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Modeling and numerical solution of the Laplace equation in 2D by the finite difference method case of the heat equation -Study of stability -

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Abstract:

The objective of this paper is to show the procedure to follow to analyze a physical phenomenon, with a simple and effective method. Mathematical stability examination was performed to define which resolution is stable and physically feasible. We have seen the approach and the steps that must be taken to go from a mathematical model to a numerical model. To do this, we used the Poisson equation as an example, and to show the results, we used the heat transfer equation. In real life, the most difficult step is not the numerical processing and analysis, but to understand the physical phenomenon and to translate it into a mathematical formulation. The finite difference approach is employed to resolving those equations numerically. The resulting schemes were resolved by the iterative GAUSS Seidel technique.These differential schemes have an approximate order $0(h^2)$, and are absolute stable.Differential schemes are a linear algebraic equation system which solution can be solved by the Gaussian techniqueof elimination. This paper provides a good basis for the analysis of physical phenomena that deal with partial differential equations in 2D. In several sectors and branches, whether in the industrial framework.

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1. Introduction

In mathematics and theoretical physics, the heat equation is a second order parabolic partial differential equation, which can be solved by numerical approaches in several ways.To describe the physical phenomenon of heat conduction, initially introduced in 1807 by Joseph Fourier [1], after experiments on the propagation of heat, followed by the modeling of the evolution of temperature with trigonometric series, since then called Fourier series and Fourier transforms, allowing a great improvement in the mathematical modeling of phenomena, especially for the foundations of thermodynamics, and which also led to very important mathematical work to make them rigorous, a real revolution both physical and mathematical.

A variant of this equation is very present in physics under the generic name of diffusion equation. It is found in the diffusion of mass in a binary medium or of electric charge in a conductor, radiative transfer, etc. It is also related to the Burgers equation and the Schrödinger equation [2]. Many engineering problems are reduced to partial differential equations which, because of their complexity, must be replaced by approximations. The theory of partial differential equations (existence, uniqueness, well-posed problem) is not as complete as that of ordinary differential equations. On the other hand, in the case where analytical solutions exist, their solutions are trivial or so simple that they are not useful in practice.Vaidya et al. [3] solved the partial differential heat equation by different methods. Obtained the same solution when used Fourier transform, Laplace transform ,MATLAB software, and separation of variables method. Ullaha et al. [4] used the iterative Laplace transform to develop the series solution for a two-dimensional wave equation involving an external source term of fractional order. Computed the series solution that converges quickly to the exact value. Benhissen et al. [5]. used the finite difference method to solve the Poisson equation. Gives an approximate solution of the real behavior of a physical phenomenon. Example the heat transfer equation.Lagrée [6] presents analytical solutions in simple cases of the heat equation in 1D. Makhtoumi [7] studied the derivation of analytical and numerical solutions of heat diffusion in one dimension using adaptive methods. Applied and analyzed a comprehensive comparison based on the perturbation method and the finite difference method. Dhoyer et al. [8] studied the boundary behavior of the distribution from the spatially averaged solution of the stochastic heating equation piloted with a Rosenblatt sheet. They proved that spatially averaged converged low, in the continuous function space.Loskor et al. [9] proposed a numerical resolution using the finite differential approach with the direct time and centered space scheme for a problem that contains one-dimensional heating equation and the scheme stability condition.Taloub et al [10-11] have developed and validated the finite differential approach to non-stationary heat transfer from a square furnace source.The finite differential approach is applied to solving the heating equation. If the analytical solve is not possible to find way to get an approximated solve, also in a number of points of the geometry only, numerical approaches are employed to reach this objective.Kahlaf et al. [12] utilized numerical approaches to locate the imprecise resolution of the two-dimensional Laplace equation with Dirichlet boundary states. Used two approaches the finite difference technique and the finite element technique to receive the numerical resolution of this equation. Found that the finite difference approach is a considerably proper technique after approximating these results with identical resolution.

The aim of the current paper is to investigate mathematically and numerically the stability of the temperature distribution in the heat equation, the case of two spatial variables with or without heat source (Laplace equation) in the furnace walls. This is important for the design work and for the effective operation of industrial furnaces.

