C_{XYYX}x_3^2x_2 + 0.5C_{XYYY}x_3^2x_4 + \\ &C_{XXYX}x_1x_3x_2 + C_{XXYY}x_1x_3x_4 \end{aligned} \right) \\
x'_3 &= x_4 \\
x'_4 &= \frac{K_s(x_7-x_3)}{2M_J} - \frac{1}{M_J} \left(\begin{aligned} &1 + K_{YX}x_1 + C_{YX}x_2 + K_{YY}x_3 + C_{YY}x_4 + 0.5 K_{YXX}x_1^2 \\ &+ 0.5 K_{YYY}x_3^2 + K_{YXY}x_1x_3 + C_{YXX}x_1x_2 + C_{YXY}x_1x_4 \\ &+ C_{YYX}x_3x_2 + C_{YYX}x_3x_4 + \frac{1}{6}K_{YXXX}x_1^3 + \frac{1}{6}K_{YYYY}x_3^3 \\ &+ 0.5K_{YXXY}x_1^2x_3 + 0.5K_{YXYX}x_1x_3^2 + 0.5C_{YXXX}x_1^2x_2 \\ &+ 0.5C_{YXXY}x_1^2x_4 + 0.5C_{YYXX}x_3^2x_2 + 0.5C_{YYYY}x_3^2x_4 + \\ &C_{YXYX}x_1x_3x_2 + C_{YXYX}x_1x_3x_4 \end{aligned} \right) \quad (52) \\
x'_5 &= x_6 \\
x'_6 &= \frac{K_s(x_1-x_5)}{M_D} \\
x'_7 &= x_8 \\
x'_8 &= \frac{2+K_s(x_3-x_7)}{M_D}
\end{aligned}$$

where $x_1 = X_J$, $x_2 = X'_J$, $x_3 = Y_J$, $x_4 = Y'_J$, $x_5 = X_D$, $x_6 = X'_D$, $x_7 = Y_D$, $x_8 = Y'_D$.

Finally, Eq.(52) can be written as

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, \bar{M}) \quad (53)$$

where $\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_5 \ x_7 \ x_8]$, $\bar{M} = M_D + 2M_J$ and $\frac{M_J}{M_D} = \frac{1}{18}$.

The system of equations in Eq.(53) can have a periodic solution $x(t) = x(t + T)$. This is occurred above what is called the threshold speed ($\bar{M} \geq \bar{M}_{th}$), where T is the periodic time of solution and $\bar{M}_{th}$ is the dimensionless threshold mass of the system. This will be discussed in detailed in the results section.

3. Results and discussion

In this section, the first, second and third order bearing coefficients are calculated using infinitesimal perturbation analysis. The bearing coefficients are presented and listed for full circular journal bearing. Perturbation analysis is used to investigate the validity range of these coefficients. Afterwards, the bearing coefficients are used to investigate the dynamics of flexible rotor supported by two symmetric journal bearings.

3.1. Journal bearing coefficients and mesh sensitivity

In this section, the bearing linear and nonlinear coefficients based on first, second and third order coefficients are evaluated. This is done using infinitesimal perturbation as discussed in analysis section. The presented results are for un-grooved full circular journal bearing with L/D ratio of 1. The linear bearing coefficients are eight and plotted in Figure 3- a. The nonlinear second order bearing coefficients are fourteen and are plotted in Figure 3 b . The third order coefficients are twenty and are plotted in Figure 4. The effect of mesh size on the bearing coefficients is also investigated for all the coefficients. Several mesh sizes are investigated but only four cases are presented to avoid over density of the figures. The presented mesh sizes are selected to vary from coarse to fine mesh. These mesh sizes are

50×100 , 100×200 , 150×600 and 200×800 . In both Figure 3 and Figure 4 the color is used to differentiate the bearing coefficients and the line type is used to differentiate the mesh size as shown in the subfigures legend and the figure general legend. The results of Figure 3 show that for the plotted coefficients the results of mesh 100×400 , 150×600 and 200×800 are identical over the full range of Sommerfeld number. The results of Figure 4 show that the bearing coefficients obtained using mesh 100×400 converge with that 200×800 in most coefficients. However, for K_{XXXX} , C_{XXXX} , K_{YXXX} and C_{YXXX} mesh 100×400 is not sufficient. In addition, the figure shows that mesh 150×600 and 200×800 results are converged for all coefficients. Therefore, mesh 150×600 was considered as minimum mesh used in coefficients evaluation during the present paper. Numerical values or the obtained parameters using mesh size of 150×600 are presented in Table B.1 and Table B.2.

Figure 3: (Color online) Loglog plot for un-grooved journal bearing stiffness and damping coefficients(absolute) versus the Sommerfeld number. $L/D = 1$, (a) First order bearing coefficient (b) Second order bearing coefficient. In this figure the dotted line for mesh 50×200 , dashed line for mesh 100×400 , centerline for mesh 150×600 and solid line for mesh 200×800 .

