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Bifurcation analysis in delayed Nicholson equation with harvest term*

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Abstract

In this paper, we investigate a delayed Nicholson equation with delay harvesting term which was proposed in open problems and conjectures formulated by Berezansky et al. (Applied Mathematical Modelling 34 (2010) 1405). The stability switching curves by taking two delays as parameters are obtained via the method introduced by An et al. (J. Differential Equations 266 (2019) 7073). The existence of Hopf singularity on a two-parameter plane is determined by the varying direction of two parameters. Furthermore, the normal form near the Hopf singularity is derived via applying the center manifolds theory and normal forms method of FDEs. Finally, some numerical simulations are carried out to illustrate the theoretical conclusions.

Keywords: Hopf bifurcation, two delays, stability switching curves, two-parameter plane, Nicholson equation.

1 Introduction

Blowfly is one of the most important model organisms in biological research due to its short growth cycle [1, 2]. In 1954, by conducting some experiments on the Australian sheep blowfly, Nicholson [3] observed oscillations of large amplitude in the blowfly population. Since then, there has been much research on the blowfly dynamics in both the biological and mathematical fields [4–9].

To investigate the blowfly population growing in an isolated laboratory, Gurney et. al. [10] proposed a simple time-delay model as follows:

$$\frac{du(t)}{dt} = pu(t - \tau)e^{-au(t-\tau)} - \delta u(t), \quad (1)$$

where $u(t)$ denotes the population of mature adults at time t , p is the maximum per capita daily egg production rate; $1/a$ is the size at which the fly population reproduces at its maximum rate; δ represents the per capita adult death rate and

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τ is the maturation time required for an individual from birth to maturity. They found that system (1) gave good quantitative agreement with Nicholson's classic blowfly data. Besides, the "hump" relationship between future adult population and the current adult population can also be explained.

System (1) has subsequently been considered in many fields by researchers, we refer the reader to read the review by Berezhansky et al. [4] and the references therein. There has been many studies on model (1) with diffusion, see [11–15] and the references therein.

At the end of the review paper [4], the authors have formulated seven open problems and conjectures. The sixth problem is as following: Assume that a harvesting function is a function of the delayed estimate of the true population. Consider Eq. (1) with a linear harvesting term:

$$\frac{dx(t)}{dt} = px(t-\tau)e^{-x(t-\tau)} - \delta x(t) - Hx(t-\sigma), \quad (2)$$

with a delayed harvesting strategy. Here H denotes the capture rate, σ represents the capture delay. p , τ , δ are the same as that in Eq.(1). Clearly, Eq.(2) is a delay equation with two delays. A nature question is how do the delays affect the dynamical behaviour of Eq.(2)? The purpose of the paper is to answer this question.

At first, we present the Hopf bifurcation curves on a two-parameter plane. We stress that for a possible Hopf singularity, the varying direction of these two delays are important as well as the transversal condition. Then, we calculate the normal form at a Hopf singularity along the varying direction $(\gamma_1\mu, \gamma_2\mu)$. The calculation of the normal form with two parameters σ , τ is transformed into the calculation with one parameter μ . This dimension reduction method can reduce the amount of the calculation. Meanwhile, we prove that for some fixed σ , the stability of the positive equilibrium may switches finite times with the increase of τ .

The paper is organized as follows. In section 2, we perform the stability analysis of the positive equilibrium under certain conditions, and calculate the stability switching curves and present the Hopf bifurcation theorem on a two-parameter plane. In section 3, we calculate the normal form at a Hopf singularity along the varying direction $(\gamma_1\mu, \gamma_2\mu)$ of (σ, τ) . In section 4, we carry out some numerical simulation to illustrate the theoretical results. Finally, we make a conclusion to sum up our paper.

2 Stability and existence of Hopf bifurcation

In this section, first we perform the stability analysis of the positive equilibrium, then we will investigate the existence of Hopf bifurcation on the two-parameter plane.

The equilibria of Eq.(2) can be obtained by solving the following equation:

$$pxe^{-x} - \delta x - Hx = 0. \quad (3)$$

Clearly, $x_1 = 0$ is an extinction equilibrium, $x^* = \ln \frac{p}{\delta+H}$ is a positive equilibrium when $p > \delta + H$.

We make the following assumption to guarantee the existence of the positive equilibrium x^* :

$$(\mathbf{H}_1) \quad p > \delta + H.$$

Throughout this paper, we focus on the dynamics analysis near the positive fixed point x^* . So we assume that $(\mathbf{H}_1)$ is always true. The characteristic equation associated with the positive equilibrium x^* is given by

$$D(\lambda; \sigma, \tau) := \lambda + \delta - (\delta + H)[1 - \ln(\frac{p}{\delta+H})]e^{-\lambda\tau} + He^{-\lambda\sigma} = 0. \quad (4)$$

We can easily draw the following conclusion.

Lemma 1. *If $\tau = \sigma = 0$ and $(\mathbf{H}_1)$ is satisfied, then the root of Eq.(4) is negative. Furthermore, x^* is asymptotically stable.*

One can see that if $\ln(\frac{p}{\delta+H}) = 1$, then the equation (4) is reduced into a equation contain only one term with exponential function. In this situation, the fixed point is $x^* = 1$. So in the following, we investigate the dynamics of (2) in the two cases of $p = e(\delta + H)$ and $p \neq e(\delta + H)$, respectively.

2.1 Stability analysis when $p = e(\delta + H)$

We have know that when $p = e(\delta + H)$, the equilibrium $x^* = 1$, and Eq. (4) becomes

$$\lambda + \delta + He^{-\lambda\sigma} = 0. \quad (5)$$

Then we conclude the following conclusions.

Theorem 1. *If $p = e(\delta + H)$, then*

- (i). *The equilibrium $x^* = 1$ is asymptotically stable when $H \leq \delta$.*
- (ii). *If $H > \delta$, then $x^* = 1$ is asymptotically stable for $\sigma \in [0, \sigma_0)$ and unstable for $\sigma \in (\sigma_0, +\infty)$. Particularly, Hopf bifurcation occurs when $\sigma = \sigma_j$, where*

$$\sigma_j = \frac{1}{\sqrt{H^2 - \delta^2}} \left[\arccos\left(-\frac{\delta}{H}\right) + 2j\pi \right], \quad j = 0, 1, 2, \dots$$

Proof. Clearly, $\lambda = 0$ is not a root of (5), and $\lambda = -(\delta + H) < 0$ when $\sigma = 0$. Let $\lambda = i\omega$ ($\omega > 0$) be the root of (5). Substitution it into (5) and separation the real and imaginary parts, it follows that

$$\begin{cases} H \sin(\omega\sigma) = \omega, \\ H \cos(\omega\sigma) = -\delta. \end{cases} \quad (6)$$

Square these two equations of (6) and sum them up, we have $\omega^2 + \delta^2 = H^2$, this equals

$$\omega^2 = H^2 - \delta^2. \quad (7)$$

- (i) If $H \leq \delta$, there is no positive ω such that $i\omega$ satisfies (5). This implies that all the roots of (5) have negative real parts. Thus x^* is asymptotically stable.
(ii) If $H > \delta$, then $\omega_0 = \sqrt{H^2 - \delta^2} > 0$. From (6), we have

$$\omega_0 \sigma_j = \arccos\left(-\frac{\delta}{H}\right) + 2j\pi, \quad j = 0, 1, 2, \dots$$

thus

$$\sigma_j = \frac{1}{\sqrt{H^2 - \delta^2}} \left[\arccos\left(-\frac{\delta}{H}\right) + 2j\pi \right], \quad j = 0, 1, 2, \dots$$

This implies that when $\sigma = \sigma_j$, equation (5) has a pair of purely imaginary roots $\pm i\omega_0$.

