Spin-1/2 one- and two- particle systems in physical space without *eigen*-algebra or tensor product.

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Abstract. A novel representation of spin 1/2 combines in a single geometric object the roles of the standard Pauli spin vector and spin state. Under the spin-position decoupling approximation it consists of the *ordered* sum of three orthonormal *vectors* comprising a gauge phase. In the one-particle case the representation: (1) is Hermitian; (2) is oriented due to ordering; (3) reproduces all standard expectation values, including the total one-particle spin modulus $S_{tot} = \sqrt{3}\hbar/2$; (4) constrains *basis* opposite spins to have same orientation or handiness; (5) allows to formalize irreversibility in spin measurement. In the two-particle case: (1) entangled spin pairs have opposite handiness and precisely related gauge phases; (2) the dimensionality of the spin space doubles due to variation of handiness; (3) the four maximally entangled states are naturally defined by the four *improper* rotations in 3D: reflections onto the three orthogonal frame planes (triplets) and inversion (singlet). The cross-product terms in the expression for the squared total spin of two particles relates to experiment and they yield the standard expectation values after measurement. Here spin is directly defined and transformed in 3D orientation space, without use of *eigen* algebra and tensor product as in the standard formulation. The formalism allows working with whole geometric objects instead of only components, thereby helping keep a clear geometric picture of 'on paper' (controlled gauge phase) and 'on lab' (uncontrolled gauge phase) spin transformations.

1. Introduction

Key developments in Quantum Mechanics (QM), such as the first phenomenological description of spin 1/2 by Pauli [1] and the first quantum relativistic description of the electron by Dirac [2], 're-invented' Clifford algebras (in the matrix representation), seemingly unaware of Grassmann's [3] and Clifford's [4] works more than half a century earlier. The promotion of vector-based Clifford algebras in Physics and in particular the development of the spacetime algebra (STA) was undertaken by Hesteness [5,6], with more scientists joining in during the last 2 – 3 decades [7-10]. Spin formalism is arguably the main application of STA in QM [7].

Here we address spin in the non-relativistic regime under the spin–position decoupling approximation [6,7,11]. The 3D physical space in this case reduces to *orientation space* at a point (the origin) and the relevant symmetry operations of interest are proper rotations and reflections. The three frame vectors $\boldsymbol{\sigma}_j$ and the scalar 1 generate a *real* 8D vector space matching the equivalent real dimensions of the 2 x 2 complex Pauli matrices. The space is endowed with the even sub-algebra of STA [5,7], which is isomorphic to the (Clifford) algebra of Pauli matrices [9], therefore the notation. A basis of the real vector space consists of one unit scalar, three orthonormal vectors, three bivectors and one trivector, or pseudoscalar (below $\delta_{jk}, \varepsilon_{jkl}$ are the Kronecker and the totally antisymmetric Levi-Civita symbols):

$$\{1\ (1), \boldsymbol{\sigma}_j\ (3), \boldsymbol{\sigma}_j\boldsymbol{\sigma}_k \equiv \boldsymbol{\sigma}_{jk}\big(= \delta_{jk} + \varepsilon_{jkl}I\boldsymbol{\sigma}_l; j,k,l = 1,2,3\big)\ (3), \boldsymbol{\sigma}_{123} \equiv I\ (1)\} \tag{1}$$

Notice that $\boldsymbol{\sigma}_j$ are *vectors*, not matrices. Geometrically, the bivectors $I\boldsymbol{\sigma}_l$, e.g. $I\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_{31}$, represent oriented surface elements, while the pseudoscalar $I = \boldsymbol{\sigma}_{123}$ is an oriented volume element. The geometric or Clifford 's product combines Hamilton's scalar (symmetric) and Grassmann's wedge (antisymmetric) products; for two nonzero vectors it is invertible and as will become clear in definition (3) it is proportional to a rotor:

$\mathbf{uv} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v} = \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\wedge\mathbf{u}$; $(\mathbf{uv})^{-1} = \mathbf{vu}/\mathbf{u}^2\mathbf{v}^2$. In terms of anti-commutator and commutator brackets: $\{\mathbf{u},\mathbf{v}\} \equiv \mathbf{uv} + \mathbf{vu} = 2\mathbf{u}\cdot\mathbf{v}$; $[\mathbf{u},\mathbf{v}] \equiv \mathbf{uv} - \mathbf{vu} = 2\mathbf{u}\wedge\mathbf{v}$. (2)

The geometric product generalizes to any dimension, whereas the cross product, valid only in 3D is related to the wedge product by: $I(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\wedge\mathbf{v}$. In the STA literature [6,7] it has become customary to represent spin either by a vector or by a bivector normal to it, both in the nonrelativistic and relativistic regimes.

Notably, such representations of spin $s = 1/2$ *do not* reproduce the standard squared total spin modulus S^2, which according to the rules of QM on angular momentum should equal $\mathrm{S}^2/\hbar^2 = s(s+1) = 3/4$.

The rest of the report comprises three parts. The new definition of spin opens Section 2. Transformation properties for the one and two particle cases will appear in this section in the *vector* form, i.e. as *two-sided* rotor (unitary) transformations. In Section 3 the transformation properties for the same two cases will be demonstrated in the *spinor* form, i.e. as *one-sided* rotor transformations. Orthogonality relations, e.g. between spin up and spin down hold only in spinor representation. The connection between the vector and the spinor forms will also be discussed in Section 3. The report closes by conclusions presented in Section 4.

