Fast geometry-based ambiguity resolution algorithm for real-time kinematic

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Fast geometry-based ambiguity resolution algorithm for real-time kinematic

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Abstract

Centimeter-level positioning using real-time kinematic technology is achievable only after overcoming the step of ambiguity resolution. Fast algorithms providing such a solution tend to have low success rates, often resulting in time to first fix too long in many real-life cases. On the other hand, other algorithms, such as the LAMBDA method, provide a better success rate at the expense of more computations. The recent growth in the number of constellations available for positioning along with the multiplication of signals per satellite is challenging slower algorithms which prove to have a complexity exponential with the number of signals. This article presents a new approach to the integer ambiguity problem, showing in an experiment a success rate of **100%** with a computational speed ten times higher than the MLAMBDA algorithm, the complexity of the search process being $\boldsymbol{\mathcal{O}(n^2)}$ with $\boldsymbol{n}$ the total number of signals. We consider here a short baseline model to introduce the algorithm. It is straightforward to extend the results to longer baselines using a more complete model, as the hypotheses for the algorithm to be successful are made explicit along the article. Classical techniques developed to improve ambiguity resolution, as well as computational tricks could be used to further improve the speed of this algorithm, and could hence lead to improved time to first fix in low-cost embedded equipment.

Keywords: GNSS, ambiguity resolution, single-epoch, multiple frequencies, multiple constellations, Real-Time Kinematic

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1 Introduction

Since the beginning of global navigation satellite systems (GNSS) positioning, accuracy has been considerably improved, up to instantaneous centimeter-level accuracy. Embedded receivers with technologies such as Precise Point Positioning and Real-Time Kinematic (RTK) positioning are becoming more accessible. These also brought with them their own theoretical issues, with probably the most well-known among them: the Integer Ambiguity Problem. Resolution of this problem allows reliable and accurate positioning. After decades of research, such a problem still present difficulties, as we require the resolution to be both fast and successful. Existing algorithm are generally a balance of those, presenting their perks in some case and their shortcoming in others.

The multiplication of the GNSS constellations (GPS, GLONASS, Galileo, Beidou...), and signals brought their own advantages and drawbacks with respect to this matter. On the positive side, it granted improved accuracy of code positioning, the possibility to combine pairs or trios of signals to deal with atmospheric deterioration of signals, and to mitigate multipaths, with the effect of improving the success rate of algorithms solving the Integer Ambiguity Problem. On the negative side, the multiplication of available signals requires more computations, making some algorithms obsolete or inoperable. Hence, it is at the moment still difficult to make such an availability of signals profitable for accurate positioning such as RTK positioning.

Different approaches in ambiguity resolution have been used over the years. Without being exhaustive, the most straightforward one seem to be the method adopted by Euler and Landau [1] where all possible ambiguities around an initial guess are evaluated. In their paper, for 6 signals and a search space 21 ambiguities, they evaluate 21^5 possible ambiguity combinations. In terms of position, this search space corresponds to a radius of search of $10 \times wavelength/(dilution\ of\ precision)$, equivalent to around 1.5 meter in good cases. With the current signal availability, such a search would be

21^n where n could reach a value of 100, showcasing the difficulties of treating that many signals. In fact, any algorithm presenting a complexity exponential with the number of signals will struggle to take advantage of this availability without specific techniques of dimensionality reduction. One of the most well-known and used Ambiguity Resolution algorithm has been pioneered by Teunissen in 1995 [2, 3] and is known as the LAMBDA method. This method allowed to improve greatly the computational time, showing a success rate high enough for some real world applications. This algorithm has been optimized by Chang, Yang and Zhou in 2005 [4], so as to improve performances of the previous method. If this algorithm allowed a fast resolution of the integer ambiguity problem for larger n, the complexity is still exponential with the number of satellites, again limiting its potential usefulness. Another shortcoming shown in the paper of Chang et al. is the unreliability of the computation time, that can vary greatly depending on the covariance matrix, and can turn out to be a problem when time is limited. Other approaches favor speed over success rate; the most computationally efficient algorithm is the rounding operation, transforming the float ambiguity in an integer vector by rounding each vector component. With an unbeatable computation time, the success rate falls sharply, but can sometimes prove to have its own perks. Integer bootstrapping improves the latter technique making use of the correlation between the measurements, at the expense of more computations. Such techniques are well described by Teunissen in [5, 6]. Since then, many improvements of such algorithms have been described, using combinations of signals, partial ambiguity solutions, advanced models.

In this paper, we introduce a new approach to the integer ambiguity resolution in the case of double-differences (RTK case) for short baselines. Using a simple hypothesis, we define a search space in a 3-dimensional subspace of radius R, and evaluate the ambiguities in this subspace using n^2 computations. The search process being in the embedded space of coordinates, the basis of this approach recalls the works of Mader [7] and Remondi and Hilla [8], in which the coordinates space is searched using an ambiguity function described by Counselman and Gourevitch [9]. The embedding of the position space in the ambiguity one for the search process, along with an insight on the lower boundary of the step size to be used, allows the research to be more efficient and successful. In terms of computations, remembering the aforementioned work of Euler and Landau, we hence have $K \sim (R/DOP)^3$ possible ambiguities to evaluate, where DOP stands for dilution of precision, instead of the K^n with n large. The full complexity of the algorithm presented is $\mathcal{O}(n^3 + K^3 n^2)$, overcoming difficulties shown by algorithm with exponential complexity. This guarantees our algorithm to be fast, even when the number of signals tends to be high. The size of the search space is directly connected to the success rate as the code positioning can be assumed to follow a normal distribution, and inside the search space, the success rate reaches 100%. The computation time also proves to be steady and predictable, allowing to adapt the duration of the algorithm in advance as the search space can be changed at any moment, making

it possible to strive for more speed or higher success rate when required. The algorithm also stays customizable and adaptable to improvement techniques developed throughout the years. Moreover, if the algorithm still requires a reasonable first positioning (for example code positioning) to be efficient, the float solution needs not to be close to any integer solution so as to find a solution. Finally, as will be made clear when introducing the main hypothesis, a short baseline is not required, as long as the general model compensates for it. At the expenses of adding some dimensions to the problem (such as ionospheric or tropospheric terms), it is straightforward to adapt the latter algorithm to medium-range or long-range problems. An application of the algorithm on a simple experiment using low-cost equipment shows results reaching 100% success rate for positioning at less than the centimeter to the true position, in a tenth of the time required by the MLAMBDA algorithm.

2 Preliminaries

In this section, we present the setting we consider throughout the paper, and the main equations of the problem. We then described the problem to solve, as well as the main hypotheses related to it. Before doing so, we introduce a few notations relevant in our work.

2.1 Notations

As we consider double differences, we will often refer to a base b, a rover r, a satellite s, and a pivot satellite p. The satellites will be shown as superscripts while the base and rover will be referred to as subscripts. When using double differences, we use the following simple notation:

$$X_{br}^{ps} = (X_b - X_r) - (X^p - X^s).$$

We will keep the letter $r$for vectors in $\mathbb{R}^3$, and use the letter d to denote the Euclidean norm of this vector: $d = \|r\|_2$. The variable P will refer to the pseudorange, while φ refers to the phase in cycles, and Φ to the phase-range in meters.
The rounding operation will be written as

$$\lfloor x \rceil = \lfloor x + 1/2 \rfloor .$$

The set of matrices of size (m, n) with elements in $\mathbb{K}$ is $\mathbb{K}^{m\times n}$. Finally, for the sake of readability we let:

$$M^{-\intercal} = (M^{-1})^{\intercal}$$

2.2 Classical setting

Consider a single constellation and signal: call $\mathcal{I}$ the set of all satellites that are not a pivot satellite and p the pivot satellite. For each satellite s in $\mathcal{I}$:

$$\begin{cases} P_{br}^{ps} = d_{br}^{ps} + \varepsilon_{br,P}^{ps} \\ \Phi_{br}^{ps} = d_{br}^{ps} + \lambda_s N_{br}^{ps} + \varepsilon_{br,\Phi}^{ps} \end{cases} \tag{1}$$

Consider now that we already computed an approximated solution r_0 by some classical means (*e.g.* solving the pseudorange equations above using a generalized non-linear least square algorithm). We then have, at the first order:

$$d_{br}^{ps} = d_{br_0}^{ps} + u_{r_0}^{ps} \cdot r_{r_0 r}, \qquad \text{with } u_{r_0}^{s} = \frac{r_{r_0}^{s}}{\|r_{r_0}^{s}\|_2} \tag{2}$$

For the sake of simplicity, we choose to divide the above system of equations by the wavelength λ_s and reason in cycles instead of meters. Although unusual for the pseudorange equations, it simplifies the further notations. Writing $p_0 = \left((P_{br}^{ps} - d_{br_0}^{ps})/\lambda_s\right)_{s\in\mathcal{I}} \in \mathbb{R}^n$, $\varphi_0 = \left(\varphi_{br}^{ps} - d_{br_0}^{ps}/\lambda_s\right)_{s\in\mathcal{I}} \in \mathbb{R}^n$, $H = \left(u_{r_0}^{ps}/\lambda_s\right)_{s\in\mathcal{I}} \in \mathbb{R}^{n\times 3}$, $N = \left(N_{br}^{ps}\right)_{s\in\mathcal{I}} \in \mathbb{Z}^n$, and $r = r_{r_0 r} \in \mathbb{R}^3$, we have the following problem:

$$\begin{cases} p_0 = Hr + \varepsilon_p \\ \varphi_0 = Hr + N + \varepsilon_\varphi \end{cases} \tag{3}$$

Moreover, when considering double differences, the fact that we use a pivot satellite implies that the measures are correlated, and we consider here:

$$Cov((\varepsilon_p, \varepsilon_\varphi)) = \begin{pmatrix} Q_p & 0 \\ 0 & Q_\varphi \end{pmatrix} \tag{4}$$

Remark 1 The previous setting can easily be adapted with a more complete model (such as atmospheric corrections), the following computations will stay true.
In the case of GLONASS, a correction should be applied to the corresponding equations (1): in general, it is sufficient to subtract $\lambda^{ps} N_{br_0}^{p}$ from the value of ϕ_0, as described in [10]).