NOMENCLATURE

C	Specific heat with pressure constant
D	Domain of resolution
F	Thermal energy source, or function of x,y
H	Coefficient of exchange by convection or step according to x,y
S	Energy source
t	Time
T	Temperature
T'	Temperature
X,x_i	Cartesian coordinate
Y,y_j	Cartesian coordinate
Ψ	Function of x,y
Ω_h	Network
θ_i	Iinternal temperature
θ_e	External te;perqture
Δ	Laplacian operator
Γ_i	Inner surface
Γ_e	Outer surface
L	Length
Lettres grecques	
α	Diffusivity thermal of wall and internal
ρ	Density
Λ	Lambda
Indices / Exposants	
∂	Partial derivative

2. Description of different forms of the heat equation

Considering the differential equation controlling the heat conduction phenomena in anoven (if the medium is uniform *ρ, C* and *K* are fixed values) and written as follows:

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + f \tag{1}$$

With: $\alpha = \frac{K}{\rho.C} = conste > 0$, $f = \frac{S}{\rho.C}$

α: is referred to as "thermal diffusivity".

To completely write the process of thermal exchange, it is required to provide the initial temperature partition in the middle (initial boundary condition) and the boundary temperature conditions (boundary condition).

It can be take into account the following various boundary conditions:

1. The temperature *T* in *Γ* is written as:

$$T|_\Gamma = T_0$$

2. The heating flux in *Γ* is written as:

$$K\frac{\partial T}{\partial n}\bigg|_\Gamma = T_1$$

3. On the border *Γ*heating transmission takes place according to Newton's law

$$K\frac{\partial T}{\partial n} + h(T - T_0)|_\Gamma = 0$$

Or *h*: coefficient of heating transmission

T_0: Temperaturefrom the surrounding environment.

The law from Newton states that the flow of heating across the frontier is proportionate to the difference in temperature on the two sides of the boundary.

For the stationary distribution, the temperature does not vary with timei.e.$\frac{\partial T}{\partial t} = 0$.

Therefore, it get the POISSON equation of the steady distribution of heat.

$$-\Delta T = \frac{f}{\alpha} \tag{2}$$

When there is no thermic source in the middle, the equation of POISSON (2) is reduced to the equation of Laplace

$$\Delta T = 0 \tag{3}$$

It is the equation for the stationary repartition within the walls of the oven.

It uses second order approximation of finite differential schemes in the space step and $O(h^2)$ and absolutely stable.

3. The stationary heat transfer problem in the walls of an oven

Consider a furnace in the plane of the straight section shown below. We call Γ_i the internal oven surface and Γ_e the outer surface.

In steady state and without source, the temperature $T(x, y)$ at a point (x, y), of the wall verifies the Laplace equation.

$$\begin{cases} \Delta T = 0 \\ T = \theta_e on \Gamma_e \\ T = \theta_i on \Gamma_i \\ \theta_e < \theta_i \end{cases} \tag{4}$$

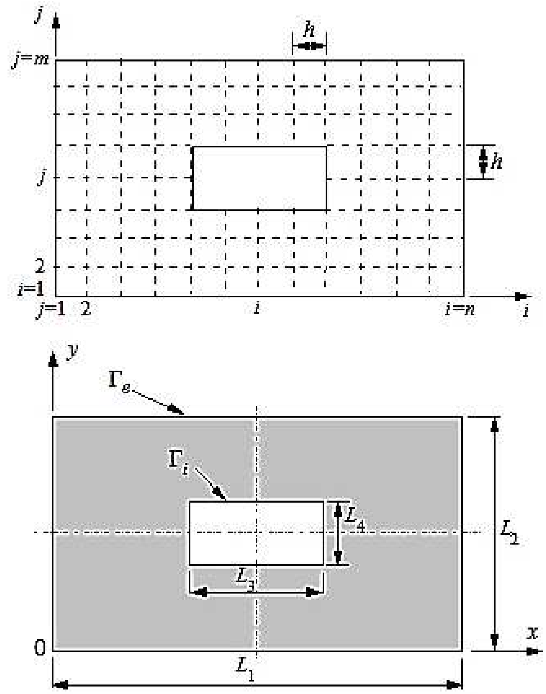

Fig 1. Geometry and mesh of the Oven walls

Where: $\Gamma = \Gamma_e \cup \Gamma_i$ the boundary of Ω

The Taylor series expansion of the function $T(x', y)$, around $T(x, y)$

Where $x' = x \pm \Delta x$ is written:

