SVD-based angular difference estimation

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RESEARCH

SVD-based angular difference estimation

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Abstract

In this paper, the estimation of angular difference between adjacent sources is investigated in the uniform linear array. In the presence of additive Gaussian noise, we use singular value decomposition to obtain the eigenvalues of the spatial covariance matrix. Thus, the expressions of eigenvalues related to the signal are obtained. The smaller one between the eigenvalues is related to the degree of difference between the two sources and we derive the expression of the angular difference estimation through it. In practice, the more samples observed, the more accurate the estimate is. Numerical results confirm our theoretical analysis and demonstrate the effectiveness of the proposed estimation method. The result in this paper has important guiding significance for array signal processing in the actual environment.

Keywords: Sensor array; Singular value decomposition; Angular difference estimation

1 Introduction

For decades, array processing of the received signal has been intensively investigated. Resolution and accurate estimation of signals and their parameters are central capabilities of adaptive sensor array processing [1]. Many techniques were the result of an attempt by researchers to go beyond the classical Fourier-limit. For example, the Maximum Entropy (ME) spectral estimation method [2] overcome the problem of “Rayleigh limit” in the conventional beam-forming method [3], which have important inspirations for the subsequent effort. The subspace-based approach such as the multiple signal classification (MUSIC) algorithm [4, 5] relies on certain geometrical properties of the assumed data model, resulting in a resolution capability which is not limited by the array aperture. The maximum likelihood (ML) method [6] including deterministic ML (DML) [7] algorithm and stochastic ML (SML) [8] algorithm can achieve asymptotic optimal estimates with multi-dimensional optimization.

Eigenspace decompositions are used in solving many signal processing problems such as source location estimation [9]. In this case, either the eigenvalue decomposition (EVD) of a covariance matrix or the singular value decomposition (SVD) of a data matrix is required [10]. At present, SVD is mainly applied to the decomposition of sample covariance matrix in order

to obtain the signal and noise subspaces in array signal processing. Then the locations of the highest peaks of spectrum are recorded as the direction-of-arrival (DOA) estimates. MUSIC algorithm is the typical one among these subspace-based methods. However, the algorithm results in the computational complexity. For two closely spaced signal sources, it’s important to accurately estimate the angular difference. Through the estimate, we can judge whether the sources are distinguishable under different conditions.

In this paper, we use the quadrature component in the transformed signal space obtained by singular value decomposition. Then we obtain the estimate of the angular difference between adjacent sources. This paper is organized as follows. The system model of a uniform linear array (ULA) detecting two narrowband far-field sources is presented in Section 2.1. In Section 2.2, the process of SVD is given and the expression of angular difference is derived. The estimation for finite samples in practice are discussed in Section 2.3. The proposed method is tested via a few simulations, which appear in Section 3. The main results of this paper are discussed and concluded in Section 4.

2 Methods

2.1 System model

Let us consider a ULA composed of M antenna elements, as shown in Figure 1 and the inter-element spacing is half wavelength. Two far-field narrow-band signals are impinging on the array. Then, the received signal by the m -th element is given by

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$$x_m(t) = \sum_{i=1}^2 s_i(t) e^{j\omega_0 \tau_m(\theta_i)} + w_m(t) \quad (1)$$

where $s_i(t)$ stands for the complex amplitude of the source signals, ω_0 is the carrier frequency and $\tau_m(\theta_i)$ describes the time delay of the signal with direction angle θ_i to the m -th element. $w_m(t)$ is the additive noise at the m -th element.

