# Theoretical and numerical analysis of the Minimal Residual Method for solving the neutron transport equation in spherical geometry

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**Research Article**

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# Theoretical and numerical analysis of the Minimal Residual Method for solving the neutron transport equation in spherical geometry

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**Abstract**

This paper focuses on an infinite dimensional adaptation of the Minimal Residual method for solving the neutron transport equation in spherical geometry. This method is based on a splitting of the collision operator strategy taking into account the characteristics of the transport operator. The theoretical and numerical analysis of this algorithm are given and compared to the methods given in [1, 2, 3].
**Keywords:**Neutron transport equation, Integro-differentials operators, Splitting, Minimal Residual method.

## 1 Introduction

In practice, neutron transport problems are solved in the industry using diffusion equations [4, 5, 6, 7]. They are very extremely expensive, but do not take into account the behavior direction of neutrons [1, 2, 3]. The physical modeling of the neutron population in several geometries and in particular in the spherical geometry is to solve an integro-differential transport equation. The numerical resolution of this equation has been the subject of several scientific works [3, 8] and the references therein. It requires the use of iterative algorithms. In effect a direct discretization would lead to the resolution of a large linear system practically full due to the presence of the integral operator. The idea is therefore to iterate over this integral term and to solve a hyperbolic partial differential equation at each iteration. Two problems then arise: Firstly, the convergence of this algorithm is often extremely slow and secondly the discretization presents a significant number of unknown flows, even when the angular variables are discretized as finely as possible. A splitting of the collision operator [2] made it possible to define new algorithms (Jacobi and Gauss-Seidel in infinite dimension) [1]. This study shows that the Gauss-Seidel method converges twice as fast as the Jacobi method and that the latter converges at least fast as the standard algorithm and diffusion synthetic acceleration algorithm [3].

Due to the non-self-adjoint nature of the operators involved in the model and due to the impossibility of using a method inspired by that of the conjugate gradient algorithm, our work investigates the solution of the neutron transport equation in spherical geometry with another iterative method independent of the choice of discitization. More specifically, it focuses on the Minimal Residual algorithm adapted to the non-self-adjoint character of the matrix of operators, in order to further speed up the convergence.

This work is divided into 5 sections and organized as follows: The second section is devoted to introducing mathematically the model of neutron population, and in order to give meaning to boundary conditions, we recall the adequate functional spaces so that the algorithm resulting from such a splitting is well posed. The functional space on which the transport equation admits a unique solution has been justified. In the third one, we the recall principle splitting method of the collision operator [2, 9]. In section 4, we improve the speed of convergence by a infinite dimensional adaptation of Minimal Residual method algorithm based on the splitting method. The theoretical aspect of this algorithm is given. In the last section, we present the numerical aspect of this algorithm. Several simulations were performed and compared with existing methods [1, 2, 3].

## 2 Mathematical model

The nuclear physics inside a nuclear reactor core is governed by the transport of neutrons and the interactions between neutrons and nucleus. Therefore, the de-sign, analysis and control of a nuclear reactor requires solving the neutron transport equation in spherical geometry. The solution of this equation determines the neutron distribution in the reactor [1, 2, 8, 9, 10, 11, 12] and the references therein. This problem is governed by the following equation

$$\frac{\mu}{r^2}\frac{\partial}{\partial r}\left(r^2 u\left(r,\mu\right)\right)+\frac{1}{r}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)u\left(r,\mu\right)\right] + \sigma u(r,\mu) = \int_{-1}^{1} k(\mu,\mu')u(r,\mu')d\mu' + S(r,\mu), \quad (1)$$

for all $(r,\mu)$ in $(0,R)\times(-1,1)$ and completed by the following boundary condition

$$u(R,\mu)=0,\ \forall\mu<0. \quad (2)$$

Equation (1) can be written

$$\mu\frac{\partial u}{\partial r}(r,\mu)+\frac{1-\mu^2}{r}\frac{\partial u}{\partial \mu}(r,\mu)+\sigma u(r,\mu)=\int_{-1}^{1}k(\mu,\mu')u(r,\mu')d\mu'+S(r,\mu). \quad (3)$$

The region occupied by the reactor media in a sphere of radius $R > 0$, $r$ is the distance from the center of the sphere, $\mu$ is the cosines of the angle the neutron velocity makes with the radius vector, $\sigma$ is the scattering cross-section accounting for neutron-domain interactions, supposed constant, $k(\mu,\mu')$ is the scattering fission kernel, and $S$ is a non negative source term given in $L^2(\Omega)$.