$$\begin{cases} T(x+\Delta x, y) = T(x, y) + \Delta x \frac{\partial T}{\partial x} + \frac{(\Delta x)^2}{2!} . \frac{\partial^2 T}{\partial x^2} + \frac{(\Delta x)^3}{3!} . \frac{\partial^3 T}{\partial x^3} + \frac{(\Delta x)^4}{4!} . \frac{\partial^4 T}{\partial x^4} + \cdots \\ T(x-\Delta x, y) = T(x, y) - \Delta x \frac{\partial T}{\partial x} + \frac{(\Delta x)^2}{2!} . \frac{\partial^2 T}{\partial x^2} - \frac{(\Delta x)^3}{3!} . \frac{\partial^3 T}{\partial x^3} + \frac{(\Delta x)^4}{4!} . \frac{\partial^4 T}{\partial x^4} + \cdots \end{cases} \tag{5}$$

Adding these two relations and dividing by $(\Delta x)^2$, we find:

$$\frac{\partial^2 T}{\partial x^2} = \frac{T(x+\Delta x, y) - 2T(x, y) + T(x-\Delta x, y)}{(\Delta x)^2} + O(\Delta x)^2 \tag{6}$$

Operating in the same way for the variable y, we find:

$$\frac{\partial^2 T}{\partial y^2} = \frac{T(x, y+\Delta y) - 2T(x, y) + T(x, y-\Delta y)}{(\Delta y)^2} + O(\Delta y)^2 \tag{7}$$

Thus, the two-dimensional Laplacian is written:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{T(x+\Delta x,y)+T(x-\Delta x,y)-4T(x,y)+T(x,y+\Delta y)+T(x,y-\Delta y)}{O(\Delta x)^2+O(\Delta y)^2} \quad (8)$$

In our case, we have$\Delta x=\Delta y=h$.

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{T(x+\Delta x,y)+T(x-\Delta x,y)-4T(x,y)+T(x,y+\Delta y)+T(x,y-\Delta y)}{h^2} \quad (9)$$

We pose:$\begin{cases} x_i=i\Delta x=ih & \longrightarrow 1\le i\le n \\ y_j=j\Delta y=jh & \longrightarrow 1\le j\le m \end{cases}$

To simplify the notation, the previous problem expression (4) is approximated by the following difference scheme:

$$\begin{cases} \frac{T_{i-1,j}+T_{i+1,j}+T_{i,j+1}+T_{i,j-1}-4T_{i,j}}{h^2}=0 \qquad \forall\,(i,j)\text{interior} \in \Omega_h^0 \\ T_{i,j}=\theta_e \quad , \quad \forall\,(i,j) \in \Gamma_e\Omega_h = \Omega_h^0\cup\Gamma_h \\ T_{i,j}=\theta_i \quad , \forall\,(i,j) \in \Gamma_i \end{cases} \quad (10)$$

It is obvious that the approximation of this scheme to differences is of order 2.

4. Stability study

The stability of difference schemes (10) is proved as follows:

Let Ω be the solution domain of the stationary heat problem (9) and Γ the boundary of Ω.

The difference scheme (10) is defined on the network: $\Omega_h=\{(x_i,y_j)=(ih,jh)\in\Omega\}$

A node (x_i,y_j)or simpler $(i,j)$is called an interior node of $\Omega_h$if all its 4 juxtaposed nodes$(i-1,j)$,$(i+1,j)$,$(i,j-1)$, $(i,j+1)$also belong toΩ_h.

We name $\Omega_h^0$the set of nodes inside Ω_h.

Then we have the decomposition

$\Omega_h=\Omega_h^0\cup\Gamma_h$ with $\Omega_h^0\cup\Gamma_h=\emptyset$ and Γ_h is the set of border nodes.