Remark 2 When considering double differences, one can consider that the covariance for each signal S from satellite i and j of constellation C is given by $\sigma_{p,C,S}^2$ if $i \neq j$, where $\sigma_{p,C,S}^2$ is the variance of the pivot satellite, and $\sigma_{p,C,S}^2 + \sigma_{i,C,S}^2$ if $i = j$. Hence, the matrix Q_P and Q_φ takes the general block form:

$$Q = diag(Q_{C_1,S_1}, Q_{C_1,S_2}, ..., Q_{C_2,S_1}, ..., Q_{C_k,S_l}), \tag{5}$$

where each Q_{C_i,S_j} is defined by:

$$Q_{C_i,S_j} = \begin{pmatrix} \sigma^2_{p,C_i,S_j} + \sigma^2_{1,C_i,S_j} & \sigma^2_{p,C_i,S_j} & \cdots & \sigma^2_{p,C_i,S_j} \\ \sigma^2_{p,C_i,S_j} & \sigma^2_{p,C_i,S_j} + \sigma^2_{2,C_i,S_j} & \cdots & \sigma^2_{p,C_i,S_j} \\ \vdots & & & \\ \sigma^2_{p,C_i,S_j} & \sigma^2_{p,C_i,S_j} & \cdots & \sigma^2_{p,C_i,S_j} + \sigma^2_{n_{C_i,S_j},C_i,S_j} \end{pmatrix} \tag{6}$$

In general, choosing the pivot such that $\sigma^2_{p,C_i,S_j} < \sigma^2_{i,C_i,S_j}$ allows the matrix Q to be well-defined, symmetric positive definite. We will assume Q to satisfy these constraints thereafter.

Without loss of generality, for the sake of simplicity and readability, we consider only one constellation with one signal in the article; the results can be adapted to any number of constellations and signals in a straightforward way.

2.3 Main problem

In the following, we will consider two different problems. In the first one, we forget about the pseudorange equations, and consider only the integer ambiguity resolution of the carrier phase equations $\varphi_0 = Hr + N + \varepsilon_\varphi$. This allows to have an easier geometric representation of the problem which is beneficial to a first approach. In a second phase, we will add the pseudorange equations. It has the beneficial effect of bringing convexity to the first problem, limiting the search space naturally to solutions close enough to the least square solution of $p_0 = Hr$.

When looking for an ambiguity, the general approach is to consider that the actual ambiguity vector minimizes the norm of the residuals ε. With the existence of correlations between the measures, the Mahalanobis norm $\|x\|_Q = \sqrt{x^\intercal Q^{-1} x}$, where Q is the covariance matrix, has been the regular choice of norm for the residuals $\varphi_0 - Hr - N$ (see [11])

Finally, in the first case, we will need to limit the space of research in terms of positions. The (bounded) search space will be called $\mathcal{P} \subset \mathbb{R}^3$. The definition of this subspace is not necessary in the second case, given the convexity argument that will be made clearer thereafter. We can now state the two different problems:

Problem 1 *The solution of the ambiguity resolution is an ambiguity vector $N \in \mathbb{Z}^n$ and a position $r \in \mathcal{P} \subset \mathbb{R}^3$ such that*

$$\|\varphi_0 - Hr - N\|_{Q_\varphi} = \min_{r' \in \mathcal{P}, N' \in \mathbb{Z}^n} \left\|\varphi_0 - Hr' - N'\right\|_{Q_\varphi} = \min_{r' \in \mathcal{P}, N' \in \mathbb{Z}^n} \|\varepsilon_\varphi\|_{Q_\varphi}, \tag{7}$$

where $\|x\|_{Q_\varphi} = \sqrt{x^\intercal Q_\varphi^{-1} x}$.

Problem 2 *The solution of the ambiguity resolution is an ambiguity vector $N \in \mathbb{Z}^n$ and a position $r \in \mathbb{R}^3$ such that*

$$\|(p_0 - Hr, \varphi_0 - Hr - N)\|_{Q_{p,\varphi}} = \min_{r' \in \mathbb{R}^3, N' \in \mathbb{Z}^n} \|\varepsilon\|_{Q_{p,\varphi}}, \tag{8}$$

where $\varepsilon = (\varepsilon_p, \varepsilon_\varphi)$ in equations (3), and $Q_{p,\varphi}$ is defined in equation (4).

2.4 Dimensionality reduction

The problem of ambiguity resolution seems at first to be an $n+3$ dimensional problem, with n integer dimensions and 3 real ones. Under the constraint of minimization of problems 1 and 2, we can see that this is not the case anymore. Indeed, considering problem 1, we can write:

$$\min_{r\in\mathcal{P},N\in\mathbb{Z}^n} \|\varphi_0 - Hr - N\|_{Q_\varphi} = \min_{N\in\mathbb{Z}^n}\left(\min_{r\in\mathcal{P}} \|\varphi_0 - N - Hr\|_{Q_\varphi}\right) \tag{9}$$

$$= \min_{r\in\mathcal{P}}\left(\min_{N\in\mathbb{Z}^n} \|\varphi_0 - Hr - N\|_{Q_\varphi}\right) \tag{10}$$

The first equation (9) fixes first the ambiguity N, and then searches a minimizing r among the subset of positions $\mathcal{P}$. For the latter, there exists at most one solution to this minimizing problem, corresponding to the oblique projection of $\varphi_0 - N$ onto the subspace $span(H)$ (there is no solution when this process gives a solution in $\mathcal{P}^{\mathrm{C}}$). Hence for each $N \in \mathbb{Z}^n$, we can associate a unique $r \in \mathcal{P} \cup \{\emptyset\}$.
The second equation (10) first fixes an $r \in \mathcal{P}$, then looks for a minimizer over the set of ambiguity vectors $\mathbb{Z}^n$. Such a solution is almost surely unique, and for each r we hence can associate an ambiguity vector in $\mathbb{Z}^n$.
The problem is therefore a problem either in an n-dimensional discrete space, or in a 3-dimensional continuous one. The method developed thereafter allows to consider this time a 3-dimensional discrete subspace of the ambiguity space under some conditions.

Historically, approaches using equation (9) have been used, such as in the work of Euler and Landau [1]. Such algorithms look for a solution in an n-dimensional discrete space, and are hence extremely inefficient and inapplicable in the case of many satellites and many signals. Actual methods are generally based on the LAMBDA algorithm [3, 12], which searches an n-dimensional ellipsoid around an initial guess, following somewhat closely the space $span(H)$ [13]. However, with the multiplication of signals and constellations, these methods require an exponential cost of computations, even for the modified LAMBDA method [4].
In the scheme we propose, we consider a discrete search space in the subspace $span(H)$. To each point of this subspace, we associate an optimal (under some conditions) ambiguity vector N. To this ambiguity vector, we can again associate a vector r by the aforementioned projection. Hence, we associate to each position r_0 a new position r, minimizing locally the residual vector, and finally keeping the best one. This process cost is proportional to the number of visited points in the search space.
To this end, we can decompose our work in three different parts. First, we need to determine a search space that allows the search to be both efficient and successful. Secondly, each searched position r needs to be associated to an ambiguity vector N efficiently. Finally, for each ambiguity vector, we have

to compute the minimizing position associated to it and its residual, so as to rank the ambiguities.

Remark 3 It is important to emphasize that this algorithm supports wide-lane strategies to browse more swiftly the search space as well as multi-epoch reduction methods. Besides, every argument previously mentioned about problem 1 stands when considering problem 2.

3 Process

In this section, we answer the main questions related to the algorithm. To simplify the approach, we start from a fixed ambiguity N and compute the associated optimal position as well as its residual. Secondly, we consider the problem of the optimal ambiguity for a given position. Then, we determine the discrete search space in the 3-dimensional space of position. Finally, we discuss the extension to the solution of problem 2, when one takes into account the pseudoranges.

3.1 Projection and residual

Recall the fundamental equation (1) for problem 1:

$$\varphi_0 = Hr + N + \varepsilon_\varphi$$

with covariance matrix Q_φ. We want to minimize the residual $\|\varepsilon_\varphi\|_{Q_\varphi}$ over r on some domain $\mathcal{P} \subset \mathbb{R}^3$ where $N \in \mathbb{Z}^n$ is fixed. Define $\mathcal{A}$ the affine subspace $\varphi_0 + span(H)$. Whatever the ambiguity vector N is, the problem corresponds to the determination of the closest point of $\mathcal{A}$ to N, under the norm $\|\cdot\|_{Q_\varphi}$. Consider the Cholesky decomposition $Q_\varphi = CC^\intercal$ of the covariance matrix. Write

$$\varphi_0^\star = H^\star r + N^\star + \varepsilon_\varphi^\star, \tag{11}$$

with $\varphi_0^\star = C^{-1}\varphi_0$, $H^\star = C^{-1}H$, $\varepsilon^\star = C^{-1}\varepsilon$ and finally $N^\star = C^{-1}N$. We then have

$$\|\varepsilon_\varphi\|_{Q_\varphi} = \|\varepsilon_\varphi^\star\|_2,$$

which allows to consider a minimization problem with the Euclidean norm. The solution to this problem is the orthogonal projection of $N^\star$ on the affine subspace $\mathcal{A}^\star = \varphi_0^\star + span(H^\star)$. Let us call natural space of ambiguity the space of ambiguity vectors $\mathbb{Z}^n \subset \mathbb{R}^n$ before transformation, and transformed space of ambiguity vectors after transformation: $\{C^{-1}K, K \in \mathbb{Z}^n\}$.
The solution of the previous problem is given by

$$r' = \left(H^{\star\intercal}H^\star\right)^{-1} H^{\star\intercal}(\varphi_0^\star - N^\star) = \left(H^\intercal Q_\varphi^{-1} H\right)^{-1} H^\intercal Q_\varphi^{-1}(\varphi_0 - N) \tag{12}$$

We can then compute the residuals from this formula. Aiming for the least number of computations, we do a QR factorization of $H^\star$, letting $H^\star = Q^\star R^\star$. In this setting, the vector $N^\star$ can be decomposed in two orthogonal vectors, one in the affine subspace going through φ_0 and spanned by $H^\star$, that corresponds to the projection onto this subspace, given by $Q^\star Q^{\star\intercal}(N^\star - \varphi_0^\star)$, and the orthogonal part given by $(I_n - Q^\star Q^{\star\intercal})(N^\star - \varphi_0^\star)$. To obtain $Q^\star$, a Gram-Schmidt process on the 3 three columns of $H^\star$ is sufficient, making the QR decomposition extremely fast. The residual vector and the solution r' of the minimizing problem is now given by:

$$\varepsilon_\varphi^\star = \varphi_Q^\star - (I_n - Q^\star Q^{\star\intercal})N^\star \tag{13}$$

$$r' = r_Q^\star - (R^\star)^{-1}Q^{\star\intercal}N^\star \tag{14}$$

where $\varphi_Q^\star = (I_n - Q^\star Q^{\star\intercal})\varphi_0^\star$ and $r_Q^\star = (R^\star)^{-1}Q^{\star\intercal}\varphi_0^\star$. For each ambiguity vector to test, we hence only need $\mathcal{O}(n^2)$ computations.

