Explicit Finite Difference Analysis of Pseudo-Hyperbolic Partial Differential Equations

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Explicit Finite Difference Analysis of Pseudo-Hyperbolic Partial Differential Equations

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Abstract

The explicit finite difference method is suggested in this publication to resolve pseudo-hyperbolic partial differential equations. An explicit finite difference scheme is developed for the suggested problem. The estimated and exact solutions are compared to create the error analysis table. The stability inequalities for this difference scheme are demonstrated using matrix analysis. An example is illustrated as an implementation of the finite difference scheme method. The results show that the proposed method is very accurate and convenient. The solutions are represented graphically. Other numerical methods can be applied to investigate pseudo-hyperbolic partial differential equations.

Keywords: Pseudo-hyperbolic equation, explicit finite difference scheme, stability, nonlocal initial condition, approximation solution.

1 Introduction

Linear and nonlinear hyperbolic partial differential equations are important in many areas of science, including physics and engineering. Examples include the wave equation and the telegraph equation, which model problems such as vibrating strings fixed at both ends, longitudinal vibrations, thermodynamics, electrodynamics waves, electromagnetic waves, and sound wave propagation. Euler's gas dynamics equations are also hyperbolic partial differential equations. These physical phenomena can be expressed mathematically as partial differential equations.

The implicit finite difference method for solving linear and nonlinear pseudo-parabolic equations with Dirichlet boundary conditions was introduced in the early 1970s [2], [14]. The proposed problem was first highlighted in [8] where they used the residual power series technique to present analytical approximate solutions to the third-order linear pseudo hyperbolic partial differential equation with non-local integral conditions.

Theoretical research into pseudo-parabolic equations with non-local boundary conditions began because of their implementations in complex problems in science and technology. The solutions to the pseudo-parabolic partial differential equation of the third order were presented in [7]. Later, many studies devoted to the modelling of dynamics using a pseudo-parabolic equation with non-local boundary conditions were published.

Over the years, academics have consistently shown an interest in numerical approaches for pseudo-parabolic and pseudo-hyperbolic equations with non-local boundary conditions. The pseudo-hyperbolic partial differential equations with non-local conditions were taken into consideration by the authors in [8]. By using an implicit finite difference technique, the linear pseudo-equivalent equations have been presented with a range of integral terms [9], [11], [13]. The implicit finite difference techniques were effectively used by the authors in [3] for various problems. Pseudo-parabola implementations are displayed in [5, 6], [9]. In this manuscript, the following pseudo-hyperbolic partial differential equations are taken into consideration:

$$u_{\tau\tau}(y,\tau) - \lambda u_{\sigma yy}(y,\sigma) - u_{yy}(y,\sigma) - f(y,\sigma) = 0, \quad y \in (0,Y), \quad \sigma \in (0,T), \tag{1}$$

with

$$u(0,y) = V_0(y), \quad u_\sigma(y,0) = V_1(y), \quad y \in [0,Y], \tag{2}$$

$$u(\sigma,0) = \alpha(\sigma) + \int_0^Y u(y,\sigma)dy, \quad \sigma \in [0,T], \tag{3}$$

$$u(\sigma,y) = \beta(\sigma) + \int_0^Y u(y,\sigma)dy, \quad \sigma \in [0,T]. \tag{4}$$

The approximate numerical solution for the case of the parabolic equation was examined in [4] using the implicit and explicit finite difference approach. The explicit finite difference solution to the pseudo-parabolic equation was addressed in [10]. This study investigates the approximate solutions of the pseudo-hyperbolic partial differential equations using the explicit finite difference technique, as well as derives stability analysis for the given problem using the suggested method.

This study is divided into several sections: in the second section, explicit finite difference schemes for pseudo-hyperbolic partial differential equations are defined. The stability estimates are presented in the third section. The fourth section discusses utilizes the finite difference method to a pseudo-hyperbolic equation. Finally, the last section discusses the main conclusion, which summarises the findings of this study.