Figure 4: (Color online) Loglog plot for un-grooved full circular journal bearing third order stiffness and damping coefficients (absolute) versus the Sommerfeld number. L/D ratio is unity. In this figure the dotted line for mesh 50×200 , dashed line for mesh 100×400 , centerline for mesh 150×600 and solid line for mesh 200×800

3.2. Perturbation Analysis

In this section, a perturbation analysis is used to validate the obtained bearing coefficients. This analysis is executed using un-grooved full circular journal bearing with L/D ratio of 1. This is done by applying displacement and velocity perturbations to the equilibrium point then evaluate the obtained forces using four different methods. These methods are based on direct integration of Reynolds equation ($\bar{F}_{Re.}$), first order coefficients ($\bar{F}_1$), second order coefficients ($\bar{F}_2$) and third order coefficients ($\bar{F}_3$). The component of these forces in X and Y directions are evaluated and plotted in Figure 5. To investigate the

validity range of the bearing linear and nonlinear coefficients in evaluating the nonlinear bearing forces around the equilibrium point, four different values of perturbations are used. The first one is ($\delta X = 0.0001, \delta X' = 0.0001, \delta Y = 0.0001$ and $\delta Y' = 0.0001$) and its results are plotted in Figure 5 a. In Figure 5 the left column represents the X component of the perturbation force and the right column represent the Y component of the perturbation force. All the subfigures of Figure 5 are drawn versus the Sommerfeld number. Figure 5 b, c and d are for perturbations ($\delta X = 0.001, \delta X' = 0.001, \delta Y = 0.001$ and $\delta Y' = 0.001$), ($\delta X = 0.01, \delta X' = 0.01, \delta Y = 0.01$ and $\delta Y' = 0.01$) and ($\delta X = 0.1, \delta X' = 0.1, \delta Y = 0.1$ and $\delta Y' = 0.1$) respectively. The figure results show that at low perturbations the forces evaluated using all the four methods are approximately the same as shown in Figure 5 a, b. However, by increasing the values of perturbation as in Figure 5-c, a significant deviation between the forces evaluated based on the nonlinear coefficients and that based on the direct Reynolds integration forces ($\bar{F}_{Re.}$) is recognized. The last case Figure 5-d shows that at very low and high Sommerfeld number the deviation between the forces is larger than when $S \in [0.1 - 0.6]$. Also, the x component of force approximation based on the third order coefficients was closer to the forces based on direct integration of Reynolds equation at $S < 0.55$ and y component of force approximation was closer to the forces based on direct integration of Reynolds equation when $S < 0.3$.

Figure 5: Semi-log plot for the dimensionless force components $\bar{F}_X$ and $\bar{F}_Y$ resulted from journal perturbation in X and Y directions for ungrooved journal bearing with $L/D = 1$ (a) $(\delta X = 0.0001, \delta X' = 0.0001, \delta Y = 0.0001$ and $\delta Y' = 0.0001)$, (b) $(\delta X = 0.001, \delta X' = 0.001, \delta Y = 0.001$ and $\delta Y' = 0.001)$, (c) $(\delta X = 0.01, \delta X' = 0.01, \delta Y = 0.01$ and $\delta Y' = 0.01)$ and (d) $(\delta X = 0.1, \delta X' = 0.1, \delta Y = 0.1$ and $\delta Y' = 0.1)$.

3.3. Dynamic results

3.3.1. Stability Analysis

In this section, the calculated bearing coefficients are used to investigate the dynamic analysis of the elastic rotor system presented in Figure 2. Initially, the stability of the autonomous dynamical system in Eq.(52) is investigated. This is done by evaluating the dimensionless threshold mass $\bar{M}_{th} = mc\Omega_{th}^2/W$ which implicitly includes the threshold speed of the system. The dimensionless threshold mass $\bar{M}_{th}$ can be evaluated using the eigenvalues of the Jacobian matrix $\mathbf{J}(\mathbf{x}, \bar{M})$ in Eq. (54) at the fixed points. The fixed points can be obtained by solving the $\mathbf{f}(\mathbf{x}, \bar{M}) = 0$ in Eq.(53). This results in $\mathbf{x}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2/K_s & 0 \end{bmatrix}^T$.

$$\mathbf{J}(\mathbf{x}, \bar{M}) = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ J_{21} & J_{22} & J_{23} & J_{24} & \frac{K_s}{2M_J} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ J_{41} & J_{42} & J_{43} & J_{44} & 0 & 0 & \frac{K_s}{2M_J} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{K_s}{M_D} & 0 & 0 & 0 & -\frac{K_s}{M_D} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{K_s}{M_D} & 0 & 0 & 0 & -\frac{K_s}{M_D} & 0 \end{bmatrix} \quad (54)$$

