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Shortest Minkowski billiard trajectories on convex bodies

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Daniel Rudolf · Stefan Krupp

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Abstract We rigorously investigate closed Minkowski/Finsler billiard trajectories on n-dimensional convex bodies. We outline the central properties in comparison and differentiation from the Euclidean special case and establish two main results for length-minimizing closed Minkowski/Finsler billiard trajectories: one is a regularity result, the other is of geometric nature. Building on these results, we develop an algorithm for computing length-minimizing closed Minkowski/Finsler billiard trajectories in the plane.

Keywords Minkowski Billiards · Generalized Billiards · Shortest Periodic Orbit · Polytopes · Algorithmic Geometry

Mathematics Subject Classification (2010) MSC 37C83

1 Introduction and main results

Minkowski/Finsler billiards are the natural extensions of Euclidean billiards to the Minkowski/Finsler setting.

Euclidean billiards are associated to the local Euclidean billiard reflection rule: The angle of reflection equals the angle of incidence (here, we assume that the relevant normal vector as well as the incident and the reflected ray lie in the same two-dimensional affine flat). This local Euclidean billiard reflection

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rule follows from the global least action principle. For a reflection in a hyperplane this principle means that a billiard trajectory segment (p_{j-1}, p_j, p_{j+1}) minimizes the Euclidean length in the space of all paths connecting p_{j-1} and p_{j+1} via a reflection at this hyperplane.

In Finsler geometry, the notion of length of vectors in $\mathbb{R}^n$, $n \geqslant 1$, is given by a convex body $T \subset \mathbb{R}^n$, i.e., a compact convex set in $\mathbb{R}^n$ which has the origin in its interior (in $\mathbb{R}^n$). The Minkowski functional

$$\mu_T(x) = \min\{t \geqslant 0 : x \in tT\},\ x \in \mathbb{R}^n,$$

determines the distance function, where we recover the Euclidean setting when T is the n-dimensional Euclidean unit ball. Then, heuristically, billiard trajectories are defined via the global least action principle with respect to μ_T–we specify this in a moment–, because in Finsler geometry, there is no useful notion of angles.

There is generally much interest in the study of billiards: Problems in almost every mathematical field can be related to problems in mathematical billiards, see for example [12], [16] and [26] for comprehensive surveys. Euclidean billiards in the plane have been investigated very intensively. Nevertheless, so far not much is known about Euclidean billiards on higher dimensional billiard tables. But even much less is known for Minkowski/Finsler billiards. Although the applications are numerous and important in many different areas, to the authors' knowledge, [2, 3, 4, 5, 6, 7, 9, 11, 13, 21] are the only publications concerning different aspects of Minkowski/Finsler billiards so far.

After Minkowski/Finsler billiards were introduced in [13], their study was intensified when in [7] the relationship to the EHZ-capacity of convex Lagrangian products in $\mathbb{R}^{2n}$ was proven. When studied isoperimetric Minkowski/Finsler billiard inequalities, this opened the connection to Viterbo's conjecture (cf. [27]) within symplectic geometry. With [8], this consequently also allowed to analyze the famous Mahler conjecture (cf. [19]) from convex geometry.

In this paper, we prove fundamental properties of Minkowski/Finsler billiards. Particular attention is paid to length-minimizing closed Minkowski/Finsler billiard trajectories. As part of the investigation of the latter, we state two main results: one mainly is a regularity result, the other is geometric in nature. Together they can be seen as the generalization of Theorem 1.2 in [17] to the Minkowski/Finsler setting. Based on these results, we derive an algorithm for computing length-minimizing closed Minkowski/Finsler billiard trajectories in the plane.

Before we state the main results of this paper, let us precisely define Minkowski/Finsler billiards, while we suggest from now on to call them just *Minkowski billiards* following [7] and [9].

In the further course of this paper, we will see that it makes sense to differentiate between weak and strong Minkoswski billiard trajectories. We begin by introducing weak Minkowski billiard trajectories.

Definition 11 (Weak Minkowski billiards) *Let $K \subset \mathbb{R}^n$ be a convex body which from now on we call the billiard table. Let $T \subset \mathbb{R}^n$ be another convex*

body and $T^\circ \subset \mathbb{R}^n$ its polar body. We say that a closed polygonal curve[1] *with vertices $q_1, ..., q_m$, $m \in \mathbb{N}_{\geqslant 2}$, on the boundary of K (denoted by ∂K) is a closed weak (K,T)-Minkowski billiard trajectory if for every $j \in \{1, ..., m\}$, there is a K-supporting hyperplane H_j through q_j such that q_j minimizes*

$$\mu_{T^\circ}(\bar{q}_j - q_{j-1}) + \mu_{T^\circ}(q_{j+1} - \bar{q}_j), \tag{1}$$

over all $\bar{q}_j \in H_j$ (cf. Figure 1). We encode this closed weak (K,T)-Minkowski billiard trajectory by $(q_1, ..., q_m)$ and call its vertices bouncing points. Its ℓ_T-length[2] *is given by*

$$\ell_T((q_1, ..., q_m)) = \sum_{j=1}^{m} \mu_{T^\circ}(q_{j+1} - q_j).$$

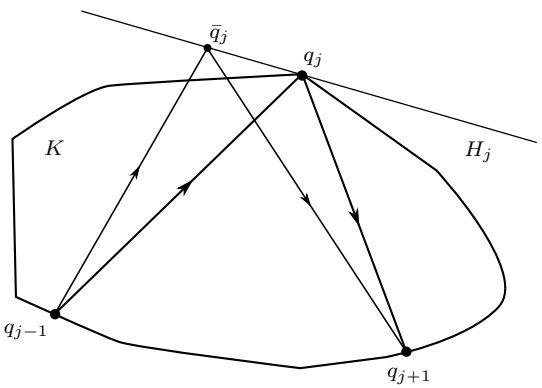

Fig. 1 The weak Minkowski billiard reflection rule: q_j minimizes (1) over all $\bar{q}_j \in H_j$, where H_j is a K-supporting hyperplane through q_j.

We call a boundary point $q \in \partial K$ *smooth* if there is a unique K-supporting hyperplane through q. We say that ∂K is *smooth* if every boundary point is smooth (we also say K is smooth while we actually mean ∂K).

Concerning the definition of weak Minkowski billiards, we remark that, in general, the K-supporting hyperplanes H_j are not uniquely determined. This is only the case for smooth and strictly convex T (cf. Proposition 33).

We note that the weak Minkowski billiard reflection rule defined in Definition 11 does not only generalize the Euclidean billiard reflection rule to Minkowski/Finsler geometries, it also extends the classical understanding of

[1] For the sake of simplicity, whenever we talk of the vertices $q_1, ..., q_m$ of a closed polygonal curve, we assume that they satisfy $q_j \neq q_{j+1}$ and q_j is not contained in the line segment connecting q_{j-1} and q_{j+1} for all $j \in \{1, ..., m\}$. Furthermore, whenever we settle indices $1, ..., m$, then the indices in $\mathbb{Z}$ will be considered as indices modulo m.

[2] This length-definition can be generalized to any closed polygonal curve.

billiard trajectories to non-smooth billiard table boundaries[3]. In the latter case, there can exist infinitely many different K-supporting hyperplanes through non-smooth bouncing points at the billiard table boundary, and consequently, from a constructive viewpoint, the weak Minkowski billiard reflection rule may produce different bouncing points following two already known consecutive ones.

In the case when T° is smooth and strictly convex, the definition of weak Minkowski billiards yields a geometric interpretation of the billiard reflection rule: On the basis of Lagrange's multiplier theorem, one derives the condition

$$\nabla_{\bar{q}_j} \Sigma_j(\bar{q}_j)_{|\bar{q}_j=q_j} = \nabla \mu_{T^\circ}(q_j - q_{j-1}) - \nabla \mu_{T^\circ}(q_{j+1} - q_j) = \mu_j n_{H_j},$$

where $\mu_j > 0$, since the strict convexity of T° implies

$$\nabla \mu_{T^\circ}(q_j - q_{j-1}) \neq \nabla \mu_{T^\circ}(q_{j+1} - q_j),$$

and where n_{H_j} is the outer unit vector normal to H_j. This implies that the weak Minkowski billiard reflection rule can be illustrated as within Figure 2. For smooth, strictly convex, and centrally symmetric $T^\circ \subset \mathbb{R}^n$, this interpretation is due to [13, Lemma 3.1, Corollary 3.2 and Lemma 3.3] (this interpretation has also been referenced in [6]). For the extension to just smooth and strictly convex $T^\circ \subset \mathbb{R}^n$, it is due to [11, Lemma 2.1]. However, from the constructive point of view, this interpretation has its limitations.

We come to the definition of strong Minkowski billiards.

Definition 12 (Strong Minkowski billiards) *Let $K, T \subset \mathbb{R}^n$ be convex bodies. We say that a closed polygonal curve q with vertices $q_1, ..., q_m$, $m \in \mathbb{N}_{\geqslant 2}$, on ∂K is a closed strong (K,T)-Minkowski billiard trajectory if there are points $p_1, ..., p_m$ on ∂T such that*

$$\begin{cases} q_{j+1} - q_j \in N_T(p_j), \\ p_{j+1} - p_j \in -N_K(q_{j+1}) \end{cases} \tag{2}$$

is satisfied for all $j \in \{1, ..., m\}$. We call $p = (p_1, ..., p_m)$ a closed dual billiard trajectory in T. We say that $q = (q_1, ..., q_m)$ is a closed strong (K,T)-Minkowski billiard trajectory with respect to $H_1, ..., H_m$, if $H_1, ..., H_m$ are the K-supporting hyperplanes through $q_1, ..., q_m$ which are normal to $n_K(q_1), ..., n_K(q_m)$, where

$$p_{j+1} - p_j = -\mu_{j+1} n_K(q_{j+1}),\ \mu_{j+1} \geqslant 0, \tag{3}$$

for all $j \in \{1, ..., m\}$.

This definition appeared implicitly in [13, Theorem 7.1], then later the first time explicitly in [7]. It yields a different interpretation of the billiard reflection rule. Without requiring a condition on T, the billiard reflection rule

[3] Classical billiard trajectories are usually understood as trajectories with bouncing points in smooth boundary points (billiard table gangs) while they stop in non-smooth boundary points (billiard table holes).

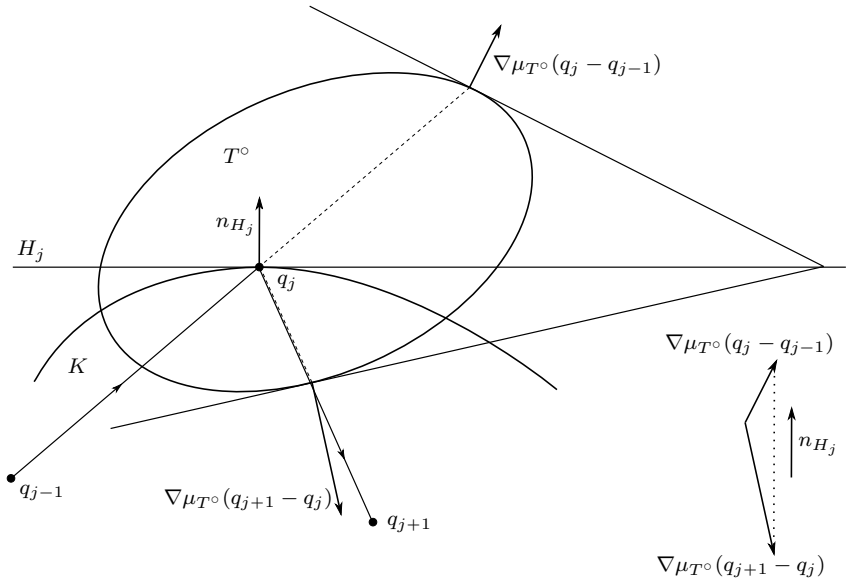

Fig. 2 T° is a smooth and strictly convex body in $\mathbb{R}^2$ and its boundary plays the role of the indicatrix, i.e., the set of vectors of unit Finsler (with respect to T°) length, which therefore is an 1-level set of μ_{T°. Note that the two T°-supporting hyperplanes intersect on H_j due to the condition $\nabla\mu_{T^\circ}(q_j - q_{j-1}) - \nabla\mu_{T^\circ}(q_{j+1} - q_j) = \mu_j n_{H_j}$.

can be represented as within Figure 3. From the constructive point of view, this interpretation is much more appropriate in comparison to the one for weak Minkowski billiards.

We remark that, in general, the closed dual billiard trajectory in T is not uniquely determined. This is only the case when T is strictly convex (cf. Proposition 32). Further, we remark that under the condition that T is strictly convex and smooth, the closed dual billiard trajectory p in Definition 12 also is a closed Minkowski billiard trajectory–we refer to Proposition 35 for the precise statement.

The natural follow-up question concerns the relationship between weak and strong Minkowski billiards. The following theorem gives an answer to this question.

Theorem 11 *Let $K, T \subset \mathbb{R}^n$ be convex bodies. Then, every closed strong (K,T)-Minkowski billiard trajectory is a weak one, more precisely, if $q = (q_1, ..., q_m)$ is a closed strong (K,T)-Minkowski billiard trajectory with respect to $H_1, ..., H_m$, then q fulfills the weak Minkowski billiard reflection rule with respect to $H_1, ..., H_m$.*

If T is strictly convex, then every closed weak (K,T)-Minkowski billiard trajectory is a strong one, more precisely, if $q = (q_1, ..., q_m)$ is a closed weak (K,T)-Minkowski billiard trajectory fulfilling the weak Minkowski billiard reflection rule with respect to the K-supporting hyperplanes $H_1, ..., H_m$, then q is a closed strong (K,T)-Minkowski billiard trajectory with respect to $H_1, ..., H_m$.

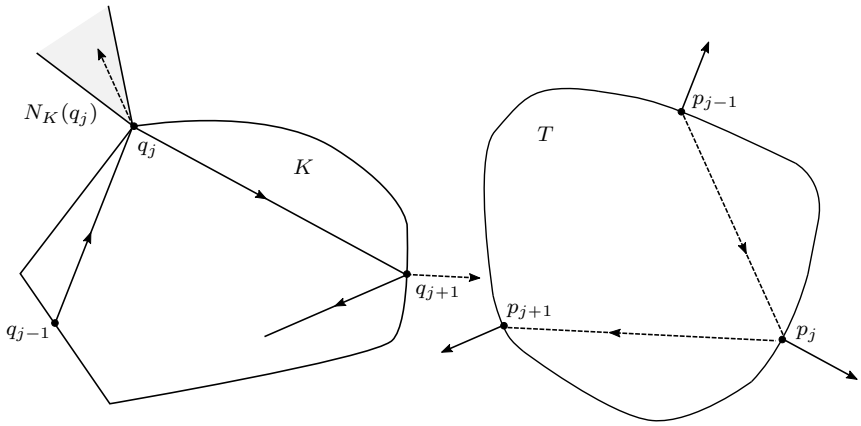

Fig. 3 The pair (q,p) fulfills (2), namely: $q_j - q_{j-1} \in N_T(p_{j-1})$, $q_{j+1} - q_j \in N_T(p_j)$, $p_j - p_{j-1} \in -N_K(q_j)$, and $p_{j+1} - p_j \in -N_K(q_{j+1})$.

Theorem 11 is sharp in the following sense: One can construct convex bodies $K, T \subset \mathbb{R}^n$ (where T is not strictly convex) and a closed weak (K,T)-Minkowski billiard trajectory which is not a strong one (cf. Example A in Section 6).

In the following, if the risk of confusion is excluded, we will call strong Minkowski billiards trajectories just Minkowski billiard trajectories. Although the connection to the least action principle, which is the basis for the definition of the weak version, is lost for non-strictly convex bodies, we make the strong version to the center of our investigations, since one can use them to compute the EHZ-capacity of convex Lagrangian products (cf. [23]).

In the following theorem, the main properties of closed Minkowski billiard trajectories are collected. We recall that for a convex body $K \subset \mathbb{R}^n$, we denote by $F(K)$ the set of subsets of $\mathbb{R}^n$ which cannot be translated into $\mathring{K}$ and by $F^{cp}_{n+1}(K)$ the set closed polygonal curves in $\mathbb{R}^n$ with at most $n+1$ vertices which cannot be translated into $\mathring{K}$. Furthermore, we denote by $M_{n+1}(K,T)$ the set of closed (K,T)-Minkowski billiard trajectories with at most $n+1$ bouncing points.

Theorem 12 *Let $K, T \subset \mathbb{R}^n$ be convex bodies.*

(i) [Proposition 34] Let q be a closed (K,T)-Minkowski billiard trajectory with dual billiard trajectory p. Then, we have

$$\ell_T(q) = \ell_{-K}(p).$$

(ii) [Proposition 35] Let T be strictly convex and smooth. Let q be a closed (K,T)-Minkowski billiard trajectory with dual billiard trajectory p. Then, p is a closed $(T,-K)$-Minkowski billiard trajectory.

(iii) [Proposition 39] Let q be a closed (K,T)-Minkowski billiard trajectory with dual billiard trajectory p. Then, we have

$$q \in F(K) \quad \text{and} \quad p \in F(T).$$

(iv) *[Proposition 310] Let $q = (q_1, ..., q_m)$ be a closed (K,T)-Minkowski billiard trajectory with respect to $H_1, ..., H_m$ and let U be the inclusion minimal linear subspace of $\mathbb{R}^n$ containing the outer unit vectors $n_K(q_1), ..., n_K(q_m)$ which are normal to $H_1, ..., H_m$. Then, there is a selection $\{i_1, ..., i_{\dim U+1}\} \subseteq \{1, ..., m\}$ such that*

$$\{q_{i_1}, ..., q_{i_{\dim U+1}}\} \in F(K).$$

(v) *[Theorem 312] Let T be strictly convex. Then, every ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory is an ℓ_T-minimizing element of $F^{cp}_{n+1}(K)$ and, conversely, every ℓ_T-minimizing element of $F^{cp}_{n+1}(K)$ can be translated in order to be an ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory. Especially, one has*

$$\min_{q \in F^{cp}_{n+1}(K)} \ell_T(q) = \min_{q \in M_{n+1}(K,T)} \ell_T(q). \tag{4}$$

We note that in [23, Theorem 2.2], we actually prove that (4) holds without the condition of strict convexity of T (in the general case, the other part of the statement in Theorem 12(v) holds slightly changed). However, for the purposes of this paper, this formulation is enough for us.

The main interest we pursue in the following part of this paper is the investigation of the subsequent questions (in analogy to the questions in [17]). Given an ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory–depending on the conditions on $K \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^n$:

(a) What can be said concerning the number of bouncing points?
(b) What can be said concerning the convex cone which is spanned by the normal vectors related to the Minkowski billiard reflection rule? Is it a subspace? What is its dimension?
(c) What can be said concerning the dimension of the inclusion minimal affine section of K containing this billiard trajectory?
(d) What can be said concerning the regularity of the bouncing points? Are they smooth? Are they smooth within the inclusion minimal section of K containing this billiard trajectory?

All these questions are important in order to develop an algorithm for computing ℓ_T-length-minimizing closed (K,T)-Minkowski billiard trajectories.

Our first main result is the following regularity result:

Theorem 13 *Let $K, T \subset \mathbb{R}^n$ be convex bodies, where T is additionally assumed to be strictly convex and smooth. Let $q = (q_1, ..., q_m)$ be an ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory (which fulfills the Minkowski billiard reflection rule with respect to $H_1, ..., H_m$) and let U be the convex cone spanned by the outer unit vectors $n_K(q_1), ..., n_K(q_m)$ which are normal to $H_1, ..., H_m$. Then, U is a linear subspace of $\mathbb{R}^n$ with*

$$\dim U = m - 1$$

and

$$\dim (N_K(q_j) \cap U) = 1 \tag{5}$$

for all $j \in \{1, ..., m\}$.

Since in general $\dim U \leqslant n$, it follows that

$$m \leqslant n+1.$$

In fact, (5) is a *regularity* result: If $m = n+1$, meaning that $U = \mathbb{R}^n$, then (5) becomes

$$\dim\left(N_K(q_j)\right) = 1$$

for all $j \in \{1, ..., m\}$, i.e., q is regular, meaning that all bouncing points of q are smooth boundary points of K.

Some special cases of Theorem 13 were already known: For T equals the Euclidean unit ball in $\mathbb{R}^n$, Theorem 13 coincides with [17, Theorem 1.2], since by [17, Proposition 2.6], in this Euclidean case, U equals V_0 which is the underlying linear subspace of the affine subspace $V \subseteq \mathbb{R}^n$ which is chosen such that $K \cap V$ is the inclusion minimal affine section of K containing q. In the situation of Theorem 13, for ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectories $q = (q_1, ..., q_m)$, it has been proven in [3] that m is bounded by $n+1$, and in [2, Theorem 4.8], that the ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectories with $n+1$ bouncing points are regular. In [6], it has been proven the statement of Theorem 13 for $n = 2$.

As in the less general Euclidean case, Theorem 13 refutes the presumption which at first appears to be intuitively correct, that every ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory with more than two bouncing points is regular within the inclusion minimal section of K containing this billiard trajectory.

For the sharpness of Theorem 13, we refer on the one hand to the sharpness of [17, Theorem 2.1]. There, for T equals the Euclidean unit ball, we showed that an ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory may not be regular within the inclusion minimal sections of K containing the billiard trajectory (which in this case is a translate of U). This can even appear for the unique ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory. Then, we also showed that an ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory can have bouncing points in vertices as well as in more than 0-dimensional faces of K.

On the other hand, we can prove that in Theorem 13, the smoothness of T is a necessary condition. For that, we refer to Example F. Furthermore, we can show that within the weaker situation of weak Minkowski billiard trajectories, in general, the statement of Theorem 13 is not true without requiring the strict convexity of T. For that, we refer to Example G (cf. Section 6). In this latter example, we will see that without requiring the strict convexity of T, even the notion of ℓ_T-minimizing closed weak (K,T)-Minkowski billiard trajectories does not make sense: There are configurations for which such minimizers do not exist.

The second main result generalizes a property of length-minimizing closed Euclidean billiard trajectories on K (cf. [17, Theorem 2.1]) to the Minkowski/Finsler setting (K,T) under the condition that both K and T are strictly convex and smooth:

Theorem 14 *Let $K, T \subset \mathbb{R}^n$ be strictly convex and smooth bodies. Let $q = (q_1, ..., q_m)$ be an ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory and V an affine subspace of $\mathbb{R}^n$ such that $K \cap V$ is the inclusion minimal affine section of K containing q. Then, we have*

$$\dim V = m - 1,$$

i.e., q is maximally spanning, meaning that the dimension of the convex hull of the bouncing points $q_1, ..., q_m$ is $m - 1$.