Equation (1) can be expressed compactly as

$$\mathbf{X}(t) = \mathbf{A}(\boldsymbol{\theta}) \mathbf{S}(t) + \mathbf{W}(t) \quad (2)$$

and the steering matrix is

$$\mathbf{A}(\boldsymbol{\theta}) = \begin{bmatrix} e^{j\omega_0 \tau_0(\theta_1)} & e^{j\omega_0 \tau_0(\theta_2)} \\ e^{j\omega_0 \tau_1(\theta_1)} & e^{j\omega_0 \tau_1(\theta_2)} \\ \vdots & \vdots \\ e^{j\omega_0 \tau_{M-1}(\theta_1)} & e^{j\omega_0 \tau_{M-1}(\theta_2)} \end{bmatrix} \quad (3)$$

In (2), we have the following definitions:

$$\begin{aligned} \mathbf{X}(t) &= [x_1(t) x_2(t) \cdots x_M(t)]^T, \\ \mathbf{S}(t) &= [s_1(t) s_2(t)]^T, \\ \mathbf{W}(t) &= [w_1(t) w_2(t) \cdots w_M(t)]^T \end{aligned} \quad (4)$$

Besides, the i -th source is given by $s_i(t) = \alpha_i e^{j\varphi_i(t)}$, where α_i denotes the known amplitude and $\varphi_i(t)$ is the unknown phase of source. We assume that $\varphi_i(t)$ is uniformly distributed in the interval $[0, 2\pi]$.

Here, we define the i -th source SNR as

$$\rho_i^2 = \frac{E(\alpha_i^2)}{N_0} \quad (5)$$

where N_0 is the power spectral density of noise. Specifically, the noise is modeled as a complex Gaussian zero-mean spatially and temporally white random process with identical power spectral density N_0 on each array element. What's more, it is uncorrelated among different array elements.

2.2 Eigenvalue decomposition of covariance matrix

The covariance matrix of $\mathbf{X}(t)$ is

$$\mathbf{R} = E[(\mathbf{X}(t) - m_x(t))(\mathbf{X}(t) - m_x(t))^H] \quad (6)$$

where $m_x(t) = E[\mathbf{X}(t)]$ and $m_x(t) = 0$. Then, under the basic assumption that the source signals and the noise are uncorrelated we have

$$\mathbf{R} = E[\mathbf{X}(t) \mathbf{X}(t)^H] = \mathbf{A}(\boldsymbol{\theta}) \mathbf{R}_S \mathbf{A}^H(\boldsymbol{\theta}) + N_0 \mathbf{I} \quad (7)$$

In (7), the sources are uncorrelated and their covariance matrix is

$$\mathbf{R}_S = E[\mathbf{S}(t) \mathbf{S}(t)^H] = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \quad (8)$$

where $P_1 = E(\alpha_1^2)$ and $P_2 = E(\alpha_2^2)$ respectively represents the known power of source 1 and source 2. We consider the situation that the sources have equal power and denote the average power as $\bar{P}$.

The singular value decomposition of the $M \times 2$ matrix $\mathbf{A}(\boldsymbol{\theta})$ is obtained by making a certain well-defined choice for U and V , which then gives rise to the following decomposition:

$$\mathbf{A}(\boldsymbol{\theta}) = U \Sigma V^H \quad (9)$$

where $U \in C^{M \times M}$ and $V \in C^{2 \times 2}$ is a unitary matrix. Σ is an $M \times 2$ diagonal matrix containing the singular values of $\mathbf{A}(\boldsymbol{\theta})$.

Let $|\lambda \mathbf{I} - \mathbf{A}^H(\boldsymbol{\theta}) \mathbf{A}(\boldsymbol{\theta})| = 0$, then we get the eigenvalues as

$$\lambda_1 = M \left(1 + \frac{\sin(M(\beta_1 - \beta_2)/2)}{M \sin((\beta_1 - \beta_2)/2)} \right) \quad (10)$$

$$\lambda_2 = M \left(1 - \frac{\sin(M(\beta_1 - \beta_2)/2)}{M \sin((\beta_1 - \beta_2)/2)} \right) \quad (11)$$

where $\beta_l = 2\pi d \sin \theta_l / \lambda$. Thus, we get the singular value matrix Σ as

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & 0 & \cdots & 0 \end{bmatrix}^T \quad (12)$$

