Proving Collatz Conjecture by finding cycles and diverging seeds

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Proving Collatz Conjecture by finding cycles and diverging seeds

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Abstract

The Collatz conjecture can be formulated in this question: Do we guarantee reaching one after applying function f (Hereinafter called Collatz function) multiple times?

$$f(x) = (3x + 1,\ x|2) \vee \left(\frac{x}{2}, x \nmid 2\right)$$

In this paper, we discuss an approach to analyze Collatz conjecture nature for all $qx + 1,\ q \in \{1,3,5,7,..\}$, this approach builds a method to know the number of potential cycles for $qx + 1,\ q \in \{1,3,5,7,..\},\ x < M_x,\ M_x \in \mathbb{C}$. Multiple non-trivial cycles (numbers that form a cycle and does not reach one) were previously found where $q \in \{5,\ 181\}$. This paper will introduce equations that find non-trivial cycles, and therefore proving or disproving Collatz conjecture for any q .

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Introduction

Sultanow, Koch, and Sean showed that non-trivial cycles exist [1] for $qx + 1$ as shown in the following table.

Table 1

Non-trivial cycles for $qx + 1$

q	x
5	13, 33, 83, 27, 17, 43
181	27, 35, 99, 611

The existence of non-trivial cycles means that after applying Collatz function multiple times on x , the result will be the same x . This is similar to the 4-2-1 loop in $3x + 1$ function. The existence of another loop (a non-trivial cycle) will immediately disprove Collatz Conjecture. Also, 5,181 and 3 are odd numbers, prime numbers, have prime twins, sum of two squares, and are the smallest hypotenuses of primitive Pythagorean triples, this observation may be used in the future for narrowing q search range, and for future research. Also, the equations in this paper can contribute to building high speed wide-range non-trivial cycles brute forcers as will be shown in the paper.

Searching for other non-trivial cycles for $q = 3$ and beyond require certain methods. This paper will introduce a new method for searching for other non-trivial cycles, by analyzing Collatz function behavior and state certain equations and conclusions that aim to:

1. narrowing the search range for both q and x .
2. introducing a reliable way of analyzing Collatz functions.
3. expect the behavior of a possible non-trivial cycle.

Solving Collatz conjecture can be broken down to two questions:

1. Is there any non-trivial cycles for $q = 3$?

2. Is there a diverging starting number of a Collatz sequence (hereinafter called a tail or a seed)?

Answering both questions partially or fully will depend on analyzing Collatz sequences, which is the main concern of this paper.

A diverging starting number is a seed which goes to infinity, there are plenty of them in $q > 3$ which we will investigate. A seed is a starting number for a Collatz sequence. A dead-end seed is a seed which goes to one. One way of formalizing a dead-end seed is $(f \circ f \circ f \circ \ldots)(x) = 2^n$, $n \in \mathbb{N}$ where x is a dead-end seed. For a seed that forms a non-trivial cycle the formula is $(f \circ f \circ f \circ \ldots)(x) = x$ where x is a seed which forms a closed cycle like what Sultanow, Koch, and Sean found.

Methods

We will start with a generalization for a Collatz function $f(x) = (qx + 1,\ x \nmid 2) \vee \left(\frac{x}{2}, x \middle| 2\right)$. The first observation is that all even numbers will collapse to an odd number. Additionally, we can form infinite number of sets that collapse to one odd number regardless of q value. If we guarantee that $q = 3,\ x = 3$ will eventually collapse to 4-2-1 loop, we guarantee that $x = 3(2^n),\ n \in \mathbb{N} \cup \{0\}$ will eventually collapse to the same loop, this is true for all values of q if we guarantee that the odd seed will eventually collapse to the usual loop. This finding concludes that there is no need to search for even number to find diverging seeds or non-trivial cycles. This finding reduces the search range by half. This conclusion was already deduced in previous research [2]. Also, the continued sequence of divisions until we reach an odd number was observed and discussed in previous research [3, 4].

Any Collatz sequence can be thought of as a connected directed series with heads and tails, where the tails are odd numbers, and the heads are even numbers originating from applying $qx + 1$ over an odd number. For example, $q = 3,\ x = 7$ sequence will have the following series: $7_T \rightarrow 22_H \rightarrow 11_T \rightarrow 34_H \rightarrow 17_T \rightarrow 52_H \rightarrow 26 \rightarrow 13_T \rightarrow 40_H \rightarrow 20 \rightarrow 10 \rightarrow 5_T \rightarrow 16_H \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1_T \rightarrow 2_H \rightarrow$.. where X_T: $Tail$, X_H: $Head$. Also, all numbers between any tail and head are even numbers that are the result of dividing the previous head by two for multiple times.

An expected diverging seed will have $T_n > T_{n-1}$, this means that every odd number in the sequence is higher than the previous odd number in the same sequence, for example, $q = 5,\ x = 45$ and the all sequences $q = 5,\ x = 45(2^n)\ ,\ n \in \mathbb{N} \cup \{0\}$ are diverging with the previous expected behavior.

Based on the Tail-Head behavior, we can construct the required equations for detecting cycles (closed loops or sequences). There are infinitely many types of cycle types. A cycle type has a level x where an x-level cycle has $x + 1$ odd numbers (tails) in it, hereinafter will be called an x-level loop.