The problem (1)-(2) can be written in the following form: Find the flux of neutrons $u : \Omega := (0,R)\times(-1,1) \to \mathbb{R}^+$, solution of

$$\begin{cases} Tu(r,\mu) = Ku(r,\mu) + S(r,\mu) & \text{for } (r,\mu)\in\Omega, \\ u \in \mathcal{W}, \end{cases} \quad (4)$$

where $T$ be the transport operator defined by

$$Tu(r,\mu) = \frac{\mu}{r^2}\frac{\partial}{\partial r}\left(r^2 u(r,\mu)\right) + \frac{1}{r}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)u(r,\mu)\right] + \sigma u(r,\mu),$$

$K$ be an integral collision operator of positive kernel $k$ given by

$$K(r,\mu) = \int_{-1}^{1} k(\mu,\mu')u(r,\mu')d\mu'.$$

and

$$\mathcal{W} := \Big\{u \in L^2(\Omega) \text{ and } \frac{\mu}{r^2}\frac{\partial}{\partial r}\left(r^2 u(r,\mu)\right)+\frac{1}{r}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)u(r,\mu)\right] \in L^2(\Omega)$$
$$\text{such that } u(R,\mu)=0,\ \forall \mu<0\Big\}. \quad (5)$$

We make the following hypothesis:

$(H_1)$ $\rho(\Theta) < \frac{c}{2}$, with $\Theta = T^{-1}K$ and $0 < c \leq 1$, where $\rho$ designate the spectral radius.

$(H_2)$ $k$ is non-negative and bounded function.

**Remark 1** *The operator $A$ defined by*

$$A := \frac{\mu}{r^2}\frac{\partial}{\partial r}\left(r^2 u(r,\mu)\right) + \frac{1}{r}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)u(r,\mu)\right]$$

*is m-accretive, then $T^{-1}$ exists and the operator $T^{-1}K$ is compact [8, 9].*

# 3 Splitting method

We denote by $T_1$ (respectively $T_2$) the neutron transport operator defined on $\Omega_1 = (0,R)\times(0,1)$ (respectively $\Omega_2 = (0,R)\times(-1,0)$). Let $K_{ij}$, $i,j \in \{1,2\}$, be the integral operators whose kernel are

$$k_{ij}(r,\mu,\mu') = k(\mu,\mu') \times \mathbb{1}_{\Omega_i}(r,\mu) \times \mathbb{1}_{\Omega_j}(x,\mu'),$$

with $\mathbb{1}_{\Omega_i}$ the indicator function of $\Omega_i$, $i \in \{0,1\}$. We deduce that

$$K_{ij}(u) = K\left(u \times \mathbb{1}_{\Omega_j}\right) \times \mathbb{1}_{\Omega_i},\ i \in \{0,1\}.$$

We obtain a splitting of the integral operator $K$ in the form

$$K = \sum_{i=1}^{2}\sum_{j=1}^{2} K_{ij}.$$

Note that $K_{ij}$ is an operator acting from $L^2(\Omega)$, using only the values of $u$ on $\Omega_j$, such that $K_{ij}u$ has its support in $\Omega_i$. The problem (4) is spited into two coupled problems defined respectively on $\Omega_1$ and $\Omega_2$, admitting respectively $u_1$ and $u_2$ as solutions. This splitting operator method resides in adjusting conditions on the border $\Gamma = (0,R)\times\{0\}$ (see [2]). The solution $u$ of (4) is given under the form $u = (u_1,u_2)$, where $u_1$ and $u_2$ are the solutions of the coupled system:

$$\begin{pmatrix} T_1 - K_{11} & -K_{12} \\ -K_{21} & T_2 - K_{22} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \tag{6}$$

with $S_i = S \times \mathbb{1}_{\Omega_i}$ for $i \in \{1,2\}$.

The idea in [1] is to introduce and study various algorithms, relying on a splitting of the collision operator, and adapted from the methods of Jacobi, Gauss-Seidel. We will seek a method that gives good rate of the convergence, but does not need any extra parameter calculation.