From Theorem 13 and a consideration in the context of Theorem 14, we immediately derive the following corollary for $n = 2$:

Corollary 15 *Let $K, T \subset \mathbb{R}^2$ be convex bodies, where T is additionally assumed to be strictly convex and smooth. Then, every ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory $q = (q_1, ..., q_m)$ has two or three bouncing points, i.e., $m \in \{2, 3\}$, where in the latter case the billiard trajectory is regular. Furthermore, when q fufills the Minkowski billiard reflection rule with respect to $H_1, ..., H_m$, then we have:*

(i) If U is the convex cone spanned by the unit vectors $n_K(q_1), ..., n_K(q_m)$ which are normal to $H_1, ..., H_m$, then U is a linear subspace of $\mathbb{R}^2$ with

$$\dim U = m - 1.$$

(ii) If $V \subseteq \mathbb{R}^2$ is an affine subspace such that $K \cap V$ is the inclusion minimal affine section of K containing q, then

$$\dim V = m - 1.$$

Here, (i) follows from Theorem 13 and (ii) from the fact that, by definition, both 2- and 3-bouncing Minkowski billiard trajectories are maximally spanning (so, for (ii), the strict convexity of K, as required in Theorem 14 for general dimension, is not necessary).

Corollary 15 can be used for the construction of ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectories when $K \subset \mathbb{R}^2$ is a convex polytope. In Section 7.1, we describe an algorithm, while in Section 7.3, we provide a detailed survey concerning the efficiency and methods used within the implementation. We remark especially that our implemented algorithm is the solution to the open problem of finding the Fagnano triangle in a Minkowski triangle stated in [6].

Let us briefly give an overview of the structure of this paper: In Section 2, we enumerate properties of the polar body and the Minkowski functional that we will use repeatedly within this paper. In Section 3, we prove fundamental properties of Minkowski billiard trajectories which we utilize in Sections 4 and 5 for the proofs of Theorems 13 and 14, respectively. In Section 6, we give various examples in order to show the sharpness of the statements in Section 3 as well as the sharpness of the main results. In Section 7, we discuss the algorithm for computing ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectories for $K, T \subset \mathbb{R}^2$. In Section 8, we present a note on the existence of closed regular Minkowski billiard trajectories with three bouncing points in obtuse triangles.

2 Preliminaries from convex geometry

In this section, we collect some useful properties of the polar body and the Minkowski functional. Since these properties are well-known, we will just state them and refer for the proofs to the usual literature on this topic.

Let $T \subset \mathbb{R}^n$ be a convex body. Then, the *polar body* of T is

$$T^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leqslant 1 \text{ for all } y \in T\} \subset \mathbb{R}^n.$$

The polar body satisfies the following properties:

Proposition 21 *Let $P, Q \subset \mathbb{R}^n$ be convex bodies. Then:*

(i) *P° is in $\mathcal{C}(\mathbb{R}^n)$.*
(ii) *For $c \neq 0$ we have $(cP)^\circ = \frac{1}{c} P^\circ$.*
(iii) *If $P \subseteq Q$, then $P^\circ \supseteq Q^\circ$.*
(iv) *It is $(P^\circ)^\circ = P$.*

The *Minkowski functional* μ_T, defined by

$$\mu_T(x) := \min\{t \geqslant 0 : x \in tT\},\ x \in \mathbb{R}^n,$$

determines a distance function, where we recover the Euclidean distance when T is the n-dimensional Euclidean unit ball. The *support function* h_T of T is given by

$$h_T(x) := \max\{\langle x, y \rangle : y \in T\},\ x \in \mathbb{R}^n.$$

The following lemma clarifies the connection between the Minkowski functional and the support function and will be useful throughout many proofs of the following sections:

Proposition 22 *Let $T \subset \mathbb{R}^n$ be a convex body.*

(i) *One has*

$$h_T(x) = \mu_{T^\circ}(x)$$

for all $x \in \mathbb{R}^n$.

(ii) *The following equivalence holds for all $x \in \mathbb{R}^n$: One has*

$$h_T(x) = \langle x, y \rangle \ \Leftrightarrow\ x \in N_T(y)$$

under the constraint $y \in \partial T$.

The next proposition collects properties of the Minkowski functional:

Proposition 23 *Let $S, T \subset \mathbb{R}^n$ be convex bodies.*

(i) *If T is additionally assumed to be strictly convex and if*

$$x, y \in \mathbb{R}^n \backslash \{0\} \ \text{ with } \ x \neq \lambda y\ \forall \lambda \in \mathbb{R},$$

then

$$\mu_{T^\circ}(x + y) < \mu_{T^\circ}(x) + \mu_{T^\circ}(y). \tag{6}$$

We note that, when just requiring convexity of T, one has $\leqslant$ in (6).

(ii) With $S \subseteq T$ we have $T^\circ \subseteq S^\circ$. This implies

$$\mu_{S^\circ}(x) \leqslant \mu_{T^\circ}(x) \quad \forall x \in \mathbb{R}^n.$$

(iii) For $c \neq 0$ we have

$$\mu_{T^\circ}(cx) = \mu_{(cT)^\circ}(x) = c\mu_{T^\circ}(x) \quad \forall x \in \mathbb{R}^n.$$

(iv) The map

$$\mu_{T^\circ} : (\mathbb{R}^n, |\cdot|) \to (\mathbb{R}_{\geqslant 0}, |\cdot|)$$

is continuous.

3 Properties of closed (K, T)-Minkowski billiard trajectories

We begin with the following lemma:

Lemma 31 *Let $T \subset \mathbb{R}^n$ be a strictly convex body and $p_i, p_j \in \partial T$. Then one has the equivalence*

$$q_i \neq q_j \Leftrightarrow N_T(q_i) \cap N_T(q_j) = \{0\}.$$

Proof From

$$N_T(q_i) \cap N_T(q_j) = \{0\},$$

together with

$$\{0\} \subsetneqq N_T(q_i) \text{ and } \{0\} \subsetneqq N_T(q_j),$$

it directly follows $q_i \neq q_j$.

Let $q_i \neq q_j$. If there is a nonzero

$$n \in N_T(q_i) \cap N_T(q_j),$$

then it follows from the definition of strict convexity that

$$\langle n, z_1 - q_i \rangle < 0 \quad \forall z_1 \in T$$

and

$$\langle n, z_2 - q_j \rangle < 0 \quad \forall z_2 \in T.$$

Choosing $z_1 = q_j$ and $z_2 = q_i$ yields

$$0 > \langle n, q_j - q_i \rangle,$$

and therefore

$$0 > \langle n, q_i - q_j \rangle = -\langle n, q_j - q_i \rangle > 0,$$

a contradiction. Therefore, it follows

$$N_T(q_i) \cap N_T(q_j) = \{0\}.$$

The subsequent proposition clarifies the uniqueness of closed dual billiard trajectories.

Proposition 32 *Let $K, T \subset \mathbb{R}^n$ be convex bodies, where T is additionally assumed to be strictly convex. Let q be a closed strong (K,T)-Minkowski billiard trajectory. Then, the closed dual billiard trajectory p in T is uniquely determined.*

Proof Referring to (2), $p = (p_1, ..., p_m)$ fulfills

$$q_{j+1} - q_j \in N_T(p_j)$$

for all $j \in \{1, ..., m\}$. Using Lemma 31 (T is assumed to be strictly convex), this implies that $p_1, ..., p_m$ are uniquely determined.

Without requiring the strict convexity of T, p is not necessarily uniquely determined (cf. Example B in Section 6).

We proceed with the proof of Theorem 11, which clarifies the relationship between weak and strong Minkowski billiards.

Proof (Proof of Theorem 11) We first prove that every closed strong Minkowski billiard trajectory is a weak one. For that, let T be an arbitrary convex body in $\mathbb{R}^n$. Given a closed strong (K,T)-Minkowski billiard trajectory $q = (q_1, ..., q_m)$ together with its dual billiard trajectory $p = (p_1, ..., p_m)$. We let $n_K(q_1), ..., n_K(q_m)$ be the unit vectors in $N_K(q_1), ..., N_K(q_m)$ for which

$$p_{j+1} - p_j = -\mu_{j+1} n_K(q_{j+1}),\ \mu_{j+1} \geqslant 0,$$

holds for all $j \in \{1, ..., m\}$. Now, let $j \in \{1, ..., m\}$ be arbitrarily chosen and let H_j be the K-supporting hyperplane through q_j which is normal to $n_K(q_j)$. Then, one has

$$\langle q_j - q_j^*, p_j - p_{j-1} \rangle = 0$$

for all $q_j^* \in H_j$ (since $p_j - p_{j-1} = -\mu_j n_K(q_j)$) and therefore together with

$$q_j - q_{j-1} \in N_T(p_{j-1}) \ \text{ and } \ q_{j+1} - q_j \in N_T(p_j)$$

and Proposition 22 that

$$\begin{aligned}
\Sigma_j(q_j) =& \mu_{T^\circ}(q_j - q_{j-1}) + \mu_{T^\circ}(q_{j+1} - q_j) \\
=& \langle q_j - q_{j-1}, p_{j-1} \rangle + \langle q_{j+1} - q_j, p_j \rangle \\
=& \langle q_j - q_{j-1}, p_{j-1} \rangle + \langle q_{j+1} - q_j, p_j \rangle + \langle q_j - q_j^*, p_j - p_{j-1} \rangle \\
=& \langle q_j^* - q_{j-1}, p_{j-1} \rangle + \langle q_{j+1} - q_j^*, p_j \rangle \\
=& \langle q_j^* - q_{j-1}, p_{j-1}^* \rangle + \langle q_j^* - q_{j-1}, p_{j-1} - p_{j-1}^* \rangle \\
&+ \langle q_{j+1} - q_j^*, p_j^* \rangle + \langle q_{j+1} - q_j^*, p_j - p_j^* \rangle \\
=& \Sigma_j(q_j^*) + \langle q_j^* - q_{j-1}, p_{j-1} - p_{j-1}^* \rangle + \langle q_{j+1} - q_j^*, p_j - p_j^* \rangle
\end{aligned}$$

for all $q_j^* \in H_j$, where $p_j^*, p_{j-1}^* \in \partial T$ were chosen (possibly not uniquely) to fulfill

$$q_j^* - q_{j-1} \in N_T(p_{j-1}^*) \ \text{ and } \ q_{j+1} - q_j^* \in N_T(p_j^*). \tag{7}$$

By (7), it follows from the definition of the normal cone and $p_j, p_{j-1} \in \partial T$ together with the convexity of T that

$$\langle q_j^* - q_{j-1}, p_{j-1} - p_{j-1}^* \rangle \leqslant 0 \text{ and } \langle q_{j+1} - q_j^*, p_j - p_j^* \rangle \leqslant 0.$$

This implies

$$\Sigma_j(q_j) \leqslant \Sigma_j(q_j^*)$$

for all $q_j^* \in H_j$. We conclude that the polygonal curve segment (q_{j-1}, q_j, q_{j+1}) fulfills the weak Minkowski billiard reflection rule in q_j with respect to the K-supporting hyperplane H_j through q_j which is normal to $n_K(q_j)$. This eventually means that $(q_1, ..., q_m)$ is a closed weak (K, T)-Minkowski billiard trajectory.

We proceed by proving that for strictly convex body $T \subset \mathbb{R}^n$, every closed weak Minkowski billiard trajectory is a strong one. So, let $q = (q_1, ..., q_m)$ be a closed weak (K, T)-Minkowski billiard trajectory. We define a closed polygonal curve $p = (p_1, ..., p_m)$ by

$$q_{j+1} - q_j \in N_T(p_j). \tag{8}$$

We note that due to the strict convexity of T (cf. Lemma 31), p is uniquely determined. Let $j \in \{1, ..., m\}$ be arbitrarily chosen. Since there is a K-supporting hyperplane H_j through q_j such that q_j minimizes $\Sigma_j(q_j^*)$ over all $q_j^* \in H_j$, we conclude by Lagrange's multiplier theorem that there is a $\mu_j \in \mathbb{R}$ with

$$\nabla_{q_j^*} \Sigma_j(q_j^*)_{|q_j^* = q_j} = \mu_j n_{H_j},$$

where n_{H_j} is the outer unit vector normal to H_j. We note that the differentiability of Σ_j follows from the strict convexity of T–cf. the following calculation or, more basically, the duality between strict convexity of T and smoothness of T° (cf. [22, Theorem 11.13]). We calculate the left side:

$$\begin{aligned}
&\nabla_{q_j^*} \Sigma_j(q_j^*)_{|q_j^* = q_j} \\
&\overset{(a)}{=} \nabla_{q_j^*} \left(\langle q_j^* - q_{j-1}, p_{j-1}(q_j^*) \rangle + \langle q_{j+1} - q_j^*, p_j(q_j^*) \rangle \right)_{|q_j^* = q_j} \\
&\overset{(b)}{=} \left(\partial_{q_j^*, i} \left[\langle q_j^* - q_{j-1}, p_{j-1}(q_j^*) \rangle + \langle q_{j+1} - q_j^*, p_j(q_j^*) \rangle \right]_{|q_j^* = q_j} \right)_{i \in \{1, ..., n\}} \\
&\overset{(c)}{=} \Bigg(\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big[\langle q_j + \varepsilon e_i - q_{j-1}, p_{j-1}(q_j + \varepsilon e_i) \rangle - \langle q_j - q_{j-1}, p_{j-1} \rangle \\
&\qquad + \langle q_{j+1} - q_j - \varepsilon e_i, p_j(q_j + \varepsilon e_i) \rangle - \langle q_{j+1} - q_j, p_j \rangle \Big] \Bigg)_{i \in \{1, ..., n\}} \\
&= \Bigg(\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big[\langle q_j - q_{j-1}, p_{j-1}(q_j + \varepsilon e_i) - p_{j-1} \rangle + \langle q_{j+1} - q_j, p_j(q_j + \varepsilon e_i) - p_j \rangle \\
&\qquad + \langle \varepsilon e_i, p_{j-1}(q_j + \varepsilon e_i) - p_j(q_j + \varepsilon e_i) \rangle \Big] \Bigg)_{i \in \{1, ..., n\}} \\
&\overset{(d)}{=} \left(\lim_{\varepsilon \to 0} \langle e_i, p_{j-1}(q_j + \varepsilon e_i) - p_j(q_j + \varepsilon e_i) \rangle \right)_{i \in \{1, ..., n\}}
\end{aligned}$$

$$\overset{(e)}{=} (\langle e_i, p_{j-1} - p_j \rangle)_{i \in \{1,\dots,n\}}$$
$$= p_{j-1} - p_j,$$

where in equality (a), for every $q_j^* \in \mathbb{R}^n$ $p_{j-1}(q_j^*)$, $p_j(q_j^*)$ (here p_{j-1} and p_j are acting as functions) are the boundary points of T (uniquely determined for $q_j^* \in \mathbb{R}^n$ near q_j due to the strict convexity of T and Lemma 31) fulfilling

$$q_j^* - q_{j-1} \in N_T(p_{j-1}(q_j^*)) \text{ and } q_{j+1} - q_j^* \in N_T(p_j(q_j^*)), \tag{9}$$

where we note

$$p_{j-1}(q_j) = p_{j-1} \text{ and } p_j(q_j) = p_j,$$

in equality (b), by $\partial_{q_j^*,i}$ we denote the i-th partial derivative with respect to q_j^*, and in equality (c), by e_i we denote the i-th standard unit vector in $\mathbb{R}^n$. In equality (d), we used

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \langle q_j - q_{j-1}, p_{j-1}(q_j + \varepsilon e_i) - p_{j-1} \rangle = 0 \tag{10}$$

for all $i \in \{1, ..., n\}$.

Indeed, if

$$\dim N_T(p_{j-1}) = 1, \tag{11}$$

then

$$\lim_{\varepsilon \to 0} \frac{p_{j-1}(q_j + \varepsilon e_i) - p_{j-1}}{\varepsilon}$$

is a tangent vector at ∂T in p_j and therefore

$$\langle q_j - q_{j-1}, \lim_{\varepsilon \to 0} \frac{p_{j-1}(q_j + \varepsilon e_i) - p_{j-1}}{\varepsilon} \rangle = 0,$$

and consequently

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \langle q_j - q_{j-1}, p_{j-1}(q_j + \varepsilon e_i) - p_{j-1} \rangle \\ =&\langle q_j - q_{j-1}, \lim_{\varepsilon \to 0} \frac{p_{j-1}(q_j + \varepsilon e_i) - p_{j-1}}{\varepsilon} \rangle \\ =&0. \end{aligned}$$

If

$$\dim N_T(p_{j-1}) > 1 \text{ and } q_j - q_{j-1} \in \mathring{N}_T(p_{j-1}), \tag{12}$$

then it follows

$$p_{j-1}(q_j + \varepsilon e_i) - p_{j-1} = p_{j-1} - p_{j-1} = 0$$

for $|\varepsilon|$ small and therefore (10). If

$$\dim N_T(p_{j-1}) > 1 \text{ and } q_j - q_{j-1} \in \partial N_T(p_{j-1}),$$

then for $\varepsilon > 0$ (10) follows from the argument either made for the case (11) or for the case (12). Similarly, for $\varepsilon < 0$ (10) follows from the argument either made for the case (11) or for the case (12).

By similar reasoning, we derive

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \langle q_{j+1} - q_j, p_j(q_j + \varepsilon e_i) - p_j \rangle = 0 \text{ for all } i \in \{1, ..., n\}.$$

In equality (e), we applied the continuity (which holds due to the strict convexity of T) of the functions p_j and p_{j-1} defined by (9). Therefore, we conclude

$$p_j - p_{j-1} = -\mu_j n_{H_j}. \tag{13}$$

It remains to show $\mu_j \geqslant 0$. For that, scalar multiplication of (13) by n_{H_j} implies

$$\langle p_j - p_{j-1}, n_{H_j} \rangle = -\mu_j.$$

From

$$\langle q_j - q_{j-1}, n_{H_j} \rangle \geqslant 0 \text{ and } \langle q_{j+1} - q_j, n_{H_j} \rangle \leqslant 0$$

together with

$$q_j - q_{j-1} \in N_T(p_{j-1}) \text{ and } q_{j+1} - q_j \in N_T(p_j),$$

it follows from the convexity of T that

$$\langle p_j - p_{j-1}, n_{H_j} \rangle \leqslant 0,$$

and therefore $\mu_j \geqslant 0$.

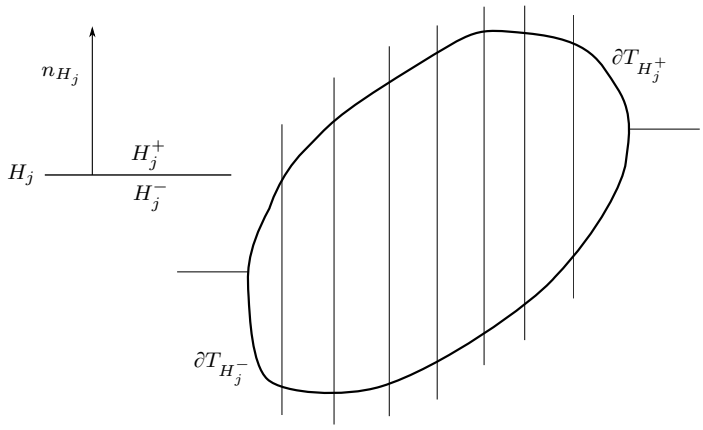

Fig. 4 Illustration of $\partial T_{H_j^+}$ and $\partial T_{H_j^-}$. By H_j^- and H_j^+ we denote the closed half-spaces of $\mathbb{R}^n$ bounded by H_j.

Indeed,

$$\langle q_j - q_{j-1}, n_{H_j} \rangle \geqslant 0 \text{ together with } q_j - q_{j-1} \in N_T(p_{j-1})$$

implies that

$$p_{j-1} \in \{p' \in \partial T : \langle n, n_{H_j} \rangle \geqslant 0 \ \forall n \in N_T(p')\} =: \partial T_{H_j^+}, \tag{14}$$

and

$$\langle q_{j+1} - q_j, n_{H_j} \rangle \leqslant 0 \ \text{ together with } \ q_{j+1} - q_j \in N_T(p_j)$$

implies

$$p_j \in \{p' \in \partial T : \langle n, n_{H_j} \rangle \leqslant 0 \ \forall n \in N_T(p')\} =: \partial T_{H_j^-}. \tag{15}$$

If $\mu_j < 0$, i.e., $p_j - p_{j-1}$ is a positive multiple of n_{H_j}, then it follows from the strict convexity of T (cf. Figure 4) that

$$p_j \in \partial T_{H_j^+} \ \text{ and } \ p_{j-1} \in \partial T_{H_j^-},$$

a contradiction to (14) and (15). Therefore it follows $\mu_j \geqslant 0$.

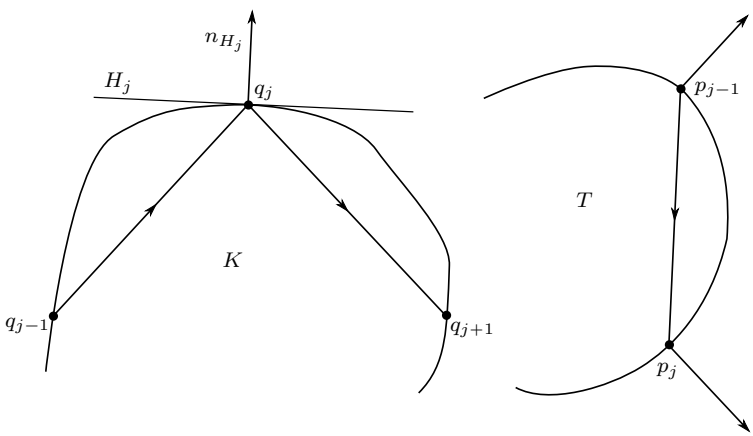

Fig. 5 We have $q_j - q_{j-1} \in N_T(p_{j-1})$, $q_{j+1} - q_j \in N_T(p_j)$ and $p_j - p_{j-1} = -\mu_j n_{H_j}$, $\mu_j \geqslant 0$.

If we define

$$n_K(q_j) := n_{H_j} \quad \forall j \in \{1, ..., m\},$$

then the pair (q, p) fulfills (2) by referring to (8) and (13).

The following rather obvious proposition is needed for the proof of Theorem 13 and 14. It follows immediately from within the proof of Theorem 11 when T is additionally required to be smooth.

Proposition 33 *Let $K, T \subset \mathbb{R}^n$ be convex bodies, where T is additionally assumed to be strictly convex and smooth. Let $q = (q_1, ..., q_m)$ be a closed (K,T)-Minkowski billiard trajectory. Then, for every $j \in \{1, ..., m\}$, there is only one K-supporting hyperplane through q_j for which the weak Minkowski billiard reflection rule in q_j is satisfied.*

Before we prove this proposition, it is important to note that the combination of Proposition 32 and Theorem 11 does not imply this statement. That is because the vertices of the closed dual billiard trajectory can coincide, and as consequence, condition (3) may not determine a unique hyperplane (cf. Example C in Section 6). We will see that the smoothness of T guarantees that the closed dual billiard trajectory is a closed polygonal curve (cf. Footnote 1).