3.2 Minimization over the ambiguity vectors

Consider equation (1):

$$\varphi_0 = Hr + N + \varepsilon_\varphi$$

with covariance matrix Q_φ. We want to minimize the residual $\|\varepsilon_\varphi\|_{Q_\varphi}$ over the N on $\mathbb{Z}^n$ where $r \in \mathbb{R}^3$ is fixed.
Consider first that the matrix Q_φ is diagonal. We have the following result:

Lemma 1 *Let $r \in \mathbb{R}^3$, $Q_\varphi \in \mathbb{R}^{n\times n}$ diagonal. Then $N = \lfloor \varphi_0 - Hr \rceil$ minimizes $\|\varphi_0 - N - Hr\|_{Q_\varphi}$ over $N \in \mathbb{N}^n$.*

The proof is straightforward, as any other choice for N leads to a larger norm of the residuals. This fact is true for any norm associated to a diagonal matrix, and in particular for any L^p-norm for $p \geq 1$. Hence, if the measurements are not correlated, minimizing N is straightforward.
In the general case, this fact is not true anymore. As an example, take the following values:

$$Q_\varphi^{-1} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}, \quad \varphi_0 - Hr = \begin{pmatrix} 0.3 \\ -0.4 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

In this case, we can see that N_1 is not the integer choice minimizing the residuals: taking N_1, the squared norm of the residuals gives 0.83, whereas taking N_2, the squared norm becomes 0.63, and is hence smaller. However, by providing a limit on the maximal residual needed, one can, as we will show, only consider the rounded values of a vector under consideration to solve ambiguities, allowing a large simplification on ambiguity resolution.

When applying a suitable model for the double-differences at the true position,

one can expect the residuals to be less than half the wavelength. In the case of short baselines, this argument stands for most elevations. When the elevation is too low, or the multipaths are too high, this might not be true anymore. For longer baselines, the residuals at the true position can grow larger than half the wavelength, as atmospheric terms play a larger role. Our approach as it is would hence not be suited. Nevertheless, by completing the model, with atmospheric terms for example, our process stands as long as the expected residuals of the signals are smaller than half their wavelength.
As an extension to the process described here, if the model can discriminate some poor signals, one can forget about these signals in a first stage, and deal with such data later. In the case where the problem is related to elevation, as the variance of this measurement would usually be much larger, the position change inferred by changing the ambiguity by ± 1 is in general extremely small, and in particular less than the expected precision, justifying such an omission.

The argument of considering only rounded values of $\varphi_0 - Hr$ is driven by the fact that if the residuals are small enough for the norm $\|\cdot\|_{Q_\varphi}$, then they are also small for the Euclidean norm. Figure 1a shows an affine subspace of dimension 1, embedded in an ambiguity vector space of dimension 2. The ambiguity vectors closer to the subspace are starred and the rounding process of three different points is shown on the left figure. Taking

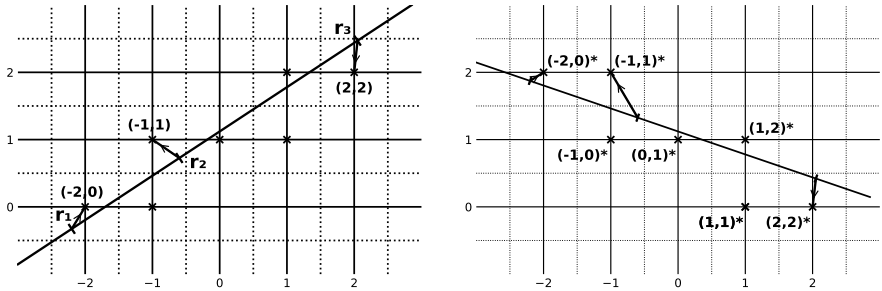
$$C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

as the Cholesky decomposition of Q_φ, the figure on the right shows the transformed space. The former ambiguities N are shown as $N^\star$ on the figure. As one can notice, the choice of ambiguity for r_2 was wrong, as the ambiguity $(-1, 0)^\star$ would minimize the distance to $r_2^\star$. However, it is easy to see that the ambiguity vectors very close to the subspace in the natural space, stay close to the affine subspace in the transformed space. Indeed, we have:

Lemma 2 *Let $r \in \mathbb{R}^3$. If there exists $N \in \mathbb{Z}^n$ such that $\|\varphi_0 - Hr - N\|_Q < l$ with $l = \left(2\sqrt{\max(\lambda(Q))}\right)^{-1}$, then $N = \lfloor \varphi_0 - Hr \rceil$.*

Proof $B_Q(\varphi_0 - Hr, l) \subset B_2(\varphi_0 - Hr, \frac{1}{2}) \subset B_\infty(\varphi_0 - Hr, \frac{1}{2})$. □

This lemma ensures that any ambiguity vectors giving a residual less than l is in its intended form. However, it does not show the existence of such a vector. In the search process, we associate to each r the ambiguity $\lfloor \varphi_0 - Hr \rceil$. If any residual is found to be less than l, then the lemma ensures that it is sufficient to consider only the ambiguity vectors of the form $\lfloor \varphi_0 - Hr \rceil$.
Observe also that the transposition is true: if the solution of the ambiguity resolution was not the rounded value of $\varphi_0 - Hr$, then this would mean that the



(a) (b)

Fig. 1 Rounding process for 3 different starting values r_i in the affine subspace. On figure 1a, the natural subspace, in which the ambiguity vectors are integer vectors. On figure 1b, the same operation in the transformed subspace, in which the former ambiguity choice is not always the minimizer of our problem. The ambiguities close to the subspace in the natural space stay close to the new subspace after transformation by C^{-1}.

residual is larger than the value in lemma 2. Hence, as soon as one ambiguity has residual less than this value, the solution is of the form $\lfloor \varphi_0 - Hr \rceil$. With a well-suited model and good quality receptors, such a constraint should be verified.

Remark 4 The limit l is not optimal. Indeed, when Q_φ is diagonal, lemma 1 shows that no bound on the norm of the residuals is necessary for the lemma to stand, in other words we should have $l = \infty$. This shows that the optimal limit depends not only on the eigenvalues of Q_φ, but also on the directions of its eigenvectors.

Remark 5 When the covariance matrix is block diagonal, the limit l can be described as a vector $l = (l_{C_i,S_j})_{i,j}$, where each constellation and signal has its own limit l, hence making an extension for a constellation by constellation and signal by signal approach straightforward.

Observe that, in the general case, one can bound the eigenvalues of matrix Q_φ using Weyl's inequality [14]. In the classical setting of double-difference calculations, from equation (6), the covariance matrix can be written as the perturbation of a diagonal matrix by a rank one matrix:

$$Q = D + \sigma_p^2 u u^\intercal, \ \textit{with } u = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{15}$$

In this case, the result by Ipsen and Nadler [15] gives a better bound. We have:

Lemma 3 *Let* $Q = D + \sigma_p uu^\intercal \in \mathbb{N}^n$, *where* $D = diag(\lambda_1, ..., \lambda_n)$, $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$. *Then:*

$$\lambda(Q) \leq \lambda_n + \frac{1}{2}\left(n\sigma_p^2 - (\lambda_n - \lambda_{n-1}) + \sqrt{\left(n\sigma_p^2 - (\lambda_n - \lambda_{n-1})\right)^2 + 4(\lambda_n - \lambda_{n-1})\sigma_p^2}\right) \tag{16}$$

$$\leq \lambda_n + n\sigma_p^2 \tag{17}$$

Proof The first equation (16) is Ipsen and Nadler result adapted to our case, while the (17) is Weyl's one. □

With this second lemma, given the covariance values of each measurements, one can now compute an acceptable limit l on the norm of the residuals for our process. A more thorough study of the perturbation of a diagonal by a rank one perturbation could however lead to a better limit l, increasing the limit of application of this argument.

Application 1 To have an idea on the bound of the residuals, let us consider the variance of the measure in a bad case. Consider seven satellites of a same constellation, and more particularly a pivot p and a satellite s, with respective elevation of 90° and 10°. For σ_φ, we consider the formula in Leick, Rapoport, Tatarnikov [11]

$$\sigma_\varphi^2 = \sigma_0^2 + \left(\frac{\sigma_1}{\varepsilon + \sin(E)}\right)^2, \tag{18}$$

where we take $\sigma_0 = \sigma_1 = 0.03$ cycles and $\varepsilon = 0.1$. In this case, $\sigma_s^2 \approx 0.0129$ and $\sigma_p^2 \approx 0.00164$. Equation (17) gives $\max(\lambda(Q_\varphi)) \leq 0.0293$ as a bound on the largest eigenvalue of the covariance matrix. Hence, the limit on the residuals is $l = 3.32$, meaning that the residuals we consider for the different satellites should amount to less than this value in terms of σ_φ. Given the number of satellites, this would lead to an average residual of around $0.5\sigma_\varphi$, which is quite restrictive. However, as explained before, the limit value is not optimal, and can be improved.