2 Finite Difference Schemes

According to the methodology of [10], we write the three-layer schemes for equations (1.1)-(1.4), for this goal. The explicit finite difference schemes can be constructed using (u_{txx}) was first used for pseudo-parabolic in [10].

We provide grids in standardised steps

$$\overline{w}^h = \{y_i : y_i = ih, \quad i = \overline{0, M}\}, \quad h = \frac{Y}{M},$$

$$\overline{w}^\sigma = \{\sigma_j : \sigma_j = j\sigma, \quad j = \overline{0, N}\}, \quad \sigma = \frac{T}{N},$$

$w^h = \{y_1, y_2, \ldots, y_{M-1}\}, \quad w^\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_{N-1}\}$. In [0,Y]×[0,T], we utilize grids $\overline{w} = \overline{w}^h \times \overline{w}^\sigma, \quad w = w^h \times w^\sigma$. We utilize $U_i^j = U(y_i, \sigma_j)$ to the function described on this grids $\overline{w}^h \times \overline{w}^\sigma$. Additionally, then we define the vectors $\overline{U} = (U_0, U_1, \ldots, U_M)^\sigma$ and $U = (U_0, U_1, \ldots, U_{M-1})^\sigma$. At any point of $(\sigma_j, y_i) \in w$, we have

$$(u_{\sigma yy})_i^j = (u_{yy\sigma})_i^j = \frac{1}{h^2}\left((u_\sigma)_{i-1}^j - 2(u_\sigma)_i^j + (u_\sigma)_{i+1}^j\right) + O(h^2),$$

we create explicit finite difference schemes for third order derivative (u_{xxt}) as follows:

$$\begin{aligned}(u_{\sigma yy})_i^j &= \frac{1}{h^2}\left(\frac{u_{i-1}^j - u_{i-1}^{j-1}}{\sigma} - 2\frac{u_i^{j+1} - u_i^j}{\sigma} + \frac{u_{i+1}^j - u_{i+1}^{j-1}}{\sigma} + O(\sigma)\right) + O(h^2) \\ &= \frac{1}{\sigma}\left(\frac{u_{i-1}^j - 2u_i^{j+1} + u_{i+1}^j}{h^2} - \frac{u_{i-1}^{j-1} - 2u_i^j + u_{i+1}^{j-1}}{h^2}\right) + O(\frac{1}{h^2} + h^2). \qquad (5)\end{aligned}$$

Now, we write the three-layer finite difference schemes for the problems (1.1)–(1.4), as

$$\begin{aligned}\frac{U_i^{j+1} - 2U_i^j + U_i^{j-1}}{\sigma^2} &= \frac{1}{\sigma}\left(\frac{U_{i-1}^j - 2U_i^{j+1} + U_{i+1}^j}{h^2} - \frac{U_{i-1}^{j-1} - 2U_i^j + U_{i+1}^{j-1}}{h^2}\right) \\ &\quad + \frac{U_{i-1}^j - 2U_i^j + U_{i+1}^j}{h^2} + F_i^j, \qquad (6)\end{aligned}$$

$$U_0^j = L(\overline{U}^j) + V_1^j, \quad t_j \in \overline{w}^\sigma, \qquad (7)$$

$$U_M^j = L(\overline{U}^j) + V_r^j, \quad t_j \in \overline{w}^\sigma, \qquad (8)$$

$$U_i^0 = (V_0)_i, \quad y_i \in \overline{w}^h, \qquad (9)$$

$$U_i^1 = (V_1)_i, \quad y_i \in \overline{w}^h, \qquad (10)$$

here, $L(\overline{U}^j) = U_0^j + U_M^j \frac{h}{2} + \sum_{i=1}^{M-1} U_i^j h$. According to the use of the three-layer layout, it should assess a supplement condition on the first layer. We should implement an implicit two-layer difference scheme in the first step of certainty $O(\sigma^2 + h^2)$ [10]. Then, we obtain

$$\begin{aligned} \sigma^2 F_i^j &= U_i^{j+1} - 2U_i^j + U_i^{j-1} - \frac{\sigma\lambda U_i^{j+1}}{h^2} + \frac{2\sigma\lambda U_{i-1}^j}{h^2} - \frac{\sigma\lambda U_{i+1}^j}{h^2} + \frac{\sigma\lambda U_{i-1}^{j-1}}{h^2} \\ &\quad - \frac{2\sigma\lambda U_i^j}{h^2} + \frac{\sigma\lambda U_{i+1}^{j-1}}{h^2} - \frac{\sigma^2 U_{i-1}^j}{h^2} + \frac{2\sigma^2 U_i^j}{h^2} - \frac{\sigma^2 U_{i+1}^j}{h^2}. \end{aligned} \tag{11}$$