Proof (Proof of Proposition 33) Since q is a closed (K,T)-Minkowski billiard trajectory, due to the proof of Theorem 11, there is a hyperplane H_j through q_j and an outer unit vector n_{H_j} normal to H_j such that

$$\nabla_{q_j^*}\Sigma_j(q_j^*)|_{q_j^*=q_j} = p_{j-1} - p_j = \mu_j n_{H_j}, \tag{16}$$

where $\mu_j \geqslant 0$ and p_{j-1} and p_j are uniquely determined by

$$q_{j+1} - q_j \in N_T(p_j) \text{ and } q_j - q_{j-1} \in N_T(p_{j-1}).$$

We aim to show that H_j, respectively n_{H_j}, is uniquely determined. Against the background of (16), it would be enough to show that $\mu_j > 0$.

If $\mu_j = 0$, then $p_{j-1} = p_j$. But this implies that $q_{j+1} - q_j$ and $q_j - q_{j-1}$ are in the same normal cone, while they are not parallel (since q_j is not on the line connecting q_{j-1} and q_{j+1}). Therefore, there are two linearly independent nonzero vectors in $N_T(p_j) = N_T(p_{j-1})$, a contradiction to the smoothness of T.

Consequently, it follows $\mu_j > 0$ and therefore the uniquess of n_{H_j} and H_j.

The statement of Proposition 33 is not true without requiring T to be both strictly convex and smooth. For the necessity of the strict convexity, we refer to Example B, for the necessity of the smoothness, we refer to Example C (cf. Section 6).

Its notion suggests that dual billiard trajectories are in fact billiard trajectories. Indeed, the following Propositions 34 and 35 show that for strictly convex and smooth body $T \subset \mathbb{R}^n$ the closed dual billiard trajectory of a closed (K,T)-Minkowski billiard trajectory q is a closed $(T,-K)$-Minkowski billiard trajectory p with

$$\ell_{-K}(p) = \ell_T(q).$$

For the case T is a strictly convex body in $\mathbb{R}^n$, this also implies that the ℓ_T-length of ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectories equals the ℓ_{-K}-length of ℓ_{-K}-minimizing closed $(T,-K)$-Minkowski billiard trajectories[4]. Later, we will use this fact for the proof of Theorem 14.

Proposition 34 *Let $K,T \subset \mathbb{R}^n$ be convex bodies. Let $q = (q_1,...,q_m)$ be a closed (K,T)-Minkowski billiard trajectory with closed dual billiard trajectory $p = (p_1,...,p_m)$ in T. Then, we have*

$$\ell_T(q) = \ell_{-K}(p).$$

[4] The existence of these minimums will be proved in Theorem 312 under the condition of strict convexity of T and in [23, Theorem 2.2] for the general case.

Proof By definition of the Minkowski billiard reflection rule, we have

$$q_{j+1} - q_j \in N_T(p_j) \text{ and } p_{j+1} - p_j \in -N_K(q_{j+1}).$$

for all $j \in \{1, ..., m\}$. By recalling Proposition 22, this implies

$$\mu_{T^\circ}(q_{j+1} - q_j) = h_T(q_{j+1} - q_j) = \langle q_{j+1} - q_j, p_j \rangle$$

and

$$\mu_{K^\circ}(p_j - p_{j+1}) = h_K(p_j - p_{j+1}) = \langle p_j - p_{j+1}, q_{j+1} \rangle$$

for all $j \in \{1, ..., m\}$. Then, we compute

$$\begin{aligned}
\ell_T(q) = \sum_{j=1}^{m} \mu_{T^\circ}(q_{j+1} - q_j) = \sum_{j=1}^{m} \langle q_{j+1} - q_j, p_j \rangle \\
=&(\langle q_2, p_1 \rangle - \langle q_1, p_1 \rangle) + (\langle q_3, p_2 \rangle - \langle q_2, p_2 \rangle) + ... \\
&... + (\langle q_m, p_{m-1} \rangle - \langle q_{m-1}, p_{m-1} \rangle) + (\langle q_{m+1}, p_m \rangle - \langle q_m, p_m \rangle) \\
=& - \langle q_1, p_1 \rangle + (\langle q_2, p_1 \rangle - \langle q_2, p_2 \rangle) + (\langle q_3, p_2 \rangle - \langle q_3, p_3 \rangle) + ... \\
&... + (\langle q_{m-1}, p_{m-2} \rangle - \langle q_{m-1}, p_{m-1} \rangle) + (\langle q_m, p_{m-1} \rangle - \langle q_m, p_m \rangle) + \langle q_{m+1}, p_m \rangle \\
=& \sum_{j=1}^{m} \langle p_j - p_{j+1}, q_{j+1} \rangle = \sum_{j=1}^{m} \mu_{K^\circ}(p_j - p_{j+1}) = \sum_{j=1}^{m} \mu_{-K^\circ}(p_{j+1} - p_j) \\
=& \sum_{j=1}^{m} \mu_{(-K)^\circ}(p_{j+1} - p_j) \\
=&\ell_{-K}(p),
\end{aligned}$$

where we used $q_{m+1} = q_1$, $p_{m+1} = p_1$ and the property

$$\mu_{K^\circ}(-\cdot) = \mu_{-K^\circ}(\cdot)$$

of the Minkowski functional utilizing Proposition 23(iii) and

$$(-K)^\circ = -K^\circ$$

(cf. Proposition 21).

Proposition 35 *Let $K, T \subset \mathbb{R}^n$ be convex bodies and T is additionally assumed to be strictly convex and smooth. Let $q = (q_1, ..., q_m)$ be a closed (K, T)-Minkowski billiard trajectory with its closed dual billiard trajectory $p = (p_1, ..., p_m)$ in T. Then, p is a closed $(T, -K)$-Minkowski billiard trajectory with*

$$-q^{+1} := (-q_2, ..., -q_m, -q_1)$$

as closed dual billiard trajectory on $-K$.

Proof Since the pair (q,p) fulfills (2), $p_1,...,p_m$ are uniquely determined (cf. Lemma 31) by the condition

$$q_{j+1}-q_j \in N_T(p_j) \quad \forall j \in \{1,...,m\}. \tag{17}$$

Since $q_1,...,q_m$ satisfy $q_j \neq q_{j+1}$ for all $j \in \{1,...,m\}$ and q_j is not contained in the line segment connecting q_{j-1} and q_{j+1} for all $j \in \{1,...,m\}$ (cf. Footnote 1),

$$q_2-q_1,...,q_m-q_{m-1},q_1-q_m \tag{18}$$

are nonzero and satisfy

$$q_j-q_{j-1} \nparallel q_{j+1}-q_j \quad \forall j \in \{1,...,m\}.$$

Then, (17) together with the strict convexity and smoothness of T implies that $p_1,...,p_m$ satisfy $p_j \neq p_{j+1}$ for all $j \in \{1,...,m\}$ and p_j is not contained in the line segment connecting p_{j-1} and p_{j+1} for all $j \in \{1,...,m\}$.

This implies

$$\begin{cases} p_{j+1}-p_j = -\mu_{j+1}n_K(q_{j+1}) \in -N_K(q_{j+1}) = N_{-K}(-q_{j+1}) = N_{-K}(-q_j^{+1}) \\ (-q_{j+1}^{+1})-(-q_j^{+1}) = (-q_{j+2})-(-q_{j+1}) \in -N_T(p_{j+1}) \end{cases} \tag{19}$$

for all $j \in \{1,...,m\}$, where we used

$$N_{-K}(-q_{j+1}) = -N_K(q_{j+1}) \quad \forall j \in \{1,...,m\}.$$

From (19), we conclude that the pair $(p,-q^{+1})$ fulfills (2) for the configuration $(T,-K)$. Therefore, p is a closed $(T,-K)$-Minkowski billiard trajectory with $-q^{+1}$ as its closed dual billiard trajectory on $-K$.

In order to show the necessity of the smoothness of T in Proposition 35, we refer to Example C (cf. Section 6)–there, one can construct a closed polygonal curve p, but which does not satisfy $p_j \neq p_{j+1}$ for all j.

We have the following proposition as generalization of [17, Proposition 2.3] to the Minkowski/Finsler setting:

Proposition 36 *Let $K,T \subset \mathbb{R}^n$ be convex bodies. Let $q=(q_1,...,q_m)$ be a closed weak (K,T)-Minkowski billiard trajectory and $V \subseteq \mathbb{R}^n$ an affine subspace such that $K \cap V$ is an affine section of K containing q. Then, q is a closed weak $(K \cap V,T)$-Minkowski billiard trajectory.*[5]

Proof Since q is a closed weak (K,T)-Minkowski billiard trajectory, there are K-supporting hyperplanes $H_1,...,H_m$ through $q_1,...,q_m$ such that q_j minimizes

$$\mu_{T^\circ}(\bar{q}_j-q_{j-1})+\mu_{T^\circ}(q_{j+1}-\bar{q}_j) \tag{20}$$

over all $\bar{q}_j \in H_j$ for all $j \in \{1,...,m\}$. Since $K \cap V$ contains q it follows that q_j minimizes (20) over all $\bar{q}_j \in H_j \cap V$ for all $j \in \{1,...,m\}$. This implies that q is a closed weak $(K \cap V,T)$-Minkowski billiard trajectory.

[5] We notice that the dimension of $K \cap V$ is possibly smaller than the dimension of T. For these cases, we consider that Definiton 11 can be easily extended to covex bodies T which are allowed to have dimension greater n.

Clearly, in general, the converse is not true: We can imagine an affine section $K \cap V$ of K that can be translated into $\mathring{K}$. Then, every closed weak $(K \cap V, T)$-Minkowski billiard trajectory q can be translated into $\mathring{K}$. But in Proposition 39, we will prove that $q \in F(K)$, a contradiction.

In [17, Examples A, B, C, and D], we have seen (for T equals the Euclidean unit ball) that in general the length minimality of a closed weak (K, T)-Minkowski billiard trajectory is not invariant under going to (inclusion minimal) affine sections of K containing the billiard trajectory. This billiard trajectory may not even locally minimize the length of closed polygonal curves in $F(K \cap V)$. We note that these examples can be easily generalized to settings when the weak Minkowski billiard reflection rule is not necessarily governed by the Euclidean unit ball.

The next two propositions make a statement concerning the positional relationship of the hyperplanes which determine the weak Minkowski billiard reflection rule.

Proposition 37 *Let $K, T \subset \mathbb{R}^n$ be convex bodies. Let $q = (q_1, ..., q_m)$ be a closed (K, T)-Minkowski billiard trajectory with respect to $H_1, ..., H_m$. Then, we have*

$$0 \in \operatorname{conv}\{n_K(q_1), ..., n_K(q_m)\},$$

where $n_K(q_1), ..., n_K(q_m)$ are the outer unit vectors normal to $H_1, ..., H_m$.

If T is assumed to be smooth and if we denote by $\widetilde{U}$ the convex cone spanned by

$$n_K(q_1), ..., n_K(q_m),$$

then $\widetilde{U}$ is a linear subspace of $\mathbb{R}^n$ with dimension less or equal than $m - 1$.

Proof Let $p = (p_1, ..., p_m)$ be a closed dual billiard trajectory of q. Then, there are $\mu_1, ..., \mu_m \geqslant 0$ with

$$p_{j+1} - p_j = -\mu_{j+1} n_K(q_{j+1}) \quad \forall j \in \{1, ..., m\}. \tag{21}$$

We first consider the case when $\mu_j > 0$ for all $j \in \{1, ..., m\}$. Then, we define

$$s_j := \frac{\mu_{j+1}}{\mu_1 + ... + \mu_m} \quad \forall j \in \{1, ..., m\}$$

and conclude

$$\sum_{j=1}^{m} s_j \frac{1}{\mu_{j+1}} (p_{j+1} - p_j) = \frac{1}{\mu_1 + ... + \mu_m} \sum_{j=1}^{m} (p_{j+1} - p_j) = 0 \tag{22}$$

while

$$\sum_{j=1}^{m} s_j = \sum_{j=1}^{m} \frac{\mu_{j+1}}{\mu_1 + ... + \mu_m} = 1. \tag{23}$$

This implies by the definition of the convex hull that

$$\begin{aligned} 0 &\in \left\{ \sum_{j=1}^{m} \widetilde{s}_j \frac{1}{\mu_{j+1}}(p_{j+1}-p_j) : \sum_{j=1}^{m} \widetilde{s}_j = 1,\ \widetilde{s}_j \geqslant 0 \right\} \\ &= \operatorname{conv}\left\{ \frac{1}{\mu_2}(p_2-p_1), ..., \frac{1}{\mu_m}(p_m-p_{m-1}), \frac{1}{\mu_1}(p_1-p_m) \right\} \end{aligned} \tag{24}$$

and therefore, together with (21),

$$0 \in \operatorname{conv}\{-n_K(q_1), ..., -n_K(q_m)\}$$

and consequently

$$0 \in \operatorname{conv}\{n_K(q_1), ..., n_K(q_m)\}. \tag{25}$$

If $\mu_j = 0$ for some $j \in \{1, ..., m\}$ (for all is impossible), then, by (21), also

$$p_j - p_{j-1} = 0$$

for all these j. But then, the vector corresponding to $p_j - p_{j-1}$ can be removed from within the set of vectors building the convex hull in (24) without influencing (22) and (23). Therefore, 0 is in the convex hull of the nonzero

$$\frac{1}{\mu_{j+1}}(p_{j+1}-p_j),$$

what implies that 0 is in the convex hull of the associated unit normal vectors $n_K(q_{j+1})$. But the latter convex hull is subset of

$$\operatorname{conv}\{n_K(q_1), ..., n_K(q_m)\}.$$

Therefore, we derive (25).

Let us assume T is smooth. Then, this implies

$$p_{j+1} - p_j \neq 0 \quad \forall j \in \{1, ..., m\} \tag{26}$$

(cf. Proposition 33). It is

$$(p_2 - p_1) + ... + (p_m - p_{m-1}) + (p_1 - p_m) = 0.$$

For

$$s_j := \frac{1}{m} \quad \forall j \in \{1, ..., m\},$$

we also have

$$s_1(p_2 - p_1) + ... + s_{m-1}(p_m - p_{m-1}) + s_m(p_1 - p_m) = 0.$$

Since

$$\sum_{j=1}^{m} s_j = 1 \ \text{ and } \ s_j \neq 0 \ \ \forall j \in \{1, ..., m\},$$

it follows that 0 lies in the relative interior of

$$\operatorname{conv}\{p_2 - p_1, ..., p_m - p_{m-1}, p_1 - p_m\}.$$

But this implies that the convex cone spanned by

$$p_2 - p_1, ..., p_m - p_{m-1}, p_1 - p_m \tag{27}$$

and therefore, by (21) and (26), also $\widetilde{U}$ is a linear subspace of $\mathbb{R}^n$. Obviously, then $\widetilde{U}$ is the inclusion minimal linear subspace containing the vectors (27) and consequently has dimension less or equal than $m-1$.

The necessity of the smoothness of T for the second statement follows by referring to Example C. In this case, the convex cone spanned by $n_K(q_1), n_K(q_2), n_K(q_3)$ is not a linear subspace of $\mathbb{R}^2$. Furthermore, for the weaker situation of closed weak (K,T)-Minkowski billiard trajectory, one can show–having in mind Theorem 11–the necessity of the strict convexity of T for the second statement by referring to Example D–and also for the first statement by referring to Example E (cf. Section 6 for the three examples).

Proposition 38 *Let $K, T \subset \mathbb{R}^n$ be convex bodies. Let $q = (q_1, ..., q_m)$ be a closed (K,T)-Minkowski billiard trajectory with respect to $H_1, ..., H_m$ and let U be the inclusion minimal linear subspace of $\mathbb{R}^n$ containing the outer unit vectors $n_K(q_1), ..., n_K(q_m)$ which are normal to $H_1, ..., H_m$. We denote by $H_1^+, ..., H_m^+$ the closed half-spaces of $\mathbb{R}^n$ which are bounded by $H_1, ..., H_m$ and contain K. Further, let W be the orthogonal complement to U in $\mathbb{R}^n$. Then, we can write*

$$H_j = (H_j \cap U) \oplus W \quad \text{and} \quad H_j^+ = (H_j^+ \cap U) \oplus W$$

for all $j \in \{1, ..., m\}$ and have that

$$\bigcap_{j=1}^{m} \left(H_j^+ \cap U\right) \text{ is nearly bounded in } U, \quad \bigcap_{j=1}^{m} H_j^+ \text{ is nearly bounded in } \mathbb{R}^n.$$

If T is assumed to be smooth, then U coincides with the convex cone spanned by $n_K(q_1), ..., n_K(q_m)$ and we have that

$$\bigcap_{j=1}^{m} \left(H_j^+ \cap U\right) \text{ is bounded in } U, \quad \bigcap_{j=1}^{m} H_j^+ \text{ is nearly bounded in } \mathbb{R}^n.$$

Proof Since U is a linear subspace of $\mathbb{R}^n$ containing $n_K(q_1), ..., n_K(q_m)$, we can write

$$H_j = (H_j \cap U) \oplus W \quad \text{and} \quad H_j^+ = \left(H_j^+ \cap U\right) \oplus W. \tag{28}$$

Let $\widetilde{U}$ be the convex cone spanned by $n_K(q_1), ..., n_K(q_m)$. By Proposition 37, we have that

$$0 \in \operatorname{conv}\{n_K(q_1), ..., n_K(q_m)\}. \tag{29}$$

Now, 0 either is an interior point or a boundary point (both with respect to U) of the convex hull in (29).

If 0 is a boundary point (with respect to U) of the convex hull in (29), then $\widetilde{U}$ is subset of a $\widetilde{U}$-supporting closed half-space $H_{\widetilde{U}}$ of U while

$$\partial H_{\widetilde{U}} \cap \widetilde{U}$$

contains a selection of unit vectors

$$\{n_K(q_{i_1}), ..., n_K(q_{i_k})\} \subseteq \{n_K(q_1), ..., n_K(q_m)\}, \quad k \leqslant m,$$

with 0 in the relative interior of

$$\operatorname{conv}\{n_K(q_{i_1}), ..., n_K(q_{i_k})\} \subseteq \partial H_{\widetilde{U}} \cap \widetilde{U}.$$

We denote by $H_{i_1}, ..., H_{i_k}$ the associated K-supporting hyperplanes through $q_{i_1}, ..., q_{i_k}$ which are normal to $n_K(q_{i_1}), ..., n_K(q_{i_k})$. It follows that

$$\bigcap_{l=1}^{k} \left(H_{i_l}^+ \cap U\right) \tag{30}$$

intersected with the convex cone spanned by the vectors $n_K(q_{i_1}), ..., n_K(q_{i_k})$ is bounded in $\partial H_{\widetilde{U}}$. We denote this intersection by I. Then, we can write

$$\bigcap_{l=1}^{k} \left(H_{i_l}^+ \cap U\right) = I \oplus I^{\perp_U},$$

where by $I^{\perp_U}$ we denote the orthogonal complement in U to the inclusion minimal linear subspace of U containing I. Clearly, the boundedness of I in $\partial H_{\widetilde{U}}$ implies the nearly boundedness of $I \oplus I^{\perp_U}$ in U: Because of the boundedness of I in $\partial H_{\widetilde{U}}$, there are two parallel hyperplanes G and $G + c$, $c \in \partial H_{\widetilde{U}}$, in $\partial H_{\widetilde{U}}$ such that I lies in-between. Then, $I \oplus I^{\perp_U}$ lies between the two hyperplanes

$$G \oplus I^{\perp_U} \ \text{ and } \ (G + c) \oplus I^{\perp_U} \ \text{ in } \ U, \tag{31}$$

i.e., it is nearly bounded in U. Then, using (28),

$$\bigcap_{l=1}^{k} H_{i_l}^+ = \bigcap_{l=1}^{k} \left(\left(H_{i_l}^+ \cap U\right) \oplus W\right) = \left(\bigcap_{l=1}^{k} \left(H_{i_l}^+ \cap U\right)\right) \oplus W$$

lies between the two parallel hyperplanes

$$\left(G \oplus I^{\perp_U}\right) \oplus W \ \text{ and } \ \left((G + c) \oplus I^{\perp_U}\right) \oplus W \ \text{ in } \ \mathbb{R}^n, \tag{32}$$

i.e., it is nearly bounded in $\mathbb{R}^n$. This implies that

$$\bigcap_{j=1}^{m} \left(H_j^+ \cap U\right) \subseteq \bigcap_{l=1}^{k} \left(H_{i_l}^+ \cap U\right)$$

lies between the two hyperplanes in (31) and

$$\begin{aligned}\bigcap_{j=1}^{m} H_j^+ &= \bigcap_{j=1}^{m} \left((H_j^+ \cap U) \oplus W \right) \\ &= \left(\bigcap_{j=1}^{m} (H_j^+ \cap U) \right) \oplus W \\ &\subseteq \bigcap_{l=1}^{k} H_{i_l}^+ \\ &= \left(\bigcap_{l=1}^{k} (H_{i_l}^+ \cap U) \right) \oplus W\end{aligned}$$

between the two hyperplanes in (32), i.e., they are nearly bounded in U and $\mathbb{R}^n$, respectively.

If 0 is an interior point of the convex hull in (29), i.e., when $\widetilde{U}$ coincides with U–and by Proposition 37 this is also the case when T is assumed to be smooth–, then this directly implies that

$$\bigcap_{j=1}^{m} (H_j^+ \cap U) \tag{33}$$

is bounded in U. From this, we conclude that there are parallel hyperplanes H and $H + d$, $d \in U$, in U such that (33) lies in-between. With (28), this implies that

$$\bigcap_{j=1}^{m} H_j^+ = \bigcap_{j=1}^{m} \left((H_j^+ \cap U) \oplus W \right) = \left(\bigcap_{j=1}^{m} (H_j^+ \cap U) \right) \oplus W$$

lies between the parallel hyperplanes

$$H \oplus W \quad \text{and} \quad (H + d) \oplus W \quad \text{in} \quad \mathbb{R}^n$$

and therefore is nearly bounded in $\mathbb{R}^n$.

Proposition 39 *Let $K, T \subset \mathbb{R}^n$ be convex bodies. Let $q = (q_1, ..., q_m)$ be a closed (K, T)-Minkowski billiard trajectory with closed dual billiard trajectory $p = (p_1, ..., p_m)$. Then, we have*

$$q \in F(K) \quad \textit{and} \quad p \in F(T).$$

Proof Let $H_1, ..., H_m$ be the K-supporting hyperplanes through $q_1, ..., q_m$ which are associated to the Minkowski billiard reflection rule and let $H_1^+, ..., H_m^+$ the closed half-spaces of $\mathbb{R}^n$ containing K and which are bounded by $H_1, ..., H_m$. By Proposition 38, we conclude that

$$H_1^+ \cap ... \cap H_m^+$$

is nearly bounded in $\mathbb{R}^n$. Then, from [17, Lemma 2.1(ii)] it follows that

$$\{q_1, ..., q_m\} \in F(K),$$

i.e., $q \in F(K)$.