3.3 Search process

Given a vector $r \in \mathbb{R}^3$, we have now defined a corresponding ambiguity vector N, which in turn is associated to a value r' minimizing the residuals for N. When searching for the minimizing ambiguity vector and position, we hence need to reach the wanted ambiguity in our search process. Observe that the position r such that $\lfloor \varphi_0 - Hr \rceil$ is in general not isolated. Recall that our initial search space is called $\mathcal{P} \in \mathbb{R}^3$. Define the set of ambiguity vectors around the affine subspace $\mathcal{A} = \varphi_0 + span(H)$:

$$\mathcal{N}_\mathcal{P} = \{N \in \mathbb{Z}^n,\ \exists r \in \mathcal{P}\ s.t.\ N = \lfloor \varphi_0 - Hr \rceil\}$$

Each ambiguity vector in $\mathcal{N}_\mathcal{P}$ reciprocally defines a set P_N in $\mathcal{P}$:

$$\forall N \in \mathcal{N}_\mathcal{P},\ P_N = \{r \in \mathcal{P}, N = \lfloor \varphi_0 - Hr \rceil\}$$

Naturally, we have that $\mathcal{P} = \cup_{N \in \mathcal{N}_\mathcal{P}} P_N$. If we define a search process such that accesses every P_N, then we would visit all our potential ambiguity vectors and could determine the couple (N, r) minimizing the norm of the residuals. However, the requirement to visit all the P_N in a limited time is not realistic as we will show; fortunately it is not needed either. The scheme we propose will ensure that we access all the P_N for which N has residual less than some fixed value.

First, let $H = QR$ be the QR decomposition of H, where $Q \in \mathbb{R}^{n \times 3}$ and $R \in \mathbb{R}^{3 \times 3}$. The columns (a, b, c) of matrix Q define an orthonormal basis of the subspace $span(H)$. Call $\mathcal{B}$ the image $\varphi_0 + H\mathcal{P} \subset \mathcal{A}$ the set of vectors of $\mathcal{P}$ after transformation. To define our search process, consider the following lattice of $\mathcal{A}$: $\mathcal{D}(\alpha) = \left\{\varphi_0 + \alpha k_1 a + \alpha k_2 b + \alpha k_3 c, (k_1, k_2, k_3) \in \mathbb{Z}^3\right\} \cap \mathcal{B}$. The value α determines the step of the lattice. It remains to choose α wisely, to ensure that for any N close enough to the subspace $\mathcal{A}$, $\varphi_0 + HP_N \cap \mathcal{D}(\alpha) \neq \emptyset$. A straightforward upper bound on α is 1. As figure 2a shows, let $\alpha > 1$, $\varphi_0 = (1 - \alpha/2, 0)$, and $a = (1, 0)$, then for $k_1 = k_2 = 0$, the associated ambiguity is $(0, 0)$, while for $k_1 = 1$, it is $(2, 0)$; we hence did not search ambiguity $(1, 0)$, which has the same residuals as any ambiguity vector $(k, 0)$.

On the other hand, on figure 2b, to make sure that we pass through a point that rounds to the ambiguity vector $(1, 1)$, α would need to be small. To visit all the ambiguity vectors of $\mathcal{N}_\mathcal{P}$, if one lets $\mathcal{P}$ tend to $\mathcal{R}^3$, α will tend to zero almost surely, showing that a lower bound on α has to be zero without further hypothesis. However, observe that the smaller the intersection between the subspace $\mathcal{S}$ and the space rounding to one specific ambiguity, the larger the norm of the residual. Since we are interested in minimizing the norm of the residuals, we can generate a lower bound on α, ensuring that the good ambiguity vectors with small residuals are accessible via our lattice, while dropping some others that would lead to high residual. Besides, observe that the larger the α, the fewer points are in the search space.

Lemma 4 *Let $0 < m \leq n \in \mathbb{N}$, and $0 < s \leq 1/2 \in \mathbb{R}$. Consider $\mathcal{C} = \overline{B_\infty}(0, 1/2)$ the n-dimensional hypercube of radius $1/2$ centered in 0, and $\overline{B_2}(0, s)$ the closed ball of radius s for the L^2-norm. Then, for any m-dimensional subspace $\mathcal{S}$ intersecting $\overline{B_2}(0, s)$ ($\mathcal{S} \cap \overline{B_2}(0, s) \neq \emptyset$), there exists a point $a \in \mathcal{S}$ and $\frac{1-2s}{2\sqrt{m}} \leq s' \leq 1$ such that $\overline{B_\infty}(a, s') \cap \mathcal{S} \subset \mathcal{C}$.*

In other words, let us consider the ambiguity space of dimension n, a given ambiguity vector (here we take 0), and our affine subspace $\mathcal{S}$ of position of dimension 3. As seen before, the position locally minimizing the residuals is the projection of the ambiguity vector 0 on $\mathcal{A}$. The norm of the residuals then corresponds to the norm of the projection of the ambiguity vector on the subspace orthogonal to $\mathcal{A}$. For this norm to be less than some upper bound s, it means that the ball of radius r for the euclidean norm intersects the subspace $\mathcal{A}$. Now, the lemma states that there exists a cube of some radius r' such that

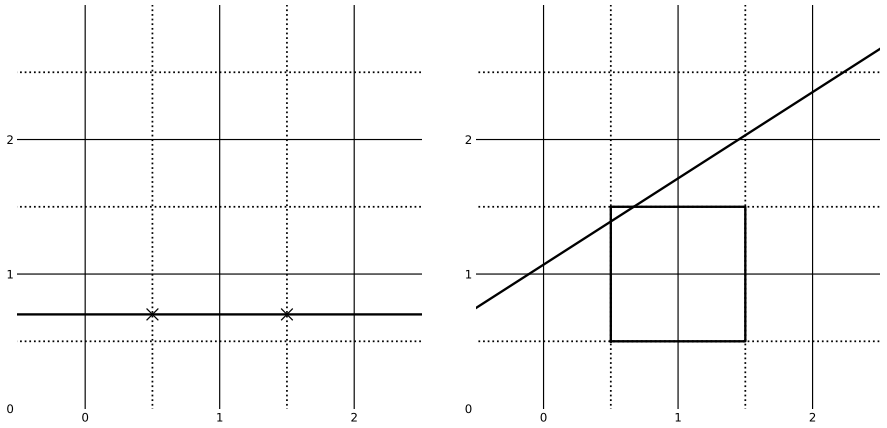


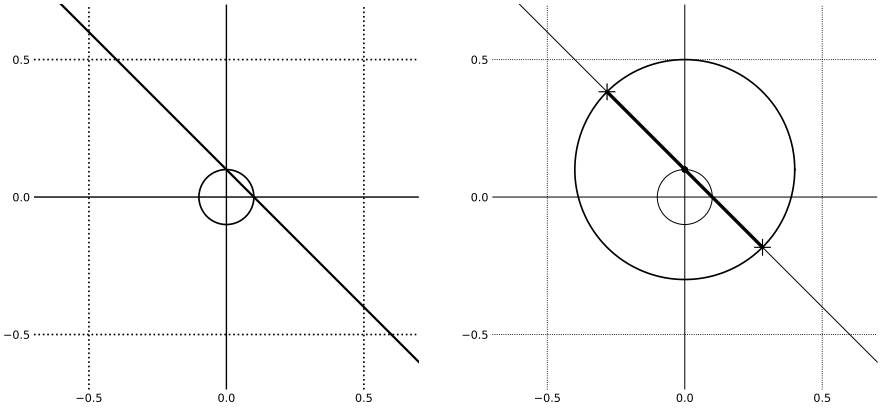
Fig. 2 Figure 2a shows that the search space lattice upper bound is 1, otherwise one might miss an ambiguity vector when rounding the elements of the lattice. Figure 2b shows that the lower bound on the lattice step has to be small enough so as to reach every ambiguity vectors. However, ambiguity vectors which require a small lattice step to be reached have high residuals, as they are far from the ambiguity vectors itself.

$\frac{\alpha}{2} = \frac{1-2s}{2\sqrt{m}} \leq s' \leq 1$, that is contained in the subspace of positions $\mathcal{S}$ as well as in the space of points that round to the considered ambiguity. This shows that if we let (a, b, c) be an orthonormal basis of $\mathcal{S}$ and we consider the set of points $\left\{k_1\alpha a + k_2\alpha b + k_3\alpha c, (k_1, k_2, k_3) \in \mathbb{Z}^3\right\}$, for any ambiguity vector N with Euclidean norm of its residuals less than s, there exists (k_1, k_2, k_3) such that $\lfloor \varphi_0 + k_1\alpha a + k_2\alpha b + k_3\alpha c \rceil = N$. Hence, we will reach such an ambiguity vector and determine its residual during the search algorithm.

Proof Let $x \in \mathcal{A} \cap \overline{B_2}(0, s)$. We have $B' = \overline{B_2}(x, 1/2 - s) \subset \mathcal{C}$. Now consider the intersection $B' \cap \mathcal{A}$, this set is $\left\{x \in \mathcal{A}, \|y - x\|_2 \leq 1/2 - s\right\}$. In such a space, it is possible to inscribe a hypercube of side length equal to $\alpha = (1 - 2s)/\sqrt{m}$, which is a ball of radius $\alpha/2$ for the infinite norm. □

The construction for the proof is given in figure 3, showing the different steps. The previous lemma works for the L^2-norm on the space of residuals. However, if we want to consider the norm $\|\cdot\|_{Q_\varphi}$, we need to make a small adjustment:

Corollary 1 *Let $0 < m \leq n \in \mathbb{N}$, and $0 < s \leq 1/2 \in \mathbb{R}$. Let Q_φ be a symmetric positive definite matrix, and $0 < \lambda = \max \lambda(Q_\varphi)$ its largest eigenvalue. Define $l = s/\sqrt{\lambda}$. Consider $\mathcal{C} = \overline{B_\infty}(0, 1/2)$ the n-dimensional hypercube of radius $1/2$ centered in 0, and $\overline{B_Q}(0, l)$ the closed ball of radius l for the norm $\|\cdot\|_{Q_\varphi}$. Then, for any m-dimensional subspace $\mathcal{A}$ intersecting $\overline{B_Q}(0, l)$ ($\mathcal{A} \cap \overline{B_Q}(0, l) \neq \emptyset$), there exists a point $a \in \mathcal{A}$ and $\frac{1-2s}{2\sqrt{m\lambda}} \leq s' \leq 1$ such that $\overline{B_\infty}(a, s') \cap \mathcal{A} \subset \mathcal{C}$.*



(a) (b)

Fig. 3 3a displays the ambiguity space of dimension 2, in which the subspace of position of dimension $m = 1$ intersects the ball of radius s, resulting in a corresponding position for the ambiguity $(0, 0)$ having residual less than s. In 3b, we built the ball of radius $1/2 - r$, its size is optimal as it is tangent to the line $y = 0.5$. In the intersection of this ball with the position space, the largest m-dimensional ball has radius $(1/2 - r)/\sqrt{m}$.