Based on the results of decomposition above, we rewrite (7) as

$$\mathbf{R} = \bar{P} U \Sigma \Sigma^H U^H + N_0 \mathbf{I} \quad (13)$$

Dividing the covariance matrix obtained by the noise power in the channel gives the following expression

$$\frac{\mathbf{R}}{N_0} = U \Lambda U^H \quad (14)$$

where $\Lambda = \overline{\rho^2} \Sigma \Sigma^H + \mathbf{I}$ is a $M \times M$ diagonal matrix and $\overline{\rho^2} = \overline{P}/N_0$ is the average SNR. The specific expressions of the eigenvalues related to signals in the eigenvalues of the spatial covariance matrix can also be obtained.

According to (12), we can obtain the specific representation of the matrix Λ .

$$\begin{aligned} \Lambda &= \begin{bmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_M \end{bmatrix} \\ &= \begin{bmatrix} 1+\overline{\rho^2}\lambda_1 & & & \\ & 1+\overline{\rho^2}\lambda_2 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix} \end{aligned} \quad (15)$$

In (15), the first two diagonal values are related to the source signals. λ_1 and λ_2 are two signal subspaces obtained by singular value decomposition. λ_1 represents the in-phase component in the transformed signal space, while λ_2 represents the quadrature component. Note that the quadrature component is related to the degree of difference between the two sources, we use λ_2 to estimate the angular difference.

When the two sources are adjacent, we exploit the Taylor expansion at $\sin \theta_1 - \sin \theta_2 = 0$ on λ_2 as

$$\lambda_2 = \frac{\pi^2 d^2 M (M^2 - 1)}{6\lambda^2} (\sin \theta_1 - \sin \theta_2)^2 + o(\sin \theta_1 - \sin \theta_2) \quad (16)$$

where $o(\cdot)$ is the higher power of $\sin \theta_1 - \sin \theta_2$ and it can be neglected. After the transformation of the trig formula, we have the following expression

$$\lambda_2 \approx \frac{\pi^2 d^2 M (M^2 - 1)}{6\lambda^2} \left(2 \sin \frac{\Delta}{2} \cos \theta_0 \right)^2 \quad (17)$$

where $\theta_0 = (\theta_1 + \theta_2)/2$ is the azimuth axis and $\Delta = \theta_1 - \theta_2$ denotes the angular difference.

Then, we can denote the angular difference by

$$\Delta = \sqrt{\frac{6\lambda^2 (\Lambda_2 - 1)}{\overline{\rho^2} \pi^2 d^2 M (M^2 - 1) \cos^2 \theta_0}} \quad (18)$$

The method enables us to estimate the angular difference of adjacent sources through the obtained eigenvalues of the covariance matrix. Therefore, the estimation method is simple and fast in practice.

2.3 Angular difference estimation for finite samples

All of the earlier formulation in the last section assume the existence of exact quantities, i.e. infinite observation time. It is clear that in practice only sample estimates which we denote by a hat, i.e., $\hat{\Lambda}$, are available. The sensor outputs are appropriately pre-processed and sampled at arbitrary time instances, labeled $t = 1, 2, \dots, N$ for simplicity.

A natural estimate of $\mathbf{R}$ is the sample covariance matrix and the specific expression is shown as (19), where $\hat{\mathbf{R}}_S$ is the source sample covariance matrix and $\hat{\mathbf{R}}_W$ is the noise sample covariance matrix.

Here, we ignore the cross term between the source and the noise in order to simplify the complexity of the problem. Hence, we can write

$$\hat{\mathbf{R}} = \mathbf{A}(\boldsymbol{\theta}) \hat{\mathbf{R}}_S \mathbf{A}^H(\boldsymbol{\theta}) + \hat{\mathbf{R}}_W \quad (20)$$

Because the source signals are assumed to be nonrandom (i.e., the same in all realizations) in this paper, $\hat{\mathbf{R}}_S$ can be expressed as $\mathbf{R}_S$.