Let $\theta_{ij} = T_i^{-1}K_{ij}$, and $\widetilde{S}_i = T_i^{-1}S_i,\ i,j \in \{1,2\}$. The system (6) can be written:

$$\begin{pmatrix} I - \theta_{11} & \theta_{12} \\ \theta_{21} & I - \theta_{22} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \widetilde{S}_1 \\ \widetilde{S}_2 \end{pmatrix}. \tag{7}$$

Applying the diagonal preconditioning method to this system, we obtain

$$\underbrace{\begin{pmatrix} I & -(I-\theta_{11})^{-1}\theta_{12} \\ -(I-\theta_{22})^{-1}\theta_{21} & I \end{pmatrix}}_{\mathcal{A}}\underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} -(I-\theta_{11})^{-1}\widetilde{S}_1 \\ -(I-\theta_{22})^{-1}\widetilde{S}_2 \end{pmatrix}}_{\mathcal{B}}. \tag{8}$$

**Remark 2** *For all $i \in \{1,2\}$, we have $\|\theta_{ij}\|_2 = \|T_i^{-1}K_{ij}\|_2 \le \rho\left(\theta_{ij} = T_i^{-1}K_{ij}\right)$. Then*

$$\|\ \theta_{ij}\ \|_2 \le \frac{c}{2}\ \textit{for}\ (i,j) \in \{1,2\}\times\{1,2\}. \tag{9}$$

**Remark 3** *From* (9), *we deduce*

$$\begin{aligned} \| (I-\theta_{ii})^{-1}\theta_{ij} \| &\leq \| (I-\theta_{ii})^{-1} \| \times \| \theta_{ij} \|, \\ &= \frac{c}{2}\sum_{p=0}^{+\infty} \| \theta_{ii}^{p} \|_2, \\ &\leq \frac{c}{2}\,\frac{1}{1-\frac{c}{2}} = \frac{c}{2-c} =: d \end{aligned} \tag{10}$$

We denote by $\langle .,. \rangle$ the scalar product usual in $L^2(\Omega)\times L^2(\Omega)$. Similarly, $\| \, . \, \|_2$ will represent the norm in $L^2(\Omega)$ associated to this scalar product.

We introduce, the Minimal Residual method. This algorithm was originally proposed by [13] in the finite dimensional case. It can be applied to symmetric indefinite systems. It minimizes $\mathcal{E}(X) = \| \mathcal{B}-\mathcal{A}X \|_2^2$ by the following algorithm:

1. Initially we choose $X^0$, calculate $r^0=\mathcal{B}-\mathcal{A}X^0$, and put $p^0=r^0,\ q^0=\mathcal{A}p^0$.
2. Compute for $k=0,1,...$ until $\| r^k \|_2 < \varepsilon$,

$$\begin{aligned} &\alpha^k = \tfrac{\langle r^k, q^k\rangle}{\langle q^k, q^k\rangle}, \\ &X^{k+1} = X^k + \alpha^k p^k, \\ &r^{k+1} = r^k - \alpha^k q^k, \\ &\beta^{k+1} = -\tfrac{\langle \mathcal{A}r^{k+1}, q^k\rangle}{\langle q^k, q^k\rangle}, \\ &p^{k+1} = r^{k+1} + \beta^{k+1}p^k, \\ &q^{k+1} = \mathcal{A}r^{k+1} + \beta^{k+1}q^k. \end{aligned}$$

In the following, we propose a theoretical aspect of the convergence to this algorithm.

# 4 The convergence

The residual term $\mathcal{E}(X)$ can be estimated by the the following proposition. It inspirited by the analysis given in [14] and it is easy to prove.

**Proposition 1** *Let $X^k$ be constructed by the preceding algorithm and starting from $X^0$. For $k \geq 0$, we have*

$$\mathcal{E}(X^{k+1}) \leq \mathcal{E}(X^k)\left(1-\frac{\langle r^k, \mathcal{A}r^k\rangle}{\langle r^k, r^k\rangle}\frac{\langle r^k, \mathcal{A}r^k\rangle}{\langle \mathcal{A}r^k, \mathcal{A}r^k\rangle}\right). \tag{11}$$

**Proposition 2** *Under the assumptions $(H_1)-(H_2)$, for all $X$, the operator $\mathcal{A}$ verifies*

$$\langle \mathcal{A}X, X\rangle \geq (1-d)\langle X, X\rangle \ \ \textit{and}\ \ \langle \mathcal{A}X, X\rangle \geq \frac{1}{1+d}\langle \mathcal{A}X, \mathcal{A}X\rangle. \tag{12}$$