Proof $B_{Q_\varphi}(a, l) \subset B_2(a, s)$. □

We can bound the maximum eigenvalue of Q_φ using Weyl's or Ipsen and Nadler's formulas, described in lemma 3. Once again, the lower bound is not optimal, as we have to go through the ball determined by the Euclidean norm to obtain it, instead of finding a bound directly. However given the complexity and the diversity of the geometry of an embedded subspace of dimension m in an n-dimensional space, such a lower bound is extremely helpful. In practice, it is possible to use a larger value for α to get less redundancy in the search process without missing the potential solution.

The search space $\mathcal{D}(\alpha)$ is explored by a lattice, directed by vectors a, b and c. We want this lattice to cover the subset $\mathcal{B}$. In the case the initial position was determined by the pseudoranges equation $p_0 = Hr + \varepsilon_p$, with $H = QR$ and measurement covariance Q_p, then the covariance of the solution is related to $(R^\intercal Q^\intercal Q_p^{-1} QR)^{-1}$, shaping the search space $\mathcal{B}$. In the general case, when the geometry of the satellite is good, the influence of Q_p over the covariance matrix should not be fundamental, most of the influence coming from the dilution of precision given by $(R^\intercal R)^{-1}$. Neglecting the influence of Q_p on the search space, and considering the vectors $r' = Rr$, the covariance matrix in the new space of position then becomes the identity matrix and the space $\mathcal{B}$ becomes a sphere. Hence, we can define a value K to simplify the definition of the search space. Define:

$$D(\alpha, K) = \left\{ k_1\alpha a + k_2\alpha b + k_3\alpha c, k = (k_1, k_2, k_3) \in \mathbb{Z}^3, \|k\|_2 < K \right\}. \quad (19)$$

Even if this definition neglects the covariance matrix Q_p, it facilitates the definition of the search space. To avoid computations, one can even use a different norm in the definition of $D(\alpha, K)$, so as to make the search algorithm as readable as possible.

Remark 6 Given the previous construction of the matrix Q and R, we can loosely relate the variable K to the radius of search in the position space. Indeed, given that the rows in the matrix H are the difference between two unit vectors divided by the wavelength, a rough estimation of the minimal radius of search is $K \times \sqrt{min(\lambda(R^\intercal R)^{-1})}$. More coarsely, the radius of search is related to $K \times PDOP$, by definition of the position dilution of precision.

3.4 Pseudorange convexity

We now extend the previous process to the full problem 2 with the pseudorange measurements. We are looking for a minimizer of

$$\|p_0 - Hr\|^2_{Q_p} + \|\varphi_0 - Hr - N\|^2_{Q_\varphi} \tag{20}$$

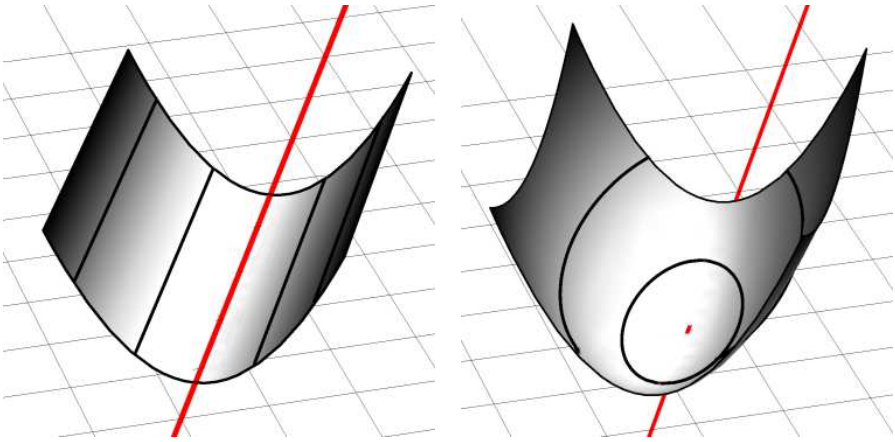
Considering only the pseudorange equation, the position minimizing the first term is the projection of p_0 on the subspace spanned by H. Call r_0 this solution, then we have $p_0 = Hr_0 + s_0$ with s_0 the projection of p_0 on the subspace orthogonal to H. For any r, we then have:

$$\begin{aligned}\|p_0 - Hr\|^2_{Q_p} &= \|s_0\|^2_{Q_p} + \|H(r - r_0)\|^2_{Q_p} \\ &= \|s_0\|^2_{Q_p} + (r - r_0)^\intercal \left(H^\intercal Q_p^{-1} H\right)(r - r_0)\end{aligned}$$

This value growing with the distance to r_0, the pseudorange equations add convexity to the previous problem, as shown on figure 4. Indeed, when far enough from the position r_0, the first part of the norm in equation (20) grows larger than the fixed limit for the norm of the residuals. The advantage of this convexity is that one does not have to define artificially the width of the search space. Hence, by taking the best computed residual γ during the algorithm, if the position tested is too far, and $\|H(r - r_0)\|_{Q_p} \geq \gamma - \|s_0\|^2_{Q_p}$, then for any $\lambda > 1$, it is not needed to test vectors $r' = r_0 + \lambda(r - r_0)$ allowing the algorithm to stop by itself when testing positions far from r_0.

It remains to check and eventually modify the results of the previous parts. First, the residuals and projections formulas (13) and (14) are changed. Indeed, given the weight of being far from r_0 on the norm, the projections on $\mathcal{A}$ will be made more oblique towards the point Hr_0. It is straightforward to compute the new projection of an ambiguity vector, as instead of considering only Q_φ we have to consider the full covariance matrix. We let

$$Q_{p,\varphi} = \begin{pmatrix} Q_p & 0 \\ 0 & Q_\varphi \end{pmatrix}, \qquad CC^\intercal = Q_{p,\varphi}$$



(a) (b)

Fig. 4 The planes $z = 0$ of the two figures represent the ambiguity space of dimension 2, while the line represents the subspace of position of dimension $m = 1$. In figure 4a, the surface drawn represents the squared norm of the residuals $z = \|\varphi_0 - Hr - N\|^2_{Q_\varphi}$. The search space should follow the subspace of position, and is not finite without further hypothesis, as along the line the norm of the residuals is null. In figure 4b, the surface drawn represents the squared norm of the residuals $z = \|\varphi_0 - Hr - N\|^2_{Q_\varphi} + \sigma \times \|r - r_0\|^2_2$. Using the effect of the pseudorange equations, the search space is now limited by the strict convexity of the squared norm of the residuals.

$$\begin{pmatrix} p_0^\star \\ \varphi_0^\star \end{pmatrix} = C^{-1} \begin{pmatrix} p_0 \\ \varphi_0 \end{pmatrix}, \qquad Q^\star R^\star = C^{-1} \begin{pmatrix} H \\ H \end{pmatrix}, \qquad \varepsilon^\star = C^{-1} \begin{pmatrix} \varepsilon_p \\ \varepsilon_\varphi \end{pmatrix}$$

We then have:

$$\varepsilon^\star = (I_n - Q^\star Q^{\star\intercal}) \begin{pmatrix} p_0^\star \\ \varphi_0^\star \end{pmatrix} - (I_n - Q^\star Q^{\star\intercal}) \begin{pmatrix} 0 \\ N^\star \end{pmatrix} \tag{21}$$

$$r' = (R^\star)^{-1} Q^{\star\intercal} \begin{pmatrix} p_0^\star \\ \varphi_0^\star \end{pmatrix} - (R^\star)^{-1} Q^{\star\intercal} \begin{pmatrix} 0 \\ N^\star \end{pmatrix} \tag{22}$$

About the minimization of the residuals over the ambiguity vectors, we can first observe that if the matrix Q_φ is diagonal, then clearly the minimizing ambiguity vector is again the vector $\lfloor \varphi_0 - Hr \rceil$. In any case, when r is fixed and consequently Hr is too, the problem of finding N is exactly the same as before, as the matrix Q_φ is unchanged and the other measurements are not correlated to these ones.

Regarding the search process, the goal was to reach all ambiguity vectors close enough to the subspace $\mathcal{A}$, with residuals less than some fixed limit. Adding the pseudorange equations, the norm of the residual increases, adding another term to it. Apart from adjusting the limit of the accepted residual by adding $\|s_0\|^2_{Q_p}$, we reach the exact same ambiguities and there is no difference between the two search processes. One could also choose to compute an irregular lattice with growing step depending on the distance to r_0. Our process is hence mostly

the same when adding the pseudorange equations, apart from the projection equations and the residual computations.

3.5 Naive example of algorithm

We can state a naive algorithm solving problem 1 by gathering our work. We do so in algorithm 1 in the style of the algorithm by Chang, Yang and Zhou [4]. We are not looking to be optimal in term of number of computations, instead we aim for readability.