Let $\lambda(\sigma) = \alpha(\sigma) + i\omega(\sigma)$ be the root of Eq.(5) satisfying $\alpha(\sigma_j) = 0$ and $\omega(\sigma_j) = \omega_0$. Applying the result due to Cooke and Grossman [17], we have that $\alpha'(\sigma_j) > 0$. This shows that the transversal condition is satisfied. Meanwhile, we have that all the roots of (5) have negative real parts when $\sigma \in [0, \sigma_0)$, and (5) has at least a couple of roots with positive real parts when $\sigma > \sigma_0$. Thus, the conclusions (ii) follows. $\square$

Theorem 1 shows that when the birth rate, death rate and the capture rate of the population meet certain relation, the mature delay τ has no influence on the stability of the population; When the capture rate is lower than the death rate, the positive equilibrium is locally stable as long as it exists; When the capture rate is higher than the death rate, the stability of the positive equilibrium is related to the capture delay σ .

2.2 Stability analysis when $p \neq e(\delta + H)$

In this subsection, we shall perform the dynamics analysis near the positive equilibrium x^* when $p \neq e(\delta + H)$ under the assumption $(\mathbf{H}_1)$. In this situation, there are two terms with exponent function in (4). In order to analyze the distribution of the roots of (4), the method introduced by An et al. [16] shall be used. We first state the following lemma due to Ruan and Wei [18].

Lemma 2. *As (σ, τ) varies continuously in $\mathbb{R}_+^2$, the number of zeros (with multiplicity counted) of $D(\lambda; \sigma, \tau)$ on $\mathbb{C}_+$ can change only if a zero appears or cross the imaginary axis.*

By direct calculation, we get that the root of Eq.(4) is given by

$$\lambda = -(\delta + H) \ln\left(\frac{p}{\delta + H}\right) < 0$$

when $(\tau, \sigma) = (0, 0)$, and $\lambda = 0$ is not a root of Eq.(4) always. Now we are going to seek purely imaginary roots of (4) to explore the stability switching curves of x^* .

Let $\lambda = i\omega$ ($\omega > 0$) be a root of Eq.(4), and substitute it into Eq.(4), we have

$$i\omega + \delta - (\delta + H)[1 - \ln(\frac{p}{\delta + H})]e^{-i\omega\tau} + He^{-i\omega\sigma} = 0. \quad (8)$$

For simplicity, denote $a_1 = -(\delta + H)[1 - \ln(\frac{p}{\delta + H})]$, Eq.(8) becomes

$$i\omega + \delta + a_1 e^{-i\omega\tau} + He^{-i\omega\sigma} = 0. \quad (9)$$

Thus we have

$$|i\omega + \delta + He^{-i\omega\sigma}|^2 = |-a_1 e^{-i\omega\tau}|^2,$$

which equals

$$\delta^2 + \omega^2 + H^2 + 2H[\delta \cos(\omega\sigma) - \omega \sin(\omega\sigma)] = a_1^2.$$

There exists some continuous function $\phi \in (-\pi, 2\pi] \cap \text{Arg}\{\delta + i\omega\}$ such that

$$2H\sqrt{\delta^2 + \omega^2} \cos(\omega\sigma + \phi) = a_1^2 - \delta^2 - \omega^2 - H^2, \quad (10)$$

the existence of non-negative σ satisfying (10) if and only if

$$2H\sqrt{\delta^2 + \omega^2} \geq |a_1^2 - \delta^2 - \omega^2 - H^2|. \quad (11)$$

Define

$$F(\omega) = 2H\sqrt{\delta^2 + \omega^2} - |a_1^2 - \delta^2 - \omega^2 - H^2|,$$

and denote the set of all positive ω 's satisfying (11) as Ω^1 . Let

$$\theta_1(\omega) = \arccos\left(\frac{a_1^2 - \delta^2 - \omega^2 - H^2}{2H\sqrt{\delta^2 + \omega^2}}\right), \quad \omega \in \Omega^1. \quad (12)$$

Then we have

$$\sigma_{k_1}^\pm = \frac{\pm\theta_1(\omega) - \phi(\omega) + 2k_1\pi}{\omega}, \quad k_1 = 0, \pm 1, \pm 2 \dots \quad (13)$$

Similarly, we get that

$$\tau_{k_2}^\pm = \frac{\pm\theta_2(\omega) - \psi(\omega) + 2k_2\pi}{\omega}, \quad k_2 = 0, \pm 1, \pm 2 \dots, \quad (14)$$

where

$$\theta_2(\omega) = \arccos\left(\frac{H^2 - \delta^2 - \omega^2 - a_1^2}{2a_1\sqrt{\delta^2 + \omega^2}}\right), \quad (15)$$

and $\psi \in (-\pi, 2\pi] \cap \text{Arg}\{\delta + i\omega\}$ such that

$$2a_1\sqrt{\delta^2 + \omega^2} \cos(\omega\tau + \psi) = H^2 - \delta^2 - \omega^2 - a_1^2.$$

Here, the condition on ω is as follows:

$$|2a_1\sqrt{\delta^2 + \omega^2}| \geq |H^2 - \delta^2 - \omega^2 - a_1^2|. \quad (16)$$

Denote the set of all positive ω 's satisfying Eq.(16) as Ω^2 . We can easily prove that (11) is equivalent to (16) by squaring both sides of the two inequalities. Thus, $\Omega^1 = \Omega^2 : \triangleq \Omega$.

We present the following lemma to clarify that in a finite region, the number of stability switching curves is finite.

Lemma 3. *The set Ω consists of a finite number of intervals of finite length.*

One can verify that $\tau = \tau_{k_2}^+$ when $\sigma = \sigma_{k_1}^+$, and $\tau = \tau_{k_2}^-$ when $\sigma = \sigma_{k_1}^-$, respectively. Therefore,

$$\begin{aligned} \mathcal{T}_{k_1, k_2}^\pm &= \{(\sigma_{k_1}^\pm(\omega), \tau_{k_2}^\pm(\omega)) : \omega \in \Omega\} \\ &= \left\{ \left(\frac{\pm\theta_1(\omega) - \phi(\omega) + 2k_1\pi}{\omega}, \frac{\pm\theta_2(\omega) - \psi(\omega) + 2k_2\pi}{\omega} \right) : \omega \in \Omega \right\}, \end{aligned} \quad (17)$$

and the set of all stability switching curves can be written as

$$\mathcal{T} = \bigcup_{k_1, k_2 \in \mathbb{Z}} (\mathcal{T}_{k_1, k_2}^{\pm k} \cap \mathbb{R}_+^2). \quad (18)$$

Proposition 1. *$\mathcal{T}$ is the set of all stability switching curves on $\sigma - \tau$ plane for Eq.(4). For any $(\sigma, \tau) \in \mathcal{T}$, Eq.(4) has at least a pair of purely imaginary roots $\pm i\omega$, with $\omega \in \Omega$.*

2.3 Hopf bifurcation theorem on two-parameter plane

Now we conclude the following Hopf bifurcation theorem.