By the definition of the Minkowski billiard reflection rule, there are factors $\lambda_1, ..., \lambda_m > 0$ (which are > 0 due to Footnote 1) and unit vectors

$$n_T(p_1), ..., n_T(p_m) \text{ in } N_T(p_1), ..., N_T(p_m)$$

such that

$$q_{j+1} - q_j = \lambda_j n_T(p_j) \quad \forall j \in \{1, ..., m\}.$$

Since q is closed, we justify

$$0 \in \operatorname{conv}\{n_T(p_1), ..., n_T(p_m)\} \tag{34}$$

in a similar way to the proof of Proposition 37. Let U' be the inclusion minimal linear subspace of $\mathbb{R}^n$ containing $n_T(p_1), ..., n_T(p_m)$. Then, as in the proof of Proposition 38, (34) implies that

$$H_1'^+ \cap ... \cap H_m'^+ \tag{35}$$

is nearly bounded in $\mathbb{R}^n$, where $H_1'^+, ..., H_m'^+$ are the closed half-spaces of $\mathbb{R}^n$ containing T and which are bounded by $H_1', ..., H_m'$ which are the T-supporting hyperplanes of $\mathbb{R}^n$ through $p_1, ..., p_m$ normal to $n_T(p_1), ..., n_T(p_m)$. By [17, Lemma 2.1(ii)], it follows from the nearly boundedness of (35) that

$$\{p_1, ..., p_m\} \in F(T),$$

i.e., $p \in F(T)$.

The first statement of Proposition 39, i.e., $q \in F(K)$, in general, is not true when q is just assumed to be a closed weak (K, T)-Minkowski billiard trajectory and T is not required to be strictly convex. To see this, we consider Example E (cf. Section 6).

Proposition 310 *Let $K, T \subset \mathbb{R}^n$ be convex bodies. Let $q = (q_1, ..., q_m)$ be a closed (K, T)-Minkowski billiard trajectory with respect to $H_1, ..., H_m$ and let U be the inclusion minimal linear subspace of $\mathbb{R}^n$ containing the outer unit vectors $n_K(q_1), ..., n_K(q_m)$ which are normal to $H_1, ..., H_m$. Then, there is a selection*

$$\{i_1, ..., i_{\dim U+1}\} \subseteq \{1, ..., m\} \tag{36}$$

such that

$$\{q_{i_1}, ..., q_{i_{\dim U+1}}\} \in F(K).$$

Proof For $m = \dim U + 1$, we can just apply Proposition 39 and nothing more is to prove. If $\dim U = n$, i.e., $U = \mathbb{R}^n$, then the claim follows immediately from Proposition 39 and [17, Lemma 2.1(i)] (cf. the equivalent expression below this Lemma).

Let

$$\dim U \leqslant \min\{n-1, m-2\}.$$

Proposition 38 implies, on the one hand, that we can write

$$H_j = (H_j \cap U) \oplus W \quad \text{and} \quad H_j^+ = (H_j^+ \cap U) \oplus W$$

for all $j \in \{1, ..., m\}$, where W is the orthogonal complement to U in $\mathbb{R}^n$ and $H_1^+, ..., H_m^+$ are the closed half-spaces of $\mathbb{R}^n$ containing K and which are bounded by $H_1, ..., H_m$, and, on the other hand, that

$$\bigcap_{j=1}^{m} \left(H_j^+ \cap U\right)$$

is nearly bounded in U. This implies by[17, Lemma 2.1(ii)] that

$$\pi_U(q) \in F\left(\bigcap_{j=1}^{m} \left(H_j^+ \cap U\right)\right),$$

where we denote by π_U the orthogonal projection onto U. Then, by [17, Lemma 2.1(i)], there is a selection

$$\{i_1, ..., i_{\dim U+1}\} \subset \{1, ..., m\}$$

such that

$$\left\{\pi_U(q_{i_1}), ..., \pi_U(q_{i_{\dim U+1}})\right\} \in F\left(\bigcap_{j=1}^{m} \left(H_j^+ \cap U\right)\right).$$

Referring again to [17, Lemma 2.1(ii)], there are $\bigcap_{j=1}^{m} \left(H_j^+ \cap U\right)$-supporting hyperplanes[6]

$$\widetilde{H}_1, ..., \widetilde{H}_{\dim U+1}$$

in U through

$$\pi_U(q_{i_1}), ..., \pi_U(q_{i_{\dim U+1}})$$

such that

$$\bigcap_{j=1}^{\dim U+1} \widetilde{H}_j^+$$

is nearly bounded in U with

$$\bigcap_{j=1}^{m} \left(H_j^+ \cap U\right) \subseteq \bigcap_{j=1}^{\dim U+1} \widetilde{H}_j^+,$$

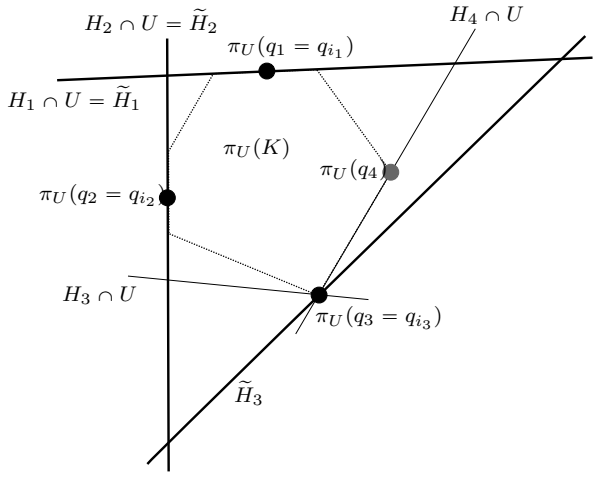

Fig. 6 Illustration of the selection of $\{\pi_U(q_{i_1}), \pi_U(q_{i_2}), \pi_U(q_{i_3})\}$ out of $\{\pi_U(q_1), \pi_U(q_2), \pi_U(q_3), \pi_U(q_4)\}$ and the choice of $\bigcap_{j=1}^m \left(H_j^+ \cap U\right)$-supporting hyperplanes $\widetilde{H}_1, \widetilde{H}_2, \widetilde{H}_3$ in U such that $\bigcap_{j=1}^3 \widetilde{H}_j^+$ is nearly bounded in U.

where $\widetilde{H}_j^+$ is the half-space bounded by $\widetilde{H}_j$ containing $\pi_U(K)$ for all $j \in \{1, ..., \dim U + 1\}$ (cf. Figure 6). Then, this implies that

$$\bigcap_{j=1}^{\dim U+1} \left(\widetilde{H}_j^+ \oplus W\right)$$

is nearly bounded in $\mathbb{R}^n$ with

$$K \subseteq \bigcap_{j=1}^{m} H_j^+ \subseteq \bigcap_{j=1}^{\dim U+1} \left(\widetilde{H}_j^+ \oplus W\right)$$

and

$$q_{i_j} \in \widetilde{H}_j \oplus W \quad \forall j \in \{1, ..., \dim U + 1\}.$$

By using [17, Lemma 2.1(ii)], this yields

$$\{q_{i_1}, ..., q_{i_{\dim U+1}}\} \in F(K).$$

We remark that the statement of Proposition 310, in general, is not true when requiring q just to be a closed polygonal curve in $F(K)$ (and not a closed (K, T)-Minkowski billiard trajectory). In [17], we gave a counterexample for T equals the Euclidean unit ball in $\mathbb{R}^n$.

In order to state/prove the upcoming Theorem 312, we recall that for a convex body $K \subset \mathbb{R}^n$, the set $F_{n+1}^{cp}(K)$ is the set of all closed polygonal curves

$$q = (q_1, ..., q_m) \in F(K)$$

[6] It does not necessarily have to be $\widetilde{H}_j = H_{i_j} \cap U$. Further, note that due to this special situation it is not necessary to proceed to an even finer selection of $\{i_1, ..., i_{\dim U+1}\}$ (and even if that were necessary, it would not impede the following argument).

with $m \leqslant n+1$.

Let (M,d) be a metric space and $P(M)$ the set of all nonempty compact subsets of M. We recall that $P(M)$ is a metric space together with the *Hausdorff metric* d_H which for $X, Y \in P(M)$ is defined by

$$d_H(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\right\}.$$

For the following Proposition 311, we denote by $\mathcal{C}(\mathbb{R}^n)$ the set of all convex bodies and by $cp(\mathbb{R}^n)$ the set of all closed polygonal curves in $\mathbb{R}^n$. For $K \in \mathcal{C}(\mathbb{R}^n)$, we will consider $(F(K), d_H)$ and $(F^{cp}_{n+1}(K), d_H)$ as well as $(cp(\mathbb{R}^n), d_H)$ as metric subspaces of the complete metric space $(P(\mathbb{R}^n), d_H)$ which is induced by the Euclidean space $(\mathbb{R}^n, |\cdot|)$.

Proposition 311 *(i) If $K, L \in \mathcal{C}(\mathbb{R}^n)$ with $K \subseteq L$, then*

$$F(L) \subseteq F(K) \quad \textit{and} \quad F^{cp}_{n+1}(L) \subseteq F^{cp}_{n+1}(K).$$

(ii) If $K \in \mathcal{C}(\mathbb{R}^n)$ and $c > 0$, then

$$F(cK) = cF(K) \quad \textit{and} \quad F^{cp}_{n+1}(cK) = cF^{cp}_{n+1}(K).$$

(iii) If $S, T \in \mathcal{C}(\mathbb{R}^n)$ with $S \subseteq T$ and $q \in cp(\mathbb{R}^n)$, then

$$\ell_S(q) \leqslant \ell_T(q).$$

(iv) If $T \in \mathcal{C}(\mathbb{R}^n)$, $q \in cp(\mathbb{R}^n)$ and $c > 0$, then

$$\ell_T(cq) = \ell_{cT}(q) = c\ell_T(q).$$

(v) If $T \in \mathcal{C}(\mathbb{R}^n)$, then

$$\ell_T : (cp(\mathbb{R}^n), d_H) \to (\mathbb{R}_{\geqslant 0}, |\cdot|)$$

is continuous.

(vi) If $q \in cp(\mathbb{R}^n)$, then

$$\Gamma_q : (\mathcal{C}(\mathbb{R}^n), d_H) \to (\mathbb{R}_{\geqslant 0}, |\cdot|), \quad \Gamma_q(C) = \ell_C(q),$$

is continuous.

Proof (i) If $F \in F(L)$, then F cannot be translated into $\mathring{L}$. With $K \subseteq L$, F also cannot be translated into $\mathring{K}$. Therefore, $F \in F(K)$. This means

$$F(L) \subseteq F(K).$$

Analogously, one argues

$$F^{cp}_{n+1}(L) \subseteq F^{cp}_{n+1}(K).$$

(ii) If $F \in F(cK)$, then F cannot be translated into $c\mathring{K}$. Scaling F and K by the factor $\frac{1}{c}$, we conclude that $\frac{1}{c}F$ cannot be translated into $\mathring{K}$. Therefore,

$$\frac{1}{c}F \in F(K),$$

and consequently $F \in cF(K)$. Analogously, we conclude that $F \in cF(K)$ implies $F \in F(cK)$. This finally implies

$$F(cK) = cF(K).$$

Analogously one argues

$$F_{n+1}^{cp}(cK) = cF_{n+1}^{cp}(K).$$

(iii) With $S \subseteq T$, we have $T^\circ \subseteq S^\circ$. Using Propositon 23(ii), this implies

$$\mu_{S^\circ}(x) \leqslant \mu_{T^\circ}(x) \quad \forall x \in \mathbb{R}^n.$$

This directly implies

$$\ell_S(q) \leqslant \ell_T(q).$$

(iv) From Proposition 23(iii), it follows

$$\mu_{T^\circ}(cx) = \mu_{(cT)^\circ}(x) = c\mu_{T^\circ}(x) \quad \forall x \in \mathbb{R}^n.$$

This directly implies

$$\ell_T(cq) = \ell_{cT}(q) = c\ell_T(q).$$

(v) It is enough to prove that

$$\mu_{T^\circ} : (\mathbb{R}^n, |\cdot|) \to (\mathbb{R}_{\geqslant 0}, |\cdot|)$$

is continuous. But this follows from Proposition 23(iv).

(vi) Let $(T_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{C}(\mathbb{R}^n)$ d_H-converging to $T \in \mathcal{C}(\mathbb{R}^n)$. This means for all $\varepsilon > 0$ that there is $j_0 = j_0(\varepsilon) \in \mathbb{N}$ with

$$(1-\varepsilon)T \subseteq T_j \subseteq (1+\varepsilon)T \tag{37}$$

for all $j \geqslant j_0$.
We consider the sequence

$$(\Gamma_q(T_j))_{j\in\mathbb{N}} = (\ell_{T_j}(q))_{j\in\mathbb{N}}.$$

Because of (37) and (iv)&(v), we have for $\varepsilon > 0$ and big enough $j_0 \in \mathbb{N}$

$$(1-\varepsilon)\ell_T(q) = \ell_{(1-\varepsilon)T}(q) \leqslant \ell_{T_j}(q) \leqslant \ell_{(1+\varepsilon)T}(q) = (1+\varepsilon)\ell_T(q)$$

for all $j \geqslant j_0$. For $\varepsilon \to 0$, this implies that

$$(\ell_{T_j}(q))_{j\in\mathbb{N}} \text{ converges to } \Gamma_q(T) = \ell_T(q).$$

Therefore, Γ_q is continuous.

For the next theorem, we denote by $M_{n+1}(K,T)$ the set of closed (K,T)-Minkowski billiard trajectories with at most $n+1$ bouncing points.

Theorem 312 *Let $K,T \subset \mathbb{R}^n$ be convex bodies, where T is additionally assumed to be strictly convex. Then, every ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory is an ℓ_T-minimizing element of $F^{cp}_{n+1}(K)$, and, conversely, every ℓ_T-minimizing element of $F^{cp}_{n+1}(K)$ can be translated in order to be an ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory.*

Especially, one has

$$\min_{q \in F^{cp}_{n+1}(K)} \ell_T(q) = \min_{q \in M_{n+1}(K,T)} \ell_T(q). \tag{38}$$

We remark that Theorem 312 is an existence result: In general, it guarantees the existence of ℓ_T-minimizing elements of $F^{cp}_{n+1}(K)$, and furthermore, under the condition of strict convexity of T, it guarantees the existence of ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectories.

We note that in [23, Theorem 2.2], we actually prove that (38) holds without the condition of strict convexity of T.

Proof (Proof of Theorem 312) It is sufficient to prove the following two points:

(i) Every closed (K,T)-Minkowski billiard trajectory is either in $F^{cp}_{n+1}(K)$ or there is a strictly ℓ_T-shorter closed polygonal curve in $F^{cp}_{n+1}(K)$.
(ii) Every ℓ_T-minimizing element of $F^{cp}_{n+1}(K)$ can be translated in order to be a closed (K,T)-Minkowski billiard tracjectory.

Ad (i): Let $q = (q_1, ..., q_m)$ be a closed (K,T)-Minkowski billiard trajectory. From Proposition 39, we conclude $q \in F(K)$. For $m \leqslant n+1$, we then have $q \in F^{cp}_{n+1}(K)$. If $m > n+1$, then, by [17, Lemma 2.1(i)], there is a selection

$$\{i_1, ..., i_{n+1}\} \subset \{1, ..., m\} \text{ with } i_1 < ... < i_{n+1}$$

such that the closed polygonal curve

$$(q_{i_1}, ..., q_{i_{n+1}})$$

is in $F^{cp}_{n+1}(K)$. Because of Proposition 23(i), we have

$$\ell_T((q_{i_1}, ..., q_{i_{n+1}})) < \ell_T(q).$$

Ad (ii): By looking only at those members of $F^{cp}_{n+1}(K)$ that lie in an n-dimensional ball $B^n_R(0)$ in $\mathbb{R}^n$ of sufficiently large radius $R > 0$ and which contains K, we get via the d_H-continuity of ℓ_T (cf. Proposition 311(v)) and a standard compactness argument, considering the identification between

$$\left(F^{cp}_{n+1}(K), d_H\right) \text{ and } \left(\left\{Q \in (\mathbb{R}^n)^{n+1} : Q \text{ cannot be translated into } \mathring{K}\right\}, ||\cdot||_2\right)$$

by identifying

$$(q_1, ..., q_m) \in F^{cp}_{n+1}(K) \text{ with } (q_1, ..., q_m, ..., q_m) \in (\mathbb{R}^n)^{n+1}$$

and

$$(q_1, ..., q_{n+1}) \in (\mathbb{R}^n)^{n+1} \text{ with } (q_1, ..., q_{n+1}) \in F^{cp}_{n+1}(K)$$

and the fact that

$$\left(F^{cp,*_R}_{n+1}(K) := \left\{q \in F^{cp}_{n+1}(K) : q \subset B^n_R(0)\right\}, d_H\right)$$

therefore can be proven to be a compact metric subspace of the complete metric space $(P(\mathbb{R}^n), d_H)$, that

$$F^{cp}_{n+1}(K)/\{\text{translations}\}$$

possesses at least one element of minimal ℓ_T-length, say $\Delta_{\min}$.

We show that there is a translate $\Delta'_{\min}$ of $\Delta_{\min}$ which is a closed (K,T)-Minkowski billiard tracjecory.

Indeed, $\Delta_{\min}$ as element of $F^{cp}_{n+1}(K)$ cannot be translated into $\mathring{K}$. Thus, with [17, Lemma 2.1(ii)] and the ℓ_T-minimality of $\Delta_{\min}$, there is a translate $\Delta'_{\min}$ of $\Delta_{\min}$ given by vertices $q'_1, ..., q'_m \in \partial K$, $m \leqslant n+1$, and there are K-supporting hyperplanes $H_1, ..., H_m$ through $q'_1, ..., q'_m$ such that

$$H^+_1 \cap ... \cap H^+_m$$

is nearly bounded in $\mathbb{R}^n$, where $H^+_1, ..., H^+_m$ are the closed half-spaces of $\mathbb{R}^n$ containing K and which are bounded by $H_1, ..., H_m$. Let $(q'_{j-1}, q'_j, q'_{j+1})$ be an arbitrary polygonal curve segment of $\Delta'_{\min}$. If this polygonal curve segment is not fulfilling the weak Minkowski billiard reflection rule with respect to H_j, meaning that q'_j is not minimizing

$$\Sigma_j(q'^*_j) = \mu_{T^\circ}(q'^*_j - q'_{j-1}) + \mu_{T^\circ}(q'_{j+1} - q'^*_j)$$

over all $q'^*_j \in H_j$, then we find a $\widetilde{q}_j \in H_j$, $\widetilde{q}_j \neq q'_j$, such that the ℓ_T-length of the polygonal curve segment $(q'_{j-1}, \widetilde{q}_j, q'_{j+1})$ is less than the ℓ_T-length of the polygonal curve segment $(q'_{j-1}, q'_j, q'_{j+1})$. We replace the polygonal curve segment $(q'_{j-1}, q'_j, q'_{j+1})$ within $\Delta'_{\min}$ by $(q'_{j-1}, \widetilde{q}_j, q'_{j+1})$. By [17, Lemma 2.1(ii)], the new closed polygonal curve

$$\widetilde{\Delta}'_{\min} = (q'_1, ..., q'_{j-1}, \widetilde{q}_j, q'_{j+1}, ..., q'_m)$$

still cannot be translated into $\mathring{K}$, while

$$\ell_T(\widetilde{\Delta}'_{\min}) < \ell_T(\Delta'_{\min}).$$

But this is a contradiction to the ℓ_T-minimality of $\Delta'_{\min}$. Therefore, every polygonal curve segment $(q'_{j-1}, q'_j, q'_{j+1})$ of $\Delta'_{\min}$ is fulfilling the weak Minkowski billiard reflection rule. Consequently, referring to Theorem 11 (T is strictly convex), $\Delta'_{\min}$ is an ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory.

4 Proof of Theorem 13

For the proof of Theorem 13, we need the following generalization of [17, Lemma 3.2].

Lemma 41 *Let $K \subset \mathbb{R}^n$ be a convex body and $\{q_1, ..., q_m\}$ a set of boundary points of K. Further, let U be the convex cone spanned by outer unit normal vecors*

$$n_K(q_1), ..., n_K(q_m) \ \ in \ \ N_K(q_1), ..., N_K(q_m).$$

Then, we have

$$N_K(q_j) \cap U = N_K(q_j) \cap N^U_{K\cap(U+q_j)}(q_j),$$

for all $j \in \{1, ..., m\}$, where

$$N^U_{K\cap(U+q_j)}(q_j) = \{n \in U : \langle n, y - q_j\rangle \leqslant 0 \text{ for all } y \in K \cap (U + q_j)\}.$$

Proof From

$$N^U_{K\cap(U+q_j)}(q_j) \subseteq U$$

for all $j \in \{1, ..., m\}$ follows

$$N_K(q_j) \cap U \supseteq N_K(q_j) \cap N^U_{K\cap(U+q_j)}(q_j)$$

for all $j \in \{1, ..., m\}$.

Let $j \in \{1, ..., m\}$ be arbitrarily chosen. Let n be a nonzero vector in

$$N_K(q_j) \cap U.$$

Then

$$n \in N_K(q_j), \text{ i.e., } \langle n, x - q_j\rangle \leqslant 0 \ \ \forall x \in K,$$

and $n \in U$. Because of

$$K \cap (U + q_j) \subseteq K,$$

this implies

$$\langle n, x - q_j\rangle \leqslant 0 \ \ \forall x \in K \cap (U + q_j), \ n \in U, \ n \in N_K(q_j).$$

From that, we conclude

$$n \in N^U_{K\cap(U+q_j)}(q_j) \ \text{ and } \ n \in N_K(q_j),$$

and therefore

$$n \in N_{K\cap(U+q_j)}(q_j) \cap N_K(q_j).$$

Consequently,

$$N_K(q_j) \cap U \subseteq N_K(q_j) \cap N^U_{K\cap(U+q_j)}(q_j)$$

for all $j \in \{1, ..., m\}$.