If one requires the p best ambiguity vectors, it is straightforward to keep them in a list, as well as the list of residuals. This algorithm, even though naive, gives an upper bound on the complexity necessary to get the best ambiguity vector out of some space of position. The position space being search, which is an ellipsoid, contains the ball $\left(\min \lambda(R^{-1})\right) K\alpha$. Before entering any loop, the preparation phase is made of matrix multiplications, Cholesky decomposition, and eventually a spectral decomposition to obtain the maximal eigenvalue of the matrix Q_φ. Hence, the complexity of the operations before starting the loops are $\mathcal{O}(n^3)$. Then, for each ambiguity we perform the product MN_0, which is $\mathcal{O}(n^2)$, making the final complexity of the algorithm and the farthest position visited in the position space given by:

$$\begin{aligned} C(K,n) &= \mathcal{O}\left(n^3 + (2K+1)^3 n^2\right) \\ d(K) &\geq K\alpha \min \lambda(R^{-1}) \end{aligned} \tag{23}$$

Remark 7 The value of α has been fixed at the lower bound of corollary 1, to make sure we visit the right ambiguity. The algorithm described here relies on a balance between the extension of the search and the success rate. Increasing the value of s decreases the extension of $\mathcal{D}(\alpha)$, while decreasing it might prevent us to reach the solution of our problem. This dilemma can be overcome by different strategies, such as changing the value of s when a new ambiguity with small residual as been found. In general, we advocate for a lower value of s (higher α), so as to avoid redundancy during the search. A better generic value can be established through trial runs after choosing a wanted percentage of success.

4 Possible extensions

We discuss here several possible extensions of the algorithm, in cases that seem to have interesting developments.

4.1 Multi-epoch case

The algorithm we developed being $\mathcal{O}(n^3)$, computing a position with a large number of satellites is computationally cheap. In the case of multiple epochs, the number of satellites to consider will increase sharply, making it harder to compute every wanted ambiguity. Moreover, observe that when doing the QR decomposition, the larger the number of satellites, the smaller the coefficients

Algorithm 1 Finds a position and integer ambiguity minimizing the norm of the residuals in the integer ambiguity problem.

Require: Let $n \in \mathbb{N}$, $n \geq 2$. Let Q_φ be a symmetric positive definite matrix in $\mathbb{R}^{n\times n}$, $\varphi_0 \in \mathbb{R}^n$, and H the measurement matrix. Choose a maximum admissible value $s < 0.5$ for the norm of the residuals $\|\varepsilon_\varphi\|_2$, and $K \in \mathbb{N} \setminus \{0\}$ defining a radius of the search space.

Ensure: Returns a position r and an integer ambiguity vector N minimizing the residual $\|\varphi_0 - Hr' - N'\|_{Q_\varphi}$ over $r' \in \mathcal{D}$ and $N' \in \mathbb{Z}^n$.

Compute the QR factorization of $\mathbf{H}$: $\mathbf{H} = \mathbf{QR}$
Call $\mathbf{a}, \mathbf{b}, \mathbf{c}$ the columns of $\mathbf{Q}$
Compute λ, the largest eigenvalue of $\mathbf{Q}_\varphi$ or a bound on it using formula (16) or (17)
$\alpha = \frac{1-2s}{\sqrt{3\lambda}}$
Compute $\mathbf{a}' = \alpha\mathbf{a}$, $\mathbf{b}' = \alpha\mathbf{b}$, $\mathbf{c}' = \alpha\mathbf{c}$
Compute the Cholesky decomposition of $\mathbf{Q}_\varphi$: $\mathbf{Q}_\varphi = \mathbf{CC}^\intercal$
Compute the QR decomposition of $\mathbf{C}^{-1}\mathbf{H}$: $\mathbf{C}^{-1}\mathbf{H} = \mathbf{Q}^\star\mathbf{R}^\star$
Compute $\mathbf{M} = (\mathbf{I_n} - \mathbf{Q}^\star\mathbf{Q}^{\star\intercal})\mathbf{C}^{-1}$
Compute $\varphi^\star_\mathbf{Q} = \mathbf{M}\varphi_\mathbf{0}$
Compute $\mathbf{L} = \mathbf{R}^{\star-1}\mathbf{Q}^{\star\intercal}\mathbf{C}^{-1}$
Compute $\mathbf{r}^\star_\mathbf{Q} = \mathbf{L}\varphi_\mathbf{0}$
Maximal residual $m = l$

for $k_1 = -K : 1 : K$ **do**
    $\mathbf{e_1} = k_1\mathbf{a}'$
    for $k_2 = -K : 1 : K$ **do**
        $\mathbf{e_2} = k_2\mathbf{b}'$
        for $k_3 = -K : 1 : K$ **do**
            $\mathbf{e_3} = k_3\mathbf{c}'$
            $\mathbf{N_0} = \lfloor \varphi_\mathbf{0} - \mathbf{e_1} - \mathbf{e_2} - \mathbf{e_3} \rceil$ ▷ Current ambiguity vector
            $\varepsilon^\star = \varphi^\star_\mathbf{Q} - \mathbf{MN_0}$
            $\|\varepsilon^\star\| = \sqrt{\varepsilon^{\star\intercal}\varepsilon^\star}$
            if $\|\varepsilon^\star\| < m$ **then**
                $l = \|\varepsilon^\star\|$
                $\mathbf{N} = \mathbf{N_0}$
            end if
        end for
    end for
end for

$\mathbf{r} = \mathbf{r}^\star_\mathbf{Q} - \mathbf{LN}$

in R. Indeed, the size of the diagonal elements $r_{i,i}$ grows as $1/\sqrt{\mu \times n}$, where μ is the number of considered epochs. Hence, the more epochs we consider, the slower in the position space our ambiguity search is. Even though this should be compensated by the reliability of the first guess, we should be entitled to ask for a more efficient algorithm, using the fact that when no cycle slip happens between two epochs, the dimension of the ambiguity space does not increase. The latter argument will be exploited in the next section, as we decide to focus for now on easy and straightforward method to decrease the number of computations.

First, observe that, if μ is the number of epochs, the number of computations in the set-up part of the algorithm is multiplied by μ^3, while those in the search process are multiplied by μ^2. The matrices computations are in general not the main problem, even in high dimensions. Indeed, the epoch by epoch nature of the problem usually simplifies the approach, as Q_φ can be considered block diagonal, and hence, the computational cost is only multiplied by μ instead of μ^3. However, in the case the residuals are time-correlated, this part could become heavy in term of computations. The main problem arises with the number of computations in the search process itself. Each searched ambiguity vector is multiplied by the matrix $M = (I_n - Q^\star Q^{\star\intercal})C^{-1}$, before computing its norm. This process can be easily improved by batch computations, considering subvectors and computing the latter product and norm on them. This method is straightforward and allows to reduce the amount of computations by some non negligible constant. Observing that the squared norm of the residuals follows a χ^2-distribution, one can eliminate ambiguity vectors such that the probability that its norm is small enough is too low given the norm of a subvector of residuals.

As previously mentioned, the eigenvalues of the matrix R obtained by QR factorization of H are inversely proportional to the square root of the number of epochs. Indeed, the more measurements we have, the more accurate our positioning, as the dilution of precision will decrease with new information. When one wants to keep the same search ratio, there exists however a heuristic technique that can help us achieve this goal. Indeed, considering the measurement matrix H, its shape is

$$H = \begin{pmatrix} H_1 \\ H_2 \\ \dots \\ H_\mu \end{pmatrix},$$

where each H_i corresponds to the measurements done at some epoch. When the epochs are relatively close together, the matrices H_i also are, apart from new or missing satellites in one epoch or the other. Consider now each column a, b, or c of Q, each of them (take a for instance) has norm 1 and can be written as $a = (e_1, ..., e_\mu)^\intercal$, each e_i being the first column of the QR factorization of H_i. Since

the e_i are very similar, we have that $a \sim (e_1, ..., e_1)^\intercal$. Now the norm of e_1 is close to $1/\sqrt{n}$. The main argument is the following: consider an n-dimensional hypercube $C^n = \bar{B}^n_\infty(0, 1)$, and a vector e; call $K = max\{k \in \mathbb{R},\ ke \in C^n\}$; consider $C^{\mu n}$ the $\mu \times n$-dimensional hypercube, the vector $e' = (e, e, ..., e)^\intercal$ and $K' = max\{k' \in \mathbb{R},\ k'e' \in C^{\mu n}\}$; then $K' = \sqrt{\mu}K$. This is easily shown by taking the coordinates of Ke for example.
In our setting, our argument states that if we want to find another ambiguity vector, we have to travel a distance $\sqrt{\mu}$ times longer than if we considered only a matrix H_i. Hence, in our algorithm 1, instead of taking $e_1 = k_1 a'$, we can take $e_1 = \alpha k_1 a'$, with $\alpha \sim \sqrt{\mu}$. The radius of research using this trick then becomes similar to the research radius with only one epoch.
If all the submatrices at each epoch were perfectly equal, this trick would be mathematically valid. However, the submatrices are never exactly the same, as the positions of the satellites change over time. The results are still convincing in many cases: for observations as long as four hours, with satellite positions subjected to a lot of changes and a high number of observations, the tests on this heuristic technique keep a very good success rate. Observe on the other hand that when the data are very degraded and the set of observed signals differs a lot from one epoch to the other, this argument becomes less relevant: when matrices H_i vary a lot between different epochs, one should not used a full multiplier $\sqrt{\mu}$, but instead could use a lower one in-between 1 and $\sqrt{\mu}$.

Remark 8 The heuristic technique basically consists in saying that our data are a repetition of μ times the same observations. Another approach, described in the next section, uses the continuity of the ambiguity to reduce the initial $\mu \times n$-dimensional problem to an n-dimensional one. In the heuristic argument, the continuity is not required at all, the argument being purely geometric, as we still solve all ambiguities for the μ epochs.