Theorem 2. *For any $p \in \mathcal{T}$ and for any smooth curve $\mathcal{S}$ intersecting with $\mathcal{T}$ transversely at p , we define the tangent of $\mathcal{S}$ at p by $\vec{l}$. If $\frac{\partial \text{Re}\lambda}{\partial l} \Big|_p \neq 0$, and all other eigenvalues of (4) have non-zero real parts at p , then system (2) undergoes a Hopf bifurcation at p when parameters (σ, τ) cross $\mathcal{T}$ at p along $\mathcal{S}$.*

Proof. Denote $p(\sigma_0, \tau_0)$. Let U be a neighbourhood of p . Suppose that the equation of the curve $\mathcal{S}$ is $\mathcal{S}(\sigma, \tau) = 0$. Introduce a mapping $J : U \rightarrow \mathbb{R}^2$, whose coordinate component function is expressed by

$$\begin{cases} \delta_1 = \delta_1(\sigma, \tau), \\ \delta_2 = \delta_2(\sigma, \tau), \end{cases} \quad (\sigma, \tau) \in O(p).$$

Suppose that J locally maps $p(\sigma_0, \tau_0)$, $\mathcal{T}$ and $\mathcal{S}$ to $p'(0, 0)$, δ_1 axis and δ_2 axis, respectively (shown in Fig.1), and the Jacobian determinant $\frac{\partial(\delta_1, \delta_2)}{\partial(\sigma, \tau)} \Big|_p$ of mapping

Figure 1: A sketch of the transformation from σ - τ plane to δ_1 - δ_2 plane.

J is not zero, then by inverse function group theorem, there exists a neighbourhood of p' , $O(p')$ such that there is a unique inverse mapping J^{-1} of J ,

$$\begin{cases} \sigma = \sigma(\delta_1, \delta_2), \\ \tau = \tau(\delta_1, \delta_2), \end{cases} \quad (\delta_1, \delta_2) \in O(p').$$

When $\delta_1 = 0$, the characteristic equation (4) of system (2) has purely imaginary root $i\omega$ at $\delta_2 = 0$. We only need to verify that $\frac{dRe\lambda}{d\delta_2} \neq 0$. Since J locally maps $\mathcal{S}$ to δ_2 axis and the δ_1 function is expressed by $\delta_1 = \delta_1(\sigma, \tau)$, then the curve $\mathcal{S}$ can be written as $\mathcal{S}(\sigma, \tau) = \delta_1(\sigma, \tau) = 0$. The tangent vector of curve $\mathcal{S}$ is $\vec{l} = (-\frac{\partial \mathcal{S}}{\partial \tau}, \frac{\partial \mathcal{S}}{\partial \sigma})^T = (-\frac{\partial \delta_1}{\partial \tau}, \frac{\partial \delta_1}{\partial \sigma})^T$. Denote $J_1 = \begin{pmatrix} \frac{\partial \delta_1}{\partial \sigma} & \frac{\partial \delta_1}{\partial \tau} \\ \frac{\partial \delta_2}{\partial \sigma} & \frac{\partial \delta_2}{\partial \tau} \end{pmatrix}$. Obviously, $\vec{e}|_{\delta_2} = (0, 1)^T = \frac{1}{\det J_1} J_1 \vec{l}$. Since

$$\begin{aligned} \frac{dRe\lambda}{d\delta_2} &= \frac{\partial Re\lambda}{d\vec{e}|_{\delta_2}} = \left(\frac{\partial Re\lambda}{\partial \delta_1}, \frac{\partial Re\lambda}{\partial \delta_2} \right) \cdot \vec{e}|_{\delta_2} \\ &= \left(\frac{\partial Re\lambda}{\partial \delta_1}, \frac{\partial Re\lambda}{\partial \delta_2} \right) \cdot \frac{1}{\det J_1} \cdot J_1 \cdot \vec{l} \\ &= \frac{1}{\det J_1} \left(\frac{\partial Re\lambda}{\partial \sigma}, \frac{\partial Re\lambda}{\partial \tau} \right) \cdot \vec{l} \\ &= \frac{1}{\det J_1} \frac{\partial Re\lambda}{\partial \vec{l}}. \end{aligned}$$

Thus the transversal condition $\frac{dRe\lambda}{d\delta_2}|_p \neq 0$ holds if $\frac{\partial Re\lambda}{\partial \vec{l}}|_p \neq 0$. Hence the conclusion follows. $\square$

For a better understanding of Theorem 2, we use Fig.2 to illustrate it as follows. If (σ, τ) cross $\mathcal{T}$ at p along the curve $\mathcal{S}_1$ with the tangent vector $\vec{l}_1$, and the

Figure 2: Two cases of the transversal condition.

transversal condition is $\frac{dRe\lambda}{dl_1}|_p = 0$, then there's no Hopf bifurcation at p along the curve S_1 ; If (σ, τ) cross $\mathcal{T}$ at p along the curve S_2 with the tangent vector $\vec{l}_2$, and the transversal condition is $\frac{dRe\lambda}{dl_2}|_p \neq 0$, then system undergoes a Hopf bifurcation at p along the curve S_2 .

For points on stability switching curves in a two parameter plane, the following case may occur. When a point $p(\sigma, \tau)$ on $\mathcal{T}$ vary along a direction $\vec{l}_1$, $\frac{dRe\lambda}{dl_1}|_p = 0$; When this point vary along another direction $\vec{l}_2$, $\frac{dRe\lambda}{dl_2}|_p \neq 0$. Therefore, whether p is a Hopf singularity depends on the direction in which (σ, τ) vary.

Now we can summarize the following theorem about the stability of x^* .

Theorem 3. *For any point $\tilde{P}(\tilde{\sigma}, \tilde{\tau})$ on the $\sigma - \tau$ plane, if there exists a curve segment $\tilde{l}$ connecting $\tilde{P}$ and the origin, such that $\tilde{l}$ does not intersect any stability switching curves, then x^* is locally asymptotically stable for $\sigma = \tilde{\sigma}, \tau = \tilde{\tau}$.*

This theorem can be proved with the aid of Lemma 2. For a better understanding of this theorem, please refer to Fig.6.

3 Normal form of the two-parameter Hopf bifurcation

To investigate the dynamics of (2) near the Hopf singularity along a direction $\vec{l}$, we calculate the normal form of the Hopf bifurcation, by applying the normal form method and the center manifold theory introduced by Faria [20].

Without loss of generality, we always assume $\tau > \sigma$ in this section. Besides, we choose the varying direction of (σ, τ) to be $\vec{l}(\gamma_1\mu, \gamma_2\mu)$, with $\gamma_1^2 + \gamma_2^2 = 1$.