where,

$$\begin{aligned}
J_{21} &= -\frac{K_s}{2M_J} \left(K_{XX} + K_{XXX}x_1 + K_{XXY}x_3 + C_{XXX}x_2 + C_{XXY}x_4 + \frac{1}{2}K_{XXXX}x_1^2 \right. \\
&\quad \left. + K_{XXXY}x_1x_3 + \frac{1}{2}K_{XXYY}x_3^2 + C_{XXXX}x_1x_2 + C_{XXXY}x_1x_4 + C_{XXYX}x_3x_2 + C_{XXYY}x_3x_4 \right) \\
J_{22} &= -\frac{1}{M_J} \left(C_{XX} + C_{XXX}x_1 + C_{XYX}x_3 + \frac{1}{2}C_{XXXX}x_1^2 + \frac{1}{2}C_{XYYX}x_3^2 + C_{XXYX}x_1x_3 \right) \\
J_{23} &= -\frac{1}{M_J} \left(K_{XY} + K_{XXY}x_1 + K_{XYY}x_3 + C_{XYX}x_2 + C_{XYY}x_4 + \frac{1}{2}K_{XYYY}x_3^2 \right. \\
&\quad \left. + \frac{1}{2}K_{XXXY}x_1^2 + K_{XXYY}x_1x_3 + C_{XYYX}x_3x_2 + C_{XYYY}x_3x_4 + C_{XXYX}x_1x_2 + C_{XXYY}x_1x_4 \right) \\
J_{24} &= -\frac{1}{M_J} \left(C_{XY} + C_{XXY}x_1 + C_{XYY}x_3 + \frac{1}{2}C_{XXXY}x_1^2 + \frac{1}{2}C_{XYYY}x_3^2 + C_{XXYY}x_1x_3 \right) \\
J_{41} &= -\frac{1}{M_J} \left(K_{YX} + K_{YXX}x_1 + K_{YXY}x_3 + C_{YXX}x_2 + C_{YXY}x_4 + \frac{1}{2}K_{YXXX}x_1^2 \right. \\
&\quad \left. + K_{YXXY}x_1x_3 + \frac{1}{2}K_{YXYY}x_3^2 + C_{YXXX}x_1x_2 + C_{YXXY}x_1x_4 + C_{YXYX}x_3x_2 + C_{YXYY}x_3x_4 \right) \\
J_{42} &= -\frac{1}{M_J} \left(C_{YX} + C_{YXX}x_1 + C_{YYX}x_3 + \frac{1}{2}C_{YXXX}x_1^2 + \frac{1}{2}C_{YYYX}x_3^2 + C_{YXYX}x_1x_3 \right) \\
J_{43} &= -\frac{K_s}{2M_J} - \frac{1}{M_J} \left(K_{YY} + K_{YXY}x_1 + K_{YYY}x_3 + C_{YYX}x_2 + C_{YYY}x_4 + \frac{1}{2}K_{YYYY}x_3^2 \right. \\
&\quad \left. + \frac{1}{2}K_{YXXY}x_1^2 + K_{YXYY}x_1x_3 + C_{YYYX}x_3x_2 + C_{YYYY}x_3x_4 + C_{YXYX}x_1x_2 + C_{YXYY}x_1x_4 \right) \\
J_{44} &= -\frac{1}{M_J} \left(C_{YY} + C_{YXY}x_1 + C_{YYY}x_3 + \frac{1}{2}C_{YXXY}x_1^2 + \frac{1}{2}C_{YYYY}x_3^2 + C_{YXYY}x_1x_3 \right)
\end{aligned}$$

Evaluating the Jacobian matrix at fixed points $\mathbf{x} = \mathbf{x}^*$, results in

$$\mathbf{J}(\bar{\mathbf{M}})_{(\mathbf{x}=\mathbf{x}^*)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{K_s}{2M_J} - \frac{K_{XX}}{M_J} & -\frac{C_{XX}}{M_J} & -\frac{K_{XY}}{M_J} & -\frac{C_{XY}}{M_J} & \frac{K_s}{2M_J} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{K_{YX}}{M_J} & -\frac{C_{YX}}{M_J} & -\frac{K_s}{2M_J} - \frac{K_{YY}}{M_J} & -\frac{C_{YY}}{M_J} & 0 & 0 & \frac{K_s}{2M_J} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{K_s}{M_D} & 0 & 0 & 0 & -\frac{K_s}{M_D} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{K_s}{M_D} & 0 & 0 & 0 & -\frac{K_s}{M_D} & 0 \end{bmatrix} \quad (55)$$

Evaluating the Jacobian matrix eigenvalues $(\lambda_1, \lambda_2, \dots, \lambda_8)$ then reorder them in descending order such as $Re(\lambda_1) \geq Re(\lambda_2) \geq \dots \geq Re(\lambda_n)$. The value of the $\bar{M}$ where $Re(\lambda_1) = 0$ is the threshold value $\bar{M}_{th}$. This analysis is repeated for the available range of Sommerfeld number.

These results are presented on Figure 6-a, b. In Figure 6, the threshold mass is plotted versus the eccentricity ratio and in Figure 6-b, the threshold mass is plotted versus the dimensionless Sommerfeld number. The plot is done for several values of dimensionless shaft stiffness ranges $K_s = 1$ to $K_s = 10000$.