2. Definition and vector (two-sided rotor) transformations for spin 1/2 of one and two particle systems.

2.1. *One-particle systems.* Spin is represented here by an ordered sum of three orthonormal vectors together with a 'gauge' rotation. Specifically, spin up $\uparrow_\sigma$ and spin down $\downarrow_\sigma$ along the $\boldsymbol{\sigma}_3$ axis is the sum of the three orthonormal vectors $\boldsymbol{\sigma}_3, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ with a two-sided rotor R_φ in the plane $\boldsymbol{\sigma}_{12} = I\boldsymbol{\sigma}_3$ (see figure 1):

$$\mathrm{S}_{\uparrow\sigma} = \frac{\hbar}{2} R_\varphi(\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)R_\varphi^\dagger = \frac{\hbar}{2}\left(\boldsymbol{\sigma}_3 + R_\varphi(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)R_\varphi^\dagger\right) \equiv \frac{\hbar}{2}\left(\boldsymbol{\sigma}_3 + (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)_\varphi\right) = \frac{\hbar}{2}\left(\boldsymbol{\sigma}_3 + (\mathbf{u}_\perp + \mathbf{u}_2)_{\varphi - \varphi_\mathrm{u}}\right);\ R_\varphi = e^{-\frac{I\boldsymbol{\sigma}_3\varphi}{2}} = \cos\frac{\varphi}{2} - I\boldsymbol{\sigma}_3 \sin\frac{\varphi}{2};$$
$$\mathrm{S}_{\downarrow\sigma} = \mathbf{u}_2 \mathrm{S}_{\uparrow\sigma}\mathbf{u}_2 = \frac{\hbar}{2}\left(-\boldsymbol{\sigma}_3 + (-\mathbf{u}_\perp + \mathbf{u}_2)_{\varphi_\mathrm{u} - \varphi}\right) = \frac{\hbar}{2}\left(-\boldsymbol{\sigma}_3 + (-\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)_{-\varphi}\right); \mathrm{S}_\uparrow\mathrm{S}_\uparrow = \mathrm{S}_\downarrow\mathrm{S}_\downarrow = \mathrm{S}_\sigma^2 = \frac{3\hbar^2}{4};\ \mathrm{S}_\sigma^\dagger = \mathrm{S}_\sigma; \mathrm{S}_{-\sigma} \neq -\mathrm{S}_\sigma; \langle \mathrm{S}_{\pm\sigma}\rangle \equiv \frac{\hbar}{2}\left(\pm\boldsymbol{\sigma}_3 + \langle R_{\pm\varphi}^2\rangle(\pm\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\right) = \pm\frac{\hbar}{2}\boldsymbol{\sigma}_3 \text{ (spin vector)} \tag{3}$$

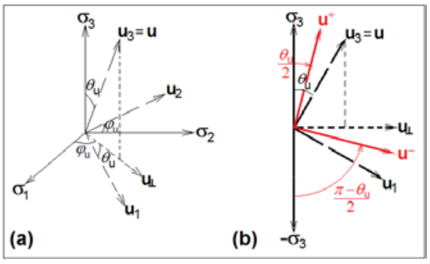

Figure 1. 3D and 2D rendering of (a) the coordinate and (b) spinor transformation.

$\mathbf{u}_2$ stands for any polar unit vector in the $\boldsymbol{\sigma}_{12}$ plane (see fig. 1a). The sum of two vectors is commutative, therefore we require an *ordered* sum (*mod* cyclic permutations), which allows to introduce orientation or handiness as discussed in the next paragraph. The Hermitian conjugate of a rotor R is also its reverse $R^\dagger = R^{-1}$. A *proper rotation* appears as a two-sided (rotor) R – (conjugated rotor) $R^\dagger$operator, which is a *unitary* transformation preserving handiness. The axial reflection $\mathbf{u}_2\mathrm{S}_{\uparrow\sigma}\mathbf{u}_2$ onto $\mathbf{u}_2$is equivalent to a π-rotation of $\mathrm{S}_{\uparrow\sigma}$ around $\mathbf{u}_2$. The gauge phase factor φ in (3) is not observable, which we express as $\langle\sin\varphi\rangle = \langle\cos\varphi\rangle = 0$. Angled brackets denote either the expected value (a scalar) or the average (a scalar, vector, etc.), as illustrated in the bottom line in (3); the average of spin $\langle S_{\pm\sigma}\rangle$ being the spin vector, and $\langle R_{\pm\varphi}^2\rangle = \langle\cos\varphi\rangle + I\boldsymbol{\sigma}_3\langle\sin\varphi\rangle = 0$. I will also use the notation $\langle\ \ \rangle_0$ for the expected value, if needed for clarity. We model spin *measurement* by extracting the *phase-insensitive* part of a spin or spin combination (see (6-7), (12-13)). The *full spin* modulus $\sqrt{S_\sigma^2}$, alike the overall phase, is not measurable; we call it an 'on-paper' quantity relating to QM angular momentum selection rules. It includes equal contributions from all the three components and exceeds by a factor of $\sqrt{3}$ the measured ('on lab') *spin vector* modulus $\sqrt{\langle S_\sigma\rangle^2} = \hbar/2$.

The handiness of the triplet $\pm\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$, in the order appearing in the expression for spin defines its *handiness*, where by convention $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$ is right-handed (R). It is preserved when two of the unit vectors reverse sign, e.g. $\boldsymbol{\sigma}_3(\uparrow) + \left(\boldsymbol{\sigma}_1 \circlearrowleft_{\pi/2} \boldsymbol{\sigma}_2\right)_{\pm\varphi}$ and $-\boldsymbol{\sigma}_3(\downarrow) + \left(-\boldsymbol{\sigma}_1 \curvearrowright_{\pi/2} \boldsymbol{\sigma}_2\right)_{\pm\varphi}$ are both R. S_σ depicts either $S_{\uparrow\sigma}$ or $S_{\downarrow\sigma}$, while S_σ and $S_{-\sigma}$ stand for opposite spins of same handiness, as in (3). S_σ and $S_{-\sigma}$ can transform onto each other by proper rotations at angles of π ($mod\ 2\pi$) around an axis lying on the $\boldsymbol{\sigma}_{12}$ plane, like $\mathbf{u}_2$ in (3). The notation $(S_\sigma)_{-\varphi}$ depicts the only substitution of φ by $-\varphi$ in the spin S_σ, with no effect on handiness. Any left-handed (L) spin-up (-down) is the *inverse* of a right-handed R spin-down (-up):

$$S_{\sigma\mathrm{L}} = IS_{-\sigma\mathrm{R}}I = -S_{-\sigma\mathrm{R}} \tag{4}$$