Let $y = x - x^*$. Then system (2) becomes

$$\dot{y}(t) = (\delta + H)[y(t - \tau) + x^*]e^{-y(t-\tau)} - \delta y(t) - Hy(t - \sigma) - (\delta + H)x^*. \quad (19)$$

Re-scale the time by $t = \tau s$ and $x(s) = y(\tau s)$ to normalize the delay τ so that system (19) can be written as

$$\dot{x}(s) = \tau \{ (\delta + H)[x(s-1) + x^*]e^{-x(s-1)} - \delta x(s) - Hx(s - \frac{\sigma}{\tau}) - (\delta + H)x^* \}. \quad (20)$$

For simplicity, we still denote s as t , then (20) becomes

$$\dot{x}(t) = \tau \{ (\delta + H)[x(t-1) + x^*]e^{-x(t-1)} - \delta x(t) - Hx(t - \frac{\sigma}{\tau}) - (\delta + H)x^* \}. \quad (21)$$

Eq.(21) can be written as

$$\dot{x}(t) = -\tau \delta x(t) + \tau(\delta + H)(1 - x^*)x(t-1) - \tau Hx(t - \frac{\sigma}{\tau}) + f, \quad (22)$$

where

$$f = \tau [\frac{1}{2}(\delta + H)(x^* - 2)x^2(t-1) + \frac{1}{6}(\delta + H)(3 - x^*)x^3(t-1) + \dots]. \quad (23)$$

Let $\mathcal{C} := C([-1, 0], \mathbb{R})$ represents the phase space with the sup norm. We write $x_t \in \mathcal{C}$ for $x_t(\theta) = x(t + \theta)$, $(-1 \leq \theta \leq 0)$.

Assume that the characteristic equation (4) has a pair of purely imaginary roots $\pm i\omega$ when $\tau = \tau^*$ and $\sigma = \sigma^*$, take $\sigma = \sigma^* + \gamma_1\mu$, $\tau = \tau^* + \gamma_2\mu$ with sufficiently small μ , system (22) may undergo a Hopf bifurcation at the equilibrium $(0, 0)$ along the direction $\vec{l}(\gamma_1\mu, \gamma_2\mu)$.

Substituting $\sigma = \sigma^* + \gamma_1\mu$, $\tau = \tau^* + \gamma_2\mu$ into (22), then we can rewrite system (22) in the space $\mathcal{C}$ as

$$\frac{dx}{dt} = L_\mu(x_t) + f(\mu, x_t), \quad (24)$$

where

$$L_\mu(\psi) = (\tau^* + \gamma_2\mu)[- \delta \psi(0) + (\delta + H)(1 - x^*)\psi(-1) - H\psi(-\frac{\sigma^* + \gamma_1\mu}{\tau^* + \gamma_2\mu})],$$

and

$$f(\mu, \psi) = (\tau^* + \gamma_2\mu)(\delta + H)[\frac{1}{2}(x^* - 2)\psi^2(-1) + \frac{1}{6}(3 - x^*)\psi^3(-1) + \dots]. \quad (25)$$

The linearized system of (24) at the origin is in the following form

$$\frac{dx(t)}{dt} = L_0(x_t). \quad (26)$$

Then system (24) can be written as

$$\frac{dx(t)}{dt} = L_0(x_t) + G(\mu, x_t), \quad (27)$$

where

$$\begin{aligned} G(\mu, x_t) = & \gamma_2 \mu \left[-\delta x_t(0) + (\delta + H)(1 - x^*)x_t(-1) - Hx_t\left(-\frac{\sigma^*}{\tau^*}\right) \right] \\ & + (\tau^* + \gamma_2 \mu)(-H) \left[x_t\left(-\frac{\sigma^* + \gamma_1 \mu}{\tau^* + \gamma_2 \mu}\right) - x_t\left(-\frac{\sigma^*}{\tau^*}\right) \right] \\ & + (\tau^* + \gamma_2 \mu)(\delta + H) \left[\frac{1}{2}(x^* - 2)x_t^2(-1) + \frac{1}{6}(3 - x^*)x_t^3(-1) \right] + \dots \end{aligned}$$

Denote the enlarged space $\mathcal{BC}$ of $\mathcal{C}$ as

$$\mathcal{BC} := \{\psi : [-1, 0] \rightarrow R, \psi \text{ is continuous on } [-1, 0], \exists \lim_{\theta \rightarrow 0^-} \psi(\theta)\},$$

formulating (27) in the extended Banach space $\mathcal{BC}$ as an abstract ordinary differential equation

$$\dot{x}_t = A_0 x_t + X_0 G(\mu, x_t), \quad (28)$$

where A_0 denotes the infinitesimal generators associated with the linear equation (26), defined by $A_0 : \mathcal{C} \rightarrow \mathcal{C}$, $A_0 \phi = \dot{\phi} + X_0[L_0(\phi) - \dot{\phi}(0)]$, and X_0 is given by $X_0(\theta) = 0$ for $\theta \in [-1, 0)$ and $X_0(0) = 1$.

Using the Riesz representation theorem, we see that there exists a bounded variation function $\eta(\theta)$ ($\theta \in [-1, 0]$) such that

$$L_0(\phi) = \int_{-1}^0 d\eta(\theta)\phi(\theta), \quad \phi \in \mathcal{C}.$$

In fact, we can choose

$$\eta(\theta) = \begin{cases} -\tau^*(\delta + H)(1 - x^*), & \theta = -1, \\ 0, & \theta \in (-1, -\frac{\sigma^*}{\tau^*}], \\ -\tau^*H, & \theta \in (-\frac{\sigma^*}{\tau^*}, 0), \\ -\tau^*(\delta + H), & \theta = 0. \end{cases} \quad (29)$$

Let A^* denotes the formal adjoint of A_0 on $\mathcal{C}^* := C([0, 1], \mathbb{R}^*)$ as

$$A^* \psi(s) = \begin{cases} \int_{-1}^0 \psi(-\theta) d\eta(\theta), & s = 0, \\ -\dot{\psi}(s), & s \in (0, 1], \end{cases}$$

under the bilinear form

$$\langle \psi, \phi \rangle = \psi(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \psi(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi. \quad (30)$$

Then $q(\theta) = e^{i\omega_0 \tau^* \theta}$ and $q^*(s) = D e^{-i\omega_0 \tau^* s}$ are the eigenvectors of A_0 and A^* corresponding to $i\omega_0 \tau^*$, respectively. And $\langle q^*, q \rangle = 1$, where

$$D = \left[1 + \tau^*(\delta + H)(1 - x^*)e^{-i\omega_0 \tau^*} - H\sigma^* e^{-i\omega_0 \sigma^*} \right]^{-1}.$$

We denote $\Phi = (q(\theta), \bar{q}(\theta))$.

Now we can decompose $\mathcal{BC}$ into a center subspace and its orthocomplement, i.e.,

$$\mathcal{BC} = \mathcal{P} \oplus \text{Ker}\pi, \quad (31)$$

where $\pi : \mathcal{C} \rightarrow \mathcal{P}$ is the projection defined by

$$\pi(\varphi) = q < q^*, \varphi > + \bar{q} < \bar{q}^*, \varphi >.$$

For simplicity, we denote

$$z_1 = < q^*, x_t >, \quad z_2 = < \bar{q}^*, x_t > \quad \text{and} \quad z = (z_1, z_2)^T \in \mathbb{C}^2.$$

According to (31), x_t can be composed as

$$x_t(\theta) = q(\theta)z_1 + \bar{q}(\theta)z_2 + w(\theta), \quad (32)$$

where $w \in \mathcal{BC} \cap \text{Ker}\pi := Q^1$ for any t . Then system (28) is equivalent to the system

$$\begin{aligned} \dot{z}_1 &= i\omega\tau^*z_1 + q^*(\theta)G[\mu, \Phi z + w(\theta)], \\ \dot{z}_2 &= -i\omega\tau^*z_2 + \bar{q}^*(\theta)G[\mu, \Phi z + w(\theta)], \\ \frac{dw}{dt} &= A_1w + (I - \pi)X_0G[\mu, \Phi z + w(\theta)], \end{aligned} \quad (33)$$

where A_1 is the restriction of A_0 on

$$Q^1 \subset \text{Ker}\pi \rightarrow \text{Ker}\pi, \quad A_1\varphi = A_0\varphi \quad \text{for } \varphi \in Q^1.$$