**Proof 1** *Let* $X = (u_1, u_2)$. *We have*

$$\langle \mathcal{A}X, X\rangle = \| X \|_2^2 - \left\langle (I-\theta_{11})^{-1}\theta_{12}u_2, u_1 \right\rangle - \left\langle (I-\theta_{22})^{-1}\theta_{21}u_1, u_2 \right\rangle .$$

*According to* (10), *we obtain*

$$\langle \mathcal{A}X, X\rangle \geq \left(1 - \frac{c}{2-c}\right) \| X \|_2^2, \tag{13}$$

*and consequently the left hand inequality of* (12). *For the proof of the right hand inequality of* (12), *we consider* $\xi > 0$ *and we explicit*

$$\begin{aligned}\langle \mathcal{A}X, X\rangle - \xi \langle \mathcal{A}X, AX\rangle = (1-\xi) \| X \|_2^2 \\ - (1-2\xi)\left[\left\langle (I-\theta_{11})^{-1}\theta_{12}u_2, u_1 \right\rangle + \left\langle (I-\theta_{22})^{-1}\theta_{21}u_1, u_2 \right\rangle\right] \\ -\xi\left[\left\langle (I-\theta_{22})^{-1}\theta_{21}u_1, (I-\theta_{11})^{-1}\theta_{21}u_1 \right\rangle + \left\langle (I-\theta_{11})^{-1}\theta_{12}u_2, (I-\theta_{11})^{-1}\theta_{12}u_2 \right\rangle\right].\end{aligned} \tag{14}$$

*Taking* $\xi > \dfrac{1}{2}$, *and using* (10), *we have*

$$\langle \mathcal{A}X, X\rangle - \xi \langle \mathcal{A}X, AX\rangle \geq \left(1 + d - \xi (1+d)^2\right) \| X \|_2^2 . \tag{15}$$

*We choose* $\xi$ *such that* $1 + d - \xi(1+d)^2 = 0$, *then* $\xi = \dfrac{1}{1+d}$. *Consequently the inequality of the right hand of* (2) *is proved.*

The next theorem illustrates the convergence of the Minimal Residual algorithm and how can occur.

**Theorem 1** *Under the assumptions* $(H_1) - (H_2)$, *the Minimal Residual method converges, and the residual decreases at least at the following rate:*

$$\mathcal{E}(X^k) \leq c^k \, \mathcal{E}(X^0). \tag{16}$$

**Proof 2** *By inserting the inequalities of proposition 2 into (11), we get*

$$\mathcal{E}(X^{k+1}) \leq c \, \mathcal{E}(X^k). \tag{17}$$

*and consequently (16). Since* $c \in (0,1)$, *then* $\mathcal{E}(X^k)$ *converges towards* 0 *when* $k$ *goes to infinity. Using the left hand inequality of (11), we get*

$$\| X^{k+1} - \mathcal{A}^{-1}\mathcal{B} \|_2^2 \leq \frac{1}{1-d} \left\langle \mathcal{A}X^{k+1} - \mathcal{B}, X^{k+1} - \mathcal{A}^{-1}\mathcal{B} \right\rangle ; \tag{18}$$

*So that*

$$\| X^{k+1} - \mathcal{A}^{-1}\mathcal{B} \|_2 \leq \frac{1}{1-d}\sqrt{\mathcal{E}(X^{k+1})}.$$

*Which means that* $X^{k+1} \longrightarrow X$ *with* $\mathcal{A}X = \mathcal{B}$. *Hence the Minimal Residual method converges.*

# 5 Numerical results

In this section, we we are mainly interested in solving the transport problem following neutron: Find $u\ :\ (0,R)\times(-1,1)\to\mathbb{R}^+$ such that

$$\begin{cases} \mu\frac{\partial u}{\partial r}(r,\mu)+\frac{1-\mu^2}{r}\frac{\partial u}{\partial \mu}(r,\mu)+\sigma u(r,\mu)=\dfrac{\sigma c}{2}\displaystyle\int_{-1}^{1}u(r,\mu')d\mu'+S(r,\mu),\\ u(R,\mu)=0,\ \forall\ \mu<0. \end{cases} \tag{19}$$