The two spaces L and R are disjoint under proper rotations, while plane *reflections* and *inversion*, also known as improper rotations, convert a spin from one space to the other. Focusing on the R space, the combined rotation $R_\mathrm{u} \equiv R_{\theta_\mathrm{u}}R_{\varphi_\mathrm{u}}$transforms 'on paper' (i.e. with conserved gauge phase φ) S_σ to S_u as follows:

$(S_\sigma)_\varphi \to R_u S_\sigma R_u^\dagger = \frac{\hbar}{2} R_{\theta_u}(\boldsymbol{\sigma}_3 + (\mathbf{u}_\perp + \mathbf{u}_2)_\varphi) R_{\theta_u}^\dagger = \frac{\hbar}{2}\left(\mathbf{u}^+ \cos\frac{\theta_u}{2} + \mathbf{u}^- \sin\frac{\theta_u}{2} + R_{\theta_u}(\mathbf{u}_\perp + \mathbf{u}_2)_\varphi R_{\theta_u}^\dagger\right) = \frac{\hbar}{2}[\mathbf{u} + (\mathbf{u}_1 + \mathbf{u}_2)_{\varphi'}] = (S_u)_{\varphi'};\ \varphi'(\mathbf{u}_{12}\text{plane}) = \varphi(\boldsymbol{\sigma}_{12}\text{plane});\ R_{\theta_u} = e^{\mathrm{B}\theta_u/2}; \mathrm{B} = \mathbf{u}_\perp\boldsymbol{\sigma}_3 = \mathbf{u}_{13} = \mathbf{u}^-\mathbf{u}^+ = -I\mathbf{u}_2$ (5)

Notice that (5) can be considered as a generalization of (3) by substituting $S_{\uparrow\sigma}$ with $S_{\uparrow u}$ and R_φ with $R_u \equiv R_{\theta_u} R_{\varphi_u}$in the first equation in (3). A spin *measurement* of S_σ along $\mathbf{u}$ by a Stern-Gerlach (*S-G*) magnet, which we call an 'on lab' transformation, differs from (5) by the two gauge angles $\varphi'$and φ becoming uncorrelated:

$$(S_\sigma)_\varphi \overset{S-G}{\to} (S_u)_{\varphi'} = \frac{\hbar}{2}[\mathbf{u} + (\mathbf{u}_1 + \mathbf{u}_2)_{\varphi'}];\ \langle \sin\varphi' \sin\varphi\rangle_{S-G} = \langle\sin\varphi'\rangle\langle\sin\varphi\rangle = 0 \quad (6)$$

In many instances, it is sufficient to apply the transformation to only the spin vectors, which are phase-insensitive:

$$\frac{\hbar}{2}\boldsymbol{\sigma}_3 \to \frac{\hbar}{2} R_{\theta_u}\boldsymbol{\sigma}_3 R_{\theta_u}^\dagger = \frac{\hbar}{2}\left(\mathbf{u}^+ \cos\frac{\theta_u}{2} + \mathbf{u}^- \sin\frac{\theta_u}{2}\right) = \frac{\hbar}{2}\mathbf{u},\ \text{where: } R_{\theta_u}\boldsymbol{\sigma}_3 = \mathbf{u}^+;\ R_{\pi-\theta_u}^\dagger(-\boldsymbol{\sigma}_3) = \mathbf{u}^- \quad (6')$$

The halfway vectors $\mathbf{u}^+, \mathbf{u}^-$(see figure 1b) depict one form of *reduced* spinor representation and as shown by the last two equalities in (6') they can be generated by the action of one-sided rotor (spinor) onto the spin vectors up and down, respectively. Of course, one can obtain both $\mathbf{u}^+$and $\mathbf{u}^-$by action of one-sided rotors on spin-up alone; even then, *two* terms are needed with distinct probability amplitudes for coincidence and anti-coincidence. In order to render the discussion more systematic, we insist that the spin basis consist of two *opposite spins*; then they must have *same handiness* in order for the spinor representation illustrated in fig. 1b to close into a proper rotation, as recounted in Section 3, eq. (16).

The expectation value for the 'on lab' transformation (6), i.e. a spin measurement is:

$$\langle S_{\sigma u}\rangle_{0(S-G)} = \langle\boldsymbol{\sigma}_3\mathbf{u}\rangle_0 = \boldsymbol{\sigma}_3 \cdot \mathbf{u} = (-\boldsymbol{\sigma}_3)\cdot(-\mathbf{u}) = \cos\theta_u = \cos^2\frac{\theta_u}{2} - \sin^2\frac{\theta_u}{2} \quad (7)$$

From (7) the probability for coincidence and anticoincidence outcomes is $\cos^2\theta_u/2$ and $\sin^2\theta_u/2$. If the relative probabilities and probability amplitudes show up in relations involving only spin vectors, like in (6'), (7) and (19), why do we then need the full spin (3) and its transformations (5) or (16)? Well, the full spin allows to account for the modulus, compatible with quantum selection rules for angular momentum of spin 1/2. In this sense, it is analogue to the standard Pauli spin vector. The full spin also constrains the form and handiness of the basis spins to comply with the 'on paper' transformations (5), (16). In addition, the explicit gauge phase in (3) permits formalizing the entrance of irreversibility in measurement as due to the loss of phase correlation, thus justifying the restricted transformations (6'), (19'). The full spin expression (3) becomes even more important in the two-spin case that will be discussed shortly. Especially the gauge *phase* and *handiness* turn out to be key concepts for understanding entanglements in the present framing.