By denoting $\Lambda T_{ij}=\frac{1}{h^2}(T_{i-1,j}+T_{i+1,j}+T_{i,j-1}+T_{i,j+1}-4T_{i,j})$

Then we can write the difference scheme (9) in the symbolic form

$$L_h T^h=f^h \quad (11)$$

$(L_hT^h)_{ij}=\Lambda T_{ij}$ in $(i,j)\in\Omega_h^0$

And $(L_hT^h)_{ij}=T_{ij}$ in $(i,j)\in\Gamma_h$

$(f^h)_{ij}=0$ in all $(i,j)\in\Omega_h^0$

$(f^h)_{ij}=\Psi_{ij}$given value, $\forall\,(i,j)\in\Gamma_h$

To prove the stability of the difference scheme (10) we first prove two following propositions:

Let T' be a network function on Ω_h and $\Lambda T'_{i,j} = \frac{1}{h^2}(T'_{i-1,j} + T'_{i+1,j} + T'_{i,j-1} + T'_{i,j+1} - 4T'_{i,j})$

***Proposition 1*:** If $\Lambda T'_{ij} > 0$ at every interior node of Ω_h^0 then the maximum component of T' is reached (realized) at least at one boundary node of Γ_h i.e., $\exists\ (i,j) \in \Gamma_h$ such that:

$T'_{ij} \geq T_{k,l}, \forall (k,l) \in \Omega_h$.

Proof: Let us reason by absurdity, i.e. if the conclusion of Proposition 1 is false then we end up with a contradiction.

Indeed, suppose that the maximal component T'_{ij} is not reached on the boundary Γ_h.

We choose an inner node (i,j) located on the leftmost side i.e. with:

a) $T'_{ij} \geq T_{kl}, \forall (k,l) \in \Omega_h$
b) $T'_{ij} \geq T_{i-1,j}$

Then we can write:

$$\Lambda T'_{i,j} = \frac{1}{h^2}\Big(\big(T'_{i-1,j} - T'_{i,j}\big) + \big(T'_{i+1,j} - T'_{i,j}\big) + (T'_{i,j-1} - T'_{i,j}) + (T'_{i,j+1} - T'_{i,j})\Big)$$

All terms are ≤ 0 and the first term $T'_{i-1,j} - T'_{i,j} < 0 \Longrightarrow \Lambda T'_{i,j} < 0$

This contradicts the assumption $\Lambda T'_{i,j} \geq 0, \forall (i,j) \in \Omega_h^0$

Therefore, Proposition 1 is true.

Proposition 2: If $\Lambda T'_{ij} \leq 0$ at any interior point (i,j) then the minimum component of T' is reached at least at one boundary node.

Demonstration of stability:

The solution T^h of the difference scheme (5) verifies both the inequalities $\Lambda T_{ij} \geq 0$ and $\Lambda T_{ij} \leq 0$ at any node $(i,j) \in \Omega_h^0$

From the previous two propositions, we obtain:

$$\|T^h\|_{U_h} = \max_{(i,j)\in\Omega_h} |T_{ij}| = \max_{(i,j)\in\Gamma_h} |T_{ij}| = \max_{(i,j)\in\Gamma_h} |\Psi_{ij}| \qquad (12)$$

On the other hand

$$\|f^h\|_{F_h} = \max_{(i,j)\in\Omega_h} |(f^h)_{ij}| = \max_{(i,j)\in\Gamma_h} |(f^h)_{ij}| = \max_{(i,j)\in\Gamma_h} |\Psi_{ij}| \qquad (13)$$

From (11) and (12) it follows:

$$\|T^h\|_{U_h} = \|f^h\|_{F_h}$$

From which we also have

$$\|T^h\|_{U_h} \leq \|f^h\|_{F_h}$$

This is the stability of schemes with differences (11).

5. Results and discussion

Here are some results: If we apply our system as a heat equation, with constant conductivity and steady state.

Thus, the system (or discretized Laplace equation) can be solved by the Gauss-Seidel method (explicit method) where the Laplace equation gives the central point of each molecular form by the expression:

$$T_{i,j} = \frac{T_{i+1,j}+T_{i-1,j}+T_{i,j+1}+T_{i,j-1}}{4} \tag{14}$$

The elaborated program allows to solve the system (14) by the explicit method. To do this, we give ourselves an arbitrary initial value (distribution) $T_{i,j}^{(0)}$, which is carried in equation (14) to the second member for each pair (i,j), gives a new value $T_{i,j}^{(1)}$, and so on. The computations stop when $\left|T_{i,j}^{(p+1)} - T_{i,j}^{(p)}\right| \leq \varepsilon$ where ε is the limit of convergence that we give ourselves.