Considering (2.1), we obtain the difference problem (2.2)–(2.6) the convergence of the (1.1)-(1.4) conditionally via $O(\sigma^2 + h^2 + \frac{\sigma}{h^2})$ as $\sigma = O(h^2)$. If $\lambda = 0$, then Equation (6) is conditionally consistent via an explicit finite difference scheme with $O(\sigma^2 + h^2 + \frac{\sigma^2}{h^2})$ for the unconditionally stable Dirichlet boundary conditions for the second order parabolic equation [10].

We construct the three-layer FDS (2.2)–(2.6) to two-layer system like in [8]. First, we consider (2.3) and (2.4) as a linear system by U_0^j and U_M^j, and

$$U_0^j = \tilde{\gamma}_0 \tilde{L}(U^j) + \tilde{V}_1^j, \quad U_M^j = \tilde{\gamma}_1 \tilde{L}(U^j) + \tilde{V}_r^j, \tag{12}$$

where

$$\tilde{\gamma}_0 = \frac{1}{d}, \quad \tilde{\gamma}_1 = \frac{1}{d}, \quad \tilde{L}(U^j) = \sum_{i=1}^{M-1} U_i^j h, \quad d = 1 - h,$$

$$\tilde{V}_1^j = \frac{V_1^j + hc^j}{d}, \quad \tilde{V}_r^j = \frac{V_r^j + hc^j}{d}, \quad c^j = \frac{V_r^j - V_1^j}{2}.$$

Here, $d > 0$ for $h < 1$ and $d < 0$ if h is small enough $(h > 1)$. Substituting the expressions (2.8) in (2.7), we can rewrite finite difference scheme (2.2)–(2.4) for each $j = \overline{1, N-1}$ as

$$AU^{j+1} + BU^j + CU^{j-1} = \sigma^2 F^j. \tag{13}$$

3 Matrix Stability for Finite Difference Schemes

Assume that $h = \frac{X}{M}$ for $x-$axis and $\sigma = \frac{T}{N}$ for $t-$axis as grid mess, then we acquire

$$x_i = ih; \quad i = 0, 1, 2, \ldots, M, \quad t_j = j\sigma; \quad j = 0, 1, 2, \ldots, N,$$

for the pseudo hyperbolic partial differential equation (1.1), we create the difference schemes by three-layer FDS (2.2), as

$$\begin{cases} \frac{U_i^{j+1}-2U_i^j+U_i^{j-1}}{\sigma^2} = \frac{1}{\sigma}\left(\frac{U_{i-1}^j-2U_i^{j+1}+U_{i+1}^j}{h^2} - \frac{U_{i-1}^{j-1}-2U_i^j+U_{i+1}^{j-1}}{h^2}\right) \\ \qquad\qquad + \frac{U_{i-1}^j-2U_i^j+U_{i+1}^j}{h^2} + F_i^j. \\ U_0^j = L(\overline{U}^j) + V_1^j, \\ U_M^j = L(\overline{U}^j) + V_r^j, \\ U_i^0 = (V_0)_i, \\ U_i^1 = (V_1)_i, \end{cases} \tag{14}$$

where $L(\overline{U}^j) = U_0^j + U_M^j \frac{h}{2} + \sum_{i=1}^{M-1} U_i^j h$.
We can rewrite equation (3.1) as