We come to the proof of Theorem 13:

Proof (Proof of Theorem 13) By Proposition 37, U is a linear subspace of $\mathbb{R}^n$ with

$$\dim U \leqslant m-1 \leqslant n,$$

where the last inequality follows from Theorem 312. By Proposition 310, there is a selection

$$\{i_1, ..., i_{\dim U+1}\} \subseteq \{1, ..., m\}$$

with

$$\left\{q_{i_1}, ..., q_{i_{\dim U+1}}\right\} \in F(K).$$

Without loss of generality, we can assume

$$i_1 < ... < i_{\dim U+1}$$

and define the closed polygonal curve

$$\widetilde{q} = \left(q_{i_1}, ..., q_{i_{\dim U+1}}\right).$$

For

$$\dim U + 1 < m,$$

it follows by Proposition 23(i) (requires strict convexity of T) that

$$\ell_T(\widetilde{q}) < \ell_T(q).$$

But with Theorem 312, this is a contradiction to the ℓ_T-minimality of q. Therefore, we conclude

$$\dim U = m-1.$$

Let us denote by $H_1^+, ..., H_m^+$ the closed half-spaces of $\mathbb{R}^n$ containing K and which are bounded by $H_1, ..., H_m$. By Proposition 38, we conclude that we can write

$$H_j = (H_j \cap U) \oplus W \quad \text{and} \quad H_j^+ = \left(H_j^+ \cap U\right) \oplus W \tag{39}$$

for all $j \in \{1, ..., m\}$, where W is the orthogonal complement to U in $\mathbb{R}^n$, and that

$$\bigcap_{j=1}^{m} \left(H_j^+ \cap U\right) \text{ is bounded in } U, \ \bigcap_{j=1}^{m} H_j^+ \text{ is nearly bounded in } \mathbb{R}^n.$$

By the definition of U, we have

$$n_K(q_j) \in N_K(q_j) \cap U \quad \forall j \in \{1, ..., m\}$$

and therefore

$$\dim\left(N_K(q_j) \cap U\right) \geqslant 1 \quad \forall j \in \{1, ..., m\}.$$

Let us assume there is an $i \in \{1, ..., m\}$ such that

$$\dim\left(N_K(q_i) \cap U\right) > 1.$$

Then, using Lemma 41, i.e.,

$$N_K(q_i) \cap U = N_K(q_i) \cap N^U_{K\cap(U+q_i)}(q_i),$$

it follows

$$\dim \left(N_K(q_i) \cap N^U_{K\cap(U+q_i)}(q_i) \right) > 1,$$

and because of [17, Lemma 3.1] (for $d = m-1$ and $k = m$), we can find a unit vector

$$n_i^{pert} \in N_K(q_i) \cap N^U_{K\cap(U+q_i)}(q_i) \ \text{ with } \ n_i^{pert} \neq n_i \tag{40}$$

such that

$$H_{i,U}^{pert,+} \cap \left(\bigcap_{j=1, j\neq i}^{m} \left(H_j^+ \cap U \right) \right) \tag{41}$$

remains bounded in U, where we denote by $H_{i,U}^{pert,+}$ the closed half-space of U that contains $\pi_U(K)$, where π_U is the orthogonal projection (projection along W) onto U, and which is bounded by $H_{i,U}^{pert}$ which is the hyperplane in U through $\pi_U(q_i)$ that is normal to n_i^{pert}. Since by Proposition 33, the weak Minkowski billiard reflection rule in q_i (cf. Theorem 11) is no longer satisfied with respect to the perturbed hyperplane

$$H_i^{pert} := H_{i,U}^{pert} \oplus W,$$

the bouncing point q_i can be moved along

$$H_{i,U}^{pert} + (q_i - \pi_U(q_i)) \subset H_i^{pert},$$

say to q_i^*, in order to reduce the length of the polygonal curve segment

$$(q_{i-1}, q_i, q_{i+1}).$$

We define the closed polygonal curve

$$q^* := (q_1, ..., q_{i_1}, q_i^*, q_{i+1}, ..., q_m)$$

and argue that $q^* \in F(K)$: With the boundedness of (41) in U, it follows with

$$H_i^{pert,+} := H_{i,U}^{pert,+} \oplus W \tag{42}$$

and (39) the nearly boundedness of

$$H_i^{pert,+} \cap \left(\bigcap_{j=1, j\neq i}^{m} H_j^+ \right) \tag{43}$$

in $\mathbb{R}^n$.

Indeed, when the intersection in (41) is bounded in U, then there is a hyperplane H in U such that the intersection lies between H and $H+d$ for an appropriate $d \in U$. Then it follows with (39) and (42) that

$$\begin{aligned}
&H_i^{pert,+} \cap \left(\bigcap_{j=1, j\neq i}^{m} H_j^+ \right) \\
&= \left(H_{i,U}^{pert,+} \oplus W \right) \cap \left(\bigcap_{j=1, j\neq i}^{m} \left(\left(H_j^+ \cap U \right) \oplus W \right) \right) \\
&= \left(H_{i,U}^{pert,+} \cap \left(\bigcap_{j=1, j\neq i}^{m} \left(H_j^+ \cap U \right) \right) \right) \oplus W
\end{aligned}$$

lies between the hyperplanes

$$H \oplus W \quad \text{and} \quad (H+d) \oplus W.$$

Since H_i^{pert} is a K-supporting hyperplane through q_i (what follows from the fact that by (40) its outer unit normal vector n_i^{pert} is an element of $N_K(q_i)$), we conclude that K is a subset of the intersection in (43). Then, it follows from the nearly boundedness (in $\mathbb{R}^n$) of the intersection in (43) together with [17, Lemma 2.1(ii)] that

$$q^* \in F(K).$$

By referring to Theorem 312, from

$$\ell_T(q^*) < \ell_T(q),$$

we derive a contradiction to the ℓ_T-minimality of q.

Therefore:

$$\dim \left(N_K(q_i) \cap U \right) = 1.$$

We remark that for the proof of Theorem 13, the smoothness of T is a necessary condition. It guarantees the application of Proposition 33 and the boundedness of

$$\bigcap_{j=1}^{m} (H_j^+ \cap U). \tag{44}$$

Without the smoothness of T, from Proposition 38, we know of (44)'s nearly boundedness, but that is not enough in order to utilize Lemma [17, Lemma 3.1].

5 Proof of Theorem 14

In the proof of Theorem 14 we will use the following proposition:

Proposition 51 *Let $K, T \subset \mathbb{R}^n$ be convex bodies. Let $q = (q_1, ..., q_m)$ be a closed weak (K,T)-Minkowski billiard trajectory. Then, $-q$ is a closed weak $(-K,-T)$-Minkowski billiard trajectory with*

$$\ell_T(q) = \ell_{-T}(-q).$$

Proof Since q is a closed weak (K,T)-Minkowski billiard trajectory, there are K-supporting hyperplanes $H_1, ..., H_m$ through $q_1, ..., q_m$ such that q_j minimizes $\Sigma_j(q_j^*)$ over all $q_j^* \in H_j$ for all $j \in \{1, ..., m\}$. More precisely, we have the following for every $j \in \{1, ..., m\}$: q_j minimizes

$$\Sigma(q_j^*) = \mu_{T^\circ}(q_j^* - q_{j-1}) + \mu_{T^\circ}(q_{j+1} - q_j^*) = \langle q_j^* - q_{j-1}, p_{j-1}^* \rangle + \langle q_{j+1} - q_j^*, p_j^* \rangle$$

over all $q_j^* \in H_j$, where $p_{j-1}^*, p_j^* \in \partial T$ (possibly not uniquely determined) fulfill

$$q_j^* - q_{j-1} \in N_T(p_{j-1}^*) \text{ and } q_{j+1} - q_j^* \in N_T(p_j^*).$$

Because of

$$\langle q_j^* - q_{j-1}, p_{j-1}^* \rangle = \langle -q_j^* - (-q_{j-1}), -p_{j-1}^* \rangle$$

and

$$\langle q_{j+1} - q_j^*, p_j^* \rangle = \langle -q_{j+1} - (-q_j^*), -p_j^* \rangle$$

as well as

$$q_j^* - q_{j-1} \in N_T(p_{j-1}^*) \Leftrightarrow -q_j^* - (-q_{j-1}) \in N_{-T}(-p_{j-1}^*)$$

and

$$q_{j+1} - q_j^* \in N_T(p_j^*) \Leftrightarrow -q_{j+1} - (-q_j^*) \in N_{-T}(-p_j^*),$$

we conclude that $-q_j$ minimizes

$$\langle -q_j^* - (-q_{j-1}), -p_{j-1}^* \rangle + \langle -q_{j+1} - (-q_j^*), -p_j^* \rangle$$

and therefore

$$\Sigma_j(-q_j^*) = \mu_{(-T)^\circ}(-q_j^* - (-q_{j-1})) + \mu_{(-T)^\circ}(-q_{j+1} - (-q_j^*))$$

over all $-q_j^* \in -H_j$. It follows that $-q$ is a closed weak $(-K,-T)$-Minkowski billiard trajectory.

We finally argue that

$$\ell_T(q) = \ell_{-T}(-q):$$

We have

$$\ell_T(q) = \sum_{j=1}^{m} \mu_{T^\circ}(q_{j+1} - q_j) = \langle q_{j+1} - q_j, p_j \rangle,$$

where $p_j \in \partial T$ (possibly not uniquely determined) fulfills

$$q_{j+1} - q_j \in N_T(p_j).$$

Using

$$q_{j+1} - q_j \in N_T(p_j) \Leftrightarrow -q_{j+1} - (-q_j) \in N_{-T}(-p_j),$$

we conclude

$$\begin{aligned}\ell_T(q) = \sum_{j=1}^{m} \langle q_{j+1} - q_j, p_j \rangle &= \sum_{j=1}^{m} \langle -q_{j+1} - (-q_j), -p_j \rangle \\ &= \sum_{j=1}^{m} \mu_{(-T)^\circ}(-q_{j+1} - (-q_j)) \\ &= \ell_{-T}(-q).\end{aligned}$$

Proof (Proof of Theorem 14) We have

$$\dim V \leqslant m - 1,$$

since in general m points can maximally span an $(m-1)$-dimensional cone/space.

Let us assume

$$\dim V < m - 1.$$

Since $K \cap V$ is the inclusion minimal affine section of K containing q, we conclude that the convex cone spanned by

$$q_2 - q_1, ..., q_m - q_{m-1}, q_1 - q_m \tag{45}$$

is V_0, where V_0 is the linear subspace of $\mathbb{R}^n$ underlying V ($\dim V_0 < m - 1$).

Indeed, we argue similarly to within the proof of Proposition 37: we show that the convex cone spanned by (45) is in fact a linear subspace of $\mathbb{R}^n$. For that, we notice that

$$(q_2 - q_1) + ... + (q_m - q_{m-1}) + (q_1 - q_m) = 0$$

and therefore that

$$s_1(q_2 - q_1) + ... + s_{m-1}(q_m - q_{m-1}) + s_m(q_1 - q_m) = 0,$$

where we defined

$$s_j := \frac{1}{m} \quad \forall j \in \{1, ..., m\}.$$

Since

$$\sum_{j=1}^{m} s_j = 1 \text{ and } n_j \neq 0 \ \forall j \in \{1, ..., m\},$$

we have that 0 is within the relative interior of the convex cone spanned by the vectors (45). Consequently, the convex cone in fact is the linear subspace V_0 of $\mathbb{R}^n$ which underlies V.

Let $p = (p_1, ..., p_m)$ be the uniquely determined closed dual billiard trajectory of q in T (cf. Proposition 32). Then, because of Proposition 33 (and

therefore, there is a $\mu_j > 0$ such that $p_j - p_{j-1} = -\mu_j n_K(q_j)$, where $n_K(q_j) \in N_K(q_j)$, for all $j \in \{1, ..., m\}$), the pair (q, p) fulfills

$$\begin{cases} q_{j+1} - q_j = \lambda_j n_T(p_j) \in N_T(p_j), \ \lambda_j > 0, \\ p_{j+1} - p_j = -\mu_{j+1} n_K(q_{j+1}), \ \mu_{j+1} > 0, \end{cases} \tag{46}$$

for all $j \in \{1, ..., m\}$. By Proposition 35, p is a closed $(T, -K)$-Minkowski billiard trajectory (which requires the strict convexity and smoothness of K). By Proposition 34, we have

$$\ell_T(q) = \ell_{-K}(p).$$

From (46), we conclude that the convex cone spanned by $n_T(p_1), ..., n_T(p_m)$ is V_0. Then, by Proposition 310, there is a selection

$$\{i_1, ..., i_{\dim V_0+1}\} \subset \{1, ..., m\} \ \text{ with } \ i_1 < ... < i_{\dim V_0+1}$$

such that the closed polygonal curve

$$\widetilde{p} = (p_{i_1}, ..., p_{i_{\dim V_0+1}})$$

is in $F(T)$. Because of the strict convexity of K and Proposition 23(i), we have

$$\ell_{-K}(\widetilde{p}) < \ell_{-K}(p).$$

Applying Theorem 312, there has to be an ℓ_{-K}-minimizing closed $(T, -K)$-Minkowski billiard trajectory p^* with

$$\ell_{-K}(p^*) \leqslant \ell_{-K}(\widetilde{p}).$$

Let q^* be its dual billiard trajectory on $-K$ which by Propositions 34 and 35 is a closed $(-K, -T)$-Minkowski billiard trajectory with

$$\ell_{-T}(q^*) = \ell_{-K}(p^*).$$

Then, it follows by Proposition 51 that $-q^*$ is a closed (K, T)-Minkowski billiard trajectory with

$$\ell_T(-q^*) = \ell_{-T}(q^*) = \ell_{-K}(p^*) \leqslant \ell_{-K}(\widetilde{p}) < \ell_{-K}(p) = \ell_T(q).$$

This is a contradiction to the ℓ_T-minimality of q.

Therefore,

$$\dim V = m - 1.$$

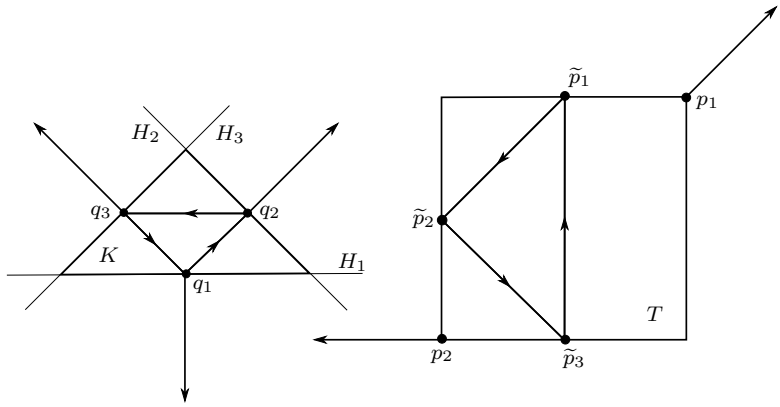

Fig. 7 Example A: $q = (q_1, q_2, q_3)$ is a closed weak (K, T)-Minkowski billiard trajectory which fulfills the weak Minkowski billiard reflection rule with respect to the K-supprting hyperplanes H_1, H_2, H_3, but it is not a strong one, i.e., there is no closed polygonal curve p such that the pair (q, p) fulfills (2).

6 Examples

Example A: We consider the following example: Let $K \subset \mathbb{R}^2$ be the triangle with vertices

$$(1,0), (0,1), (-1,0)$$

and $T \subset \mathbb{R}^2$ the square with vertices

$$(1,1), (-1,1), (-1,-1), (1,-1).$$

Let $q = (q_1, q_2, q_3)$ be a closed polygonal curve with

$$q_1 = (0,0),\ q_2 = \left(\frac{1}{2}, \frac{1}{2}\right),\ q_3 = \left(-\frac{1}{2}, \frac{1}{2}\right).$$

We denote by H_1, H_2, H_3 the K-supporting hyperplanes through q_1, q_2, q_3. We claim that (q_1, q_2, q_3) is a closed weak (K, T)-Minkowski billiard trajectory fulfilling the weak Minkowski billiard reflection rule with respect to the hyperplanes H_1, H_2, H_3. Exemplary, we show that the weak Minkowski billiard reflection rule is satisfied for the polygonal curve segment from q_1 over q_2 to q_3. For that, we show that q_2 minimizes

$$\Sigma_2(q_2^*) = \mu_{T^\circ}(q_2^* - q_1) + \mu_{T^\circ}(q_3 - q_2^*)$$

over all $q_2^* \in H_2$: We have

$$\Sigma_2(q_2) = \langle q_2 - q_1, p_1 \rangle + \langle q_3 - q_2, p_2 \rangle$$

with $p_1 = (1,1)$ and $p_2 = (-1,-1)$ (the idea behind this example is that T allows to choose p_1 and p_2 on ∂T such that their connecting line is othogonal to H_2). Since

$$\langle q_2 - q_2^*, p_2 - p_1 \rangle = 0 \ \text{ for all } q_2^* \in H_2,$$

we conclude for any $q_2^* \in H_2$

$$\begin{aligned}\Sigma_2(q_2) &= \langle q_2 - q_1, p_1\rangle + \langle q_3 - q_2, p_2\rangle + \langle q_2 - q_2^*, p_2 - p_1\rangle \\ &= \langle q_2^* - q_1, p_1\rangle + \langle q_3 - q_2^*, p_2\rangle \\ &= \langle q_2^* - q_1, p_1^*\rangle + \langle q_3 - q_2^*, p_2^*\rangle + \langle q_2^* - q_1, p_1 - p_1^*\rangle + \langle q_3 - q_2^*, p_2 - p_2^*\rangle,\end{aligned}$$

where $p_1^*, p_2^* \in \partial T$ are chosen to fulfill

$$q_2^* - q_1 \in N_T(p_1^*) \text{ and } q_3 - q_2^* \in N_T(p_2^*)$$

(by this condition possibly not uniquely determined). From the convexity of T, it follows

$$\langle q_2^* - q_1, p_1 - p_1^*\rangle \leqslant 0 \text{ and } \langle q_3 - q_2^*, p_2 - p_2^*\rangle \leqslant 0$$

and therefore

$$\Sigma_2(q_2) \leqslant \langle q_2^* - q_1, p_1^*\rangle + \langle q_3 - q_2^*, p_2^*\rangle = \mu_{T^\circ}(q_2^* - q_1) + \mu_{T^0}(q_3 - q_2^*) = \Sigma_2(q_2^*),$$

where we used Proposition 22. Consequently, q_2 minimizes $\Sigma_2(q_2^*)$ over all $q_2^* \in H_2$.

Similarly, one could prove that the polygonal curve segment from q_2 over q_3 to q_1–by choosing $p_2 = (-1, 1)$ and $p_3 = (1, -1)$–as well as the one from q_3 over q_1 to q_2–by choosing $p_3 = (1, -1)$ and $p_1 = (1, 1)$–fulfills the weak Minkowski billiard reflection rule. This gives us the idea behind choosing this example: For every polygonal curve segment of q consisting of three consecutive bouncing points, the weak Minkowski billiard reflection rule is satisfied, but it is not possible to find p_1, p_2 and p_3 uniformly in order to construct a dual billiard trajectory.

In fact, we claim that there is no closed polygonal curve $p = (\widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3)$ with vertices on ∂T such that the pair (q, p) fulfills (2). If this would be the case, then there would be $\widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3 \in \partial T$ with

$$(\widetilde{p}_2 - \widetilde{p}_1) + (\widetilde{p}_3 - \widetilde{p}_2) + (\widetilde{p}_1 - \widetilde{p}_3) = 0 \tag{47}$$

and, additionally, there would be outer unit vectors $n_K(q_1), n_K(q_2), n_K(q_3)$ at K normal to H_1, H_2, H_3 such that

$$\begin{aligned} q_2 - q_1 &\in N_T(\widetilde{p}_1) & \widetilde{p}_2 - \widetilde{p}_1 &= -\mu_2 n_K(q_2) \\ q_3 - q_2 &\in N_T(\widetilde{p}_2) & \widetilde{p}_3 - \widetilde{p}_2 &= -\mu_3 n_K(q_3) \\ q_1 - q_3 &\in N_T(\widetilde{p}_3) & \widetilde{p}_1 - \widetilde{p}_3 &= -\mu_1 n_K(q_1) \end{aligned} \tag{48}$$

where $\mu_1, \mu_2, \mu_3 \geqslant 0$. From $\widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3 \in \partial T$ together with (47) and the second column in (48), we then could conclude

$$\widetilde{p}_1 = (0, 1),\ \widetilde{p}_2 = (-1, 0),\ \widetilde{p}_3 = (0, -1).$$

But this would imply that the conditions within the first column in (48) cannot be satisfied. Consequently, the pair (q, p) does not fulfill (2).

Summarized, this example shows that, without requiring T to be strictly convex, it can happen that a closed weak (K,T)-Minkowski billiard trajectory is not a closed strong (K,T)-Minkowski billiard trajectory. □

Example B: Let $K \subset \mathbb{R}^2$ be the square given by the vertices

$$(1,0),(0,1),(-1,0),(0,-1)$$

and $T \subset \mathbb{R}^2$ the union of the square given by the vertices

$$(1,1),(-1,1),(-1,-1),(1,-1)$$

and the two balls

$$B_1^2+(1,0) \text{ and } B_1^2+(-1,0).$$

Then, T is smooth, but not strictly convex.

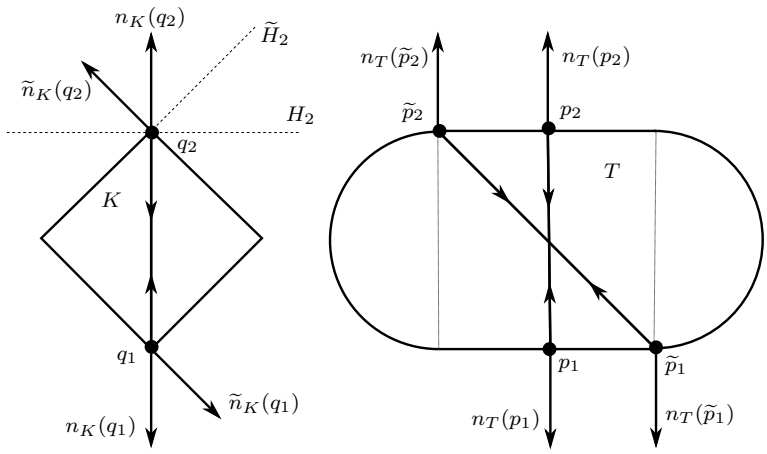

Fig. 8 Example B: The closed (K,T)-Minkowski billiard trajectory q has $p=(p_1,p_2)$ as well as $\widetilde{p}=(\widetilde{p}_1,\widetilde{p}_2)$ as closed dual billiard trajectory in T. Furthermore, q_2 minimizes (49) over all $\bar{q}_2 \in H_2$ as well as over all $\bar{q}_2 \in \widetilde{H}_2$.

Then, we can easily check that

$$q=(q_1,q_2) \text{ with } q_1=(0,-1) \text{ and } q_2=(0,1)$$

is a closed (K,T)-Minkowski billiard trajectory: (2) is satisfied for the pair (q,p) for

$$p=(p_1,p_2) \text{ with } p_1=q_1 \text{ and } p_2=q_2$$

with the corresponding outer unit normal vectors

$$n_K(q_1)=n_T(p_1)=(0,-1),\ n_K(q_2)=n_T(p_2)=(0,1).$$

Since every closed (strong) Minkowski billiard trajectory is also a weak one (cf. Theorem 11), q_2 minimizes

$$\mu_{T^\circ}(\bar{q}_2-q_1)+\mu_{T^\circ}(q_1-\bar{q}_2) \tag{49}$$

over all $\bar{q}_2 \in H_2$, where H_2 is the K-supporting horizontal line through q_2.