For reference in the following discussion of the two-particle case, let spend a few lines to formalize improper rotations (reflections and inversion). As already noted the spin handiness does not change under axial reflection, as in (3), i.e. $\boldsymbol{\sigma}_k S_u \boldsymbol{\sigma}_k$ can be obtained by proper rotations of S_u or S_σ:

$$R_{\pi,\sigma_k} S_u R_{\pi,\sigma_k}^\dagger \equiv e^{-\frac{I\sigma_k\pi}{2}} S_u e^{\frac{I\sigma_k\pi}{2}} = -I\boldsymbol{\sigma}_k S_u I\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_k S_u \boldsymbol{\sigma}_k \quad (8)$$

Had we chosen plane reflections $I\boldsymbol{\sigma}_k S_u I\boldsymbol{\sigma}_k = -\boldsymbol{\sigma}_k S_u \boldsymbol{\sigma}_k$ instead, the handiness would have reversed. Handiness reverses in 3D by plane reflections and by inversion (i.e. 'reflection' onto the directed volume unit *I*), which can be shown by the unified expression (taking $\boldsymbol{\sigma}_0 = 1$):

$$S_u \to I\boldsymbol{\sigma}_\mu S_u I\boldsymbol{\sigma}_\mu = -\boldsymbol{\sigma}_\mu S_u \boldsymbol{\sigma}_\mu = \begin{cases} \text{Inversion} & \text{for } \mu = 0 \\ \text{Three basic plane reflections} & \text{for } \mu = j = 1,2,3 \end{cases} \quad (9)$$

2.2. *Two-particle systems.* The above discussion for the one-particle case generalizes in the following way to the two-particle case, again preserving a clear geometric picture of spin. First, the total spin and its square are now (see (2) for the definition of scalar product in STA):

$$\mathrm{S_{tot}} = \mathrm{S_{(1)}} + \mathrm{S_{(2)}};\ \mathrm{S_{tot}^2} = \mathrm{S_{(1)}^2} + \mathrm{S_{(2)}^2} + \mathrm{S_{(1)}S_{(2)}} + \mathrm{S_{(2)}S_{(1)}} = \frac{3\hbar^2}{2} + 2\mathrm{S_{(1)}} \cdot \mathrm{S_{(2)}} \tag{10}$$

$\mathrm{S_{tot}}$ is Hermitian and symmetric with respect to the spin swap $\mathrm{S_{(1)}} \leftrightarrows \mathrm{S_{(2)}}$. Therefore in the following discussion, which will revolve around equation (10), the shorthand $\mathrm{S_{(1)}}$and $\mathrm{S_{(2)}}$ will stand for $\mathrm{S_{(1or2)}}$ and $\mathrm{S_{(2or1)}}$ respectively. Similarly to the squared modulus for one-particle spin given in (3), $\mathrm{S_{tot}^2}$ for two particles in (10) is not measurable. It can be calculated as illustrated below for the maximally entangled pairs, where the spins are specular or inverse images of each-other (see (9)). The four maximally entangled two-particle states, also known as Bell states [12] can be defined (below take $(\mathrm{S_\sigma} \cdot \boldsymbol{\sigma}_0)\boldsymbol{\sigma}_0 = \mathrm{S_\sigma}$) as:

$$\Upsilon_{\sigma(\mu)}:\ \left\{\mathrm{S_{(1)}} = (\mathrm{S_\sigma})_\varphi;\ \mathrm{S_{(2)}} = \left(I\boldsymbol{\sigma}_\mu \mathrm{S_\sigma} \boldsymbol{\sigma}_\mu I\right)_\varphi;\ \mu = 0,1,2,3 \text{ for } \Psi^-, \Phi^-, \Phi^+, \Psi^+, \text{respectively}\right\} \tag{11}$$

$$\mathrm{S}_{tot(\mu)} = \mathrm{S_\sigma} + I\boldsymbol{\sigma}_\mu \mathrm{S_\sigma} I\boldsymbol{\sigma}_\mu = (\mathrm{S_\sigma})_\varphi - \left(\boldsymbol{\sigma}_\mu \mathrm{S_\sigma} \boldsymbol{\sigma}_\mu\right)_\varphi = \begin{cases} 0 & \text{for } \mu = 0 \quad (\Psi^-) \\ 2\left[\mathrm{S_\sigma} - \left(\mathrm{S_\sigma} \cdot \boldsymbol{\sigma}_j\right)\boldsymbol{\sigma}_j\right];\ \left(\mathrm{S}_{tot(j)}\right)^2 = 2\hbar^2 & \text{for } \mu = j = 1,2,3\ (\Phi^-, \Phi^+, \Psi^+) \end{cases} \tag{11a}$$

$$2\left(\mathrm{S_{(1)}} \cdot \mathrm{S_{(2)}}\right)_{(\mu)} = -2(\mathrm{S_\sigma})_\varphi \cdot \left(\boldsymbol{\sigma}_\mu \mathrm{S_\sigma} \boldsymbol{\sigma}_\mu\right)_\varphi = 2\mathrm{S_\sigma} \cdot \left(\mathrm{S_\sigma} - 2\left(\mathrm{S_\sigma} \cdot \boldsymbol{\sigma}_\mu\right)\boldsymbol{\sigma}_\mu\right) = 2\left(\mathrm{S_\sigma^2} - 2\left(\mathrm{S_\sigma} \cdot \boldsymbol{\sigma}_\mu\right)^2\right) = \begin{cases} -2\mathrm{S_\sigma^2} = -3\hbar^2/2 & \text{for } \mu = 0 \quad (\Psi^-) \\ 2[3\hbar^2/4 - \hbar^2/2] = \hbar^2/2 & \text{for } \mu = j = 1,2,3\ (\Phi^-, \Phi^+, \Psi^+) \end{cases} \tag{11b}$$

Ψ^-, where the spin pair relate by inversion is a singlet state, while the spin pairs in the three triplet states Ψ^+, Φ^-, Φ^+ relate by reflections onto the three frame planes defined by $\boldsymbol{\sigma}_{12}, \boldsymbol{\sigma}_{23}, \boldsymbol{\sigma}_{31}$, respectively. Relations (11) define the two spins and their relative phases in 3D, therefore together with the spin swap symmetry for the cross-product in (10) uniquely determine the two-particle states. Spinor representation of the same states is presented in Section 3, see relations (20-21). $\mathrm{S_{tot}^2} = 2\hbar^2$(or 0) for triplet (singlet) in (11a) correspond to angular momentum of $\hbar$ (or 0) for each entangled pair. It stands clear from the expressions above that entangled states are not just combinations of spin vector pairs up and down; the spins' relative handiness and the gauge phases have to satisfy precise relations as well. The vectors for $\mathrm{S_{tot}}$ of the triplet states in (11a) 'live' in the mutually perpendicular planes $I\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_{23}, I\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_{31}, I\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_{12}$, respectively. Relations (11, a-b) demonstrate in a nutshell why and how the full spin (3) is necessary to define two-spin entanglements.