$$\begin{cases} \sigma^2 F_i^j = U_i^{j+1} - 2U_i^j + U_i^{j-1} - \frac{\sigma\lambda U_i^{j+1}}{h^2} + \frac{2\sigma\lambda U_{i-1}^j}{h^2} - \frac{\sigma\lambda U_{i+1}^j}{h^2} + \frac{\sigma\lambda U_{i-1}^{j-1}}{h^2} \\ \qquad - \frac{2\sigma\lambda U_i^j}{h^2} + \frac{\sigma\lambda U_{i+1}^{j-1}}{h^2} - \frac{\sigma^2 U_{i-1}^j}{h^2} + \frac{2\sigma^2 U_i^j}{h^2} - \frac{\sigma^2 U_{i+1}^j}{h^2}. \\ U_0^j = L(\overline{U}^j) + V_1^j, \\ U_M^j = L(\overline{U}^j) + V_r^j, \\ U_i^0 = (V_0)_i, \\ U_i^1 = (V_1)_i. \end{cases} \tag{15}$$

Utilizing the initial conditions, (3.2) yields:

$$\begin{cases} AU^{j+1} + BU^j + CU^{j-1} = \sigma^2 F^j \\ \sigma^2 F_i^j = \left(1 + \frac{2\sigma\lambda}{h^2}\right)U_i^{j+1} + \left(-2 - \frac{2\sigma\lambda}{h^2} + \frac{2\sigma^2}{h^2}\right)U_i^j + \left(-\frac{\sigma\lambda}{h^2} - \frac{\sigma^2}{h^2}\right) \\ \qquad U_{i+1}^j + \left(-\frac{\sigma\lambda}{h^2} - \frac{\sigma^2}{h^2}\right)U_{i-1}^j + U_i^{j-1} + \frac{\sigma\lambda U_{i-1}^{j-1}}{h^2} + \frac{\sigma\lambda U_{i-1}^{j+1}}{h^2}. \\ U_0^j = L(\overline{U}^j) + V_1^j, \\ U_M^j = L(\overline{U}^j) + V_r^j, \\ U_i^0 = (V_0)_i, \\ U_i^1 = (V_1)_i, \end{cases} \tag{16}$$

where $F_i^j = F(x_i, t_j)$, and $U_i^j = U(x_i, t_j)$ and

$$A = \begin{bmatrix} Z & 0 & 0 & \dots & 0 & 0 \\ 0 & Z & 0 & \dots & 0 & 0 \\ 0 & 0 & Z & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & Z & 0 \\ 0 & 0 & 0 & \dots & 0 & Z \end{bmatrix}_{(M-1)\times(M-1)},$$

where $Z = \left(1 + \frac{2\sigma\lambda}{h^2}\right)$,

$$B = \begin{bmatrix} p & k & 0 & 0 & \dots & 0 \\ g & p & k & 0 & \dots & 0 \\ 0 & g & p & k & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \dots & g & p & k \\ 0 & 0 & \dots & 0 & g & p \end{bmatrix}_{(M-1)\times(M-1)},$$

where $p = \left(-2 - \frac{2\sigma\lambda}{h^2} + \frac{2\sigma^2}{h^2}\right)$, $k = \left(-\frac{\sigma\lambda}{h^2} - \frac{\sigma^2}{h^2}\right)$ and $g = \left(-\frac{\sigma\lambda}{h^2} - \frac{\sigma^2}{h^2}\right)$,

$$C = \begin{bmatrix} 1 & a & 0 & 0 & \dots & 0 \\ a & 1 & a & 0 & \dots & 0 \\ 0 & a & 1 & a & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \dots & a & 1 & a \\ 0 & 0 & \dots & 0 & a & 1 \end{bmatrix}_{(M-1)\times(M-1)},$$

where $a = \left(\frac{\sigma\lambda}{h^2}\right)$. We write

$\|A\| = \|A\|_\infty \max_{1\leq i\leq(M-1)} \left\{ \sum_{i=1}^{M-1} |a_{jm}| \right\}$, where $A = [ajm]_{(M-1)\times(M-1)}$, and I is unit matrix.

If $-2 - \frac{2\sigma\lambda}{h^2} + \frac{2\sigma^2}{h^2} > 0$, then $\|A^{-1}B\| \leq 3$.