However, (2) is also satisfied for the pair $(q, \widetilde{p})$ for

$$\widetilde{p} = (\widetilde{p}_1, \widetilde{p}_2) \text{ with } \widetilde{p}_1 = (1, -1) \text{ and } \widetilde{p}_2 = (-1, 1)$$

with the corresponding outer unit normal vectors

$$\widetilde{n}_K(q_1) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \ \widetilde{n}_K(q_2) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$$n_T(\widetilde{p}_1) = n_T(p_1), \ \widetilde{n}_T(p_2) = n_T(p_2).$$

Then, again referring to Theorem 11, q_2 minimizes (49) over all $\bar{q}_2 \in \widetilde{H}_2$, where $\widetilde{H}_2$ is the K-supporting line through q_2 with slope 1.

Summarized, this example shows that, without requiring T to be strictly convex, it can happen that the closed dual billiard trajectory of a closed (K, T)-Minkowski billiard trajectory is not uniquely determined. Furthermore, it shows that, without requiring T to be strictly convex, it can happen that the K-supporting hyperplanes corresponding via the weak Minkowski billiard reflection rule to a closed weak (K, T)-Minkowski billiard trajectory are not uniquely determined. □

Example C: Let $K \subset \mathbb{R}^2$ be the triangle given by the vertices

$$(1, 0), (-1, 2), (-1, -2)$$

and T the intersection of the two balls

$$B_1^2 + \left(-\frac{1}{2}, 0\right) \text{ and } B_1^2 + \left(\frac{1}{2}, 0\right).$$

Then, T is strictly convex, but not smooth.

Then, we can easily check that

$$q = (q_1, q_2, q_3) \text{ with } q_1 = (-1, -2), \ q_2 = (-1, 2) \text{ and } q_3 = (1, 0)$$

is a closed (K, T)-Minkowski billiard trajectory: (2) is satisfied for the pair (q, p) for

$$p = (p_1, p_2, p_3) \text{ with } p_1 = (0, x) \text{ and } p_2 = p_3 = (0, -x)$$

with the corresponding outer unit normal vectors

$$n_K(q_1) = (0, -1), \ n_K(q_2) = (0, 1), \ n_K(q_3) = (1, 0),$$

$$n_T(p_1) = (0, 1), \ n_T(p_2) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \ n_T(p_3) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

However, we notice that (2) is also satisfied when we replace $n_K(q_3) = (1, 0)$ by any other outer unit normal vector within the normal cone $N_K(q_3)$.

Summarized, this example shows that, without requiring T to be smooth, it can happen that the closed dual billiard trajectory corresponding to a closed (K, T)-Minkowski billiard trajectory is not a polygonal curve in the sense of

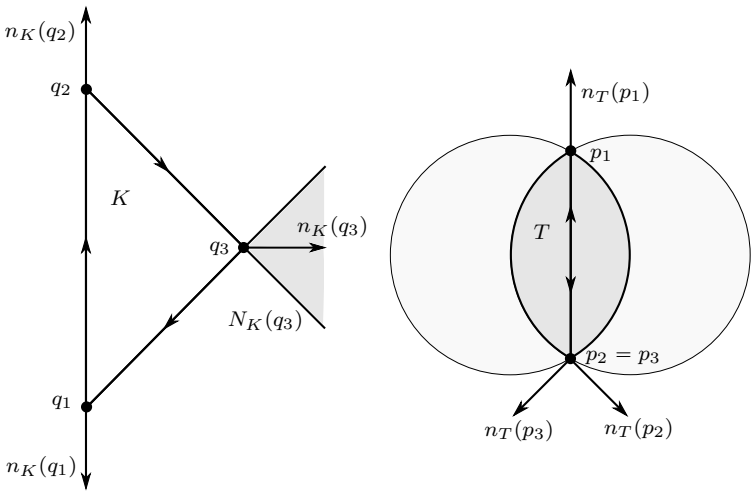

Fig. 9 Example C: $q = (q_1, q_2, q_3)$ is a closed (K, T)-Minkowski billiard trajectory, no matter which outer normal unit vector in q_3 is chosen in order to show that (q, p) fulfills (2). The vertices p_2, p_3 of q's closed dual billiard trajectory p coincide. The convex cone spanned by $n_K(q_1), n_K(q_2), n_K(q_3)$ does not span the whole $\mathbb{R}^2$.

Footnote 1. This implies that, without requiring T to be smooth, it can happen that the K-supporting hyperplanes corresponding via the weak Minkowski billiard reflection rule to a closed weak (K, T)-Minkowski billiard trajectory are not uniquely determined. Furthermore, this example shows that, without requiring T to be smooth, it can happen that the convex cone spanned by the outer unit vectors normal to the hyperplanes which correspond via the weak Minkowski billiard reflection rule to a closed weak (K, T)-Minkowski billiard trajectory does not is the whole $\mathbb{R}^n$.

Example D: Let $K \subset \mathbb{R}^2$ be the square given by the vertices

$$(1, 1), (-1, 1), (-1, -1), (1, -1)$$

and $\widetilde{T} \subset \mathbb{R}^2$ the triangle given by the vertices

$$(2, 1), (-2, 1), (0, -1).$$

By rounding off these vertices, $\widetilde{T}$ can be made smooth; after that we denote it by T.

Then, we can check that $q = (q_1, q_2, q_3)$ with

$$q_1 = (0, -1),\ q_2 = (0, 1),\ q_3 = (1, 0)$$

is a closed weak (K, T)-Minkowski billiard trajectory.

Indeed, let H_1, H_2, H_3 be the uniquely determined K-supporting hyperplanes through q_1, q_2, q_3. We show that q_j minimizes

$$\Sigma_j(q_j^*) = \mu_{T^\circ}(q_j^* - q_{j-1}) + \mu_{T^\circ}(q_{j+1} - q_j^*)$$

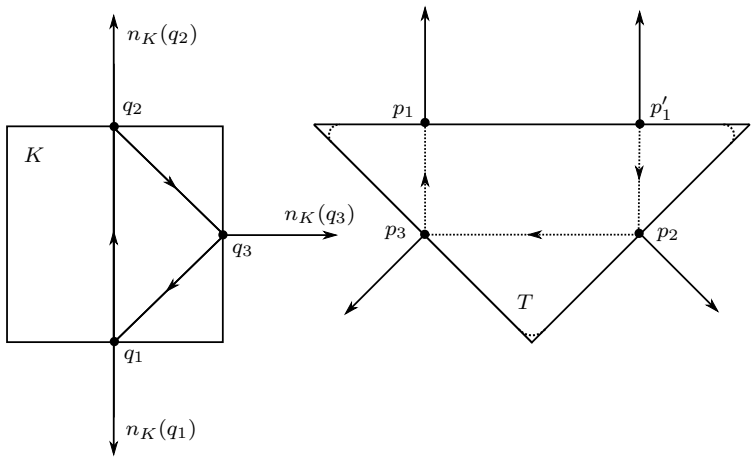

Fig. 10 Example D: $T \subset \mathbb{R}^2$ is smooth (the vertices are rounded off). q is a closed weak (K,T)-Minkowski billiard trajectory, but the convex cone spanned by $n_K(q_1), n_K(q_2), n_K(q_3)$ is not a linear subspace of $\mathbb{R}^2$.

over all $q_j^* \in H_j$ for all $j \in \{1,2,3\}$. We have

$$\Sigma_1(q_1) = \langle q_1 - q_3, p_3\rangle + \langle q_2 - q_1, p_1\rangle$$

for

$$p_1 = (-1,1) \ \text{ and } \ p_3 = (-1,0).$$

Since

$$\langle q_1 - q_1^*, p_1 - p_3\rangle = 0 \ \text{ for all } q_1^* \in H_1,$$

we conclude for any $q_1^* \in H_1$

$$\begin{aligned}\Sigma_1(q_1) &= \langle q_1 - q_3, p_3\rangle + \langle q_2 - q_1, p_1\rangle + \langle q_1 - q_1^*, p_1 - p_3\rangle \\ &= \langle q_1^* - q_3, p_3\rangle + \langle q_2 - q_1^*, p_1\rangle \\ &= \langle q_1^* - q_3, p_3^*\rangle + \langle q_2 - q_1^*, p_1^*\rangle + \langle q_1^* - q_3, p_3 - p_3^*\rangle + \langle q_2 - q_1^*, p_1 - p_1^*\rangle,\end{aligned}$$

where $p_1^*, p_3^* \in \partial T$ (possibly not uniquely determined) fulfill

$$q_1^* - q_3 \in N_T(p_3^*) \ \text{ and } \ q_2 - q_1^* \in N_T(p_1^*).$$

From the convexity of T, it follows

$$\langle q_1^* - q_3, p_3 - p_3^*\rangle \leqslant 0 \ \text{ and } \ \langle q_2 - q_1^*, p_1 - p_1^*\rangle \leqslant 0$$

and therefore

$$\Sigma_1(q_1) \leqslant \langle q_1^* - q_3, p_3^*\rangle + \langle q_2 - q_1^*, p_1^*\rangle = \Sigma_1(q_1^*).$$

Consequently, q_1 minimizes $\Sigma_1(q_1^*)$ over all $q_1^* \in H_1$. The same argumentation yields

$$\Sigma_2(q_2) = \langle q_2 - q_1, p_1'\rangle + \langle q_3 - q_2, p_2\rangle \leqslant \Sigma_2(q_2^*)$$

for all $q_2^* \in H_2$, where

$$p_1' = (1,1) \text{ and } p_2 = (1,0),$$

and also

$$\Sigma_3(q_3) = \langle q_3 - q_2, p_2 \rangle + \langle q_1 - q_3, p_3 \rangle \leqslant \Sigma_3(q_3^*)$$

for all $q_3^* \in H_3$.

We note that the convex cone spanned by the outer unit vectors

$$n_K(q_1) = (0,-1),\ n_K(q_2) = (0,1),\ n_K(q_3) = (1,0)$$

which are normal to H_1, H_2, H_3 is not a linear subspace of $\mathbb{R}^2$.

Summarized, this example shows that, without requiring T to be strictly convex, it can happen that the convex cone of the outer unit vectors normal to the K-supporting hyperplanes which correspond via the weak Minkowski billiard reflection rule to a closed weak (K,T)-Minkowski billiard trajectory does not is the whole $\mathbb{R}^n$. □

Example E: Let $K \subset \mathbb{R}^2$ be the trapezoid given by the vertices

$$(1,-1), (4,2), (-4,2), (-1,-1)$$

and $T \subset \mathbb{R}^2$ the triangle given by the vertices

$$\left(\frac{1}{2}, 2\right), \left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, -2\right).$$

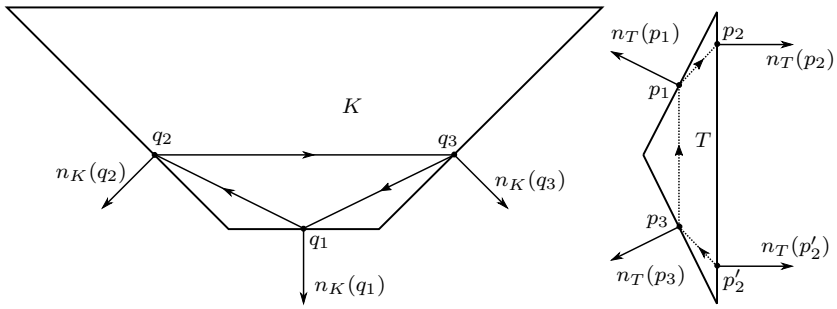

Fig. 11 Example E: T is not strictly convex. $q = (q_1, q_2, q_3)$ is a closed (K,T)-Minkowski billiard trajectory which can be translated into the interior of K. Accordingly, the convex hull of $n_K(q_1), n_K(q_2), n_K(q_3)$ does not include the origin.

Then, $q = (q_1, q_2, q_3)$ with

$$q_1 = (0,-1),\ q_2 = (-2,0),\ q_3 = (2,0)$$

is a closed weak (K,T)-Minkowski billiard trajectory.

Indeed, we denote the uniquely determined K-supporting hyperplanes through q_1, q_2, q_3 by H_1, H_2, H_3 and show that q_j minimizes

$$\Sigma_j(q_j^*) = \mu_{T^\circ}(q_j^* - q_{j-1}) + \mu_{T^\circ}(q_{j+1} - q_j^*)$$

over all $q_j^* \in H_j$ for all $j \in \{1, 2, 3\}$. We have

$$\Sigma_2(q_2) = \langle q_2 - q_1, p_1 \rangle + \langle q_3 - q_2, p_2 \rangle$$

for

$$p_1 = (0, 1) \text{ and } p_2 = \left(\frac{1}{2}, \frac{3}{2}\right).$$

Since

$$\langle q_2 - q_2^*, p_2 - p_1 \rangle = 0 \text{ for all } q_2^* \in H_2,$$

we conclude for any $q_2^* \in H_2$ that

$$\begin{aligned}\Sigma_2(q_2) &= \langle q_2 - q_1, p_1 \rangle + \langle q_3 - q_2, p_2 \rangle + \langle q_2 - q_2^*, p_2 - p_1 \rangle \\ &= \langle q_2^* - q_1, p_1 \rangle + \langle q_3 - q_2^*, p_2 \rangle \\ &= \langle q_2^* - q_1, p_1^* \rangle + \langle q_3 - q_2^*, p_2^* \rangle + \langle q_2^* - q_1, p_1 - p_1^* \rangle + \langle q_3 - q_2^*, p_2 - p_2^* \rangle,\end{aligned}$$

where $p_1^*, p_2^* \in \partial T$ (possibly not uniquely determined) fulfill

$$q_2^* - q_1 \in N_T(p_1^*) \text{ and } q_3 - q_2^* \in N_T(p_2^*).$$

From the convexity of T, it follows

$$\langle q_2^* - q_1, p_1 - p_1^* \rangle \leqslant 0 \text{ and } \langle q_3 - q_2^*, p_2 - p_2^* \rangle \leqslant 0$$

and therefore

$$\Sigma_2(q_2) \leqslant \langle q_2^* - q_1, p_1^* \rangle + \langle q_3 - q_2^*, p_2^* \rangle = \Sigma_2(q_2^*).$$

Consequently, q_2 minimizes $\Sigma_2(q_2^*)$ over all $q_2^* \in H_2$. The same argumentation yields

$$\Sigma_3(q_3) = \langle q_3 - q_2, p_2' \rangle + \langle q_1 - q_3, p_3 \rangle \leqslant \Sigma_3(q_3^*)$$

for all $q_3^* \in H_3$, where

$$p_2' = \left(\frac{1}{2}, -\frac{3}{2}\right) \text{ and } p_3 = (0, -1),$$

and also

$$\Sigma_1(q_1) = \langle q_1 - q_3, p_3 \rangle + \langle q_2 - q_1, p_1 \rangle \leqslant \Sigma_1(q_1^*)$$

for all $q_1^* \in H_1$.

Finally, firstly, we note that 0 is not within the convex hull of the unit normal vectors $n_K(q_1), n_K(q_2), n_K(q_3)$, and, secondly, that q can be translated into the interior of K; for example

$$q + \left(0, \frac{1}{2}\right) \in \mathring{K}.$$

Summarized, this example shows that, without requiring T to be strictly convex and smooth, it can happen that the convex hull of the outer unit vectors normal to the K-supporting hyperplanes which correspond via the weak Minkowski billiard reflection rule to a closed weak (K,T)-Minkowski billiard trajectory does not include the origin. Furthermore, it shows that in the situation of non-strictly convex T, it can happen that closed weak (K,T)-Minkowski billiard trajectories can be translated into the interior of K. □

Example F: Let $K \subset \mathbb{R}^2$ be the convex polytope given by the vertices

$$\left(\frac{1}{2}, 0\right), (0,1), (-2,1), (-2,-1), (0,-1)$$

and $\widetilde{T} \subset \mathbb{R}^2$ the rhombus given by the vertices

$$(1,0), (0,1), (-1,0), (0,-1).$$

Then, we can find a strictly convex body $T \subset \mathbb{R}^2$ satisfying

$$\widetilde{T} \subseteq T \quad \text{and} \quad (1,0), (0,1), (-1,0), (0,-1) \in \partial T$$

such that $N_T((0,1))$ equals the convex cone spanned by the vectors $(1,2)$ and $(-1,2)$ and

$$N_T((0,-1)) = -N_T((0,1)).$$

Then, T is strictly convex, but not smooth.

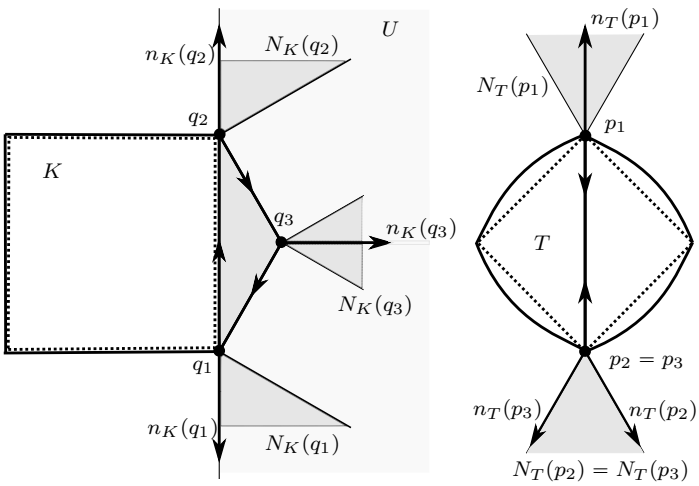

Fig. 12 Example F: In the figure, the dashed lines belong to $\widetilde{K}$ and $\widetilde{T}$, respectively. $q = (q_1, q_2, q_3)$ is an ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory, where $p = (p_1, p_2, p_3)$ is its dual billiard trajectory in T. We neither have that U (the convex cone spanned by $n_K(q_1), n_K(q_2), n_K(q_3)$) is a linear subspace of $\mathbb{R}^2$ with $\dim U = 2$, nor that $\dim(N_K(q_j) \cap U) = 1$ for all $j \in \{1,2,3\}$.

We claim that $q = (q_1, q_2, q_3)$ with

$$q_1 = (0, -1),\ q_2 = (0, 1),\ q_3 = \left(\frac{1}{2}, 0\right)$$

is an ℓ_T-minimizing closed (K, T)-Minkowski billiard trajectory. Then, if we denote by U the convex cone spanned by the outer unit vectors

$$n_K(q_1), n_K(q_2), n_K(q_3)$$

which are normal to H_1, H_2, H_3, then

$$U = \mathbb{R}_{\geqslant 0} \times \mathbb{R}$$

and

$$\dim\left(N_K(q_j) \cap U\right) > 1 \quad \forall j \in \{1, 2, 3\}.$$

So, let us prove that q is an ℓ_T-minimizing closed (K, T)-Minkowski billiard trajectory. First, q is a closed (K, T)-Minkowski billiard trajectory by checking that $p = (p_1, p_2, p_3)$ with

$$p_1 = (0, 1),\ p_2 = p_3 = (0, -1)$$

is its dual billiard trajectory in T. One has

$$\ell_T(q) = \langle q_2 - q_1, p_1\rangle + \langle q_3 - q_2, p_2\rangle + \langle q_1 - q_3, p_3\rangle = 2 + 1 + 1 = 4.$$

Now, let $\widetilde{K}$ be the convex polytope defined by the vertices

$$(0, -1), (0, 1), (-2, 1), (-2, -1).$$

We have

$$\widetilde{K} \subseteq K \ \text{ and } \ \widetilde{T} \subseteq T.$$

Therefore, we conclude that the $\ell_{\widetilde{T}}$-length of the $\ell_{\widetilde{T}}$-minimizing closed $(\widetilde{K}, \widetilde{T})$-Minkowski billiard trajectories is less or equal than the ℓ_T-length of the ℓ_T-minimizing closed (K, T)-Minkowski billiard trajectories. One easily checks that the former is 4. This implies that the latter cannot be less than 4. Since the ℓ_T-length of q is 4, this implies that q, in fact, is an ℓ_T-minimizing closed (K, T)-Minkowski billiard trajectory.

Summarized, this example shows that, without requiring T to be smooth, it can happen that there is an ℓ_T-minimizing closed (K, T)-Minkowski billiard trajectory which violates all the statements made in Theorem 13. □

Example G: Let $K \subset \mathbb{R}^2$ be the triangle given by the vertices

$$(1, 1), (-1, 1), (0, -1)$$

and $T \subset \mathbb{R}^2$ the rectangle given by the vertices

$$(1, -2), (1, 2), (-1, 2), (-1, -2).$$

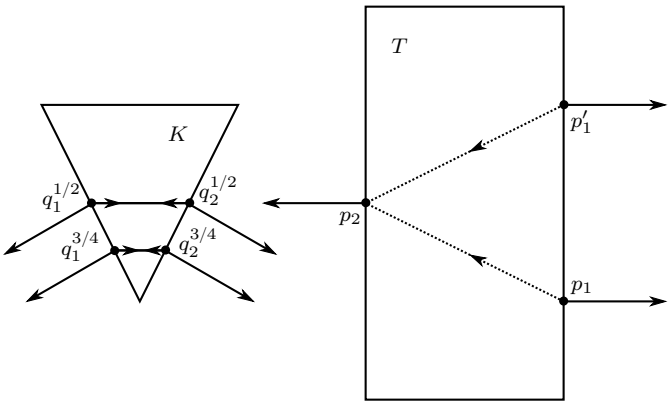

Fig. 13 Example G: For $a \to 1$ the closed weak (K,T)-Minkowski billiard trajectory $q^a = (q_1^a, q_2^a)$ d_H-converges to the point $(0,-1)$.

Then, we claim that for $a \in [0,1)$, $q^a = (q_1^a, q_2^a)$ with

$$q_1^a = (-1+a, 1-2a) \text{ and } q_2^a = (1-a, 1-2a)$$

is a closed weak (K,T)-Minkowski billiard trajectory which fulfills the weak Minkowski billiard reflection rule with respect to the uniquely determined K-supporting hyperplanes H_1, H_2 through q_1^a, q_2^a.