To investigate the stability of the periodic solution when the rotor speed exceeds the threshold speed the Monodromy matrix are evaluated from the Floquet multipliers. For the details of the Monodromy analysis, see for example(Nayfeh and Balachandran 1995, Seydel 2010, Miraskari et al. 2018b). If the eigenvalues of Monodromy matrix is less than one the periodic solution is stable limit cycle i.e. supercritical bifurcation and if the eigenvalue of the Monodromy matrix is greater than one the periodic solution is unstable i.e. subcritical bifurcation. This is plotted in Fig.6 using solid and dotted lines where the solid lines indicate

supercritical bifurcation, and the dotted lines represent subcritical bifurcation solution. The results of the figure show that for the present dynamical model decreasing the stiffness of the rotor shaft increase the possibility of subcritical Hopf bifurcation.

Figure 6: (Color online) Rotor stability curves: (a) Threshold mass versus the eccentricity ratio. (b) Threshold mass versus Sommerfeld No. .

Table 1: selected points for dynamical analysis init= [0 0 0 0.1 0 0 2/ K_s 0.1]

Operating Point	K_s	$\bar{M}$	Sommerfeld number S	Eccentricity ratio ε	Bifurcation type
I	1	$\bar{M}_{th} - 1$	0.148	0.55	Supercritical
II	1	$\bar{M}_{th} + 0.5$	0.148	0.55	Supercritical
III	1	$\bar{M}_{th} - 1$	0.507	0.24	subcritical
IV	1	$\bar{M}_{th} + 0.5$	0.507	0.24	subcritical
V	20	$\bar{M}_{th} - 1$	0.507	0.24	Supercritical
VI	20	$\bar{M}_{th} + 0.5$	0.507	0.24	Supercritical

3.3.2. Time response comparative study

In this section, selected points from the stability diagram I, II, III, IV, V and VI are investigated using orbit plots of the shaft geometrical center. The input data for these cases are shown in Table 1 and presented in Figure 6. The comparative analysis is done using four different analyses. The first analysis is based on the linear first order approximation of the bearing forces (LA). This analysis can be done using Eq.(52) while eliminating the second and the third order bearing coefficients. The second analysis is based on the nonlinear second order approximation of the bearing forces and this is done while reducing the third order coefficients to zero (NLA2). The third analysis is based on the nonlinear third order force approximation (NLA3) as stated in Eq.(52). The last analysis is based on the nonlinear force evaluated directly from Reynolds equation (RNL). In this analysis, the Reynolds equation is solved every time step to evaluate the bearing force.

The six selected operating points are listed in Table 1. All the cases are for $L/D = 1$. The first two cases I, II are at $S = 0.148$, the second two cases III, IV are at $S = 0.507$ and the last two cases V, VI are at $S = 0.507$ and $K_s = 20$. The cases I, III and V are below the threshold speed and the cases II, IV and VI are above the threshold speed.

Figure 7: Orbit plot for the journal geometrical center³⁴ at different operating points, point I, II, III and IV as shown in Table 1. For all cases (a)1st order (LA) (b)2nd order (NLA2) (c) 3rd order(NLA3) (d) nonlinear (RNL). $K_s = 1, Init = [0 \ 0 \ 0 \ 0.1 \ 0 \ 0 \ 2/K_s \ 0.1]$

Figure 8: Orbit plot for the journal geometrical center two operating conditions V and VI as shown in Table 1. For all cases (a) 1st order (LA) (b) 2nd order (NLA2) (c) 3rd order (NLA3) (d) nonlinear (RNL). $K_s = 1, Init = [0 \ 0 \ 0 \ 0.1 \ 0 \ 0 \ 2/K_s \ 0.1]$

For case I and case III where both of them are below the threshold speed the four analyses give the same response as shown in the first and third row of Figure 7. Case II and case IV are above the threshold velocity for supercritical and subcritical cases according to the Monodromy bifurcation analysis. For case II, the response based on the linear analysis shows instability as the rotor hits the journal bearing. Both analyses based on the second order coefficients (NLA2) and third order coefficients (NLA3) show a stable limit cycle. However, the analysis based on direct solution of Reynolds (RNL) shows bifurcation leads to unstable limit cycle which then jumps to a larger stable limit cycle. This behavior for journal bearings is previously reported in (Chouchane and Amamou 2011, Sghir and Chouchane 2014).

For case IV which is subcritical according to the Monodromy analysis. The first order

analysis shows unstable behavior above threshold speed which is basically expected because analysis based on first order (LA) is insensitive to bifurcation type (Miraskari et al. 2018b). In this case, the analyses based on NLA2 and NLA3 do not show the same response. The analysis based on second order coefficients shows a stable limit cycle while the analysis based on third order coefficients show unstable limit cycle. In this case, the nonlinear analysis RNL agrees with (NLA3) in showing unstable limit cycle. However, the RNL response shows a jump from unstable limit cycle to bigger stable limit cycle as shown Figure 7-IV-d. The explanation for the difference between the third order coefficient and RNL can be explained by the fact that the coefficients based on NLA3 is only valid in the neighborhood of the steady state point and as the rotor receding from this point, these coefficients become not accurate as shown in the perturbation analysis.