Proof:

$$\|A^{-1}B\| \leq \|A^{-1}\|\|B\| \leq \frac{1}{\min\limits_{1\leq i\leq (M-1)} \mid a_{mm} \mid - \left\{ \sum_{i=1}^{M-1} a_{jm} \right\}} \|B\|$$

$$\leq \frac{\mid -2 - \frac{2\sigma\lambda}{h^2} + \frac{2\sigma^2}{h^2} \mid + \mid -\frac{\sigma\lambda}{h^2} - \frac{\sigma^2}{h^2} \mid + \mid -\frac{\sigma\lambda}{h^2} - \frac{\sigma^2}{h^2} \mid}{\mid 1 + \frac{2\sigma\lambda}{h^2} \mid}$$

$$= \frac{2 + \frac{2\sigma\lambda}{h^2} - \frac{2\sigma^2}{h^2} + \frac{2\sigma\lambda}{h^2} + \frac{2\sigma^2}{h^2}}{1 + \frac{2\sigma\lambda}{h^2}}$$

$$= \frac{3(1 + \frac{2\sigma\lambda}{h^2})}{1 + \frac{2\sigma\lambda}{h^2}} - \frac{1}{1 + \frac{2\sigma\lambda}{h^2}}$$

$$= 3 - \frac{1}{1 + \frac{2\sigma\lambda}{h^2}} < 3.$$

$$\|A^{-1}C\| = 1.$$

Proof:

$$\|A^{-1}C\| \leq \|A^{-1}\|\|C\| \leq \frac{1}{\min\limits_{1\leq i\leq (M-1)} \mid a_{mm} \mid - \left\{ \sum_{i=1}^{M-1} |a_{jm}| \right\}} \|C\|$$

$$= \frac{1 + 2 \mid \frac{\sigma\lambda}{h^2} \mid}{\mid \frac{1+2\sigma\lambda}{h^2} \mid} = 1.$$

If $\|A^{-1}B\| \leq 3$ and $\|A^{-1}C\| = 1$, then, the equation (3.3) is stable.

Proof: Applying the same procedure of [12], and applying Lemmas 3.1 and 3.2 the proof of the theorem is completed.

4 Numerical Experiments

For this aim, we provide the finite difference method applications to the pseudo hyperbolic partial differential equation. We take into consideration

$$u_{tt}(x,t) - \lambda u_{txx}(x,t) - u_{xx}(x,t) - e^t(2-\lambda)\sin x = 0, x \in (0,\pi), \quad t \in (0,1), \tag{17}$$

with the initial conditions

$$u(0,x) = \sin x, \quad u_t(x,0) = -\sin x, \quad x \in [0,\pi], \tag{18}$$

and the boundary conditions

$$u(t,0) = \int_0^{\pi} u(x,t)dx - 2e^{-t}, \quad t \in [0,1],$$

$$u(t,\pi) = \int_0^\pi u(x,t)dx - 2e^{-t}, \quad t \in [0,1].$$

Solution. The exact solution is presented as $u(t,x) = e^{-t}\sin x$. We apply difference schemes method to investigate the problem (4.1) depends on initial condition (4.2). For difference equations, we employ a modified Gauss elimination strategy. The numerical solution's maximum norm of the mistake:

$$\varepsilon = \max_{\substack{i=0,1,\ldots,M \\ j=0,1,\ldots,N}} |u(t,x) - u(t_j,x_i)|.$$

Where $u_i^j = u(t_i,x_j)$ is the approximate solution, $u(t,x)$ is the exact solution and ε is the error analysis. In the following table, we explain the approximate solution, the exact solution and error analysis of the difference scheme (2.2).

Utilizing Matlab programming, we can obtain the following simulations.