Indeed, for $a \in [0,1)$ we show that q_j^a minimizes

$$\Sigma_j(q_j^*) = \mu_{T^\circ}(q_j^* - q_{j-1}^a) + \mu_{T^\circ}(q_{j+1}^a - q_j^*)$$

(note: $q_{j+1}^a = q_{j-1}^a$) over all $q_j^* \in H_j$ for $j \in \{1,2\}$. The following holds for all $a \in [0,1)$: We have

$$\Sigma_2(q_2^a) = \langle q_2^a - q_1^a, p_1 \rangle + \langle q_1^a - q_2^a, p_2 \rangle$$

for

$$p_1 = (1,-1) \text{ and } p_2 = (-1,0).$$

Since

$$\langle q_2^* - q_2^a, p_2 - p_1 \rangle = 0 \text{ for all } q_2^* \in H_2,$$

we conclude for any $q_2^* \in H_2$ that

$$\begin{aligned}\Sigma_2(q_2^a) &= \langle q_2^a - q_1^a, p_1 \rangle + \langle q_1^a - q_2^a, p_2 \rangle + \langle q_2^* - q_2^a, p_2 - p_1 \rangle \\ &= \langle q_2^* - q_1^a, p_1 \rangle + \langle q_1^a - q_2^*, p_2 \rangle \\ &= \langle q_2^* - q_1^a, p_1^* \rangle + \langle q_1^a - q_2^*, p_2^* \rangle + \langle q_2^* - q_1^a, p_1 - p_1^* \rangle + \langle q_1^a - q_2^*, p_2 - p_2^* \rangle,\end{aligned}$$

where $p_1^*, p_2^* \in \partial T$ (possibly not uniquely determined) fulfill

$$q_2^* - q_1^a \in N_T(p_1^*) \text{ and } q_1^a - q_2^* \in N_T(p_2^*).$$

From the convexity of T, it follows

$$\langle q_2^* - q_1^a, p_1 - p_1^* \rangle \leqslant 0 \text{ and } \langle q_1^a - q_2^*, p_2 - p_2^* \rangle \leqslant 0$$

and therefore

$$\Sigma_2(q_2^a) \leqslant \langle q_2^* - q_1^a, p_1^* \rangle + \langle q_1^a - q_2^*, p_2^* \rangle = \Sigma_2(q_2^*).$$

Consequently, q_2^a minimizes $\Sigma_2(q_2^*)$ over all $q_2^* \in H_2$. The same argumentation yields

$$\Sigma_1(q_1^a) = \langle q_1^a - q_2^a, p_2 \rangle + \langle q_2^a - q_1^a, p_1' \rangle \leqslant \Sigma_1(q_1^*)$$

for all $q_1^* \in H_1$, where $p_1' = (1,1)$.

We have

$$\ell_T(q^a) = \langle q_2^a - q_1^a, p_1 \rangle + \langle q_1^a - q_2^a, p_2 \rangle = 2 - 2a + 2 - 2a = 4 - 4a$$

which for $a \to 1$ goes to 0. Therefore, there is no ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory (cf. Footnote 1).

We note that T can be made smooth without loosing the above mentioned properties.

Summarized, this example shows that there are configurations (K,T) for which T is not strictly convex and there is no ℓ_T-minimizing closed weak (K,T)-Minkowski billiard trajectory.

7 Constructing shortest Minkowski billiard trajectories on convex polytopes

7.1 General construction in two dimensions

In this first subsection, we describe the general construction of ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectories for the case of a convex polytope $K \subset \mathbb{R}^2$ and a strictly convex and smooth body $T \subset \mathbb{R}^2$. For determining the ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectories, we use Corollary 15, i.e., the ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectories have two or three bouncing points, where in the latter case the billiard trajectories are regular.

In [6], the algorithm for finding closed (K,T)-Minkowski billiard trajectories with two bouncing points has already been described. For details concerning the implementation, we refer to Section 7.3. In [6], it was stated as open problem to find an algorithm for determining closed regular (K,T)-Minkowski billiard trajectories with three bouncing points. While there, for T equals the Euclidean unit ball in $\mathbb{R}^2$, they could use the uniqueness of Fagnano triangles in acute triangles in order to find the closed regular Euclidean billiard trajectories on K, in the Minkowski/Finsler setting one has to find a different approach, since there are no obvious analogues of the Fagnano triangles at first. Now, this will be the task of the remainder of this subsection.

We do the following (cf. Figure 14):

(a) Choose 3 facets F_1, F_2, F_3 of K (considering their order) such that the convex cone spanned by the associated outer normal unit vectors $n_{F_1}, n_{F_2}, n_{F_3}$ is $\mathbb{R}^2$.

(b) Construct the uniquely determined (up to scaling and translation) closed polygonal curve $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_{i+1} - \gamma_i$ given by a negative multiple of n_{F_i} for all $i \in \{1, 2, 3\}$.

(c) Find the uniquely determined $\lambda > 0$ and $c \in \mathbb{R}^n$ such that

$$\lambda\{\gamma_1, \gamma_2, \gamma_3\} + c \subset \partial T.$$

(d) Let n_i be the outer normal unit vector ar ∂T in the point

$$\lambda\gamma_i + c$$

for all $i \in \{1, 2, 3\}$. If the convex cone spanned by n_1, n_2, n_3 is $\mathbb{R}^2$, then construct the uniquely determined (up to scaling and translation) closed polygonal curve (ξ_1, ξ_2, ξ_3) with $\xi_{i+1} - \xi_i$ given by a positive multiple of n_{i+1} for all $i \in \{1, 2, 3\}$. Otherwise: If possible: Go back to (a) and start with a choice not yet made. Otherwise: End.

(e) If possible: Find $\mu > 0$ and $e \in \mathbb{R}^n$ such that

$$\mu\xi_i + e \in \mathring{F}_i \quad \forall i \in \{1, 2, 3\}.$$

Otherwise: If possible: Go back to (a) and start with a choice not yet made. Otherwise: End.

(f) Define a closed polygonal curve

$$q = (q_1, q_2, q_3)$$

by

$$q_i := \mu\xi_i + e \quad \forall i \in \{1, 2, 3\}.$$

By construction: q is a maximally spanning, closed, regular (K, T)-Minkowski billiard trajectory with three bouncing points and with closed dual billiard trajectory

$$p = (p_1, p_2, p_3)$$

given by

$$p_i := \lambda\gamma_{i+1} + c \quad \forall i \in \{1, 2, 3\}.$$

Add q to $B_3(K, T)$.

(g) If possible: Go back to (a) and start with a choice not yet made. Otherwise: End.

Finally, the set $B_3(K, T)$ contains all closed regular (K, T)-Minkowski billiard trajectories with three bouncing points whose ℓ_T-length can be easily calculated:

$$\ell_T(q) = \sum_{j=1}^{3} \mu_{T^\circ}(q_{j+1} - q_j) = \sum_{j=1}^{3} \langle q_{j+1} - q_j, p_j \rangle.$$

Let us now turn to the explanation of the individuel steps, while for the detailed justification, we refer to Section 7.3:

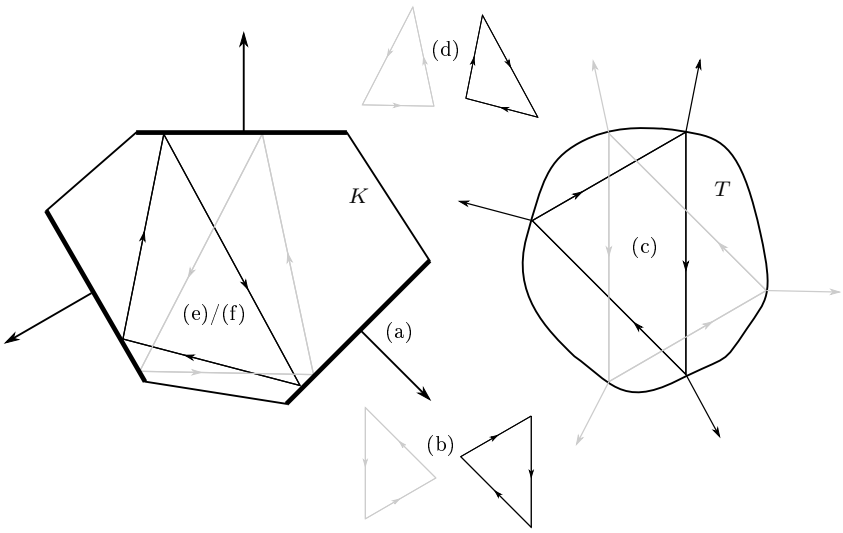

Fig. 14 Illustration of the construction of closed regular (K, T)-Minkowski billiard trajectories with three bouncing points.

Ad (a): If there is a closed regular (K, T)-Minkowski billiard trajectory with three bouncing points, then we know from Proposition 37 that the bouncing points lie in the interiors of three different facets of K whose associated outer normal unit vectors span $\mathbb{R}^2$.

Ad (b): Since the convex cone spanned by $n_{F_1}, n_{F_2}, n_{F_3}$ is $\mathbb{R}^2$, solving a system of linear equations yields a uniquely determined (up to scaling and translation) 3-tuple

$$(\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{R}^2)^3$$

and a uniquely determined (up to scaling–depending on the factor by which $(\gamma_1, \gamma_2, \gamma_3)$ will be scaled) 3-tuple

$$(\alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}_{<0})^3$$

fulfilling

$$\gamma_{i+1} - \gamma_i = \alpha_i n_{F_i} \quad i \in \{1, 2, 3\}.$$

We understand the 3-tuple $(\gamma_1, \gamma_2, \gamma_3)$ as a closed polygonal curve.

Ad (c): Because of the strict convexity of T, there is a unique combination

$$(\lambda, c) \in \mathbb{R}_{>0} \times \mathbb{R}^2$$

such that

$$\lambda\{\gamma_1, \gamma_2, \gamma_3\} + c \subset \partial T.$$

Ad (d): If the convex cone spanned by the unit vectors n_1, n_2, n_3 is $\mathbb{R}^2$, then, as in step (b), solving a system of linear equations yields a uniquely determined (up to scaling und translation) 3-tuple

$$(\xi_1, \xi_2, \xi_3) \in (\mathbb{R}^2)^3$$

and a uniquely determined (up to scaling–depending on the factor by which (ξ_1, ξ_2, ξ_3) will be scaled) 3-tuple

$$(\beta_1, \beta_2, \beta_3) \in (\mathbb{R}_{>0})^3$$

fulfilling

$$\xi_{i+1} - \xi_i = \beta_{i+1} n_{i+1} \quad \forall i \in \{1, 2, 3\}$$

We understand the 3-tuple (ξ_1, ξ_2, ξ_3) as a closed polygonal curve.

Ad (e) & (f): There is at most one combination

$$(\mu, e) \in \mathbb{R}_{>0} \times \mathbb{R}^2$$

such that

$$\mu\{\xi_1, \xi_2, \xi_3\} + e \subset \partial K.$$

By checking whether

$$\mu\xi_i + e \in \mathring{F}_i \quad \forall i \in \{1, 2, 3\},$$

we make sure that the closed polygonal curve

$$q = (q_1, q_2, q_3)$$

defined by

$$q_i := \mu\xi_i + e \quad \forall i \in \{1, 2, 3\}$$

has its vertices in the interiors of the facets F_1, F_2, F_3. q is a closed (K, T)-Minkowski billiard trajectory, where

$$p = (p_1, p_2, p_3)$$

with

$$p_i := \lambda\gamma_{i+1} + c \quad \forall i \in \{1, 2, 3\}$$

is its closed dual billiard trajectory on T. Indeed, we define

$$\lambda_i := \mu\beta_i > 0 \text{ and } \mu_i := -\lambda\alpha_i > 0$$

and notice that the pair (q, p) fulfills (2):

$$\begin{cases} q_{i+1} - q_i = (\mu\xi_{i+1} + e) - (\mu\xi_i + e) = \mu(\xi_{i+1} - \xi_i) = \mu\beta_{i+1} n_{i+1} = \lambda_{i+1} n_{i+1} \in N_T(p_i), \\ p_{i+1} - p_i = (\lambda\gamma_{i+2} + c) - (\lambda\gamma_{i+1} + c) = \lambda(\gamma_{i+2} - \gamma_{i+1}) = -\mu_{i+1} n_{F_{i+1}} \in -N_K(q_{i+1}). \end{cases}$$

7.2 A note concerning the general construction for higher dimensions

Let $K \subset \mathbb{R}^n$ be a convex polytope and $T \subset \mathbb{R}^n$ a strictly convex and smooth body. We know from Theorems 13 and 14 that there is always an ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory which is maximally spanning, has at most $n+1$ bouncing points and whose corresponding outer unit normal vectors span a cone which has the same dimension as the inclusion minimal section containing this trajectory.

Instead of that the Euclidean unit ball is replaced by T and one has to take into account that the linear subspaces underlying the inclusion minimal affine sections containing relevant Minkowski billiard trajectories can differ from the convex cone spanned by the corresponding outer unit normal vectors, these are the same preconditions as within the algorithms for the Euclidean setting. The necessary aspects which one has to consider for the adjustment to the Minkowski setting are indicated in Section 7.1 for two dimensions.

We leave the detailed execution of these adjustments to further research.

7.3 Efficiency and used methods within the implementation

We now turn our attention to the implementation of the algorithm for two dimensions which is described in Section 7.1.

Even though we focus on the case $n = 2$, we state some of the results in this subsection for arbitrary n if they hold in any dimension. In Section 7.1, we applied Corollary 15 and therefore required T to be strictly convex and smooth. Implementing such a set can be a difficult problem because we can only make finitely many inputs. Therefore, we assume that both K and T are convex polytopes (in particular neither strictly convex nor smooth) in this subsection. In the following this has to be justified.

We proved in Theorem 13 for the case when T is assumed to be strictly convex and smooth and when considering the closed (K,T)-Minkowski billiard trajectories with $n+1$ bouncing points, that, when searching for length minimizers, it is enough to just concentrate on the ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectories with $n+1$ bouncing points which are regular, i.e., whose normal cones in the bouncing points are one-dimensional. However, in [18, Section 4.3.2], it is shown that in the case when T is assumed to be a convex polytope, then, the bouncing points of an ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory may be in nonsmooth boundary points of K, but one can assume that the normal cones appearing in the system (2) can be replaced by the rays which are the one-dimensional normal cones of the neighbouring facets.[7] Therefore, when we look for a boundary point in some facet of K, we allow it to lie in the boundary of this facet, but we only consider the normal cone for some point in the relative interior of this facet.

[7] This can be proved by approximating the convex polytope T by a sequence of strictly convex and smooth bodies, using a line of argumentation which is similar to the one appearing in the proof of [23, Theorem 2.1].

Before we analyze the algorithm in greater detail, we argue that it is sufficient to compute finitely many pairs (q,p) of closed polygonal curves fulfilling (2) to find one, where q is ℓ_T-minimizing. More precisely, we show that if (q,p) and (q',p') are pairs fulfilling (2) and if their vertices lie on the same faces of K (resp. T), then q and q' have the same ℓ_T-length.

Proposition 71 (Theorem 4.3.6 in [18]) *Let $K,T \subseteq \mathbb{R}^n$ be convex polytopes, where $F_1,...,F_m$ and $G_1,...,G_m$ are the faces of K and T, respectively. Further, let*

$$q = (q_1,...,q_m) \quad \text{and} \quad q' = (q'_1,...,q'_m)$$

be closed polygonal curves with vertices on ∂K. Assume, there are closed polygonal curves

$$p = (p_1,...,p_m) \quad \text{and} \quad p' = (p'_1,...,p'_m)$$

with vertices on ∂T such that (q,p) and (q',p') fulfill (2). Further, assume for each $j \in \{1,...,m\}$ that

$$q_j, q'_j \in \operatorname{relint}(F_j) \quad \text{and} \quad p_j, p'_j \in \operatorname{relint}(G_j)$$

(unless F_j is a vertex, in which case we assume $q_j, q'_j \in F_j$ instead. The same applies for G_j). Then

$$\ell_T(q) = \ell_T(q').$$

Proof We start the proof by stating a simple fact. If F is a face of a convex polytope P with $\dim(F) \geqslant 1$ and $y_1, y_2 \in F$, then

$$\langle y_1 - y_2, v\rangle = 0\,, \ \forall v \in N_P(z) \tag{50}$$

holds for any $z \in \operatorname{relint}(F)$. To see this, consider the affine hull of F and shift it, such that it becomes a linear space. Then the vector $y_1 - y_2$ is an element of this space and $N_P(z)$ with $z \in \operatorname{relint}(F)$ is contained in the corresponding orthogonal space. Note that (50) also holds if F is a vertex in which case we have $\operatorname{relint}(F) = F$. Then $y_1, y_2 \in F$ implies $y_1 = y_2$ and the statement follows immediately. Now recall (2):

$$\begin{aligned} q_{j+1} - q_j &\in N_T(p_j)\,, & q'_{j+1} - q'_j &\in N_T(p'_j)\,, \\ p_{j+1} - p_j &\in -N_K(q_{j+1})\,, & p'_{j+1} - p'_j &\in -N_K(q'_{j+1}). \end{aligned}$$

The following calculation completes the proof:

$$\begin{aligned} \ell_T(q') &= \sum_{j=1}^m \langle q'_{j+1} - q'_j, p'_j\rangle \\ &= \sum_{j=1}^m \langle q_{j+1} - q_j, p'_j\rangle + \sum_{j=1}^m \langle q_j - q'_j, p'_j\rangle - \sum_{j=1}^m \langle q_{j+1} - q'_{j+1}, p'_j\rangle \\ &= \sum_{j=1}^m \langle q_{j+1} - q_j, p'_j\rangle + \sum_{j=1}^m \langle q_j - q'_j, p'_j\rangle - \sum_{j=1}^m \langle q_j - q'_j, p'_{j-1}\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{m} \langle q_{j+1} - q_j, p'_j \rangle + \sum_{j=1}^{m} \langle q_j - q'_j, p'_j - p'_{j-1} \rangle \\
&= \sum_{j=1}^{m} \langle q_{j+1} - q_j, p'_j \rangle \\
&= \sum_{j=1}^{m} \langle q_{j+1} - q_j, p_j \rangle + \sum_{j=1}^{m} \langle q_{j+1} - q_j, p'_j - p_j \rangle \\
&= \sum_{j=1}^{m} \langle q_{j+1} - q_j, p_j \rangle \\
&= \ell_T(q).
\end{aligned}
$$

Note that we used (50) to show that

$$\langle q_j - q'_j, p'_{j-1} - p'_j \rangle = 0 = \langle q_{j+1} - q_j, p'_j - p_j \rangle.$$

For the remainder of this section, we fix $n = 2$. We proceed with the case $m = 2$. In other words, for each choice of faces F_1, F_2 of K and for each choice of faces G_1, G_2 of T, we compute closed polygonal curves

$$q = (q_1, q_1) \text{ and } p = (p_1, p_2)$$

fulfilling (2) such that $q_j \in F_j$ and $p_j \in G_j$ for $j \in \{1, 2\}$. Note that Proposition 71 suggests that we ask for

$$q_j \in \operatorname{relint}(F_j) \text{ and } p_j \in \operatorname{relint}(G_j).$$

Instead, for the sake of simplicity, we allow $q_j \in \partial F_j$ and replace $N_K(q_j)$ with $N_K(z)$ for some $z \in \operatorname{relint}(F_j)$ (this applies analogously to $p_j \in G_j$). The proof of Proposition 71 extends directly to this case.

After considering all choice of F_1, F_2, G_1, G_2, we compare the ℓ_T-length of all found closed polygonal curves. Before starting the calculation, it is beneficial to check whether

$$N_K(q_1) \cap -N_K(q_2) \text{ and } N_T(p_1) \cap -N_T(p_2)$$

are nonempty. The reason for this is, that the existence of a pair (q, p) of closed polygonal curves satisfying (2) implies:

$$
\begin{aligned}
-N_K(q_2) \ni p_2 - p_1 = -(p_1 - p_2) \in N_K(q_1), \\
N_T(p_1) \ni q_2 - q_1 = -(q_1 - q_2) \in -N_T(p_2).
\end{aligned}
$$

Note that the normal cones $N_K(q_j), N_T(p_j)$ only depend on the faces F_j, G_j. So, in the following, we can assume that these intersections are indeed nonempty. The goal is now to calculate a pair of suitable polygonal curves (q, p) if possible. For this, it is helpful to distinguish whether the faces F_j, G_j are facets (i.e., edges) or vertices. We consider the following cases:

1) F_1, F_2, G_1, G_2 are vertices.

2) F_1, F_2 are vertices and among G_1, G_2 there is at least one facet.
3) Among F_1, F_2 as well as among G_1, G_2 there is at least one facet.

All remaining cases can be covered by switching the roles of K and T.

The first case is easy. If all chosen faces are vertices, the resulting closed polygonal curves are unique and (2) can be checked directly.

We start the second case by assuming that both G_1 and G_2 are facets. Therefore, $N_T(p_1)$ and $N_T(p_2)$ are one-dimensional cones. Let

$$w_j \in N_T(p_j)\backslash\{0\} \text{ for } j \in \{1,2\}.$$

We can ensure that

$$q_2 - q_1 \in N_T(p_1)$$

holds by checking whether w_1 is a positive multiple of $q_2 - q_1$. If this is the case,

$$q_1 - q_2 \in N_T(p_2)$$

follows directly since we assume that

$$N_T(p_1) \cap -N_T(p_2)$$

is nonempty. Alternatively, we can check whether w_2 is a positive multiple of $q_1 - q_2$ and get

$$q_2 - q_1 \in N_T(p_1)$$

for free. It remains to solve the following problem:

$$\begin{aligned} &\text{Find } p_1, p_2 \text{ such that:} \\ &\quad p_1 \in G_1\,,\ p_2 \in G_2, \\ &\quad p_2 - p_1 \in -N_K(q_2), \\ &\quad p_1 - p_2 \in -N_K(q_1). \end{aligned}$$

The constraints can be expressed with linear equations and inequalities. For this, recall the definition of the outer normal cone of a convex set C at $z \in \partial C$:

$$N_C(z) = \{v\colon \langle v, y - z\rangle \leqslant 0\,,\ \forall\ y \in C\}.$$

If C is a convex polytope, it is sufficient to demand

$$\langle v, y - z\rangle \leqslant 0$$

for every vertex y of C. Therefore, membership of $N_C(z)$ can be modeled by finitely many linear inequalities. Altogether, finding suitable points p_1, p_2, can be realized by using linear programming techniques. The same approach can be used if either G_1 or G_2 is a vertex. In this case, the linear program remains unchanged except for the fact that one of the two variable vectors is replaced by a constant vector.

In the third case, we start with the assumption that F_1, F_2, G_1, G_2 are facets. Then, all relevant normal cones are one-dimensional and we let

$$u_j \in N_K(q_j)\backslash\{0\} \text{ and } w_j \in N_T(p_j)\backslash\{0\} \text{ for } j \in \{1,2\}.$$

Now, we solve the following problem:

$$\begin{aligned}
&\text{Find } q_1, q_2, p_1, p_2, \alpha_1, \alpha_2 \text{ such that:}\\
&\quad q_1 \in F_1\,,\ q_2 \in F_2\,,\ p_1 \in G_1\,,\ p_2 \in G_2,\\
&\quad \alpha_1, \alpha_2 \geqslant 0,\\
&\quad q_2 - q_1 = \alpha_1 w_1,\\
&\quad p_2 - p_1 = -\alpha_2 u_2.
\end{aligned}$$

Similar to the previous case, this problem is a linear program. Note that the last two constraints suffice to imply (2) since

$$N_T(p_1) \cap -N_T(p_2) \ \text{ and } \ N_K(q_1) \cap -N_K(q_2)$$

are nonempty. If not all chosen faces are facets, for instance if G_1 is a vertex, the linear program has to be changed in two ways. First, much like in the second case, the corresponding variable vector, here p_1, is replaced by a constant vector. Second, if G_1 is a vertex, then the normal cone $N_T(p_1)$ is no longer one-dimensional and the definition of w_1 does not make sense any more. However, in this case, G_2 is a facet and we replace the constraint

$$q_2 - q_1 = \alpha_1 w_1 \ \text{ with } \ q_1 - q_2 = \alpha_1 w_2.$$

We apply this reasoning also when F_1 or F_2 is not a facet.