Any measurement of spin along a chosen axis $\mathbf{u}$ will produce either $+\frac{\hbar}{2}\mathbf{u}$ or $-\frac{\hbar}{2}\mathbf{u}$. Therefore the *measured* square value of total spin for two particles cannot exceed $\hbar^2$, which as seen in (11a) falls short by a factor of 2 relative to the 'on paper' value of $\mathrm{S_{tot}^2}$ for the triplet states. The same reason as for the one-particle case is valid here, the measurement projecting out two of each spin components. Applying the 'on lab' rule of transformation to the cross-product terms in (10), which is the part of the squared total spin affected by a *measurement*, one obtains the correlation expectation values:

$$\langle\Upsilon_{(\mu)}\rangle_{S-G} \equiv \langle\tfrac{2}{\hbar^2}\left(\mathrm{S_{(1)}S_{(2)}} + \mathrm{S_{(2)}S_{(1)}}\right)_{(\mu)}\rangle_{S-G} = -\tfrac{4}{\hbar^2}\left(R_\mathrm{u}\mathrm{S_\sigma}R_\mathrm{u}^\dagger\right)_{S-G} \cdot \left(\boldsymbol{\sigma}_\mu R_\mathrm{v}\mathrm{S_\sigma}R_\mathrm{v}^\dagger\boldsymbol{\sigma}_\mu\right)_{S-G} = -\mathbf{u} \cdot \left(\boldsymbol{\sigma}_\mu \mathbf{v} \boldsymbol{\sigma}_\mu\right) = \mathbf{u} \cdot \mathbf{v} - 2\left(\mathbf{u} \cdot \boldsymbol{\sigma}_\mu\right) \cdot \left(\mathbf{v} \cdot \boldsymbol{\sigma}_\mu\right) = \begin{cases} -(\mathbf{u} \cdot \mathbf{v}) = -\cos\vartheta_\mathrm{uv} = \sin^2\frac{\vartheta_\mathrm{uv}}{2} - \cos^2\frac{\vartheta_\mathrm{uv}}{2}; & \mu = 0 \\ \mathbf{u} \cdot \mathbf{v} - 2(\mathbf{u} \cdot \boldsymbol{\sigma}_k)(\mathbf{v} \cdot \boldsymbol{\sigma}_k) = \cos\vartheta_\mathrm{uv} - 2u_k v_k = \cos\vartheta_\mathrm{uv} - 2\cos\vartheta_\mathrm{uk}\cos\vartheta_\mathrm{vk}; & \mu = k = 1,2,3 \end{cases} \tag{12}$$

u_k, v_k(in italics) are the scalar components of $\mathbf{u}$ and $\mathbf{v}$ along $\boldsymbol{\sigma}_k$. (12) renders the experimental bounds explicit, i.e. $-\hbar^2/2 \le \left(2\mathrm{S_{(1)}} \cdot \mathrm{S_{(2)}}\right)_{S-G} \le \hbar^2/2$ for all two-particle states. It stands also clear from (12) that the expectation values remain unchanged under swapped reflection $\mathrm{S_u} \cdot \left(I\boldsymbol{\sigma}_\mu \mathrm{S_v} I\boldsymbol{\sigma}_\mu\right) = \left(I\boldsymbol{\sigma}_\mu \mathrm{S_u} I\boldsymbol{\sigma}_\mu\right) \cdot \mathrm{S_v}$, as they should. Notice that both 'on paper' and 'on lab' satisfy: $\sum_\mu\left(\mathrm{S_{(1)}} \cdot \mathrm{S_{(2)}}\right)_{(\mu)} = 0$.

The advantage of (12) is that the two terms render the spin swap symmetry explicit and directly yield the scalar product of the two spins; however, for calculation purposes, we could also have defined the expectation values by the scalar part of only one term:

$$\langle\Upsilon_{(\mu)}\rangle_{S-G} \equiv -\frac{4}{\hbar^2}\langle S_{\mathrm{u}}\boldsymbol{\sigma}_\mu S_{\mathrm{v}}\boldsymbol{\sigma}_\mu\rangle_{S-G} = \begin{cases} -(\mathbf{u}\cdot\mathbf{v}) \;\; for \;\; \langle\Psi^-\rangle \;\; (\mu=0) \\ \left(\mathbf{u}\cdot\mathbf{v} - 2(\mathbf{u}\cdot\boldsymbol{\sigma}_k)(\mathbf{v}\cdot\boldsymbol{\sigma}_k)\right) \; (\mu = k = 1,2,3) \end{cases}; \; \sum_\mu \langle\Upsilon_{(\mu)}\rangle = 0 \tag{13}$$

The form of correlation expectation value in (13) hints to a possible spinor representation of the four maximally entangled states that is discussed in Section 3.2 (see relations (20-21)).

Remembering the discussion on the R and L handiness in connection to definitions (3), (4) one can schematically characterize the four states as the four possible sign combinations in front of the basis vectors for RL pairs, where R is taken by convention to be (+++) and the first column below stands for $\mathrm{S}_{(1)}$:

$$\text{Singlet:}\begin{cases}\boldsymbol{\sigma}_1: + \; - \\ \boldsymbol{\sigma}_2: + \; - \\ \boldsymbol{\sigma}_3: + \; -\end{cases}\!\!, \text{Triplets:}\begin{cases}\boldsymbol{\sigma}_1: + \; - \; + \; + \\ \boldsymbol{\sigma}_2: + \; + \; - \; + \\ \boldsymbol{\sigma}_3: + \; + \; + \; -\end{cases}\!; \text{ or in general (sum) } \boldsymbol{\sigma}_{j(1)}\boldsymbol{\sigma}_{j(2)} = \begin{cases}-3 \;\; singlet \\ +1 \;\; triplets\end{cases} \tag{14}$$

The last equality is a succinct expression for the 'on paper' scalar $\mathrm{S}_{(1)}\cdot\mathrm{S}_{(2)}$ in units of $\hbar^2/4$. What about the *orthogonality* of the maximally entangled states? Alike the one-particle case (remember $\mathbf{u}^+, \mathbf{u}^-$ in fig. 1b), orthogonality is experienced in the spinor (one-sided rotor) representation, as described by equations (22-24) and the relative discussions in Section 3.2.