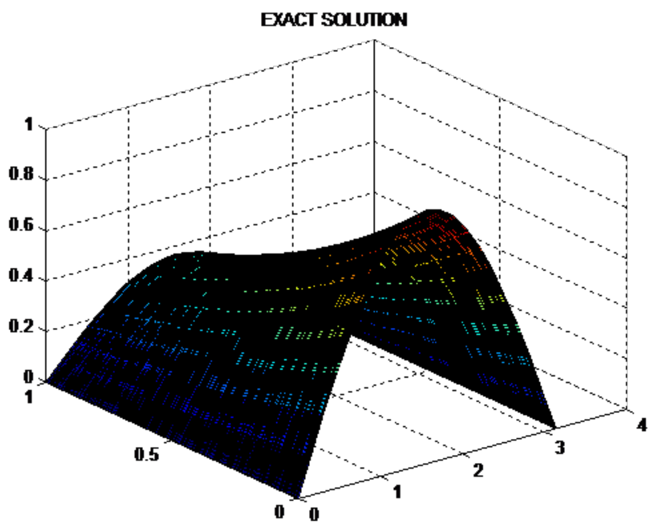

Figure 1: Shows the exact solution of equation (4.1), for $N = 400, \quad M = 20, \quad and \quad \lambda = 0.0000001$

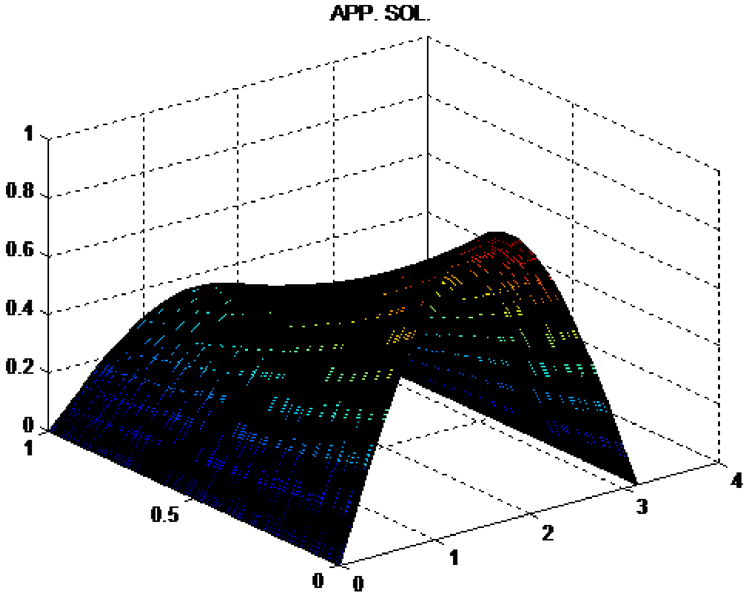

Figure 2: Gives the approximate solution of equation (4.1), for $N = 400, \quad M = 20, \quad and \quad \lambda = 0.0000001$

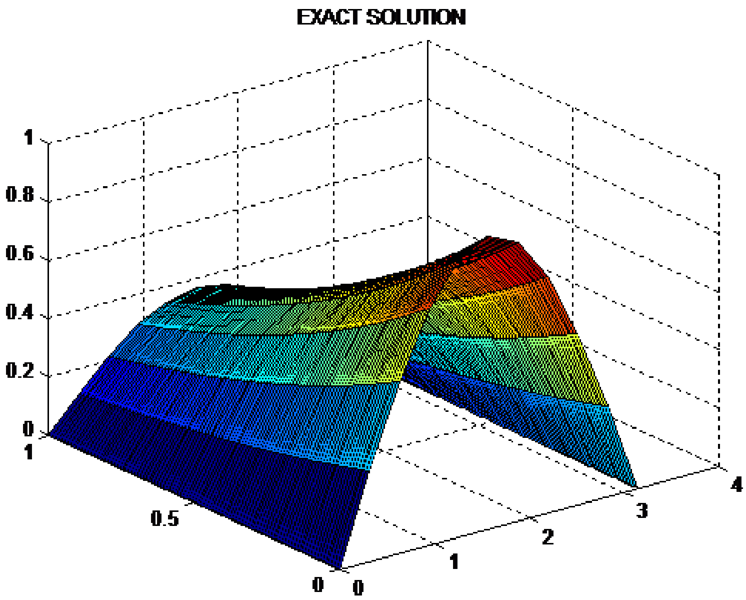

Figure 3: Exact solution of equation (4.1), for $N = 400, \quad M = 10, \quad and \quad \lambda = 0.01$