Figures (2, 3) represent the contours from the isotherms at various Δx and Δy values and three dimensional x, y and T in the case where the stationary heat transfer problem in the walls of an Oven.From the boundary conditions imposed, the diffusion of the isothermal provides a very good justification for the physics meaning for those conditions.

We notice that the smaller the spatial pitch, the more the flow penetrates inside, with a concentration in the middle. The propagation of the temperature is intense in the middle of the end to all regions. This has the effect of giving a good accuracy on some sides or regions.

With points defined by the grids in figure 2 and figure 3, starting from an arbitrary estimate *(θ = 100)*. For the interior points, the method converges in 33 iterations. A better choice of estimate would have reduced this number of iterations. Note that we could have used symmetries to decrease the number of calculations.

Instead of using a usual test of the type $\| r \| < \varepsilon$, we used the fact that the furnace has two axes of symmetry and thus the temperature distribution has this property. Since the calculation process itself is asymmetric, two symmetric points like $T_{2,2}$ and $T_{n-1,n-1}$ will have same temperature only after convergence . This test allows us to avoid the costly calculation of the norm of the residual $\| r \|$.

Knowing that the discretization of the Laplacian introduces an error of order Δx, the results with different values of Δx are significantly different at the common points between two grids. The difference decreasing with Δx.

Consider L_1=1, L_2=1, L_3=0.4, L_4=0.4, T_0 = 100K, θ_e = 50K, θ_i = 1150K, Δx = Δy = 0.1

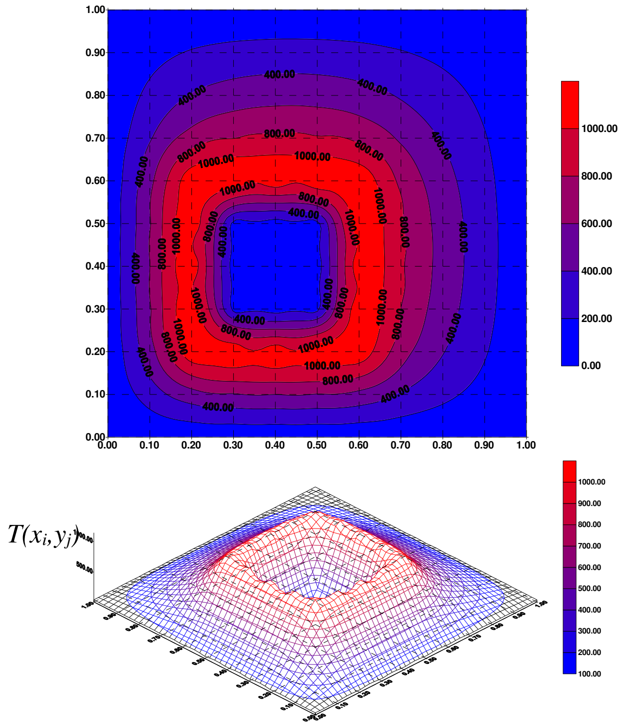

Figure 2. Temperature variation as a function of x_i, yj(stationary case) for Δx=Δy=0.1