After singular value decomposition of $\mathbf{A}(\boldsymbol{\theta})$, the representation of $\hat{\mathbf{R}}$ is similar to $\mathbf{R}$, i.e.

$$\hat{\mathbf{R}} = \overline{P} \mathbf{U} \Sigma \Sigma^H \mathbf{U}^H + \hat{\mathbf{R}}_W \quad (21)$$

Thus, we obtain

$$\hat{\Lambda}_2 = 1 + \frac{\overline{P}}{\frac{1}{N} \sum_{t=1}^N \sigma_n^2(t)} \lambda_2 \quad (22)$$

where $\sigma_n^2(t)$ is the variance of noise in the different snapshot. The above equation is similar to equation (16).

Based on the discussion of infinite snapshots in the last section, we give the following expression for finite N :

$$\hat{\Delta} = \sqrt{\frac{6\lambda^2 (\hat{\Lambda}_2 - 1) \frac{1}{N} \sum_{t=1}^N \sigma_n^2(t)}{\overline{P} \pi^2 d^2 M (M^2 - 1) \cos^2 \theta_0}} \quad (23)$$

This representation can be used to estimate the angular difference in the case of finite samples. Next, we will discuss the estimation variance of angular difference.

Because the probability density of noise obeys gaussian distribution, the variance of noise σ_n^2 is exponential distributed with a mean of N_0 and a variance of N_0^2 . Based on the law of large numbers, the sample mean of noise variance approximately follows the normal distribution when the number of samples N is sufficiently large.

$$\begin{aligned} \hat{\mathbf{R}} &= \frac{1}{N} \sum_{t=1}^N \mathbf{X}(t) \mathbf{X}^H(t) = \mathbf{A}(\boldsymbol{\theta}) \hat{\mathbf{R}}_S \mathbf{A}^H(\boldsymbol{\theta}) + \mathbf{A}(\boldsymbol{\theta}) \frac{1}{N} \sum_{t=1}^N \mathbf{S}(t) \mathbf{W}^H(t) \\ &+ \frac{1}{N} \sum_{t=1}^N \mathbf{W}(t) \mathbf{S}^H(t) \mathbf{A}^H(\boldsymbol{\theta}) + \hat{\mathbf{R}}_W \end{aligned} \quad (19)$$

$$\begin{aligned} E(x) &= \int_0^\infty x f(x) dx = \frac{1}{\sqrt{2\pi N_0^2/N}} \int_0^\infty x \exp\left(-\frac{(x-N_0)^2}{2N_0^2/N}\right) dx \\ &= \frac{1}{2} \frac{N_0(\sqrt{\pi} \operatorname{erf}(\sqrt{2N}/2) N \exp(N/2) + \sqrt{\pi} N \exp(N/2) + \sqrt{2N}) \exp(-N/2)}{N\sqrt{\pi}} \end{aligned} \quad (24)$$

If $x = \sum_{t=1}^N \sigma_n^2(t) / N$, then x is Gaussian with a mean of N_0 and a variance of N_0^2/N . Then the mean value for the random variable x is the expectation value, as shown in equation (24), where $\operatorname{erf}(\cdot)$ is error function.

When the number of samples is increased to infinity, we have the following expression

$$\begin{aligned} \lim_{N \rightarrow \infty} E(x) &= \lim_{N \rightarrow \infty} \frac{1}{2} \frac{N_0 \left(2\sqrt{\pi}N + \sqrt{2N} \exp\left(-\frac{N}{2}\right) \right)}{N\sqrt{\pi}} \\ &= N_0 \end{aligned} \quad (25)$$

leading to

$$\lim_{N \rightarrow \infty} E \left[\left(\hat{\Delta} - \Delta \right)^2 \right] = \lim_{N \rightarrow \infty} E \left(\hat{\Delta}^2 + \Delta^2 - 2\hat{\Delta}\Delta \right) = 0 \quad (26)$$

Thus, the estimation variance tends to zero as the number of samples increases, that is the estimate of angular difference in practice approaches the true value with large samples.