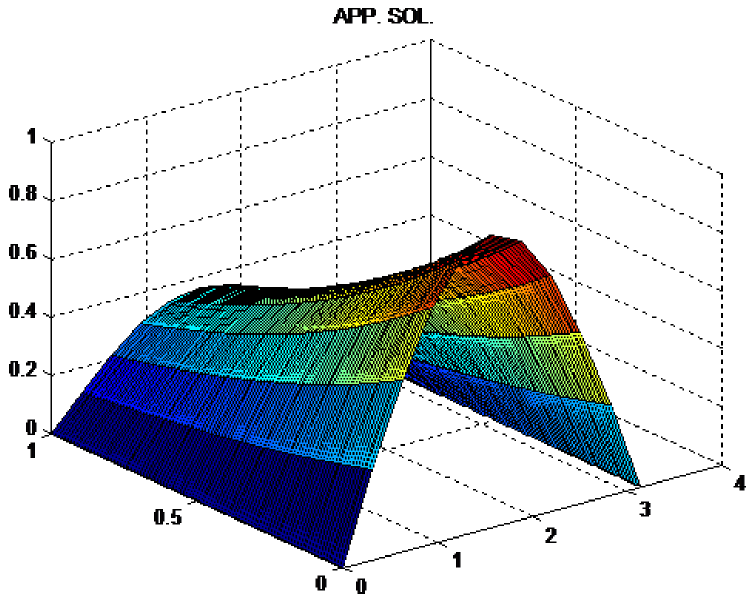

Figure 4: Gives the approximate solution of equation (4.1), for $N = 400, \quad M = 10, \quad and \quad \lambda = 0.01$

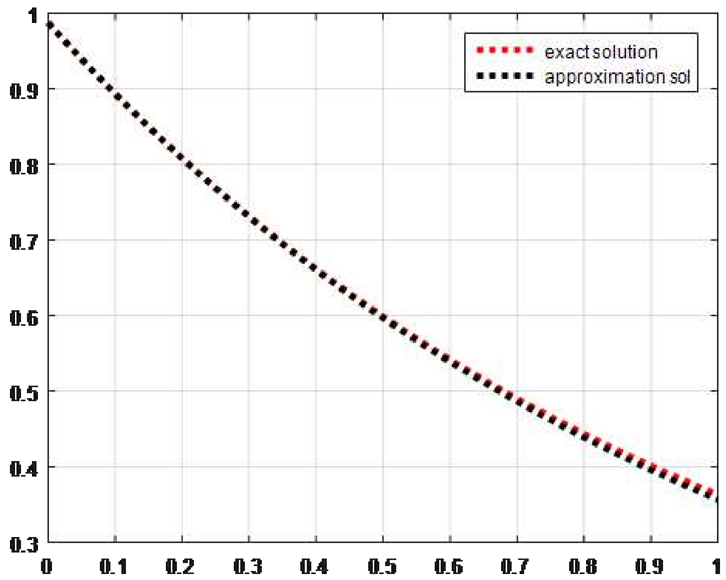

Figure 5: gives the comparison between the approximate and exact solution of equation (4.1), for $N = 400, \quad M = 20, \quad and \quad \lambda = 0.01$ and $0 \leq t \leq 1$