4.2 Multi-epoch reduction

In this section, we consider again the multi-epoch case, but we take advantage of the continuity of the ambiguities between different epochs. We describe here a scheme in the static case that can be extended to the dynamic one when relative motion is known.
We consider the observation of n satellites without interruption for μ epochs. Call $\varphi_{s,i}$ the observation at epoch i of satellite s. The ambiguity associated to the double difference is $N^{ps}_{fm} = N^{ps}$. Without interruption or cycle slip for this signal, when the pivot satellite stays the same, the ambiguities do not depend on the epoch of observation i. Writing H_i the measurement matrix at epoch

i, we have:

$$\begin{pmatrix} \varphi_{1,1} \\ \varphi_{2,1} \\ \vdots \\ \varphi_{n,\mu} \end{pmatrix} = \begin{pmatrix} H_1 \\ \vdots \\ H_\mu \end{pmatrix} r + \Lambda \begin{pmatrix} N^{p1} \\ N^{p2} \\ \vdots \\ N^{pn} \end{pmatrix} + \varepsilon, \qquad \Lambda = \begin{pmatrix} I_n \\ I_n \\ ... \\ I_n \end{pmatrix}, \tag{24}$$

with an appropriate covariance matrix Q_φ. When the pivot changes or a cycle slip occurs, we can change the form of equation (24) to take the change into account. First, define a reference pivot q, *e.g.* the pivot at the first epoch. When the pivot changes, the ambiguity N^{ps} can be written as

$$N^{ps} = N^p - N^s = N^p - N^q + N^q - N^s = N^{qs} - N^{qp}$$

If both satellites had such a measurement at the previous epoch, then upon a change of Λ, we can still write equation (24). The following equation shows a matrix Λ when the pivot changes at the second epoch:

$$\begin{pmatrix} \varphi_{1,1} \\ \varphi_{2,1} \\ \vdots \\ \varphi_{n,\mu} \end{pmatrix} = \begin{pmatrix} H_1 \\ \vdots \\ H_\mu \end{pmatrix} r + \Lambda \begin{pmatrix} N^{p1} \\ N^{p2} \\ \vdots \\ N^{pn} \end{pmatrix} + \varepsilon, \qquad \Lambda = \begin{pmatrix} I_n \\ A \\ ... \\ A \end{pmatrix}, \tag{25}$$

where A is now a matrix with -1 on the column corresponding to the pivot satellite p, a 1 on the column i corresponding to any other satellite i present at the wanted epoch, and 0 elsewhere.

Remark 9 When a cycle slip occurs for a satellite k, its previous ambiguity cannot be observed anymore. Hence, N^{pk} should not appear anymore; instead we define $N^{pk'}$, new ambiguity for the satellite k.

Remark 10 Observe that there only needs to be one satellite (apart from the pivot) with a measurement at each epoch to compute the ambiguities in the way described before. Indeed, with new satellites p' and s', if s is a satellite present at epoch m and $m-1$ and no cycle slip occurred for s, then we have $N^{p's} = N^{qs} - N^{qp'}$ which defines the ambiguity of p', and $N^{p's'} = N^{qs'} - N^{qp'}$. As long as the epochs are connected by one satellite without a cycle slip in the measurements, we can compute the ambiguity with respect to the reference satellite q. Once there exists a discontinuity for every satellite, one has to change reference satellite, and hence define new ambiguities with respect to the new reference, starting a new block with new columns in Λ.

The matrix Λ is left invertible: there exists a matrix Θ such that $\Theta\Lambda = I_{n'}$. We have:

$$\begin{pmatrix} \varphi'_1 \\ \vdots \\ \varphi'_n \end{pmatrix} = H'r + \begin{pmatrix} N^{p1} \\ \vdots \\ N^{pn} \end{pmatrix}, \quad \begin{pmatrix} \varphi'_1 \\ \vdots \\ \varphi'_n \end{pmatrix} = \Theta \begin{pmatrix} \varphi_{1,1} \\ \varphi_{2,1} \\ \vdots \\ \varphi_{n,\mu} \end{pmatrix} + \varepsilon', \quad H' = \Theta \begin{pmatrix} H_1 \\ \vdots \\ H_\mu \end{pmatrix}, \tag{26}$$

with covariance matrix $Q'_\varphi = \Theta Q_\varphi \Theta^\intercal$.

We are now in a situation where we can apply the previous algorithm, on a problem that is n-dimensional, and not $\mu \times n$-dimensional anymore (in fact, with cycle slips and other events, the problem is n'-dimensional, with $n \leq n' \leq \mu \times n$). The computations for each ambiguity vectors are hence μ^2 times faster. However, as the matrices Λ and Θ are not invertible, the previous process does not allow us to conclude. Indeed, consider a simple one-dimensional example with $\Lambda = \begin{pmatrix} 1 & 1 \end{pmatrix}^\intercal$, we then have $\Theta = \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}$ and $\varepsilon' = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)$. Observe that a whole subspace of $(\varepsilon_1, \varepsilon_2)$ would give the exact same residual, without any condition on the size of ϵ_1 and ϵ_2 independently. However, $\|\varepsilon\| \ll 1 \implies \|\varepsilon'\| \ll 1$, or in our case, if the norm of ε' is not small, then the norm of ε is not small either. Hence, we can discard ambiguity vectors quickly by looking at their residual ε', and then look at the residual ε to determine if they are an interesting ambiguity vector for the non-reduced problem.

The construction of the algorithm should hence undergo a few changes. The starting point is to consider the problem after reduction given by equation (26). Before starting the loop, as we will need to check the residual ε, we need to compute the Cholesky decomposition of the full matrix Q_φ, and the QR decomposition of H so as to compute the norm of ε. Before starting the loop, one chooses a maximal value h for the residual ε', such that, while in the loop, if $\|\varepsilon'\| < h$, then we check if the value of $\|\epsilon\|$ is lower than a previously found ambiguity vector. The limit h should be small enough to get rid of the majority of the vectors that will not minimize our problem, but high enough to not get rid of the right ambiguity vector. This second check happening in the unreduced space of higher dimension, one can again use the batch strategy of section 4.1 to improve computation time. Finally, at the end of the algorithm, one can compute the position solution of the minimizing problem with the unreduced matrices.

Such an algorithm will again speed up the search process and should be more efficient than the previous multi-epoch algorithm, as we avoid inconsistent ambiguity vectors, relying on the continuity of our observations. On large amount of epochs, with steady satellite presence, this algorithm can deal with ambiguity resolution almost as efficiently as the single epoch case.

4.3 Signals combinations

With receivers being able to make observations of several signals of a constellation, combinations of pairs or trios of signals can be used (see [16]), as some of them keep the integer nature of the ambiguity vector and are hence suitable for algorithm 1, making them potentially more efficient. As an example, in the case of short baselines, if one can neglect the atmospheric terms, then using the wide-laning combination allows to browse swiftly a large position space, as the related wavelength is much larger.
Using the wide-laning combination in the algorithm, one can browse a determined search space more swiftly as the value of K needs to be much lower. As an example, in the GPS-only case, where the wide-laning wavelength using L1 and L2 is around 4 times larger than the one of L1, the computing time for the same range in the position space would be $4^3 = 64$ times shorter. The downside of using wide-lane combinations is that the observations might lead to a less precise position, as they are more noisy. However, one can easily modify our main algorithm, so as to obtain rough positions and ambiguity vectors by wide-laning, and improve these positions by taking either narrow-lane observations, or all the L1, L2 and L5 observations.
An example of modified algorithm is to start with algorithm 1 for the wide-lane combination and $K' = K/4$. Then, keeping the l best solutions, one could operate again algorithm 1 on each of these positions, with a smaller K'' (*e.g.* $K'' = K/8$), as we know we are close to a solution. The highest coefficient of the computational cost of the search part then becomes $K^3 n^2 \left(1 + l/2\right)/8$. In comparison, for $2n$ signals, the classic algorithm would need $\sim 32K^3n^2$ computations. The use of the wide-lane combination hence needs 128 times less computations if $l = 2$, or 32 times less if $l = 14$. Hence we see that wide-lane combination can substantially reduce the computation time.

5 Application

We apply here our algorithm 1 to some real-world data, so as to show its performances. We first describe our setup, and then give results seeming relevant for its evaluation.

5.1 Setup

Receivers. To obtain the data, we consider two probes made of the same components. The GNSS receiver used is the u-blox ZED-F9P, supplied by an embedded antenna Tallysman TW1889. Using these, we obtain raw data from the following signals: GPS (L1C/A, L2C), GLONASS (L1OF, L2OF), Galileo (E1B/C, E5b) and BeiDou (B1I, B2I).

Experiment. The two probes are placed in a location with a clear sky above 15°. They are positioned side by side, at 10.8cm. We set the rate at 1s for 15 minutes, so as to obtain 900 epochs of data.

The choice of putting the probes side by side allows us to focus on the main problem: the fast ambiguity search with a good enough model. Given the proximity of the probes, the atmospheric terms can be neglected, and we hence put our focus on the success rate and the speed of the algorithm.

Algorithm. To write our algorithm, we use a homemade program in Python (run on a laptop). The algorithm is designed to find the solution of problem 1. We perform two tests: one on all raw data for all elevations and another one for elevations higher than 10°; we choose to not filter the poor SNR, nor filter poor signals by any other mean, so as to show a minimum expected success rate before any kind of pre-processing filters. Using satellites at all elevations maximizes the number of satellites at the cost of having poorer data, but it allows us to show the timing difference when more satellites are used.
We split the algorithm in three different parts for timing purposes.
First, using the pseudoranges, we determine a raw position and the clock offsets for each signals between the receivers, and we apply the time corrections to the raw data. To this end, we use the noise formula described earlier with σ depending only on the elevation, as described in equation (18). This algorithm is made as simple as possible, and is not timed as it is not part of our problem.
We then compute all the matrices needed for the search part. For each epoch, we perform two QR factorization, a Cholesky decomposition, a matrix inversion (on the matrice C), and the needed matrices multiplications. We however do not compute the eigenvalues of Q_φ; as explained in the remark following algorithm 1, the value of α can be chosen higher, so as to make the search more efficient. We take advantage here of the fact that the matrix Q_φ is block diagonal to compute the matrix C^{-1}.
Finally, we perform the search process, using the raw position values and the computed matrices. Instead of considering k_i going from $-K$ to K, we consider all the $k = (k_1, k_2, k_3)$ such that $\|k\|_2 \leq K$, implying a lower number of ambiguities to search, but in a more appropriate space.