In actual signal processing, we can easily judge whether the sources are distinguishable or not by the estimate value of the angular difference. When the estimate value tends to be constant with the decrease of the actual angular difference, it is the case that the two sources are indistinguishable.

3 Results and discussion

In this section, we demonstrate via simulations the results of the estimation of angular difference for adjacent sources.

In the following simulation, a uniform linear array of $M = 32$ antennas spaced half a wave length apart from each other is used. The power of both sources is

Figure 2 Estimated value of the angular difference varies with actual value.

set as 1. In addition, we set the element-level SNR as 5dB.

In Fig. 2, the comparison of the estimated value of the angular difference with the actual value is plotted for large samples ($N = 1000$). Note that when the true value Δ_0 is less than 0.2 degrees, the estimated value Δ deviates significantly from the true value and the estimation is not accurate. This is due to the fact that the array cannot distinguish between the two sources at this point. While in the other interval of angular difference, the estimated value is close to the true value. The phenomenon just reflects the effectiveness of the proposed estimation method.

Fig. 3 presents the angular difference estimation for different samples. We set the actual value to 0.5° . Obviously, as the number of samples N increases, the estimated value of the angular difference will tend to the actual value. This is consistent with the results of previous theoretical analysis. In this case, the more observed values obtained, the less interference caused by noise and thus the estimation is more accurate.

The computing time of angular difference estimation is measured by simulations and the results of different

Figure 3 Estimated value of the angular difference varies with snapshots.

Table 1 Average run times for various algorithms.

Algorithm	Run times (sec)
Music	1.369946
Esprit	1.518598
Algorithm in this paper	0.165054

algorithms are displayed in Table 1. We set the actual value of angular difference to 0.5° and the number of samples is 1000000. MUSIC and ESPRIT are typical ones among DOA estimation algorithms, so we compare them with the algorithm in this paper. Here, we directly estimate the DOAs of two sources and calculate the difference between them. The algorithm proposed in this paper is faster than other algorithms for the omission of search of peaks. We obtain the angular difference directly by the eigenvalue and the process of calculation is simple.

4 Conclusions

In this paper, the estimation method of angular difference between two adjacent sources in the uniform linear array is investigated. In the case of infinite observation time, we derive the theoretical expression of angular difference under the condition that source signals are nonrandom. We obtain the eigenvalues of the spatial covariance matrix by singular value decomposition of the steering matrix. Particularly, the specific expressions of two eigenvalues related to the signals are obtained.

Then we call the small eigenvalue the quadrature component and find that the eigenvalue is related to the degree of difference between the two sources. After a series of approximate calculations, the theoretical expression of angular difference is derived. Compared with other DOA estimation methods based on subspace, the computation of the method in this paper is greatly reduced. Through the estimate of angular difference, we can judge whether the sources are distinguishable under different conditions.

Besides, the estimation for finite samples is discussed. Similarly, we give the result of the estimation in practice based on the sample covariance matrix. It proves that the estimated value of the angular difference tends to the actual value as the number of samples increases.

Finally, we can generalize our research to more complicated scenario, such as extended signal models and generalized multiple sources. All these problems are worthy of further investigations.

Abbreviations

DOA: direction-of-arrival; ESPRIT: estimation of signal parameters via rotational invariance techniques; EVD: eigenvalue decomposition; ME: Maximum Entropy; ML: maximum likelihood; MUSIC: multiple signal classification; SVD: singular value decomposition; ULA: uniform linear array

Availability of data and materials

The data and materials are available for reviewers and readers.

Competing interests

The authors declare that they have no competing interests.

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Authors' contributions

DZ proposed the original idea of the full text. YZ and WT designed and implemented the simulation experiments. CS analyzed the results. YZ drafted the manuscript.

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Figures

Figure 1

Uniform linear array system model in two sources scenario.

Figure 2

Estimated value of the angular difference varies with actual value.

Figure 3

Estimated value of the angular difference varies with snapshots.