If the spins are not entangled relations (12, 13) are invalid. In this case the gauge components of the spins $\mathrm{S}_{(1)}$, $\mathrm{S}_{(2)}$ are not correlated before measurement and the two spins form a mixed state. For example, instead of the singlet state in (12), there will be a pair of antiparallel spins along the $\boldsymbol{\sigma}_3$ axis, while the total spin will lie in the $\boldsymbol{\sigma}_{12}$ plane and vary randomly between the limiting values of $\pm\hbar$. The expectation value in this case will be the product of the separate expectation values:

$$\langle S_{(1)}S_{(2)}\rangle = \frac{4}{\hbar^2}\langle S_\sigma\mathbf{u}\rangle\langle S_{-\sigma}\mathbf{v}\rangle = -(\boldsymbol{\sigma}_3\cdot\mathbf{u})(\boldsymbol{\sigma}_3\cdot\mathbf{v}) = -\cos\theta_{\mathrm{u}}\cos\theta_{\mathrm{v}} \tag{15}$$

One future scope could be to adapt the presented STA formalism to describe spin pairs with degrees of entanglement between the maximally entangled states from relations (11)-(13) and the mixed states from (15), as depending on the uncertainty of the relative gauge phase.

3. Spinor (one-sided rotor) transformation of spin 1/2 one and two particle systems.

In spinor form, the transformation of a given spin appears as a sum of one-sided rotor transformations of spin-up and spin-down with the standard probability amplitudes for coincidence and anti-coincidence.

3.1. ***One-particle systems.*** The vector transformation (5) looks like a 'deterministic' spin transformation from a given orientation in the $\boldsymbol{\sigma}$ system into a definite orientation in the $\mathbf{u}$ system. One can improve on this impression by applying the STA *spinor* transformation, which uses one-sided rotors to transform the spin, as illustrated in Fig. 1b:

$$\begin{gathered} S_\sigma \to R_{\theta_{\mathrm{u}}}S_{\sigma(\varphi)}\cos\frac{\theta_u}{2} + R^\dagger_{\pi-\theta_{\mathrm{u}}}S_{-\sigma(-\varphi)}\sin\frac{\theta_u}{2} = R_{\theta_{\mathrm{u}}}S_{\sigma(\varphi)}\cos\frac{\theta_u}{2} + R^\dagger_{(\pi-\theta)_{\mathrm{u}}}R_{\pi_{\mathrm{u}}}S_{\sigma(\varphi)}R^\dagger_{\pi_{\mathrm{u}}}\sin\frac{\theta_u}{2} = \\ R_{\theta_{\mathrm{u}}}S_{\sigma(\varphi)}\left(\cos\frac{\theta_u}{2} + I\mathbf{u}_2\sin\frac{\theta_u}{2}\right) = R_{\theta_{\mathrm{u}}}S_{\sigma(\varphi)}R^\dagger_{\theta_{\mathrm{u}}} = S_{\mathrm{u}(\varphi-\varphi_{\mathrm{u}})}, \\ \text{or: } S_\sigma \to \frac{\hbar}{2}\left[\left(\mathbf{u}^+ + R_{\theta_{\mathrm{u}}}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)_\varphi\right)\cos\frac{\theta_u}{2} + \left(\mathbf{u}^- - R^\dagger_{\pi-\theta_{\mathrm{u}}}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)_{-\varphi}\right)\sin\frac{\theta_u}{2}\right] = \frac{\hbar}{2}\Big(\mathbf{u}^+\cos\frac{\theta_u}{2} + \\ \mathbf{u}^-\sin\frac{\theta_u}{2} + R_{\theta_{\mathrm{u}}}\left((\mathbf{u}_\perp+\mathbf{u}_2)_{\varphi-\varphi_{\mathrm{u}}}\cos\frac{\theta_u}{2} - I\mathbf{u}_2(\mathbf{u}_\perp-\mathbf{u}_2)_{-\varphi+\varphi_{\mathrm{u}}}\sin\frac{\theta_u}{2}\right)\Big) = \frac{\hbar}{2}\Big(\mathbf{u} + R_{\theta_{\mathrm{u}}}(\mathbf{u}_\perp + \\ \mathbf{u}_2)_{\varphi-\varphi_{\mathrm{u}}}\left(\cos\frac{\theta_u}{2} + I\mathbf{u}_2\sin\frac{\theta_u}{2}\right)\Big) = \frac{\hbar}{2}\left(\mathbf{u} + R_{\theta_{\mathrm{u}}}(\mathbf{u}_\perp+\mathbf{u}_2)_{\varphi-\varphi_{\mathrm{u}}}R^\dagger_{\theta_{\mathrm{u}}}\right) = \frac{\hbar}{2}\left(\mathbf{u} + (\mathbf{u}_1+\mathbf{u}_2)_{\varphi-\varphi_{\mathrm{u}}}\right) = (S_{\mathrm{u}})_{\varphi-\varphi_{\mathrm{u}}} \end{gathered} \tag{16}$$

By insisting that a basis consist of *opposite* spins then in order for transformation (16) to complete into a proper rotation as it does, the two spins must have same handiness. The plane for the phase angle in the subscripts in (16) follows the plane of the vectors in brackets. The concise form of the transformation in the top two lines of (16) is typical for STA, where full geometric objects transform as a whole, with none or minimal need to work with components. Vector components appear explicitly in the second form of transformation. The full *spinor representations* for spin up and spin down are:

$$R_{\theta_{\mathrm{u}}} S_{\sigma(\varphi)} = \mathbf{u}^{+} + R_{\theta_{\mathrm{u}}} (\mathbf{u}_{\perp} + \mathbf{u}_2)_{\varphi-\varphi_{\mathrm{u}}} \text{ and } R_{\theta_{\mathrm{u}}} S_{\sigma(\varphi)} I\mathbf{u}_2 = \mathbf{u}^{-} + R_{\theta_{\mathrm{u}}} (\mathbf{u}_{\perp} + \mathbf{u}_2)_{\varphi-\varphi_{\mathrm{u}}} I\mathbf{u}_2 \tag{17}$$