Independence between epochs. Each epoch is computed completely independently from the others. There are no interactions between consecutive epochs. The raw position is computed each time from scratch. We are hence in a situation where the position and ambiguities are computed instantaneously.

3 modes. We use three different modes of computations of the ambiguities.
We call "Single" the mode where no combination of signal is done. We treat the two signals (when they are present) of each satellite as completely independent. The number of satellites used in our computation is then the sum of all eight independent (constellation, signal) couples used.
We call "Wide" the mode where we only consider the wide combinations of signals for each constellation. This divides the number of satellites at the cost of noisier data, and also allows to expand the radius of search since the wavelength is much larger. The results of the wide mode are shown to introduce

the last mode, which is a combination of the two previous ones.
The "Mixed" mode comprises two steps. First, we fix a limit for the residuals of the wide lane and compute Q_φ the covariance matrix for the wide-lane combinations. We then run our search process, keeping every ambiguity vector such that $\|\epsilon^\star\|$ is smaller than our limit. For such ambiguity vectors, we perform a second search in single mode around the position given by the wide lane ambiguity resolution. To do so, it is necessary to correct the value of ϕ_0 by the change in range at the new position. In such a mode, we need two values of K for each search. A couple (K_1, K_2) can hence be chosen for the wide lane search, and single search. A greater value of K_1 will quickly raise the radius of the search space, while a greater value of K_2 will increase the chance of finding the right ambiguity in a small space around the wide lane solution.

Factor α. The value of α is buffed so as to make the search process more efficient. Choosing a maximal residual $l = 0.2$ cycle, we then choose $\alpha_{single} = \sqrt{2} \times (1 - 2l)$ and $\alpha_{wide} = 1 - 2l$. The factor $\sqrt{2}$ is introduced as we consider 2 frequencies for each satellite, and hence have twice the same unit vector in H for almost all satellites. This argument is the same as the one developed in the multi-epoch case in section 4.1. As for the factor $1 - 2l$, in practice such a value has shown excellent results, allowing to browse a larger space of ambiguities while maintaining the same rate of success. In the following, it has been checked that reducing the value of α left unchanged the results.

5.2 Results

We give here the results of the algorithm when we consider all satellites. When it seems relevant, we also give the results obtained by considering only satellites above 10°.
During the experiment, the number of usable signals (in this context a signal with both pseudorange and phase value with an LLI of 0 or 1) stayed mostly constant, with an average of 60.6 signals for the whole sky, and 53.4 above the 10° elevation, as shown in table 1. Observe that when computing double differences, these numbers will go down by exactly 8, as we consider one pivot per (constellation, signal) couple. In terms of distribution of signals between bands, the higher frequency signals (L1, B1, E1) have an average of 33.9 satellite signals, when the lower ones (L2, E5b, B2) have an average of 26.8 satellite signals. In the case of signal combinations, after computing double differences, the average of wide lane signals is 22.8.
Pseudorange positioning at each epoch with that amount of signals gives good results. When using all the satellite above the horizon, the average distance to the real position is 22.3cm with a standard deviation of 9.4cm. The maximum distance computed reached a value of 52.4cm. When considering satellites above the 10° elevation, the results are slightly less good, with a maximum reached value of 63.3cm, as shown in table 2.

In the following, we collect the results using the three different modes, as well as different values of K (and in the wide mode, of couples (K_1, K_2)).

Table 1 Average number of satellite signals used for the tests: total number of signals tracked, number of signals L1/B1/E1 tracked, number of signals L2/E5b/B2 tracked, and number of available wide lane combination signals (after double difference), depending on the elevation constraint.

Elevation	N signals	L1/E1/B1	L2/E5b/B2	Wide Lane
$> 0°$	60.64	33.88	26.76	22.76
$> 10°$	53.35	29.72	23.63	19.63

Table 2 Distance between the position computed using pseudorange and true position: average distance of all epochs, standard deviation (STD) of the distance, maximal distance.

Elevation	Average (cm)	STD (cm)	Max (cm)
$> 0°$	22.3	9.5	52.4
$> 10°$	23.2	10.5	63.3

In table 3, we collect the average duration of the algorithm in ms for the matrix computations and then for the search process, as well as the number of ambiguities searched. In the mixed case, we call k the number of searched single ambiguities, as it depends on the limit of the residuals that we fixed. We also put between parentheses some values of the times in the case we only consider the satellites above $10°$ elevation.
The matrix computation part takes less than 2ms in each cases, which is much less than the duration of the search process. This preparatory work for the search process can be considered to have a negligible duration in the algorithm. One can see that the search process grows linearly with the number of searched ambiguities. Each ambiguity computed takes around 1.5×10^{-5}s, apart from the mixed case $K = (5, 3)$ with a value of k impacting a lot the result. As expected, we see that the time per ambiguity is slightly less when we remove some satellites due to the elevation (values between parentheses), and for 12% less satellite we have a slight gain in computation time.
On the same data set, using all the signals on each frequencies of each constellations, applying the MLAMBDA approach gives a reduction time of 18.20ms, and a search time 79.47ms. Comparing these to the single process, we see that the reduction takes a larger time than our computations on matrices, while the search duration would correspond to a number of searched ambiguity around 5680, or equivalently to the case $K \sim 11$.

In terms of success rate, it is necessary to choose an appropriate way to measure if the ambiguity resolution was successful or not. We choose here to compare the positions given by the ambiguity process to determine if the solution is successful or not. If the distance between the position given by the ambiguity algorithm and the true position does not exceed some value, we

consider the resolution successful. For the single wavelengths, we consider the cases where this distance does not exceed 1cm and 3cm. For the wide case, where data are noisier, we give the results for 3cm and 10cm. The success rates are listed in table 4.
In the single case, we reach a success rate above 50% with a very low search radius, this result being improved when considering only satellites above 10°. At $K = 10$, we are already very close to a perfect success rate for both 1cm and 3cm positioning. When $K = 15$, the results cannot be improved anymore. The wide case is mostly interesting as it gives an insight on the mixed case. Indeed, if some of the best solution given by the wide case are close enough to the true position, then it will follow that the single search occurring after has a lot more chance to be successful. The success rate for a distance to the true position of less than 10cm is almost 100% when $K = 5$, and does not change with higher K, showing that the search space was large enough already when $K = 5$ as the raw positioning was good enough. Observe also that when we consider only the satellites above 10° elevation, the results slightly deteriorate, probably because of the diminution of the number of satellites. In the mixed case, the good results given by the wide case are confirmed, and for $K = (5, 1)$, the success rate is 100%. We do not bother to give the results for higher values of K, as they are obviously the same.
Applying the classical MLAMBDA algorithm [4] gives a success rate of 88.9% at 3cm, dropping to 15.1% at 1cm. The drop in the rate can be explained by the presence of some wrong ambiguities for the solution at 3cm.

Additionally, we can give a measure of confidence of the given solution. A first possibility is to use the Neyman-Pearson lemma (see [17]), so as to compare the residuals obtained by the present algorithm, and the ones that would be obtained in the case the residuals are equally distributed on the range $]-\lambda/2, \lambda/2]$. A second possibility is to compare the residuals of the solution to the second best set of residuals. The ratio between these two values increases as the square root of the number of satellites. When successful, we find that for the single case this ratio is close to 4.3, with a standard deviation of less than 0.3. As for the wide case, this ratio is 2.21 with a standard deviation lower than 0.2. When unsuccessful, we have instead ratio very close to 1 (*e.g.* as low as 1.0005 in the single case for $K = 10$). It hence seems that this ratio is an appropriate internal measure to determine when the solution can be considered valid or not. Other classical methods in the field of ambiguity resolution can be applied (see for example [18] for a better overview on the subject).

Finally, let us give an insight on the search space in the position space. As explained earlier, the search space is an ellipsoid determined by the geometry of the satellites. The furthest searched position grows rather linearly with the value of K, as it is related to the value $K \times PDOP$. In the single case, the furthest searched position from the initial guess averages to 33cm when $K = 5$, 60cm when $K = 10$ and 88cm. In the wide case, we obtain the averages of 140cm when $K = 5$, 239cm when $K = 10$ and 342cm when $K = 15$. The

difference between the width of the search spaces in this two cases explains the difference in success rates observed. A wider research implies a better success rate, and is easily reachable using the wide lane combination, making the mixed algorithm both quick and successful.

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Table 3 Timings of the two parts of the ambiguity resolution algorithm for different modes (single, wide, mixed) and different values of K. The number of searched ambiguity is shown, as well as the duration of the matrix computations, the search process, and the average duration of an ambiguity search. Between parentheses is shown the time in the case we consider only the satellites above 10°.

Mode	K	t matrix comp. (ms)	N searched	t search (ms)	t / N search (μs)
Single	5	1.10	515	7.38	14.3
Single	10	1.09	4169	58.16	14.0
Single	15	1.08 (0.96)	14147	203.06 (191.04)	14.4 (13.5)
Wide	5	0.56	515	6.20	12.0
Wide	10	0.55	4169	50.30	12.1
Wide	15	0.55 (0.52)	14147	172.06 (170.63)	12.2
Mixed	(5, 1)	1.67	$515 + 7k$	7.25	14.1
Mixed	(5, 3)	1.66	$515 + 123k$	12.91	25.1
Mixed	(10, 1)	1.60	$4169 + 7k$	52.75	12.7
Mixed	(10, 3)	1.65	$4169 + 123k$	57.79	13.9
Mixed	(15, 3)	1.63 (1.48)	$14147 + 123k$	184.91 (173.31)	13.1

Table 4 Success rate of the ambiguity resolution algorithm: success corresponds to a distance between the given solution and the true solution of respectively less than 1cm, 3cm or 10cm.

Mode	K	Success < 1cm	Success < 3cm	Success < 10cm
Single	5	51.9% (58.0%)	59.0% (64.2%)	-
Single	10	98.9%	99.4%	-
Single	15	100%	100%	-
Wide	5	-	52.3% (45.9%)	99.9% (99.6%)
Wide	10	-	52.3%	99.9%
Wide	15	-	52.3%	99.9%
Mixed	(5, 1)	100% (100%)	100.0% (100%)	-