## 5.1 Discritization

Let the integers $N\geq 1$ and $M\geq 1$. We discretize the intervals $(-1,1)$ and $(0,R)$ as follows:

$$\begin{array}{l} -1=\mu_{-M}<\mu_{-M+1}<...<\mu_0=0<\mu_1<...<\mu_{M-1}<\mu_M=1,\\ 0=r_0<r_1<r_2<...<r_{N-1}<r_N=R. \end{array}$$

We adopt the following notations:

$$\begin{array}{l} \overline{\Omega}=\ \cup_{i,j}K_{ij}=[r_i,r_{i+1}]\times[\mu_j,\mu_{j+1}]\,,\\ \tau=\mu_{j+1}-\mu_j,\ \mu_{j+\frac{1}{2}}=\frac{\mu_{j+1}+\mu_j}{2},\\ h=r_{i+1}-r_i,\ r_{i+\frac{1}{2}}=\frac{r_{i+1}+r_i}{2},\\ m_{ij}=\frac{1}{h\tau}\displaystyle\int_{r_i}^{r_{i+1}}\int_{\mu_j}^{\mu_{j+1}}u\left(r,\mu\right)drd\mu,\\ \gamma_{i,j}=\frac{1}{\tau}\displaystyle\int_{\mu_j}^{\mu_{j+1}}u\left(r_i,\mu\right)d\mu, \end{array}$$

for all $(i,j)\in[\![0,N-1]\!]\times[\![0,M-1]\!]$. It easy to prove that

$$m_{i,j}=\frac{1}{2}\left(\gamma_{i+1,j}+\gamma_{i,j}\right)\ \forall(i,j)\in[\![O,N-1]\!]\times[\![O,M-1]\!]. \tag{20}$$

Let $V_h$ the space of functions whose restriction to each rectangle $K_{ij}$ is a function written in the form $a+br\mu$ with $a$ and $b$ are the reals numbers. It can be written by

$$u(r,\mu)=\frac{2}{h}\left(r_{i+1}-\frac{r\mu}{\mu_{j+\frac{1}{2}}}\right)m_{i,j}-\frac{2}{h}\left(r_{i+\frac{1}{2}}-\frac{r\mu}{\mu_{j+\frac{1}{2}}}\right)\gamma_{i+1,j}. \tag{21}$$

## 5.2 Resolution on $\Omega_2$

In the processing on $\Omega_2$, the values of $\gamma_{i+1,j}$ are known, so we will look for the values of $\gamma_{i,j}$ and $m_{ij}$.
we consider the following algorithm: Find $u^{k+1}\in V$ such that

$$\mu\frac{\partial u^{k+1}}{\partial r}(r,\mu)+\frac{1-\mu^2}{r}\frac{\partial u^{k+1}}{\partial \mu}(r,\mu)+\sigma u^{k+1}(r,\mu)=\frac{\sigma c}{2}\int_{-1}^{0}u^{k+1}\left(r,\mu'\right)d\mu'$$

$$+\frac{\sigma c}{2}\int_{0}^{1}u^{k}\left(r,\mu'\right)d\mu'+S\left(r,\mu\right). \quad (22)$$

This is equivalent to

$$\mu\frac{\partial u^{k+1}}{\partial r}(r,\mu)+\frac{1-\mu^2}{r}\frac{\partial u^{k+1}}{\partial \mu}(r,\mu)+\sigma u^{k+1}(r,\mu)=\frac{\sigma c}{2}\sum_{p=-M}^{p=-1}\int_{\mu_p}^{\mu_{p+1}}u^{k+1}\left(r,\mu'\right)d\mu'$$

$$+\frac{\sigma c}{2}\sum_{p=0}^{p=M-1}\int_{\mu_p}^{\mu_{p+1}}u^{k}\left(r,\mu'\right)d\mu'+S\left(r,\mu\right). \quad (23)$$