Another two cases ,V and VI, using higher rotor dimensionless stiffness are investigated. The dimensionless stiffness in this condition is $K_s = 20$. The stability diagram shows that all bifurcation for this case is supercritical. The orbit plot for the journal geometrical center is plotted at $S = 0.507$. the first case V is below the threshold speed and the second case VI is above the threshold speed. The results show that for case V below the threshold speed all the four analysis show the same response. For case VI which is higher than the threshold speed the analysis based on first order coefficients show growing of the response until the journal hits the bearing. The analyses NLA2, NLA3 and RNL show stable limit cycle. The size of the limit cycle is not the same for the three cases. This can be explained by the fact that the bearing coefficients are evaluated to be correct in the vicinity of the steady state point.

Conclusion

In the present paper, the linear and nonlinear second and third order bearing coefficients are obtained using infinitesimal perturbation method. The obtained equations are

solved using finite difference method. The effect of the mesh sensitivity on the obtained parameters are investigated. The validity range of bearing coefficients are investigated using perturbation analysis.

The obtained parameters are used to investigate the stability of a flexible rotor supported by two symmetric journal bearings. Eigenvalue analysis is used to evaluate the threshold speed. Monodromy analysis is used to obtain the bifurcation type above the threshold speed. The results of the orbit plot analysis show that below the threshold speed the analysis based on linear and nonlinear coefficients is compatible with that based on direct solution of Reynolds equation. Above the threshold speed the linear analysis is insensitive to limit cycles. The second and third bearing coefficient can predict the existence of limit cycles. The analysis based on third order coefficient giving better prediction than the second order. The analysis based on the bearing coefficients are closer to the analysis based on the complete solution of Reynolds equation at higher rotor dimensionless stiffness.

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Conflict of interest

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Appendix A. Derivative of dimensionless oil-film thickness

In this appendix, a list of important formulae for evaluating the oil-film thickness derivation which are required for first, second and third order coefficient evaluation

$$\frac{\partial H_0}{\partial \phi} = -\varepsilon_0 \sin \varphi_0 \quad (\text{A.1})$$

$$H_X = \frac{\partial H}{\partial X} = \sin \phi \quad (\text{A.2})$$

$$H_Y = \frac{\partial H}{\partial Y} = \cos \phi \quad (\text{A.3})$$

$$H_{\phi X} = \frac{\partial^2 H}{\partial \phi \partial X} = \cos \phi \quad (\text{A.4})$$

$$H_{\phi Y} = \frac{\partial^2 H}{\partial \phi \partial Y} = -\sin \phi \quad (\text{A.5})$$

$$H_{XX} = \frac{\partial^2 H}{\partial X^2} = \frac{1}{\varepsilon_0} (2 \sin \theta_0 \cos \theta_0 \sin \varphi_0 - \cos^2 \theta_0 \cos \varphi_0) \quad (\text{A.6})$$

$$H_{YY} = \frac{\partial^2 H}{\partial Y^2} = \frac{1}{\varepsilon_0} (-2 \sin \theta_0 \cos \theta_0 \sin \varphi_0 - \sin^2 \theta_0 \cos \varphi_0) \quad (\text{A.7})$$

$$H_{XY} = \frac{\partial^2 H}{\partial X \partial Y} = \frac{1}{\varepsilon_0} (\sin \theta_0 \cos \theta_0 \cos \varphi_0 + \cos^2 \theta_0 \sin \varphi_0 - \sin^2 \theta_0 \sin \varphi_0) \quad (\text{A.8})$$

$$H_{\phi XX} = \frac{\partial^3 H}{\partial \phi \partial X^2} = \frac{1}{\varepsilon_0} (2 \sin \theta_0 \cos \theta_0 \cos \varphi_0 + \cos^2 \theta_0 \sin \varphi_0) \quad (\text{A.9})$$

$$H_{\phi YY} = \frac{\partial^3 H}{\partial \phi \partial Y^2} = \frac{1}{\varepsilon_0} (-2 \sin \theta_0 \cos \theta_0 \cos \varphi_0 + \sin^2 \theta_0 \sin \varphi_0) \quad (\text{A.10})$$

$$H_{\phi XY} = \frac{\partial^3 H}{\partial \phi \partial X \partial Y} = \frac{1}{\varepsilon_0} (-\sin \theta_0 \cos \theta_0 \sin \varphi_0 + \cos^2 \theta_0 \cos \varphi_0 - \sin^2 \theta_0 \cos \varphi_0) \quad (\text{A.11})$$

$$H_{XXX} = \frac{\partial^3 H}{\partial X^3} = \frac{1}{\varepsilon_0^2} (-3 \sin \theta_0 \cos^2 \theta_0 \cos \varphi_0 - \cos^3 \theta_0 \sin \varphi_0) \quad (\text{A.12})$$