The ordinary 4-2-1 cycle in $q = 3,\ x > 0$ has all its numbers belong to $S = \{2^n, n \in \mathbb{N}\}$ with one odd number in the series, this is a zero-level loop.

A zero-level loop is a loop that contains one tail and its divisions. This can be represented by $t = \frac{qt+1}{2^n}$. For any zero-level loop in a specific q , all variables $t,\ n$ must be integers. Also, $n + 1$ is the number of connected numbers in the closed sequence. Furthermore, $n = \log_2\left(\frac{qt+1}{t}\right)$, for $q = 3,\ x = 1$, $n = 2$. To find the number of potential solutions for some certain q , we can solve $\left\lceil \lim_{t\to\infty} \log_2\left(\frac{qt+1}{t}\right) \right\rceil$. For higher level loops, we can construct similar equations: $t_0 = \frac{qt_0+1}{2^n},\ t_1 = \frac{q\left(\frac{qt_0+1}{2^n}\right)+1}{2^m}, \ldots$

Until proved otherwise, the only way to detect x-level loops for certain q is to brute force variables $m, n,\ t_0,\ t_1,$ .... The following table contains some examples of equations for x-level loop.

Level	Tail equation	Last variable equation*
0	$t = \frac{1}{2^a - q}$	$a = \log_2\left(\frac{qt+1}{t}\right)$
1	$t = \frac{-2^a - q}{-2^{a+b} + q^2}$	$b = \log_2\left(\frac{2^a + q + q^2 t}{t}\right) - a$
2	$t = \frac{-2^{a+b} - 2^a q - q^2}{-2^{a+b+c} + q^3}$	$c = \log_2\left(\frac{2^{a+b} + 2^a q + q^2 + q^3 t}{t}\right) - a - b$
3	$t = \frac{-2^{a+b+c} - 2^{a+b} q - 2^a q^2 - q^3}{-2^{a+b+c+d} + q^4}$	$d = \log_2\left(\frac{2^{a+b+c} + 2^{a+b} q + 2^a q^2 + q^3 + q^4 t}{t}\right) - a - b - c$

*: The last variable equation is used to know the number of potential solutions given the inputs as explained before, the ceil of the result is called the last variable upper boundary.

For diverging seeds, the last variable must be higher than the last variable upper boundary and all variables are integers.

Results and discussion

We have successfully generated existing cycles where $q = 5,\ q = 181$. The following tables are results of computational brute force[1]. The tables only contain some of the integer solutions.

Zero-level loops for $3 \leq q < 11\ .x = \frac{qx+1}{2^a}$.

q	x	a
3	-1	1
3	1	2
5	-1	2
7	1	3
9	-1	3

First-level loops for $3 \leq q < 11$

q	x	a	b
3	-5	1	2
3	-7	2	1
5	1	1	4
5	3	4	1
7	1	3	3

Second-level loops for $3 \le q < 11$

q	x	a	b	c
3	5	2	0	3
5	13	3	2	0
5	13	1	1	5
5	17	1	3	3
5	27	3	1	3
5	43	3	3	1
5	83	5	1	1

Some diverging seeds for $q = 5$

x	a	b	$c > Max(c)$	$Max(c)$
5003	145	50	5	-188
218467	145	51	3	-189
1251	143	53	4	-189
-13037	143	5	47	-141
-2781	141	5	50	-139
563	141	2	57	-136
223	135	59	5	-187

Depending on the equation presented in Methods section on the number of potential solutions for zero-level loop for a given q . We can deduce that there are two and only two zero-level loops for $q = 3$, $\left\lceil \lim_{x \to \infty} \left[\log_2 \left(\frac{3x+1}{x} \right) \right] \right\rceil = \lceil \log_2(3) \rceil = 2.$

Conclusion

We have presented a method to detect cycles and diverging seeds. Furthermore, we have regenerated the existing cycles at $q = 5,\ q = 181$, and explored some diverging seeds mathematically.

The two questions presented in Introduction section for solving Collatz conjecture can be answered using the method above, the only limitation to solve Collatz conjecture is a computational limitation.

References

[1]: Koch, C., Sultanow, E., & Cox, S.D. (2020). Divisions by Two in Collatz Sequences: A Data Science Approach.

[2]: Furuta, M. (2022) Proof of Collatz Conjecture Using Division Sequence. Advances in Pure Mathematics, 12, 96-108.

[3]: Kamal Barghout, "On the Probabilistic Proof of the Convergence of the Collatz Conjecture", Journal of Probability and Statistics, vol. 2019, Article ID 6814378, 11 pages, 2019. https://doi.org/10.1155/2019/6814378

[4]: Jan Kleinnijenhuis, Alissa M. Kleinnijenhuis, and Mustafa G. Aydogan: The Collatz tree as a Hilbert hotel: a proof of the 3x + 1 conjecture, arXiv.org-math-arXiv:2008.13643.

Declarations

Footnotes

[1]: Source code can be found here: https://github.com/Zelakolase/CycleBruteForce.

Availability of data and materials

Brute force code is found in footnotes.

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Authors' contributions

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Competing interests

The author declares that he has no conflict of interest.