By integrating (23) over $K_{ij}$, then dividing by $h\tau$ , we get:

$$\frac{1}{h\tau}\int_{r_i}^{r_{i+1}}\int_{\mu_j}^{\mu_{j+1}}\mu\frac{\partial u^{k+1}}{\partial r}(r,\mu)drd\mu+\frac{1}{h\tau}\int_{r_i}^{r_{i+1}}\int_{\mu_j}^{\mu_{j+1}}\frac{1-\mu^2}{r}\frac{\partial u^{k+1}}{\partial \mu}(r,\mu)drd\mu$$

$$+\sigma m_{ij}^{k+1}=\frac{\sigma c}{2}\sum_{p=-M}^{p=-1}m_{ip}^{k+1}+S_{ij}^{k}, \quad (24)$$

where

$$S_{ij}^{k}=\frac{\sigma c}{2}\sum_{p=0}^{p=M-1}m_{ip}^{k}+\frac{1}{h\tau}\int_{r_i}^{r_{i+1}}\int_{\mu_j}^{\mu_{j+1}}S\left(r,\mu\right)drd\mu.$$

Using the equations (20) and (21), the equality (24) becomes:

$$\underbrace{\left(\sigma-\frac{2}{h\mu_{j+\frac{1}{2}}}\right)}_{C_j^{-1}>0}m_{ij}^{k+1}+\frac{2}{h\mu_{j+\frac{1}{2}}}\gamma_{i+1,j}^{k+1}=\frac{\sigma c}{2}\sum_{p=-M}^{p=-1}m_{ip}^{k+1}+S_{ij}^{k},$$

hence the expression $m_{ij}^{k+1}$ :

$$m_{ij}^{k+1}=\frac{\sigma c}{2}C_j\sum_{p=-M}^{p=-1}m_{ip}^{k+1}+C_j\left(-\frac{2}{h\mu_{j+\frac{1}{2}}}\gamma_{i+1,j}^{k+1}+S_{ij}^{k}\right). \quad (25)$$

We put

$$X_i^k=\sum_{p=-M}^{p=-1}m_{ip}^{k}$$
$$\widetilde{S}_{ij}^{k}=C_j\left(-\tfrac{2}{h\mu_{j+\frac{1}{2}}}\gamma_{i+1,j}^{k+1}+S_{ij}^{k}\right).$$

So we have

$$m_{ij}^{k+1} = C_j \frac{\sigma c}{2} X_i^{k+1} + \tilde{S}_{i,j}^{k}. \tag{26}$$

Remember that all the term $\tilde{S}_{i,j}^{k+1}$ are known.
Before determining the unknowns $m_{ij}^{k+1}$, we first propose the determination of the auxiliary unknowns $X_i^{k+1}$ by summing over the index $j$ $(-M \leq j \leq -1)$, we then deduce:

$$\left(\underbrace{1 - \frac{\sigma c}{2} \sum_{j=-M}^{j=-1} C_j}_{\omega^{-1}}\right) X_i^{k+1} = \sum_{j=-M}^{j=-1} \widetilde{S}_{i,j}^{k}. \tag{27}$$

Consequently

$$X_i^{k+1} = \omega \sum_{j=-M}^{j=-1} \widetilde{S}_{i,j}^{k}. \tag{28}$$

For $i$ fixed in $[0, N-1]$, equation (28) gives the values of $X_i^{k+1}$, then using (26), we determine the terms $m_{ij}^{k+1}$. Finally, we determine the values of $\gamma_{i,j}^{k+1}$ by the equality (20). The numerical resolution on the sub-domain $\Omega_2$ is then completed.

## 5.3 Resolution on $\Omega_1$

Similarly, for the resolution on $\Omega_1$, we will first use the symmetry equality $u(0,\mu) = u(0,-\mu)$, which will allow us to know the $\gamma_{0,j} = \gamma_{0,-j}$ for $j \geq 0$, then we proceed as on $\Omega_2$ except that the direction of travel on the characteristics is opposite. This statement therefore means that the terms $\gamma_{i,j}$ are known and the terms $\gamma_{i+1,j}$ and $m_{ij}$ are unknown. In this case, we take the expression of $u$ in the form following form

$$u(r,\mu) = \frac{2}{h}\left(-r_i + \frac{1}{\mu_{j+\frac{1}{2}}} r\mu\right) m_{ij} + \frac{2}{h}\left(r_{i+\frac{1}{2}} - \frac{1}{\mu_{j+\frac{1}{2}}} r\mu\right)\gamma_{i,j}.$$

For more details, the reader can be to refer for example to [2, 9].

