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A coupled system of q -fractional Lane-Emden problem involving three sequential q -fractional derivatives with stability inequality

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Abstract

In the current manuscript, we consider a sequential q -fractional Lane-Emden system involving Riemann-Liouville and Caputo type fractional q -derivatives. We prove the uniqueness of solutions by means Banach fixed point theorems. Then, to demonstrate the existence of at least one solution, we use Schaefer's fixed point theorem. Also we define and prove two types of Ulam-Hyers for the proposed system. Finally, an illustrative example is proposed.

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Keywords: Lane-Emden problem; fractional q -difference equations; existence; Ulam-Hyers stability; singular differential equation.

1 Introduction

Quantum calculus and differential equations with quantum calculus are of great importance since they can be used in mathematical physical problems, dynamical system and quantum models, see for instance [1, 10, 12]. On the other hand, the differential equations involving fractional q -calculus plays an important role in quantum calculus, recently, there has been a very important progress in the study of the theory of fractional q -differential equations, see for example [2, 9, 18, 20, 28] and the references cited therein.

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Recently, Many scholars have studied the existence and uniqueness and Ulam-stability (U-\$\mathbb{S}\$) of solutions of differential equations involving fractional quantum calculus (\$\mathbb{FQC}\$), see the works [5, 9, 19, 25] and the references cited therein. The singular fractional \$q\$-differential equations (\$\mathbb{F}q - \mathbb{DE}\$) are also very important in applied sciences, see for example [4, 21, 27, 28]. So, in this current paper, we study the existence and U-\$\mathbb{S}\$ of solutions for sequential \$q\$-fractional Lane-Emden system. The Lane-Emden equation (L-EE) describe a variety of phenomena in physics, mathematical physics and astrophysics such as aspects of the stellar structure, for more informations and applications, the reader can consult the research papers [3, 11, 23, 24, 34]. The standard L-E problem [33] can be displayed by

$$\begin{cases} D^2\mathfrak{s}(\tau) + \frac{\eta}{\tau}D^1\mathfrak{s}(\tau) = F(\tau) - \aleph(\tau, \mathfrak{s}(\tau)), & \tau \in (0, 1], \eta \geq 0, \\ \mathfrak{s}(0) = A_1, \quad D^1\mathfrak{s}(0) = A_2, & A_i \in \mathbb{R}, i = 1, 2, \end{cases}$$

where \$\aleph, F\$ are continuous real functions. In [15], the authors solved coupled L-EEs arising in catalytic diffusion reaction by reproducing kernel Hilbert space method, described by

$$\begin{cases} D^2\mathfrak{s}_1(\tau) + \frac{\eta_1}{\tau}D^1\mathfrak{s}_1(\tau) = \aleph_1(\mathfrak{s}_1, \mathfrak{s}_2), \\ D^2\mathfrak{s}_2(\tau) + \frac{\eta_2}{\tau}D^1\mathfrak{s}_2(\tau) = \aleph_2(\mathfrak{s}_1, \mathfrak{s}_2), \\ D^1\mathfrak{s}_i(0) = 0, \mathfrak{s}_i(1) = A_i, & i = 1, 2, \end{cases}$$

for \$\tau \in (0, 1]\$, where \$\eta_i\$ is constant and \$\aleph_i(\mathfrak{s}_1, \mathfrak{s}_2)\$ is analytic function in \$\mathfrak{s}_i, i = 1, 2\$. The L-EE involving the fractional calculus have recently been discussed by several researchers, see [6, 14, 17, 31, 32] and the references cited therein. In [22], the authors have used the collocation method to study the following fractional L-EE

$$\begin{cases} {}_C\mathcal{D}^\gamma \mathfrak{s}(\tau) + \frac{\eta}{\tau^{\gamma-\delta}} {}_C\mathcal{D}^\delta \mathfrak{s}(\tau) = F(\tau) - \aleph(\tau, \mathfrak{s}(\tau)), & \tau \in (0, 1], \eta \geq 0, \\ \mathfrak{s}(0) = A_1, \quad D^1\mathfrak{s}(0) = A_2, & A_i \in \mathbb{R}, i = 1, 2, \end{cases}$$