Notice that the spinor representation is not a spin according to definition (3). As already mentioned, the midway vectors $\mathbf{u}^{+}$and $\mathbf{u}^{-}$can be taken as *reduced* spinor representations of the two spins; they are manifestly orthonormal. Spinor representations are not unique. Another reduced spinor representation comprises the factors in front of the trigonometric functions of the rotor $R_{\theta_{\mathrm{u}}}$, a scalar 1 and a bivector $I\mathbf{u}_2$, respectively, which also appear in (17). These are even-grade elements of the 3D algebra and a standard choice in the STA literature [7]. In order for the reduced representation to be an orthonormal spinor basis one must have a zero scalar (grade 0) for $\langle 1(I\mathbf{u}_2)\rangle_0 = 0$, which is indeed the case. The orthogonality relation for the full spinor representation is (remember that S_σ is Hermitian):

$$\langle (R_{\theta_{\mathrm{u}}} S_{\sigma(\varphi)} I\mathbf{u}_2)^{\dagger} (R_{\theta_{\mathrm{u}}} S_{\sigma(\varphi)})\rangle_0 = -\langle \tfrac{3\hbar^2}{4} I\mathbf{u}_2\rangle_0 = 0, \text{ which is clearly satisfied} \tag{18}$$

The orthogonality condition for the even-grade reduced representation is a normalized version of (18). There are two things to notice in (16). First, it is clear that the new spin has been expressed by the two basis spins in $\boldsymbol{\sigma}$, each contributing an amplitude given by the trigonometric factors. The basis spins, in addition to being opposite must comply with precise phase angle differences and have same handiness in order for the transformation to complete. Second, it is probably more significant to read transformation (16) backwards. Start with S_{u} and transform it with probability amplitudes of $\cos\frac{\theta_u}{2}$ and $\sin\frac{\theta_u}{2}$ into S_σ (coincidence) or $S_{-\sigma}$ (anticoincidence), respectively:

$$S_{\mathrm{u}} = R_{\theta_{\mathrm{u}}} S_\sigma \cos\frac{\theta_u}{2} + R_{\theta_{\mathrm{u}}} S_\sigma I\mathbf{u}_2 \sin\frac{\theta_u}{2} = R_{\theta_{\mathrm{u}}} S_\sigma \cos\frac{\theta_u}{2} + R_{\theta_{\mathrm{u}}} R^{\dagger}_{\pi_{\mathrm{u}}} S_{-\sigma} \sin\frac{\theta_u}{2} \tag{16'}$$

From (16'), if S_{u} is spin-up (down) then $R_{\theta_{\mathrm{u}}} S_\sigma$ represents spin-up (down) and $I\mathbf{u}_2 R_{\theta_{\mathrm{u}}} S_{-\sigma} = R_{\theta_{\mathrm{u}}} S_\sigma I\mathbf{u}_2$ spin-down (up). (16') is symmetric relative to the exchange $\uparrow_\sigma \rightleftarrows \downarrow_\sigma$ and $R_{\theta_{\mathrm{u}}} S_\sigma$ or 1 stand for coincidence, while $R_{\theta_{\mathrm{u}}-\pi} S_{-\sigma}$ or $I\mathbf{u}_2$ stand for anti-coincidence. It is also relevant to point out that the controlled phase angle in (16) actually ensures the reversibility of the 'on paper' spin transformation, ultimately enabling to read it backwards as in (16'). On the other side, the action of a *S-G* magnet makes the transformation irreversible relative to the gauge plane; in that case (16') cannot remain an equality, changing to ($\varphi, \varphi', \varphi''$ uncorrelated; $\overset{\langle\varphi\rangle}{=}$ means average wrt gauge phases):

$$S_{\mathrm{u}(\varphi)} \xrightarrow{S-G} R_{\theta_{\mathrm{u}}} S_{\sigma(\varphi')} \cos\frac{\theta_u}{2} + R^{\dagger}_{\pi-\theta_{\mathrm{u}}} S_{-\sigma(\varphi'')} \sin\frac{\theta_u}{2} \overset{\langle\varphi\rangle}{=} \frac{\hbar}{2}\left(\mathbf{u}^{+}\cos\frac{\theta_u}{2} + \mathbf{u}^{-}\sin\frac{\theta_u}{2}\right) \tag{19}$$

In the 'on lab' case it is straightforward to apply the spinor transformation to the spin vector alone:

$$\frac{\hbar}{2}\mathbf{u} = \frac{\hbar}{2} R_{\theta_{\mathrm{u}}} \boldsymbol{\sigma}_3 \left(\cos\frac{\theta_u}{2} + I\mathbf{u}_2 \sin\frac{\theta_u}{2}\right) = \frac{\hbar}{2}\left(\mathbf{u}^{+}\cos\frac{\theta_u}{2} + \mathbf{u}^{-}\sin\frac{\theta_u}{2}\right) \tag{19'}$$

The final expression in (19') shows explicitly the halfway vectors $\mathbf{u}^{+}, \mathbf{u}^{-}$as an orthonormal basis for the spinor representation. As already mentioned in Section 2.1, being insensitive to gauge phase, the spin vector is well suited to represent spin transformation by *S-G* measurement.