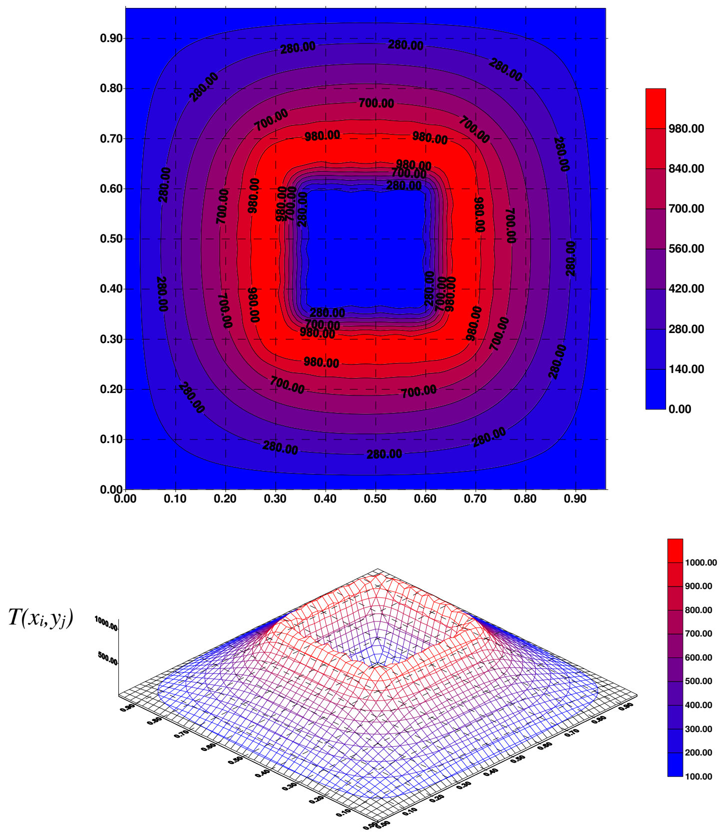

Figure 3. Temperature variation as a function of x_i, y_j (stationary case) for Δx=Δy=0.06

6. Conclusion

The numerical results demonstrate the stability of the convergent difference schemes, the conformity to the physical phenomena of the problem, and the developed algebraic calculation.The developed solving algorithm and the difference schemes presented in the present paper are solvable to the domains in any geometrical pattern with slight changes.It is possible to develop, and generalized the problem studied in the present paper into an optimum control method in which the control factor (the heat source) has to be chosen in order to get the required heat flow (stable).It is a highly common problem in the conception and functioning for industrial ovens.

Finally, this paper allowed us to work in groups and to apply mathematical theory in computer programming. This paper also shows the importance of mathematics as a tool to understand, explain and predict natural phenomena.

In the next work, we will move to 3D. With the same procedure and approach and there we will see the efficiency of the resolution algorithm, and especially the limits of capacity of sequential computers.

Declarations

Conflicts of interest The authors declare that they have no conflict of interest

Références

[1] Joseph Fourier, Théorie analytique de la chaleur, 1822

[2] Jean Zinn-Justin, Intégrale de chemin en mécanique quantique : introduction, EDP Sciences, 2003, 296 p. (ISBN 978-2-86883-660-1,

[3] N. V. Vaidya , A. A. Deshpande and S. R. Pidurkar. Solution of heat equation (Partial Differential Equation) by various methods. Journal of Physics: Conference Series. 1913 (2021) 012144.

[4] A.Ullaha , K. Shaha,b. R. A. Khan. Series Form Solution to Two Dimensional Heat Equation of Fractional Order. Results in Nonlinear Analysis. 2 (2019) No. 4, 193–199.

[5] N. Benhissen and A. Khechekhouche. Modeling and numerical resolution of the Poisson equation in 2D by the finite difference method case of the heat transfer equation. International Journal of Chemical and Petroleum Sciences. 2012, 1(1), 1-9.

[6] P. Y. Lagrée. Solving the Unsteady Heat Equation. 6.19- version March 11, 2010.

[7] M. Makhtoumi. Numerical Solutions of Heat Diffusion Equation Over One Dimensional Rod Region. International Journal of Science and Advanced Technology. 7 No 3, 2017, 10-13.

[8] R. Dhoyer, C. Tudor. Spatial average for the solution to the heat equation with Rosenblatt noise. Stochastic Analysis and Applications. 40 (6), 2022.

[9] Loskor, W. and Sarkar, R. (2022) A Numerical Solution of Heat Equation for Several Thermal Diffusivity Using Finite Difference Scheme with Stability Conditions. Journal of Applied Mathematics and Physics, 10, 449-465.

[10] D. Taloub, A Beghidja. Resolution of the heat equation in the square form four. Energy Procedia

6, 2011, 487-492.

[11] D. Taloub, A. Bouras, Z. Driss. Numerical-Resolution-of-the-Heat-Equation-in-the-Square-Form-Four-Part-II. Mathematical Modelling of Engineering Problems. 7(2), 2020, 205-211.

[12] S. J. Kahlaf, A. A. Mhassin. Numerical Solution of a Two-Dimensional Laplace Equation with Dirichlet Boundary Conditions. Journal of Physics: Conference Series. 1818, 2021, 012193.