$$H_{YYY} = \frac{\partial^3 H}{\partial Y^3} = \frac{1}{\varepsilon_0^2} (\sin^3 \theta_0 \sin \varphi_0 - 3 \sin^2 \theta_0 \cos \theta_0 \cos \varphi_0) \quad (\text{A.13})$$

$$H_{XYY} = \frac{\partial^3 H}{\partial X \partial Y^2} = \frac{1}{\varepsilon_0^2} (2 \cos^2 \theta_0 \sin \theta_0 \cos \varphi_0 - \sin^3 \theta_0 \cos \varphi_0 - \sin^2 \theta_0 \cos \theta_0 \sin \varphi_0) \quad (\text{A.14})$$

$$H_{XXY} = \frac{\partial^3 H}{\partial Y \partial X^2} = \frac{1}{\varepsilon_0^2} (2 \sin^2 \theta_0 \cos \theta_0 \cos \varphi_0 - \cos^3 \theta_0 \cos \varphi_0 + \cos^2 \theta_0 \sin \theta_0 \sin \varphi_0) \quad (\text{A.15})$$

$$H_{\phi XXX} = \frac{\partial^4 H}{\partial \phi \partial X^3} = \frac{1}{\varepsilon_0^2} (3 \sin \theta_0 \cos^2 \theta_0 \sin \varphi_0 - \cos^3 \theta_0 \cos \varphi_0) \quad (\text{A.16})$$

$$H_{\phi YYY} = \frac{\partial^4 H}{\partial \phi \partial Y^3} = \frac{1}{\varepsilon_0^2} (\sin^3 \theta_0 \cos \varphi_0 + 3 \sin^2 \theta_0 \cos \theta_0 \sin \varphi_0) \quad (\text{A.17})$$

$$H_{\phi XYY} = \frac{\partial^4 H}{\partial \phi \partial X \partial Y^2} = \frac{1}{\varepsilon_0^2} (-2 \cos^2 \theta_0 \sin \theta_0 \sin \varphi_0 + \sin^3 \theta_0 \sin \varphi_0 - \sin^2 \theta_0 \cos \theta_0 \cos \varphi_0) \quad (\text{A.18})$$

$$H_{\phi XXY} = \frac{\partial^4 H}{\partial \phi \partial Y \partial X^2} = \frac{1}{\varepsilon_0^2} (-2 \sin^2 \theta_0 \cos \theta_0 \sin \varphi_0 + \cos^3 \theta_0 \sin \varphi_0 + \cos^2 \theta_0 \sin \theta_0 \cos \varphi_0) \quad (\text{A.19})$$

Appendix B. Bearing first, second and third order coefficients

Table B.1: List of the first and second order bearing coefficients for fully circular journal bearing at $L/D = 1$