**Remark 4** *The above numerical scheme can be generated for collision kernels of the form:*

$$k(r,\mu,\mu') = C(r) \sum_{l=1}^{l=N_k} \alpha_l(\mu)\alpha_l(\mu)$$

*with positive measurable and bounded function $C$ and $N_k \in \mathbb{N}^\star$. One of the kernel taken this forme is the Thomson's kernel:*

$$k(\mu,\mu') = \frac{9\sigma}{16}\left[\left(1 - \frac{\mu^2}{3}\right)\left(1 - \frac{\mu u'^2}{3}\right) + \frac{8}{9}\mu^2\mu'^2\right].$$

### 5.4 Numerical tests

For the presented numerical results, we consider $\Omega = (0,1)\times(-1,1)$, $h = \frac{1}{500}$, $\tau = \frac{1}{10}$. We choose a source term of the following form

$$S\left(r,\mu\right) = 1 + \mu + \sigma\left[\mu\left(r - e^{-\sigma r}\right) + \left(1-c\right)\left(1 + r + e^{-\sigma r}\right)\right].$$

For this choice of $S$, the exact analitic solution of our test is given by

$$u\left(r,\mu\right) = 1 + r + r\mu + e^{-\sigma r}$$

and the boundary conditions is

$$\varphi(1,\mu) := 2 + \mu + e^{-\sigma}$$

for all $\mu < 0$.

For every iterative methods tested there, iterations are stopped when

$$\frac{\|\varphi^{k+1} - \varphi^{k}\|_2}{\|\varphi^{k}\|_2} < 10^{-9}.$$

In the left hand of the Figure 1, we plot an exact solution and in the right hand, we the source term associated using $\sigma = 10$ and $c = 0.9$.

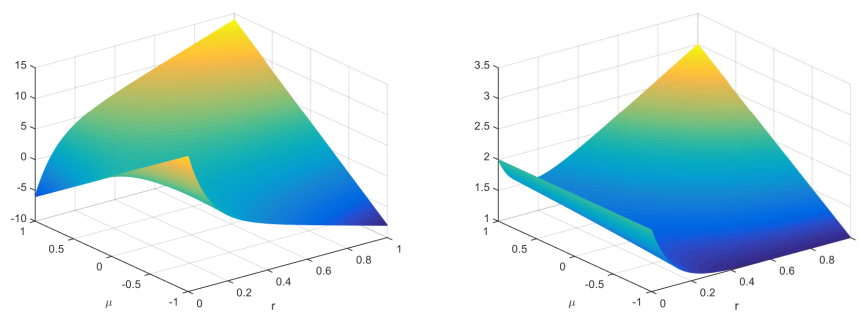


Figure 1: On the left hand source term and on the right hand the density at a fixed $\sigma = 10$ and $c = 0.9$.

There is two sets of tests: one at fixed $\sigma$, another for fixed $c$. For each case, we compare the iterations number of the convergence of the Minimal Residual algorithm with the Gauss-Seidel method [1].

The Figure 2 represent as a function of $c$ ($0 < c \leq 1$), the iterations number necessary to the convergence corresponding to Gauss-Seidel, and to this new algorithm. The $\sigma$ is fixed at 50. The Figure 3 represent as a function of $c$ neat of 1 and at fixed $\sigma = 50$, the iterations number necessary to the convergence corresponding to Gauss-Seidel, and to this new algorithm. The figure 4 illustrates the iteration number with respect $\sigma$ at fixed $c = 0.98$ of algorithms for the same example used earlier.

Observing the numerical results, we deduce that our algorithm is very efficient than the Gauss-Seidel method and Consequently is better then the methods given in [1, 2, 3].

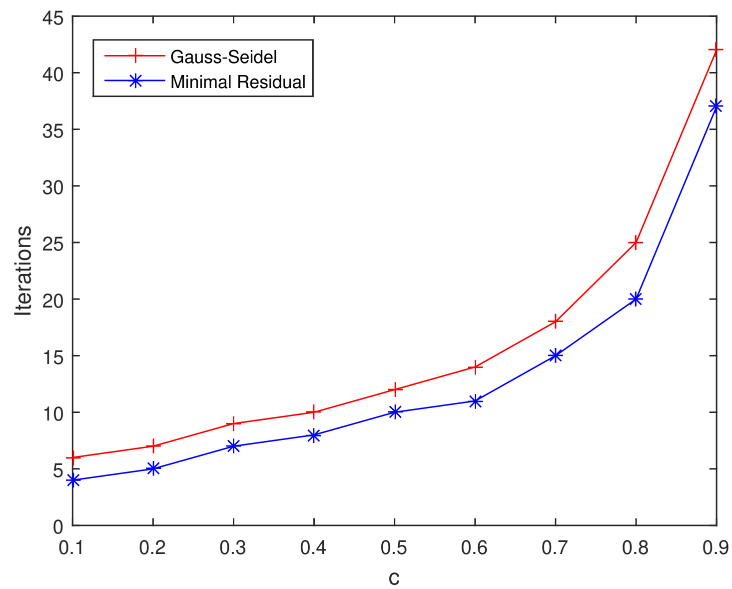


Figure 2: Comparison at a fixed $\sigma = 50$, with the iterations number $(0 < c < 1)$.