where \$0 < \gamma \leq 2, 0 < \delta \leq 1\$ and \${}_C\mathcal{D}^\varkappa, \varkappa \in \{\gamma, \delta\}\$ is the Caputo fractional derivative. Also in [16], the authors studied the existence, uniqueness and Ulam-Hyers stability (U-H-\$\mathbb{S}\$) of fractional L-EE involving two Caputo fractional derivatives, given by

$$\begin{cases} {}_C\mathcal{D}^\gamma \left[{}_C\mathcal{D}^\delta + \frac{\eta}{\tau} \right] \mathfrak{s}(\tau) = F(\tau) - \aleph(\tau, \mathfrak{s}(\tau)), & \tau \in (0, 1], \eta \geq 0, \\ \mathfrak{s}(0) = A_1, \quad \mathfrak{s}(1) = A_2, & A_i \in \mathbb{R}, i = 1, 2, \end{cases}$$

where \${}_C\mathcal{D}^\varkappa\$ denotes the Caputo fractional derivative of order \$\varkappa\$ (\$\varkappa = \gamma, \delta\$), \$\aleph\$ is a continuous function and \$k \in C([0, 1])\$. For more information on L-EE with fractional calculus, we refer the reader to the works [7, 8, 13].

In this present manuscript, we discuss the existence, uniqueness, U-\$\mathbb{S}\$ of solutions for the following sequential \$q\$-fractional Lane-Emden system involving Riemann-Liouville and

Caputo type fractional q -derivatives

$$\left\{ \begin{array}{l} {}_{\text{R.L}}\mathcal{D}_q^{\vartheta_1} \left[{}_{\text{C}}\mathcal{D}_q^{\gamma_1} \left[{}_{\text{C}}\mathcal{D}_q^{\delta_1} + \frac{\eta_1}{\tau^{\mu_1}} \right] \right] \mathfrak{s}_1(\tau) \\ = F_1(\tau) - \theta_1 \aleph_{11}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), {}_{\text{C}}\mathcal{D}_q^{\alpha_1} \mathfrak{s}_1(\tau)) \\ - \Lambda_1 \aleph_{12}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), \mathcal{I}_q^{\beta_1} \mathfrak{s}_1(\tau)), \quad \tau \in (0, 1], \\ \\ {}_{\text{R.L}}\mathcal{D}_q^{\vartheta_2} \left[{}_{\text{C}}\mathcal{D}_q^{\gamma_2} \left[{}_{\text{C}}\mathcal{D}_q^{\delta_2} + \frac{\eta_2}{\tau^{\mu_2}} \right] \right] \mathfrak{s}_2(\tau) \\ = F_2(\tau) - \theta_2 \aleph_{21}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), {}_{\text{C}}\mathcal{D}_q^{\alpha_2} \mathfrak{s}_2(\tau)) \\ - \Lambda_2 \aleph_{22}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), \mathcal{I}_q^{\beta_2} \mathfrak{s}_2(\tau)), \quad \tau \in (0, 1], \\ \\ {}_{\text{C}}\mathcal{D}_q^{\gamma_i} \left[{}_{\text{C}}\mathcal{D}_q^{\delta_i} + \eta_i \right] \mathfrak{s}_i(1) = 0, \quad {}_{\text{C}}\mathcal{D}_q^{\gamma_i} \left[{}_{\text{C}}\mathcal{D}_q^{\delta_i} \mathfrak{s}_i(0) \right] = 0, \quad \mathfrak{s}_i(1) = \mathcal{I}_q^{\omega_i} \mathfrak{s}_i(v_i), \end{array} \right. \quad (1.1)$$