There may be multiple ways to choose (p, q) for given faces F_1, F_2, G_1, G_2. If this is the case, our algorithm chooses (if possible) p such that $N_K(q_j)$ is one-dimensional for $j \in \{1, 2\}$ (or equivalently such that q_1, q_2 are not vertices of K). This is achieved in the following way. If F_1 or F_2 is a vertex, the resulting closed polygonal curve q always contains a vertex of K. So, we assume both F_1 and F_2 are facets. If q_1 and q_2 are smooth points (i.e., lie in the interior of F_1 and F_2), then there is nothing to do. Otherwise, we denote

$$N_K(q_1) = \mathbb{R}_+ u \ \text{ and } \ N_K(q_2) = \mathbb{R}_+(-u)$$

for some vector $u \in \mathbb{R}^2$. Let $v \neq 0$ be a vector orthogonal to u. Moving q_j along the facet F_j can only be done in at most two directions: v or $-v$. If we can move both q_1 and q_2 in the same direction, we simply translate the closed polygonal curve q. If q_1 and q_2 can only be moved in opposite directions, it is necessary to check whether the normal cones $N_T(p_1)$ and $N_T(p_2)$ allow such movement. If not, it is not possible to find suitable points

$$q_1 \in \operatorname{relint}(F_1) \ \text{ and } \ q_2 \in \operatorname{relint}(F_2).$$

This concludes the algorithm for the case $m = 2$. Next, we will discuss the algorithm for $m = 3$. We start as described in the previous subsection and choose facets F_1, F_2, F_3 of K. For each $j \in \{1, 2, 3\}$, we let $n_{K,j}$ be the outer unit normal vector of K at some point in the relative interior of F_j. If possible, we construct a triangle Δ by only using negative multiples of these three vectors. Here a triangle is the convex hull of three affinely independent points. This can easily be done by solving a system of linear equations. The task is now to find all (Δ, T)-inbodies which we define by:

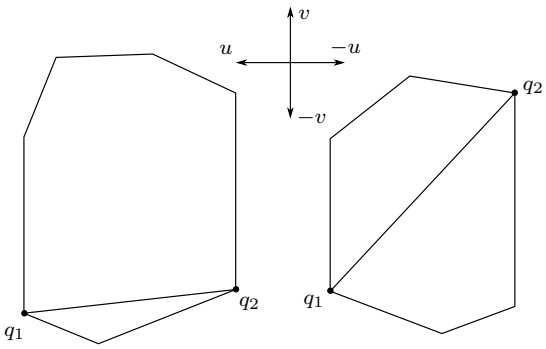

Fig. 15 Two closed polygonal curves such that q_1, q_2 are vertices of K. On the left, we can translate q_1, q_2 upwards (in direction v). On the right, we need to move q_1 upwards and q_2 downwards. Whether this is possible depends on $N_T(p_1)$ and $N_T(p_2)$.

Definition 71 ((Δ, T)-inbody) *Let $\Delta \subseteq \mathbb{R}^2$ be a triangle and $T \subseteq \mathbb{R}^2$ be a nonempty convex set. A (Δ, T)-inbody is a set $S \subseteq \mathbb{R}^2$ which fulfills:*

(i)
$$S = \lambda\Delta + u$$
for some $\lambda > 0$ and $u \in \mathbb{R}^2$.

(ii) All three vertices of S are contained in ∂T.

(iii) If
$$\{v_1, v_2, v_3\} = V(S),$$
then there is no hyperplane H through the origin, such that
$$N_T(v_1) \cup N_T(v_2) \cup N_T(v_3)$$
is contained in one of the two closed halfspaces defined by H.

Here, $V(S)$ denotes the set of vertices of S. The idea behind such a (Δ, T)-inbody is to find the closed polygonal curve p. More precisely, we will choose p as the closed polygonal curve having v_1, v_2, v_3 as vertices. (i) ensures that the pair (p, q) fulfills the second line of (2). Later on, in this algorithm, we need to construct another triangle from outer normal vectors at the points v_1, v_2, v_3. Property (iii) ensures that this is possible. Finding all (Δ, T)-inbodies is simple due to the following proposition.

Proposition 72 (Lemma 4.3.8 in [18]) *Let $T \subset \mathbb{R}^2$ be a convex polytope and $\Delta \subset \mathbb{R}^2$ be a triangle. If there is a (Δ, T)-inbody, then*
$$S^* = \lambda^*\Delta + u^*$$
is the only (Δ, T)-inbody, where λ^*, u^* is a solution of
$$\begin{aligned} \max \quad & \lambda \\ \text{such that} \quad & \lambda \geqslant 0, \ u \in \mathbb{R}^2, \\ & \lambda\Delta + u \subseteq T. \end{aligned} \tag{51}$$

Proof Let

$$S = \operatorname{conv}\{v_1, v_2, v_3\}$$
$$S^* = \operatorname{conv}\{w_1, w_2, w_3\}$$

as in the claim. Here, we choose the names for the vertices such that there are $\mu > 0$ and $x \in \mathbb{R}^2$ with

$$w_j = \mu v_j + x$$

for $j \in \{1, 2, 3\}$. Note that such a naming is possible since S is a scaled translate of S^*. We start by letting H_1, H_2, H_3 be three lines defined by

$$w_2, w_3 \in H_1\,,\ \ w_1, w_3 \in H_2 \ \text{ and } \ w_1, w_2 \in H_3.$$

Each of these lines is the affine hull of a facet of S^*. Furthermore, each line H_j devides the plane $\mathbb{R}^2$ in two halfspaces. We denote the halfspace which contains S^* by H_j^+. If $S = S^*$ there is nothing to show. So, we assume $S \neq S^*$. Because S is a smaller (or equal size) version of S^*, it is contained in H_j^+ for some $j \in \{1, 2, 3\}$. Without loss of generality we assume $S \subset H_1^+$, as the other cases can be treated similarly. This situation is depicted in Figure 16.

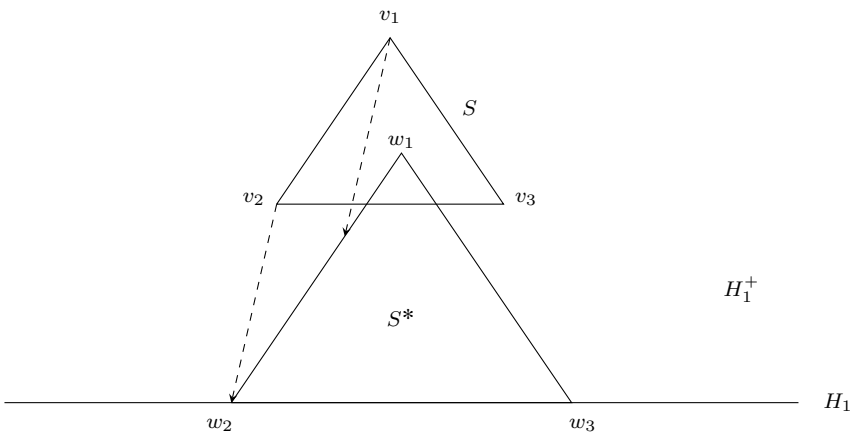

Fig. 16 Depiction of the (Δ, T)-inbody S with vertices v_1, v_2, v_3 and the triangle S^* with vertices w_1, w_2, w_3. The dashed arrows indicate the location of the line segment $[v_1, v_2]$ after shifting it by $w_2 - v_2$.

Consider the triangle with vertices v_1, w_2, w_3. Neither v_2 nor v_3 can be contained in the interior of this triangle. The reason for this is that by property (ii) of Definition 71 v_2 and v_3 are boundary points of T but

$$\operatorname{conv}\{v_1, w_2, w_3\} \subseteq T.$$

We will now show that property (iii) is violated for S. This contradiction implies $S = S^*$ and finishes the proof. More precisely, we claim that $N_T(v_j)$ is contained in the halfspace

$$I = \{x \in \mathbb{R}^2 \colon \langle w_2 - v_2, x\rangle \leqslant 0\}.$$

for every $j \in \{1,2,3\}$. By definition of the outer normal cone it follows immediately that $N_T(v_2) \subset I$. Because S is a smaller (or equal size) version of S^* we have $w_2 - w_3 = \alpha(v_2 - v_3)$ for some $\alpha \geqslant 1$. For any $x \in N_T(v_3)$ this implies:

$$\begin{aligned}\langle w_2 - v_2, x\rangle &= \langle w_2 - v_2 + w_3 - w_3 + v_3 - v_3, x\rangle \\ &= \langle (w_2 - w_3) - (v_2 - v_3), x\rangle + \langle w_3 - v_3, x\rangle \\ &\leqslant \langle w_2 - w_3, x\rangle - \langle v_2 - v_3, x\rangle \\ &= (\alpha - 1)\langle v_2 - v_3, x\rangle \\ &\leqslant 0.\end{aligned}$$

Thus,

$$N_T(v_3) \subseteq I.$$

Next, we observe that if we shift S by $w_2 - v_2$, then the face $[v_1, v_2]$ of S is contained in $[w_1, w_2]$ (see Figure 16). So,

$$v_1 + (w_2 - v_2)$$

is contained in $S^* \subseteq T$. Now, for any $y \in N_T(v_1)$ we get:

$$0 \geqslant \langle v_1 + (w_2 - v_2) - v_1, y\rangle = \langle w_2 - v_2, y\rangle$$

As desired this yields

$$N_T(v_1) \subseteq I.$$

We point out that there is not always a (Δ, T)-inbody. For example, if (51) has multiple optimal solutions, the proof shows that there is no (Δ, T)-inbody. An example for this situation is depicted in Figure 17.

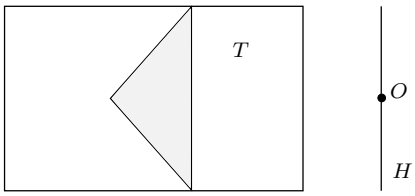

Fig. 17 A situation, where there is no (Δ, T)-inbody. The gray area is a scaled translate of Δ. Also shown is a hyperplane H through the origin O.

As we can see, there are multiple optimal solutions for (51), since we can shift the gray area to the left and right. The only way to have all vertices of this area on ∂T is to shift it to the left. Then all the corresponding normal vectors are contained in the halfspace on the left of H.

With Proposition 72, we can reduce the search of (Δ, T)-inbodies to a simple maximization problem which we can formulate as a linear problem (LP). It is clear that this problem has an optimal solution as long as T is compact. After we found a solution λ^*, u^*, we check whether

$$S^* = \lambda^* \Delta + u^*$$

fulfills properties (i)-(iii) in Definition 71. As pointed out before, it suffices to consider any optimal solution. If S^* does not meet properties (i)-(iii), then there is no (Δ, T)-inbody and we proceed with the next choice of F_1, F_2, F_3. Otherwise, we take a unit normal vector from $N_T(v)$ for each vertex v of S. We construct another triangle as before, using only positive multiples of these normal vectors. It is notable, that v may be a vertex of T. In this case, $N_T(v)$ is not one-dimensional and the choice of the corresponding normal vector is not unique. As mentioned earlier, one way of handling this case is to slightly perturb the vertices of T. As follows from Proposition 71, it is sufficient to find one vector in $N_T(v)$ such that the remaining steps of the algorithm are carried out successfully. So, another way is to sample $N_T(v)$, i.e., only consider finitely many unit normal vectors. The remainder of the algorithm is straight forward and only uses strategies which have been discussed before.

Regarding efficiency, we point out that the algorithm for finding closed polygonal curves with 2 vertices takes

$$\mathcal{O}(|V(K)|^2 \cdot |V(T)|^2)$$

iterations before it terminates. This is clear since the number of facets of a two dimensional convex polytope equals the number of its vertices. For each of the two convex polytopes K and T, the algorithm considers at most one choice for F_1, F_2, G_1, G_2 per iteration. In each iteration, we search for the points q_1, q_2, p_1, p_2. In the worst case (i.e., if F_1, F_2, G_1, G_2 are facets), we solve an LP with 10 variables and

$$2(|V(K)| + |V(T)| + 3)$$

constraints. In order to solve LPs, we use the `conelp` solver of CVXOPT. This solver relies on a primal-dual path-following method. It is well known that linear problems can be solved in polynomial time (cf. [24]).

For finding closed polygonal curves with 3 vertices, the algorithm takes

$$\mathcal{O}(|V(K)|^3)$$

iterations to consider every choice of faces F_1, F_2, F_3. In each iteration, we solve the maximization problem stated in Proposition 72. This is an LP with 3 variables and

$$3|V(T)| + 1$$

constraints. The remainder of the loop for F_1, F_2, F_3 can be realized with running time $\mathcal{O}(1)$. Finally, we note that the calculations for each choice of faces are independent of each other. Therefore, we use parallel computing to speed up the calculations.

In the following Table 1, we examine the running time of the algorithm outlined in the Sections 7.1 and 7.3. To do so, we let K and T be two-dimensional convex polytopes and consider three different cases. First, we regard the case where both K and T have the same number of vertices. In the second case, K will have a small number of vertices and, in the third case, we chose T to

have few vertices. Each time, the convex polytopes K and T have been chosen randomly in the following way. We take a number of normally distributed points and compute their convex hull. Since many of these points will be close to the origin and are unnecessary, we scale each of these normally distributed points to have a random length in $[1, 3]$ before we calculate the convex hull. As the number of points grows, the convex hull resembles a ball of radius 3 due to the normal distribution. So, to accelerate this process, we reduced the interval $[1, 3]$ for convex polytopes with many vertices ($\geqslant 30$).

We compare the running time for finding an ℓ_T-minimizing closed polygonal curve with 2 vertices to the running time for finding an ℓ_T-minimizing closed polygonal curve with 3 vertices. As we can see the running time for 2 vertices is approximately symmetric in $|V(K)|$ and $|V(T)|$. In contrast to this, the running time for 3 vertices mainly depends on $|V(K)|$.

All calculations have been done on a Dell Latitude E6530 laptop with Intel Core i7-3520M processor, 2.9 GHz (capable of running four threads). The algorithm and a detailed description on how to choose the input is available on the website www.github.com/S-Krupp/EHZ-capacity-of-polytopes.

8 A note on Minkowski billiard trajectories on obtuse triangles

It is an open problem for already a long time whether obtuse triangles $\Delta_{>\frac{\pi}{2}} \subset \mathbb{R}^2$ possess closed regular $(\Delta_{>\frac{\pi}{2}}, B_1^2)$-Minkowski billiard trajectories (cf. [14]), i.e., closed classical Euclidean billiard trajectories. The strongest result so far is the existence of a closed classical Euclidean billiard trajectory on triangles with angles not greater than 100° (cf. [25]).

Obviously, there cannot exist closed regular $(\Delta_{>\frac{\pi}{2}}, B_1^2)$-Minkowski billiard trajectories with two bouncing points (cf. [17, Proposition 2.6]). Using our algorithm described in Section 7, we can reason that there cannot be closed regular $(\Delta_{>\frac{\pi}{2}}, B_1^2)$-Minkowski billiard trajectories with three bouncing points neither: searching for closed regular $(\Delta_{>\frac{\pi}{2}}, B_1^2)$-Minkowski billiard trajectories with three bouncing points means, among other aspects, searching for closed polygonal curves (one for every choice of order of the edges of $\Delta_{>\frac{\pi}{2}}$) with three vertices consisting of line segments given by negative multiples of the outer unit normal vectors at the edges of $\Delta_{>\frac{\pi}{2}}$ that have all three vertices on the sphere S^1. But since $\Delta_{>\frac{\pi}{2}}$ is obtuse, for geometrical reasons neither of these two closed polygonal curves with vertices (which are meant to be the closed dual billiard trajectories on B_1^2) on S^1 are in $F(B_1^2)$. With Proposition 39, this implies that there is no closed regular $(\Delta_{>\frac{\pi}{2}}, B_1^2)$-Minkowski billiard trajectory.

However, instead of solving the original problem, we can use our algorithm from Section 7 in order to determine the family $\mathcal{T}$ of all convex bodies $T \subset \mathbb{R}^2$ admitting the existence of a closed regular $(\Delta_{>\frac{\pi}{2}}, T)$-Minkowski billiard trajectory with three bouncing points: Let $D(\alpha)$ be the rotation matrix in $\mathbb{R}^2$ rotating counter clockwise by angle α. Then, $\mathcal{T}$ is the set of all convex bodies

$\lvert V(K)\rvert$	$\lvert V(T)\rvert$	time for 2 vert. in s.	time for 3 vert. in s.
5	5	1.28237	0.22158
10	10	7.85068	2.54632
15	15	27.39097	11.71706
20	20	54.75007	25.03621
25	25	82.30637	60.11260
30	30	125.05111	110.61238
35	35	170.22497	181.65273
40	40	259.88731	302.30844
45	45	266.73415	385.03827
50	50	361.56254	609.04153
55	55	451.56054	786.54793
5	10	3.02675	0.22164
5	15	5.57299	0.22637
5	20	11.40925	0.22114
5	25	16.91015	0.23931
5	30	19.89903	0.21554
5	35	23.96365	0.39383
5	40	29.05107	0.21106
5	45	32.18348	0.54072
5	50	36.41029	0.38885
5	55	49.02657	0.57020
5	65	59.80655	0.79811
5	75	67.33951	0.72834
10	5	3.39280	1.19793
15	5	5.74532	4.32675
20	5	10.50168	11.31948
25	5	14.59203	24.99738
30	5	17.76183	45.60183
35	5	20.62535	90.62127
40	5	23.89690	137.52914
45	5	25.73543	170.43779
50	5	30.23246	266.66650
55	5	33.68478	345.84228
65	5	41.49229	558.22820
75	5	51.92742	937.36931

Table 1 Running times for the calculations of an ℓ_T-minimizing closed (K,T)-Minkowski billiard trajectory with 2 (resp. 3) vertices as described in Section 7. All numbers are given in seconds.

$T \subset \mathbb{R}^2$ for which there are either $\lambda^+ > 0$ and $\xi^+ \in \mathbb{R}^2$ with

$$(\lambda^+ D(\pi/2)\Delta_{>\frac{\pi}{2}} + \xi^+) \cap \partial T = \{p_1, p_2, p_3\} \in F(T)$$

and maximally spanning unit normal vectors $n_T(p_1), n_T(p_2), n_T(p_3)$ in the outer normal cones at p_1, p_2, p_3, or there are $\lambda^- > 0$ and $\xi^- \in \mathbb{R}^2$ such that the same holds for

$$\lambda^- D(-\pi/2)\Delta_{>\frac{\pi}{2}} + \xi^-.$$

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References

1. Abbondandolo, A., Majer, P.: A non-squeezing theorem for convex symplectic images of the Hilbert ball. Calc. Var. Partial Differ. Equ. 54, 1469–1506 (2015)
2. Akopyan, A., Balitskiy, A.: Billiards in convex bodies with acute angles. Isr. J. Math. 216, 833–845 (2016)
3. Akopyan, A., Balitskiy, A., Karasev, R., Sharipova, A.: Elementary approach to closed billiard trajectories in asymmetric normed spaces. Proc. Amer. Math. Soc. 144, 4501–4513 (2015)
4. Akopyan, A., Karasev, R.: When different norms lead to same billiard trajectories?. Eur. J. Math. (2020)
5. Akopyan, A., Schwartz, R., Tabachnikov, S.: Billiards in ellipse revisited. Eur. J. Math. (2020)
6. Alkoumi, N., Schlenk, S.: Shortest closed billiard orbits on convex tables. Manuscr. Math. 147, 365–380 (2015)
7. Artstein-Avidan, S., Ostrover, Y.: Bounds for Minkowski billiard trajectories in convex bodies. Int. Math. Res. Not. IMRN 1, 165–193 (2014)
8. Artstein-Avidan, S., Karasev, R., Ostrover, Y.: From symplectic measurements to the Mahler conjecture. Duke Math. J. 163(11), 2003–2022 (2014)
9. Artstein-Avidan, S., Florentin, D.I., Ostrover, Y., Rosen, D.: Duality of caustics in Minkowski billiards. Nonlinearity 31, 1197ff (2018)
10. Bezdek, D., Bezdek, K.: Shortest billiard trajectories. Geom. Dedicata 141, 197–206 (2009)
11. Blagojevic, P., Harrison, M., Tabachnikov, S., Ziegler, G.M.: Counting periodic trajectories of Finsler billiards SIGMA 16, (2020)
12. Gutkin, E.: Billiard dynamics: An updated survey with the emphasis on open problems. Chaos (2012)
13. Gutkin, E., Tabachnikov, S.: Billiards in Finsler and Minkowski geometries. J. Geom. Phys. 40, 277–301 (2002)
14. Halbeisen, L., Hungerbühler, N.: On Periodic Billiard Trajectories in Obtuse Triangles. SIAM Rev. 42, 657–670 (2000)
15. Halpern, B.: Strange Billiard Tables. Trans. Am. Math. Soc. 232, 297–305 (1977)
16. Katok, A.: The billiard table as a mathematician's playground. MCCME (2002)
17. Krupp, S., Rudolf, D.: A regularity result for shortest generalized billiard trajectories in convex bodies in $\mathbb{R}^n$. Geom. Dedicata 216(55) (2022)
18. Krupp, S.: Calculating the EHZ Capacity of Polytopes. Dissertation to download: https://kups.ub.uni-koeln.de/36196/, Universität zu Köln (2020)
19. Mahler, K.: Ein Übertragungsprinzip für konvexe Körper. Časopis Pěst. Mat. Fys. 68, 93–202 (1939)
20. Ostrover, Y.: When symplectic topology meets Banach space geoemtry. Proceedings of the ICM, Seoul 2, 959–981 (2014)
21. Radnovic, M.: A note on billiard systems in Finsler plane with elliptic indicatrices. Publ. de l'Institut Math. 74, 97–102 (2003)
22. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Springer (2004)
23. Rudolf, D.: The Minkowski Billiard Characterization of the EHZ-Capacity of Convex Lagrangian Products. J. Dyn. Differ. Equ. (2022)
24. Schrijver, A.: Theory of linear and integer programming. John Wiley & Sons Inc. (1998)
25. Schwarz, R.E.: Obtuse Triangular Billiards II: One Hundred Degrees Worth of Periodic Trajectories. Exp. Math. 18 (2009)
26. Tabachnikov, S.: Geometry and billiards. Amer. Math. Soc. (2005)
27. Viterbo, C.: Metric and isoperimetric problems in symplectic geometry. J. Amer. Math. Soc. 13(2), 411–431 (2000)