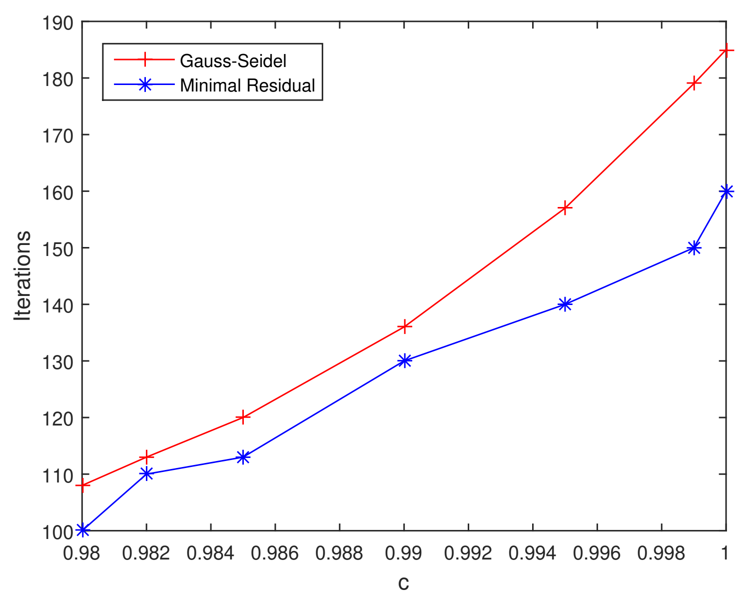


Figure 3: Comparison at a fixed $\sigma = 50$, with the iterations number $(0 \approx 1)$.

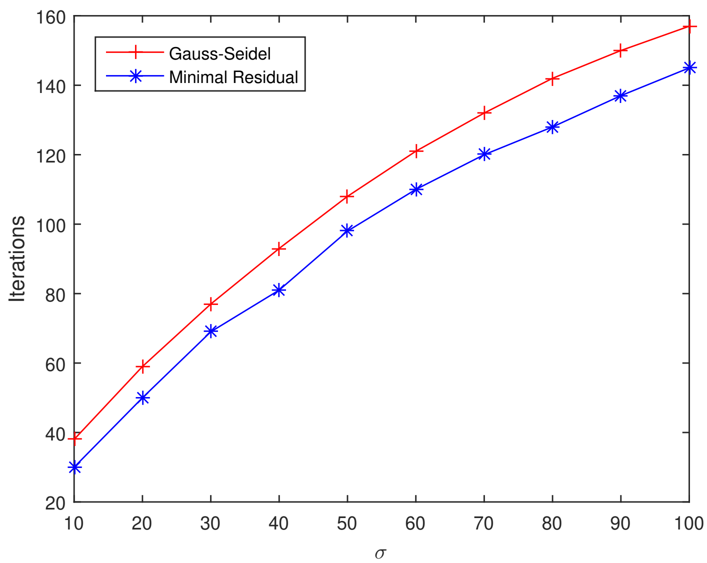


Figure 4: Comparison at a fixed $c = 0.98$, with the iterations number.

# 6 Conclusion

In the present work, a infinite adaptation of the Minimal Residual method for solving the neutron transport stationary equation in spherical geometry is presented. Theoretical proof of the rate convergence of this algorithm is made and the numerical results are given in order to illustrates this advantages the existence methods [1, 2, 3]. The convergence of this method is independent of the choice of discretization. This points is one steps to accelerate the speed of the convergence. A work is in progress to accelerate this algorithm by using the preconditioning method [15, 16] and confirm theses by the numerical results.


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# Declarations

- The authors have no relevant financial or non-financial interests to disclose.

- The authors have no competing interests to declare that are relevant to the content of this article.
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