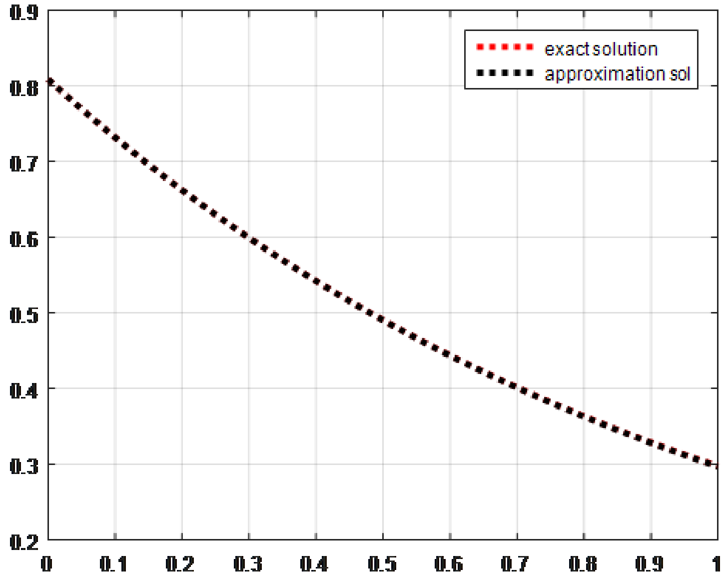

Figure 6: Comparison between the approximate and exact solution of equation (4.1), for $N = 900, \quad M = 30, \quad and \quad \lambda = 0.01$ and $0 \leq t \leq 1$

It can be seen from the above figures that the exact solution and the approximate solutions are close to each other. However, we give the following error analysis table to make a definite distinction.

Table 1: Numerical Results

$\sigma = \frac{1}{N}, \quad h = \frac{\pi}{M}$	λ	Error(ε)
$N = 100, \quad M = 10$	$\lambda = 0.5$	0.028107430087899
	$\lambda = 0.1$	0.007909466520303
	$\lambda = 0.01$	0.002181874884325
$N = 400, \quad M = 20$	$\lambda = 0.5$	0.248458473959889
	$\lambda = 0.1$	0.006794799568062
	$\lambda = 0.01$	0.001040060956417
$N = 900, \quad M = 30$	$\lambda = 0.5$	0.026761174233341
	$\lambda = 0.1$	0.006592582475333
	$\lambda = 0.01$	0.060663970068598
$N = 1600, \quad M = 40$	$\lambda = 0.5$	0.026688764570917
	$\lambda = 0.1$	0.006522117686502
	$\lambda = 0.01$	7.6528×10^{-4}
$N = 2500, \quad M = 50$	$\lambda = 0.5$	0.026655292283168
	$\lambda = 0.1$	0.006489557803639
	$\lambda = 0.01$	7.3248×10^{-4}
	$\lambda = 0.001$	1.2604×10^{-4}
	$\lambda = 0.0001$	7.1181×10^{-5}

5 Conclusion

In this study, nonlocal boundary value conditions are used to examine pseudo-hyperbolic partial differential equations. The explicit finite difference techniques were developed for this equation. A matrix stability method was used to demonstrate stability estimations for this difference scheme method. The algorithm of this problem is written for the numerical solution. Matlab programme was used for the numerical solutions. The numerical solutions were created using Matlab software. The test problem shows the effectiveness and suitability of this approach for approximating the solution of pseudo-hyperbolic partial differential equations. The simulations for this problem are displayed both the exact and approximate solutions.

Based on data analysis table 1. Generally, there is good agreement between the absolute error results produced using the explicit finite difference technique for various values of λ and the values chosen by M and N. When we choose $\lambda = 0.0001$, $\lambda = 0.001$, $\lambda = 0.01$, $\lambda = 0.5$, $M = 50$, and

$N = 2500$, the error analysis is excellent because the precise and approximate solutions are then closer to one another.

The exact solution in figure 1 and the approximate solution in figure 2 for $\lambda = 0.0000001$, $M = 20$ and $N = 400$ are quite similar, as can be seen from the graphical representation of the solutions. Additionally, for $\lambda = 0.01$, $M = 20$ and $N = 400$, the exact solution in Figure 3 and the approximate solution in Figure 4 are close to one another, Figures 5. and 6., which compare the exact and approximate solutions to equation (4.1), can be observed to be in the same line for the values of $N = 400, \quad M = 20, \quad and \quad \lambda = 0.01$ and $0 \leq t \leq 1$, $N = 900, \quad M = 30, \quad and \quad \lambda = 0.01$ and $0 \leq t \leq 1$, respectively. The tabular descriptions and graphical representations provided above indicate the effectiveness of the technique used.

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