Consider the formal Taylor expansion

$$G(\mu, \varphi) = \frac{1}{2!}G_2(\mu, \varphi) + \frac{1}{3!}G_3(\mu, \varphi) + \cdots,$$

where

$$\begin{aligned} G_2(\mu, \varphi) &= 2\gamma_2\mu \left[-\delta\varphi(0) + (\delta + H)(1 - x^*)\varphi(-1) - H\varphi\left(-\frac{\sigma^*}{\tau^*}\right) \right] \\ &\quad + 2\tau^*(-H) \left[\varphi\left(-\frac{\sigma^* + \gamma_1\mu}{\tau^* + \gamma_2\mu}\right) - \varphi\left(-\frac{\sigma^*}{\tau^*}\right) \right] + f_2(0, \Phi z + w). \end{aligned} \quad (34)$$

Notice that

$$-\frac{\sigma^* + \gamma_1\mu}{\tau^* + \gamma_2\mu} = -\frac{\sigma^*}{\tau^*} + \frac{\sigma^*}{(\tau^*)^2}\gamma_2\mu - \frac{1}{\tau^*}\gamma_1\mu,$$

combining (31) and the Taylor expansion of exponential function, we have

$$\tau^* \left[x_t\left(-\frac{\sigma^* + \gamma_1\mu}{\tau^* + \gamma_2\mu}\right) - x_t\left(-\frac{\sigma^*}{\tau^*}\right) \right] = i\omega\mu(\sigma^*\gamma_2 - \tau^*\gamma_1) \left[e^{-i\omega\sigma^*}z_1 - e^{i\omega\sigma^*}z_2 \right].$$

We can also get $f_2(0, \Phi z + w) = f_2(0, z, w)$ in (34) as follows

$$f_2(0, z, w) = f_{20}z_1^2 + f_{11}z_1z_2 + f_{02}z_2^2 + R_2(\Phi z + w) + o(|w|^2), \quad (35)$$

where

$$\begin{aligned} f_{20} &= \tau^*(\delta + H)(x^* - 2)e^{-2i\omega\tau^*}, \quad f_{11} = 2\tau^*(\delta + H)(x^* - 2), \\ f_{02} &= \tau^*(\delta + H)(x^* - 2)e^{2i\omega\tau^*}, \\ R_2(\Phi z + w) &= 2\tau^*(\delta + H)(x^* - 2)[e^{-i\omega\tau} z_1 w(-1) + e^{i\omega\tau} z_2 w(-1)] \end{aligned}$$

Besides, the term $G_3(0, z, w)$ can be obtained as follows

$$G_3(0, z, w) = f_{30}z_1^3 + f_{21}z_1^2z_2 + f_{12}z_1z_2^2 + f_{03}z_2^3 + R_3(\Phi z + w), \quad (36)$$

where

$$\begin{aligned} f_{30} &= \tau^*(\delta + H)(3 - x^*)e^{-3i\omega\tau^*}, & f_{21} &= 3\tau^*(\delta + H)(3 - x^*)e^{-i\omega\tau^*}, \\ f_{12} &= 3\tau^*(\delta + H)(3 - x^*)e^{i\omega\tau^*}, & f_{03} &= \tau^*(\delta + H)(3 - x^*)e^{3i\omega\tau^*}. \end{aligned}$$

Now Eq.(33) can be written as

$$\begin{aligned} \dot{z} &= Bz + \sum_{j \geq 2} \frac{1}{j!} f_j^1(z, w, \mu), \\ \frac{d}{dt} w &= A_1 w + \sum_{j \geq 2} \frac{1}{j!} f_j^2(z, w, \mu), \end{aligned} \quad (37)$$

where $w \in Q^1$, $B = \text{diag}\{i\omega\tau^*, -i\omega\tau^*\}$ and $f_j = (f_j^1, f_j^2)^T$, $j \geq 2$, are defined by

$$\begin{aligned} f_j^1(z, w, \mu) &= \begin{pmatrix} q^*(0)G_j[\mu, \Phi(\theta)z + w(\theta)] \\ \bar{q}^*(0)G_j[\mu, \Phi(\theta)z + w(\theta)] \end{pmatrix}, \\ f_j^2(z, w, \mu) &= (I - \pi)X_0G_j[\mu, \Phi(\theta)z_x + w(\theta)]. \end{aligned} \quad (38)$$

Similar to [19,20], use the notations in it, define the operator $M_j = (M_j^1, M_j^2)$, $j \geq 2$ by

$$\begin{aligned} M_j^1 &: V_j^{2+1}(\mathbb{C}^2) \rightarrow V_j^{2+1}(\mathbb{C}^2), \\ (M_j^1 p)(z, \mu) &= D_z p(z, \mu)Bz - Bp(z, \mu), \\ M_j^2 &: V_j^{2+1}(\mathcal{Q}^1) \subset V_j^{2+1}(\text{Ker}\pi) \rightarrow V_j^{2+1}(\text{Ker}\pi), \\ (M_j^2 h)(z, \mu) &= D_z h(z, \mu)Bz - \mathcal{A}_1 h(z, \mu). \end{aligned} \quad (39)$$

By a recursive transformations of variables

$$(z, w, \mu) = (\hat{z}, \hat{w}, \mu) + \frac{1}{j!} (U_j^1(\hat{z}, \mu), U_j^2(\hat{z}, \mu), 0) \quad (40)$$

with $U_j = (U_j^1, U_j^2) \in V_j^{2+1}(\mathbb{C}^4) \times V_j^{2+1}(\mathcal{Q}^1)$, we can obtain the normal forms of (37). We conclude that this recursive process transforms of (37) into the following equations

$$\begin{aligned} \dot{z} &= Bz + \sum_{j=2}^{\infty} \frac{1}{j!} g_j^1(z, w, \sigma), \\ \frac{d}{dt} w &= A_1 w + \sum_{j=2}^{\infty} \frac{1}{j!} g_j^2(z, w, \sigma), \end{aligned} \quad (41)$$

where $g_j = (g_j^1, g_j^2)$, $j \geq 2$ have the following form

$$g_j(z, w, \mu) = \bar{f}_j(z, w, \sigma) - M_j U_j(z, \sigma),$$

and $U_j \in V_j^{2+1}(\mathbb{C}^2) \times V_j^{2+1}(\mathcal{Q}^1)$ satisfying

$$U_j(z, \mu) = (M_j)^{-1} \text{Proj}_{\text{Im}(M_j^1) \times \text{Im}(M_j^2)} \bar{f}_j(z, 0, \mu), \quad (42)$$

where $\bar{f}_j = (\bar{f}_j^1, \bar{f}_j^2)$ stand for the terms of order j in (z, w) , which are obtained after the computation of normal forms up to order $j - 1$. The normal form truncated to the third order has the following form:

$$\dot{z} = Bz + \frac{1}{2!} g_2^1(z, 0, \sigma) + \frac{1}{3!} g_3^1(z, 0, 0) + \text{h.o.t.} \quad (43)$$