ε	θ_0	S	K_{XX}	K_{XY}	K_{YX}	K_{YY}	C_{XX}	$C_{XY} =$ C_{YX}	C_{YY}	K_{XXX}	K_{XXY}	K_{XYX}	K_{YXX}	K_{YXY}	K_{YYX}	C_{XXX}	C_{XXY}	C_{XYX}	C_{YXX}	C_{YYX}	C_{XYX}	C_{YYX}	
0.05	86.86	2.63	2.14	-19.25	20.11	1.10	38.64	2.12	40.18	86.06	-384.15	-94.13	3.77	43.50	-399.61	8.44	86.02	-767.36	-44.06	803.24	88.15	72.81	804.28
0.10	83.78	1.30	2.12	-9.54	10.22	1.13	19.33	2.10	20.35	43.43	-93.69	-47.01	3.83	22.33	-99.36	6.89	43.48	-187.87	-20.99	203.34	44.90	36.83	204.20
0.15	80.70	0.85	2.12	-6.24	7.00	1.16	12.85	2.10	13.85	29.55	-39.58	-31.40	3.96	15.49	-43.65	6.40	29.64	-80.16	-13.28	92.38	30.95	25.16	93.19
0.20	77.62	0.62	2.11	-4.54	5.44	1.21	9.56	2.10	10.70	22.79	-20.42	-23.61	4.14	12.30	-24.04	6.12	22.90	-42.18	-9.29	53.70	24.39	19.50	54.48
0.25	74.52	0.48	2.09	-3.49	4.56	1.28	7.58	2.10	8.88	18.91	-11.44	-18.94	4.38	10.59	-14.84	6.00	19.05	-24.45	-6.76	35.97	20.80	16.28	36.75
0.30	71.40	0.39	2.08	-2.75	4.01	1.37	6.23	2.10	7.74	16.44	-6.37	-15.80	4.69	9.69	-9.67	5.88	16.61	-14.57	-4.96	26.54	18.80	14.25	27.32
0.35	68.27	0.32	2.06	-2.21	3.66	1.47	5.26	2.10	6.99	14.82	-3.16	-13.54	5.07	9.28	-6.35	5.83	15.02	-8.41	-3.55	21.07	17.77	12.95	21.86
0.40	65.11	0.26	2.05	-1.77	3.43	1.60	4.51	2.09	6.48	13.72	-0.88	-11.79	5.54	9.27	-3.93	5.78	13.97	-4.14	-2.38	17.77	17.46	12.10	18.59
0.45	61.91	0.22	2.03	-1.41	3.28	1.76	3.92	2.09	6.15	12.99	0.90	-10.36	6.12	9.60	-1.93	5.75	13.28	-0.92	-1.36	15.81	17.77	11.57	16.64
0.50	58.66	0.18	2.01	-1.10	3.20	1.96	3.43	2.09	5.96	12.54	2.42	-9.11	6.84	10.29	-0.02	5.73	12.89	1.74	-0.43	14.74	18.69	11.29	15.61
0.55	55.36	0.15	1.99	-0.82	3.18	2.21	3.02	2.09	5.86	12.31	3.87	-7.94	7.73	11.41	2.06	5.71	12.73	4.13	0.46	14.36	20.31	11.21	15.28
0.60	51.97	0.12	1.97	-0.56	3.20	2.51	2.67	2.09	5.86	12.31	5.33	-6.77	8.89	13.08	4.60	5.74	12.82	6.50	1.36	14.62	22.84	11.36	15.59
0.65	48.47	0.10	1.95	-0.32	3.28	2.91	2.36	2.09	5.96	12.52	6.99	-5.43	10.39	15.56	8.10	5.77	13.12	9.05	2.29	15.50	26.62	11.70	16.54
0.70	44.82	0.08	1.93	-0.08	3.41	3.44	2.08	2.09	6.16	12.99	9.01	-3.76	12.44	19.31	13.37	5.85	13.72	12.08	3.33	17.17	32.37	12.31	18.32
0.75	40.96	0.06	1.92	0.18	3.62	4.17	1.83	2.10	6.49	13.81	11.69	-1.35	15.38	25.24	22.14	5.96	14.68	16.01	4.55	19.95	41.45	13.26	21.25
0.80	36.78	0.04	1.91	0.48	3.93	5.27	1.58	2.11	7.00	15.12	15.63	2.67	19.86	35.40	38.64	6.12	16.16	21.64	6.07	24.53	56.84	14.69	26.06
0.85	32.14	0.03	1.91	0.84	4.44	7.07	1.34	2.14	7.83	17.36	22.16	10.50	27.61	55.11	75.02	6.41	18.68	30.96	8.29	32.82	86.54	17.11	34.71
0.90	26.68	0.02	1.94	1.39	5.34	10.66	1.10	2.18	9.29	21.55	35.58	30.91	43.67	103.47	182.38	6.83	23.18	49.51	11.88	50.09	157.65	21.42	52.71
0.95	19.51	0.01	2.01	2.54	7.53	21.28	0.81	2.28	12.73	32.31	77.77	121.72	94.91	303.28	777.02	7.75	34.54	107.07	20.40	105.03	444.04	32.28	109.82

Table B.2: List of the third order bearing coefficients for fully circular journal bearing at $L/D = 1$