3.2. ***Two-particle systems.*** By writing the non-normalized expression in (13) for $\mathbf{u} = \mathbf{v} = \boldsymbol{\sigma}_3$ as:

$$\langle S_\sigma I\boldsymbol{\sigma}_\mu S_\sigma I\boldsymbol{\sigma}_\mu\rangle_0 = \langle \left(-I\boldsymbol{\sigma}_\mu \mathrm{S}_\sigma\right)^{\dagger} \left(\mathrm{S}_\sigma I\boldsymbol{\sigma}_\mu\right)\rangle_0 \text{ (notice that } \mathrm{S}_\sigma I\boldsymbol{\sigma}_\mu \text{is not Hermitian)} \tag{20}$$

one realizes that a non-normalized spinor form for the entangled pairs consists of the conjugate pair:

$$\Upsilon_{(\mu)(1or2)} \equiv -I\boldsymbol{\sigma}_\mu \mathrm{S}_\sigma;\ \Upsilon_{(\mu)(2or1)} \equiv \mathrm{S}_\sigma I\boldsymbol{\sigma}_\mu \tag{21}$$

The spinor representations (21) for the four states comprise only terms that are either even (bivectors) for the singlet state ($\mu = 0$) or odd (vectors and pseudoscalar) for the triplet states ($\mu = j$). Therefore:

$$\langle \Upsilon^{\dagger}_{(0)} \Upsilon_{(j)} \rangle_0 = \langle \Upsilon^{\dagger}_{(j)} \Upsilon_{(0)} \rangle_0 = 0 \text{ for both "on paper" and "on lab"} \tag{22}$$

From (22) the singlet state is orthogonal to the triplet states. The orthogonality test for pairs of different triplet states ($j \neq k$) in the 'on paper' spinor representations (21) takes the form:

$$\langle \Upsilon_{(j)} \Upsilon_{(k)} \rangle_0 = \langle (\mathrm{S}_\sigma)_\varphi I \boldsymbol{\sigma}_j (\mathrm{S}_\sigma)_{\pm\varphi} I \boldsymbol{\sigma}_k \rangle_0 = -\langle (\mathrm{S}_\sigma)_\varphi \left(\boldsymbol{\sigma}_j \mathrm{S}_\sigma \boldsymbol{\sigma}_j \right)_\varphi \boldsymbol{\sigma}_{jk} \rangle_0 = \langle R_\varphi \mathrm{S}_\sigma \left(\mathrm{S}_\sigma - 2\boldsymbol{\sigma}_j \left(\mathrm{S}_\sigma \cdot \boldsymbol{\sigma}_j \right) \right) R^{\dagger}_\varphi \boldsymbol{\sigma}_{jk} \rangle_0 =$$
$$-\frac{\hbar^2}{2} \langle R_\varphi \boldsymbol{\sigma}_{kj} R^{\dagger}_\varphi \boldsymbol{\sigma}_{jk} \rangle_0 = \begin{cases} -\frac{\hbar^2}{2} \text{ for } j + k = 3 \\ -\frac{\hbar^2}{2} \langle R^2_\varphi \rangle_0 = -\frac{\hbar^2}{2} \cos\varphi \text{ for } j + k > 3 \end{cases} \tag{23}$$

From (23) the 'on paper' spinor representations (i.e. with correlated gauge phases) (21) *do not pass* the orthogonality test for $j + k = 3$, or pass it only on average w.r.t. $\cos\varphi$ for $j + k > 3$. This relates to the well-known geometric property that in 3D orthogonal planes always share a common line. Finally, I check for orthogonality among the triplet states 'on lab':

$$\langle \Upsilon_{(j)} \Upsilon_{(k)} \rangle_{0(S-G)} = \langle \boldsymbol{\sigma}_3 I \boldsymbol{\sigma}_j \boldsymbol{\sigma}_3 I \boldsymbol{\sigma}_k \rangle_0 = 0 \text{ for } j \neq k \tag{24}$$

From (24) the 'on lab' spinor representations for the maximally entangled states are *mutually orthogonal*. The corresponding reduced spinor representations $\pm I \boldsymbol{\sigma}_\mu$ are manifestly orthonormal for the four entanglement states $\Upsilon^\mu, \mu = 0,1,2,3$.

A swift comparison of the two-particle spinor representation (21) with the vector representation (11) reveals that inversion and reflection operations apply to *one* of the spins in (11); the same operations are split between the *two* spinor representations in (21). Therefore, given that the two spins are experimentally monitored at different locations, one might get the impression that the spinor representation is 'more nonlocal' than the vector representation (11). However, the two types of transformations are equivalent; in the vector form we assume spin (1) to be the same, i.e. as a reference spin and transform spin (2) *relative* to it. However, as pointed out in relation to expression (10) there is full symmetry under the exchange $\mathrm{S}_{(1)} \leftrightarrows \mathrm{S}_{(2)}$. In addition, it is relevant to remember here that we are working in the decoupling approximation of spin from position and further development of the STA formalism is necessary to describe nonlocality, which necessarily requires coupling of spin and position, and that most correctly has to be treated relativistically [7]. By construction the expression for the expectation value is the same in both representations and as a correlation expectation value, it takes account of both entangled spins under the same angled brackets. This would represent a nonlocal operation, as the two measurements actually take place at different locations. However, again in the present context, due to decoupling it is *local*. The common angled bracket here only refers to the *statistical dependen*ce between the two measurements, directly connected to the *correlation* between gauge phases before measurement.

4. Conclusions

All the above results were obtained by direct use and transformation (rotation, reflection) of spin in 3D physical orientation space, without invoking concepts like *eigen* algebra and tensor product, which seem so fundamental in the standard formulation of QM. This also provides a clear geometric meaning for the four Bell states: one singlet state of spin pairs related by inversion and three triplet states, each comprising spin pairs related by reflection onto one of the three coordinate planes.

In conclusion, the STA spin model in (3) – an ordered sum of vectors with a gauge phase, displays many attractive features in the spin-position decoupling approximation. For one and two particle systems the same 3D geometric object (3) shows the correct representation of spin 1/2 relative to both 'on paper' and 'on lab' calculations. In a STA context (3) merges the roles played by the Pauli spin matrix-vector and spin states in the standard quantum formalism. The explicit gauge phase in (3) allows formalizing the irreversibility introduced by measurement ('on lab' transformation) as due to loss of phase correlation. Spin pair entanglements emerge from correlations of the gauge phases and opposite handiness in the expression for the total spin. A clear geometric picture can be maintained throughout the process of 'on paper' and 'on lab' transformations, not the least due to STA's distinguished capability to work with whole geometric objects instead of components.

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