Here, $g_3^1(z, 0, 0) = \text{Proj}_{\text{Ker}(M_3^1)} \bar{f}_3^1(z, 0, 0)$ and

$$\begin{aligned} \bar{f}_3^1(z, 0, 0) = & f_3^1(z, 0, 0) + \frac{3}{2} [D_z f_2^1(z, 0, 0) U_2^1(z, 0)] \\ & + D_w f_2^1(z, 0, 0) U_2^2(z, 0) - D_z U_2^1(z, 0) g_2^1(z, 0, 0). \end{aligned} \quad (44)$$

The calculations of $g_2^1(z, 0, \sigma)$ and $g_3^1(z, 0, 0)$ are in Appendix A. After the calculations, the normal form truncated to the third order on the center manifold for the Hopf singularity is obtained as follows:

$$\begin{aligned} \dot{z}_1 &= i\omega\tau^* z_1 + K_{11}\mu z_1 + K_{21}z_1^2 z_2, \\ \dot{z}_2 &= -i\omega\tau^* z_2 + \bar{K}_{11}\mu z_2 + \bar{K}_{21}z_1 z_2^2. \end{aligned} \quad (45)$$

Make the polar coordinate transformation

$$z_1 = \rho \cos \theta + i\rho \sin \theta, \quad z_2 = \rho \cos \theta - i\rho \sin \theta,$$

we obtain the simplified system of (45) as follows

$$\dot{\rho} = \rho(a_0\mu + b_0\rho^2), \quad (46)$$

where

$$a_0 = \text{Re}\{K_{11}\}, \quad b_0 = \text{Re}\{K_{21}\}.$$

The dynamics of (46) reflects the dynamics of (45), correspondingly, the dynamics near the Hopf bifurcation of (2) is obtained.

Once the varying direction $\vec{l}(\gamma_1\mu, \gamma_2\mu)$ of (σ, τ) is fixed, whether the possible Hopf singularity (σ^*, τ^*) is a Hopf singularity will also be determined. Furthermore, the normal form near a Hopf singularity can be calculated, thus the dynamics of the system near a Hopf singularity can be obtained.

4 Numerical simulations

In this section, we carry out some simulations to illustrate the above theoretical results.

Example 1 Let $\delta = 1.4$, $H = 0.6$, $p = 2e$, then assumption $(\mathbf{H}_1)$ holds, $x^* = 1$ is a unique positive equilibrium of (2). Since $H \leq \delta$, according to Theorem 1(a), x^* is asymptotically stable for any σ and τ . This is shown in Fig.3, where the initial values are $x_0(t) = -0.625(t + 0.4)^2 + 1$, $t \in [-0.4, 0]$ for (a), $x_0(t) = -0.1(t + 1.5)^2 + 1$, $t \in [-1.5, 0]$ for (b) and $x_0(t) = -10^{-5}(t + 2.2)^2 + 1$, $t \in [-2.2, 0]$ for (c).

Figure 3: Numerical simulations of system (2) for $\delta = 1.4$, $H = 0.6$, $p = 2e$. The positive equilibrium x^* is asymptotically stable.

Figure 4: Numerical simulations of system (2) for $\delta = 0.6$, $H = 1.4$, $p = 2e$. The positive equilibrium x^* is asymptotically stable when $\sigma < \sigma_0$; when $\sigma > \sigma_0$, x^* becomes unstable and the bifurcating periodic solutions from x^* is stable

Example 2 Choose $\delta = 0.6$, $H = 1.4$, $p = 2e$, the assumption $(\mathbf{H}_1)$ is satisfied. Hence $x^* = 1$ is a unique positive equilibrium of (2). Since $H > \delta$, by a few calculations, we have

$$\sigma_j \approx 1.5920 + 4.9673j, \quad j = 0, 1, 2, \dots$$

According to Theorem 1(b), x^* is asymptotically stable for $\sigma < \sigma_0 \approx 1.5920$ and unstable for $\sigma > \sigma_0$. These are shown in Fig.4. Without loss of generality, we set $\tau = 2.2$. We can see that when $\sigma = 1.2 < \sigma_0, \tau = 2.2$, $x^* = 1$ is locally asymptotically stable, see Fig.4(a); When $\sigma = 1.8 \in (\sigma_0, \sigma_1)$, $\tau = 2.2$, system exhibits periodic oscillation, see Fig.4(b). In Fig.4, the initial values are $x_0(t) = 0.025(t + 2.2)^2 + 0.9$, $t \in [-2.2, 0]$ for (a) and $x_0(t) = 0.025(t + 2.2)^2 + 1.1$, $t \in [-2.2, 0]$ for (b).

Example 3 Let $\delta = 0.4$, $H = 1.8$, $p = 3$. Then $(\mathbf{H}_1)$ holds, $p \neq (\delta + H)e$ and $x^* \approx 0.3102$. To illustrate the dynamics of the system in the presence of delays, we follow the process given in Section 2. As shown in Fig.5(a), $F(\omega) = 0$ has a positive root $\omega \approx 3.2935$, the crossing set is $\Omega = [0, 3.2935]$. From the previous discussion, we get the stability switching curves $\mathcal{T}$ in $[0, 10] \times [0, 15]$ corresponding to Ω , see Fig.5(b).

Figure 5: (a) The figure of $F(\omega)$. (b) The stability switch curves on σ - τ plane.

For a better understanding, we enlarge some of the areas in Fig.5(b) and redraw them in Fig.6. According to Theorem 3, all the regions connected with the origin are parameter regions of (σ, τ) that make x^* locally stable.

In the following, we fix $\sigma = 0.48$, choose the varying direction of (σ, τ) as $\vec{l}(\gamma_1, \gamma_2) = (0, 1)$, and investigate the stability of x^* with the varying of τ .

When (σ, τ) is chosen at $P_1(0.48, 1)$, from Theorem 3 and Fig.6, x^* is asymptotically stable. We illustrate this in Fig.7(a), where the initial value is $x_0(t) = 0.1(t + 1)^2 + 0.4$, $t \in [-1, 0]$. Similarly, when (σ, τ) is chosen at $P_3(0.48, 3.3)$, x^* is asymptotically stable. We illustrate this in Fig.7(b), where the initial value is $x_0(t) = 0.31015$, $t \in [-3.3, 0]$.

Choose critical points $Q_1(0.48, 1.52883)$, $Q_2(0.48, 2.93784)$ and $Q_3(0.48, 3.45538)$ on the stability switching curves $\mathcal{T}_{0,1}^+$, $\mathcal{T}_{0,1}^+$ and $\mathcal{T}_{0,2}^+$, respectively. By some calculation, we can get the following values in Table 1, since $\text{Re}\lambda'(\tau) \neq 0$, Q_1 , Q_2 , Q_3 are Hopf bifurcation singularities.

Figure 6: The stability switch curves on σ - τ plane.

Table 1: Parameter values at Q_i .

Point	σ	τ	ω	$\frac{1}{\lambda'(\tau)}$	K_{11}	K_{21}
Q_1	0.48	1.52883	3.26128	0.1465+0.224i	0.4486+1.1443i	-7.2130-1.0721i
Q_2	0.48	2.93784	2.01640	-0.1591+0.2378i	-0.1613+0.2981i	-10.0572-1.0722i
Q_3	0.48	3.45538	3.26129	0.1465+0.224i	0.3041+0.6005i	-9.0130-0.4738i

For Q_1 , we get that $\text{Re}K_{11} > 0$, and $\text{Re}K_{21} < 0$ from Table 1. The simplified system of (45) is in the following form

$$\dot{\rho} = \rho(0.4486\mu - 7.2130\rho^2).$$

Obviously, when $\mu > 0$, system (2) undergoes Hopf bifurcation, the direction of the bifurcation is forward, and the bifurcating periodic solutions are orbitally stable. We choose $\mu = 0.00017$, (σ, τ) becomes $P_2(0.48, 1.529)$, we illustrate the dynamics of system (2) at P_2 in Fig.7(c), where the initial value is $x_0(t) = 0.3, t \in [-1.529, 0]$.