ε	θ_0	S	K_{XXXX}	K_{XYYY}	K_{XXXY}	K_{XXYY}	C_{XXXX}	C_{XXXY}	C_{XYXX}	C_{XYYX}	C_{XXYX}	C_{XXYY}	K_{YXXX}	K_{YYYY}	K_{YXXY}	K_{YXYX}	C_{YXXX}	C_{YXXY}	C_{YYYX}	C_{YYXX}	C_{YXYX}	C_{YXYY}
0.05	86.86	2.63	41.59	7484.01	51.34	-2937.50	-91.91	3442.44	-15261.74	-4608.90	3178.17	-15214.64	-112.80	-2603.33	1742.78	-7935.82	-11.80	150.19	1757.04	-15911.93	201.47	1753.75
0.10	83.78	1.30	38.22	821.02	41.85	-742.79	-39.76	869.38	-1821.28	-1144.12	802.75	-1797.50	-52.38	-655.42	442.04	-964.14	-10.07	76.02	463.23	-1948.45	100.14	460.13
0.15	80.70	0.85	37.75	180.42	34.26	-336.31	-22.65	395.53	-491.13	-504.72	365.54	-474.95	-31.76	-295.85	203.31	-270.47	-9.15	52.01	223.50	-553.20	68.01	220.42
0.20	77.62	0.62	38.09	34.73	30.91	-193.57	-13.62	230.38	-174.74	-280.11	213.22	-162.26	-20.55	-170.31	120.82	-103.73	-8.29	40.58	140.62	-215.25	52.72	137.48
0.25	74.52	0.48	39.12	-12.41	29.59	-127.10	-7.48	154.91	-65.51	-175.42	143.68	-55.14	-12.88	-112.27	83.74	-44.87	-7.36	34.35	103.45	-94.50	44.34	100.19
0.30	71.40	0.39	40.52	-31.64	30.03	-90.32	-2.94	114.68	-17.93	-117.19	106.66	-8.89	-6.77	-80.77	64.93	-18.43	-6.28	30.86	84.82	-39.45	39.49	81.41
0.35	68.27	0.32	42.51	-40.75	31.53	-67.47	0.95	91.54	6.40	-80.74	85.45	14.61	-1.26	-61.61	55.18	-4.02	-4.99	29.20	75.45	-8.99	36.98	71.83
0.40	65.11	0.26	45.00	-46.08	34.25	-51.62	4.34	77.63	21.23	-55.13	72.77	28.91	4.17	-48.78	50.87	5.75	-3.45	28.94	71.76	11.62	36.16	67.86
0.45	61.91	0.22	48.17	-49.89	38.21	-39.43	7.52	69.44	32.04	-35.08	65.41	39.43	9.97	-39.16	50.51	14.13	-1.59	29.98	72.29	28.87	36.88	68.03
0.50	58.66	0.18	52.18	-53.17	43.69	-28.86	10.64	65.18	41.62	-17.33	61.72	48.92	16.61	-30.70	53.73	23.21	0.68	32.44	76.78	46.63	39.19	72.03
0.55	55.36	0.15	57.26	-56.26	51.21	-18.28	13.76	63.91	51.73	0.65	60.84	59.12	24.64	-21.43	60.94	34.96	3.47	36.58	85.76	68.35	43.36	80.38
0.60	51.97	0.12	64.06	-59.14	61.50	-6.33	17.26	65.59	63.97	21.40	62.78	71.68	35.04	-8.65	73.40	51.95	7.10	43.16	100.86	98.37	50.18	94.59
0.65	48.47	0.10	73.03	-61.10	76.17	9.55	21.08	70.12	80.22	48.75	67.47	88.51	49.12	12.83	93.90	79.09	11.81	53.17	125.21	144.21	60.66	117.70
0.70	44.82	0.08	85.64	-60.32	97.85	33.09	25.74	78.58	103.74	89.17	75.97	112.97	69.60	54.31	127.99	125.78	18.38	68.94	165.39	220.63	77.22	156.00
0.75	40.96	0.06	104.09	-51.03	132.04	72.90	31.61	92.59	140.38	156.24	89.93	151.17	101.62	145.36	187.94	214.32	27.95	94.76	235.30	361.19	104.37	222.95
0.80	36.78	0.04	132.98	-14.41	191.03	150.19	39.43	115.86	202.98	282.65	113.00	216.38	156.72	377.22	304.06	405.14	42.90	140.31	368.93	654.77	152.19	351.35
0.85	32.14	0.03	184.69	118.38	307.95	327.63	51.85	159.35	327.97	568.56	156.00	346.38	267.67	1101.76	567.87	900.12	70.22	233.92	670.43	1399.17	250.19	642.10
0.90	26.68	0.02	295.72	751.32	604.51	883.72	73.51	252.97	640.80	1428.25	248.54	670.69	552.06	4423.63	1367.18	2705.01	129.35	473.55	1564.75	4016.66	500.17	1507.48
0.95	19.51	0.01	672.26	6836.43	1906.12	4235.83	130.46	571.84	2007.63	6432.67	563.87	2081.42	1833.02	40862.76	6038.53	16839.24	337.27	1550.00	6681.07	23751.36	1617.92	6477.31

Figures

Figure 1

Full circular journal bearing (a) schematic of the journal bearing coordinates (b) journal bearing mesh (c) perturbed journal bearing in dimensionless coordinates X and Y

Figure 2

(a) Elastic rotor supported on two symmetric journal bearings. (b) Bearing coordinates.

Figure 3

(Color online) Loglog plot for un-grooved journal bearing stiffness and damping coefficients(absolute) versus the Sommerfeld number. $L/D = 1$, (a) First order bearing coefficient (b) Second order bearing coefficient. In this gure the dotted line for mesh 50×200 , dashed line for mesh 100×400 , centerline for mesh 150×600 and solid line for mesh 200×800 .

Figure 4

(Color online) Loglog plot for un-grooved full circular journal bearing third order stiffness and damping coefficients (absolute) versus the Sommerfeld number. L/D ratio is unity. In this gure the dotted line for mesh 50×200 , dashed line for mesh 100×400 , centerline for mesh 150×600 and solid line for mesh 200×800

Figure 5

"Please see the Manuscript PDF file for the complete figure caption".

Figure 6

(Color online) Rotor stability curves: (a) Threshold mass versus the eccentricity ratio. (b) Threshold mass versus Sommerfeld No. .

Figure 7

Orbit plot for the journal geometrical center at different operating points, point I, II, III and IV as shown in Table 1. For all cases (a)1st order (LA) (b)2nd order (NLA2) (c) 3rd order(NLA3) (d) nonlinear (RNL). $K_s = 1$; Init = [0 0 0 0:1 0 0 2/ K_s 0:1]

Figure 8

Orbit plot for the journal geometrical center two operating conditions V and VI as shown in Table 1. For all cases (a) 1st order (LA) (b) 2nd order (NLA2) (c) 3rd order(NLA3) (d) nonlinear (RNL). $K_s = 1$; $Init = [0 \ 0 \ 0 \ 0:1 \ 0 \ 0 \ 2/K_s \ 0:1]$