Similarly, for Q_3 , we can draw $\text{Re}K_{11} > 0$, $\text{Re}K_{21} < 0$ from Table 1. The simplified system of (45) is in the following form

$$\dot{\rho} = \rho(0.3041\mu - 9.0130\rho^2).$$

Obviously, when $\mu > 0$, system (2) undergoes Hopf bifurcation, the direction of the bifurcation is forward, and the bifurcating periodic solutions are orbitally stable.

Figure 7: The dynamics near the Hopf singularities.

We choose $\mu = 0.00062$, (σ, τ) becomes $P_4(0.48, 3.456)$, we illustrate the dynamics of system (2) at P_4 in Fig.7(d), where the initial value is $x_0(t) = 0.3, t \in [-3.456, 0]$.

For Q_2 , we can draw $\text{Re}K_{11} < 0$, $\text{Re}K_{21} < 0$ from Table 1. The simplified system of (45) is in the following form

$$\dot{\rho} = \rho(-0.1613\mu - 10.0752\rho^2).$$

Obviously, when $\mu < 0$, system (2) undergoes Hopf bifurcation, the direction of the bifurcation is backward, and the bifurcating periodic solutions are orbitally stable; when $\mu > 0$ and μ is sufficiently small, there's no bifurcating periodic solutions and the positive equilibrium is locally stable. This conclusion is coincide with the simulation of P_3 .

To sum up the above conclusion, we get that when σ is fixed at 0.48, x^* is locally asymptotically stable when $\tau \in [0, 1.52883)$; then x^* becomes unstable and system exhibits periodic oscillation when $\tau \in (1.52883, 2.93784)$; and then x^* turns to be locally asymptotically stable again when $\tau \in (2.93784, 3.45538)$ and finally, x^* is unstable for enough large τ . These indicate that for some fixed σ^* , the stability of x^* switches finite times with the increasing of τ .

5 Conclusion

In this paper, a scalar model with mature delay and harvest delay is considered. We conclude that when the birth rate, death rate and capture rate of the population meet a certain relation, the mature delay τ has no effect on the stability of the population. Furthermore, when the capture rate is greater than the death rate, the locally stable positive equilibrium would become unstable with the increase of capture delay σ . This indicates that large capture delay can affect people's accurate prediction of population size and lead to over-capture, which may result in the population extinction.

By taking two delays as parameters, the stability switching curves of x^* are calculated. Then the two-parameter Hopf bifurcation theorem at a singularity along a direction $\vec{l}$ is presented. Analysis indicates that whether a point is a Hopf singularity depends on the varying direction of two delays.

We emphasize that the varying direction of (σ, τ) is set to be $(\gamma_1\mu, \gamma_2\mu)$ instead of (μ_1, μ_2) , thus the calculation of the normal form near the Hopf singularity with two parameters is transformed into one parameter. This dimension reduction method greatly reduces the amount of the calculation. Besides, this method can also be applied to other multi-parameter bifurcation studies.

By aid of the numerical simulations, we verify the theoretical analysis by choosing the varying direction as $\vec{l}(\gamma_1\mu, \gamma_2\mu) = (0, \mu)$. We obtain that for a fixed σ , the stability of the positive equilibrium may switches finite times with the varying of the τ . This indicates that the sensitivity of τ has great influence on the stability of the system.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Informed consent Informed consent was obtained from all individual participants included in the study.

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Appendix

Since we do not consider the strongly resonant cases, from (38) and (39), it is easy to verify that

$$\text{Im}(M_2^1)^c = \text{span}\{\mu z_1 e_1, \mu z_2 e_2\} \triangleq S_2, \quad (47)$$

from (38), we have

$$f_2^1(z, 0, \mu) = \begin{pmatrix} \frac{q^*(0)G_2[\mu, \Phi(\theta)z]}{q^*(0)G_2[\mu, \Phi(\theta)z]} \end{pmatrix}.$$

From Eq.(40), (41) and (34), we obtain

$$\frac{1}{2!}g_2^1(z, 0, \mu) = \frac{1}{2!}\text{Proj}_{S_2}f_2^1(z, 0, \mu) = \begin{pmatrix} \frac{K_{11}\mu z_1}{K_{11}\mu z_2} \end{pmatrix}, \quad (48)$$

with

$$K_{11} = q^*(0)\{\gamma_2[(\delta + H)(1 - x^*)e^{-i\omega\tau^*} - \delta] - He^{-i\omega\sigma^*}[\gamma_2 + i\omega(\sigma^*\gamma_2 - \tau^*\gamma_1)]\}. \quad (49)$$

The calculation of $g_3^1(z, 0, 0)$

Here we focus on the calculation of the third order normal form of the Hopf bifurcation. To do this, we neglect the high order terms of the perturbation parameters. By direct calculation, we get

$$\text{Im}((M_3^1)^c) = \text{span}\{z_1^2 z_2 e_1, z_1 z_2^2 e_2\} \triangleq S_3. \quad (50)$$

Since $g_2^1(z, 0, 0) = 0$, from (44), we divide the computation of the third term $g_3^1(z, 0, 0)$ into three parts: $\text{Proj}_{S_3}(f_3^1(z, 0, 0))$, $\text{Proj}_{S_3}(D_z f_2^1(z, 0, 0)U_2^1(z, 0))$ and $\text{Proj}_{S_3}(D_w f_2^1(z, 0, 0)U_2^2(z, 0))$.

Part 1 The calculation of $\text{Proj}_{S_3}(f_3^1(z, 0, 0))$.

From (34), we have

$$f_3^1(z, 0, 0) = \begin{pmatrix} q^*(0)G_3[0, \Phi(\theta)z] \\ \overline{q^*}(0)G_3[0, \Phi(\theta)z] \end{pmatrix}. \quad (51)$$

therefore, we have

$$\frac{1}{3!}\text{Proj}_{S_3}f_3^1(z, 0, 0) = \begin{pmatrix} \frac{L_{21}z_1^2z_2}{L_{21}z_1z_2^2} \end{pmatrix}, \quad (52)$$

where

$$L_{21} = \frac{1}{6}q^*(0)f_{21}, \quad \overline{L_{21}} = \frac{1}{6}\overline{q^*}(0)f_{12},$$

Part 2 The calculation of $\text{Proj}_{S_3}(D_z f_2^1(z, 0, 0)U_2^1(z, 0))$.

By (34), we can write $f_2^1(z, 0, 0)$ as

$$\begin{aligned} f_2^1(z, 0, 0) &= \begin{pmatrix} q^*(0)f_2(0, \Phi(\theta)z) \\ \overline{q^*}(0)f_2(0, \Phi(\theta)z) \end{pmatrix} \\ &= \begin{pmatrix} f_{20}^{1(1)}z_1^2 + f_{11}^{1(1)}z_1z_2 + f_{02}^{1(1)}z_2^2 \\ f_{20}^{1(2)}z_1^2 + f_{11}^{1(2)}z_1z_2 + f_{02}^{1(2)}z_2^2 \end{pmatrix} \triangleq \begin{pmatrix} f_2^{1(1)} \\ f_2^{1(2)} \end{pmatrix} \end{aligned} \quad (53)$$

where

$$\begin{aligned} f_{20}^{1(1)} &= q^*(0)f_{20}, \quad f_{11}^{1(1)} = q^*(0)f_{11}, \quad f_{02}^{1(1)} = q^*(0)f_{02}, \\ f_{20}^{1(2)} &= \overline{q^*}(0)f_{20}, \quad f_{11}^{1(2)} = \overline{q^*}(0)f_{11}, \quad f_{02}^{1(2)} = \overline{q^*}(0)f_{02}. \end{aligned}$$

From Eq.(42) and (53), we can calculate $U_2^1(z, 0)$ from the following formula

$$U_2^1(z, 0) = \begin{pmatrix} U_2^{11} \\ U_2^{12} \end{pmatrix} = (M_2^1)^{-1} \text{Proj}_{S_3}f_2^1(z, 0, 0),$$

then we have

$$\begin{aligned} U_2^{11} &= \frac{1}{i\omega\tau^*}f_{20}^{1(1)}z_1^2 - \frac{1}{i\omega\tau^*}f_{11}^{1(1)}z_1z_2 - \frac{1}{3i\omega\tau^*}f_{02}^{1(1)}z_2^2, \\ U_2^{12} &= \frac{1}{3i\omega\tau^*}f_{20}^{1(2)}z_1^2 + \frac{1}{i\omega\tau^*}f_{11}^{1(2)}z_1z_2 - \frac{1}{i\omega\tau^*}f_{02}^{1(2)}z_2^2. \end{aligned}$$

Thus

$$\frac{1}{3!}\text{Proj}_{S_3}(D_z f_2^1(z, 0, 0)U_2^1(z, 0)) = \begin{pmatrix} \frac{M_{21}z_1^2z_2}{M_{21}z_1z_2^2} \end{pmatrix}, \quad (54)$$

where

$$M_{21} = \frac{1}{6}[f_{20}^{1(1)}\frac{1}{-i\omega\tau^*}f_{11}^{1(1)} + f_{11}^{1(1)}\frac{1}{i\omega\tau^*}f_{11}^{1(2)} + f_{02}^{1(1)}\frac{2}{3i\omega\tau^*}f_{20}^{1(2)}].$$

Part 3 The calculation of $\text{Proj}_{S_3}(D_w f_2^1(z, 0, 0)U_2^2(z, 0))$.
First we calculate the Fréchet derivative $D_w f_2^1(z, 0, 0) : \mathcal{Q}^1 \rightarrow X_{\mathbb{C}}$. From (34), $G_2(z, w, 0)$ can be expressed as

$$\begin{aligned} G_2(z, w, 0) &= R_2(\Phi(\theta)z, w) + o(z^2, w^2) \\ &= R_{wz_1}(w)z_1 + R_{wz_2}(w)z_2 + o(z^2, w^2) \end{aligned}$$

where $R_{wz_i}(i = 1, 2)$ are linear operators from $\mathcal{Q}^1$ to $X_{\mathbb{C}}$. Now we have

$$D_w G_2(z, 0, 0)(\varphi) = R_{wz_1}(\varphi)z_1 + R_{wz_2}(\varphi)z_2.$$

Let $U_2^2(z, 0) = h(z) = h_{20}(\theta)z_1^2 + h_{11}(\theta)z_1z_2 + h_{02}(\theta)z_2^2$, From Eq.(39) and (40), we have

$$\begin{aligned} D_w f_2^1(z, 0, 0)(U_2^2(z, 0)) &= \left(\frac{q^*(0)D_w G_2(z, 0, 0)(U_2^2(z, 0))}{q^*(0)D_w G_2(z, 0, 0)(U_2^2(z, 0))} \right) \\ &= \left(\frac{q^*(0)[R_{wz_1}(h)z_1 + R_{wz_2}(h)z_2]}{q^*(0)[R_{wz_1}(h)z_1 + R_{wz_2}(h)z_2]} \right) \end{aligned}$$

Therefore,

$$\frac{1}{3!} \text{Proj}_{S_3}(D_w f_2^1(z, 0, 0)(U_2^2(z, 0))) = \left(\frac{N_{21}z_1^2z_2}{N_{21}z_1z_2^2} \right), \quad (55)$$

where

$$N_{21} = \frac{1}{6}q^*(0)[R_{wz_1}(h_{11}) + R_{wz_2}(h_{20})].$$

Thus, we only need to calculate h_{11}, h_{20} . According to (39), we have

$$\begin{aligned} M_2^2 U_2^2(z, 0)(\theta) &= M_2^2 h(z)(\theta) = D_z h(z)Bz - \mathcal{A}^1(h(z)) \\ &= \begin{cases} D_z h(z)Bz - D_\theta h(z), & \theta \neq 0, \\ D_z h(z)Bz - L_0(h(z)), & \theta = 0. \end{cases} \end{aligned}$$

Besides, we have

$$f_2^2(z, 0, 0) = \begin{cases} -q(\theta)f_2^{1(1)}(z, 0, 0) - \bar{q}(\theta)f_2^{1(2)}(z, 0, 0), & \theta \neq 0, \\ G_2(z, 0, 0) - q(0)f_2^{1(1)}(z, 0, 0) - \bar{q}(0)f_2^{1(2)}(z, 0, 0), & \theta = 0. \end{cases}$$

Since

$$M_2^2 U_2^2(z, 0) = f_2^2(z, 0, 0),$$

By matching the coefficients of z_1^2, z_1z_2 in the above equation, we can draw the formulas of h_{11} and h_{20} .

$$\begin{aligned} h_{11}(\theta) &= f_{11} \left[-\frac{1}{L_0(1)} + \frac{q^*(0)e^{i\omega\tau^*\theta}}{i\omega\tau^*} - \frac{\bar{q}^*(0)e^{-i\omega\tau^*\theta}}{i\omega\tau^*} \right], \\ h_{20}(\theta) &= f_{20} \left[\frac{e^{2i\omega\tau^*\theta}}{2i\omega\tau^* - L_0(e^{2i\omega\tau^*\theta})} - \frac{q^*(0)e^{i\omega\tau^*\theta}}{i\omega\tau^*} - \frac{\bar{q}^*(0)e^{-i\omega\tau^*\theta}}{3i\omega\tau^*} \right], \end{aligned}$$

Therefore, by (52), (54) and (55), we get that

$$\frac{1}{3!}g_3^1(z, 0, 0) = \frac{1}{3!}\text{Proj}_{S_2}\overline{f_3^1}(z, 0, 0) = \left(\begin{array}{c} K_{21}z_1^2z_2 \\ K_{21}z_1z_2^2 \end{array} \right),$$

with

$$K_{21} = L_{21} + \frac{3}{2}(M_{21} + N_{21}).$$

Figures

Figure 1

See the Manuscript Files section for the complete figure caption.

Figure 2

See the Manuscript Files section for the complete figure caption.

Figure 3

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Figure 4

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Figure 5

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Figure 6

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(a) $\sigma = 0.48, \tau = 1$

(b) $\sigma = 0.48, \tau = 3.3$

(c) $\sigma = 0.48, \tau = 1.529$

(d) $\sigma = 0.48, \tau = 3.456$

Figure 7

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