for $\eta_i, \theta_i, \Lambda_i > 0$, $0 < \mu_i, v_i \leq 1$, $\omega_i \geq 0$, and $0 < \vartheta_i, \gamma_i, \delta_i < 1$, $\alpha_i < \delta_i$, $\Omega := [0, 1]$, where ${}_{\text{R.L}}\mathcal{D}_q^{\vartheta_i}$ and ${}_{\text{C}}\mathcal{D}_q^{\varkappa}, \varkappa \in \{\alpha_i, \gamma_i, \delta_i\}$ are the Riemann-Liouville and Caputo fractional q -derivatives respectively, $\mathcal{I}_q^{\beta_i}$ is the fractional q -integral of the Riemann-Liouville type, $F_i : \Omega \rightarrow \mathbb{R}$ and $\aleph_{ij} : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}, i, j = 1, 2$ are given continuous functions.

In Section 2, we recall some essential definition of fractional quantum calculus. Section 3 contains our main results in this work, while an example is presented to support the validity of our obtained results. stability results are extensively discussed in Section 4. An illustrative example with some needed algorithms for the problem are given in Section 5. Finally, in Section 6, conclusion are presented.

2 Fractional quantum calculus

The operator ${}_{\text{R.L}}\mathcal{D}_q^\varsigma$ is the fractional q -derivative of the Riemann-Liouville type [5, 26], defined by

$$\left\{ \begin{array}{l} {}_{\text{R.L}}\mathcal{D}_q^\varsigma [\mathfrak{s}(\tau)] = \mathcal{D}_q^{[\varsigma]} [\mathcal{I}_q^{[\varsigma]-\varsigma} [\mathfrak{s}(\tau)]] , \quad \varsigma > 0, \\ {}_{\text{R.L}}\mathcal{D}_q^0 [\mathfrak{s}(\tau)] = \mathfrak{s}(\tau), \end{array} \right.$$

where $[\varsigma]$ is the smallest integer greater than or equal to ς . The fractional q -derivative of the Caputo type of order \varkappa is given by

$$\left\{ \begin{array}{l} {}_{\text{C}}\mathcal{D}_q^\varkappa [\mathfrak{s}(\tau)] = \mathcal{I}_q^{[\varkappa]-\varkappa} [\mathcal{D}_q^{[\varkappa]} [\mathfrak{s}(\tau)]] , \quad \varkappa > 0, \\ {}_{\text{C}}\mathcal{D}_q^0 [\mathfrak{s}(\tau)] = \mathfrak{s}(\tau), \end{array} \right.$$

the fractional q -integral of the Riemann-Liouville type [5, 26] is defined by

$$\left\{ \begin{array}{l} \mathcal{I}_q^\varpi [\mathfrak{s}(\tau)] = \frac{1}{\Gamma_q(\varpi)} \int_0^\tau (\tau - qr)^{(\varpi-1)} \mathfrak{s}(r) d_q r, \quad \varpi > 0, \\ \mathcal{I}_q^0 [\mathfrak{s}(\tau)] = \mathfrak{s}(\tau), \end{array} \right.$$

where the q -gamma function is defined by $\Gamma_q(\varpi) = \frac{(1-q)^{(\varpi-1)}}{(1-q)^{\varpi-1}}$, $0 < q < 1$ and satisfies $\Gamma_q(\varpi + 1) = [\varpi]_q \Gamma_q(\varpi)$, such that

$$[\varpi]_q = \frac{1 - q^\varpi}{1 - q}, (1 - q)^{(0)} = 1, \quad (1 - q)^{(n)} = \prod_{l=0}^{n-1} (1 - q^{l+1}), n \in \mathbb{N}.$$

We recall the the following lemmas [5, 26]. See [29] for related MATLAB algorithms.

Lemma 2.1. *Let $\nu, \varpi \geq 0$ and s be a function defined in $[0, 1]$. Then*

$$\mathcal{I}_q^\nu [\mathcal{I}_q^\varpi [s(\tau)]] = \mathcal{I}_q^{\nu+\varpi} [s(\tau)], \quad D_q^\nu \mathcal{I}_q^\nu [s(\tau)] = s(\tau).$$

Lemma 2.2. *Suppose that $\nu > 0$, and ς be a positive integer. Then, we have*

$$\mathcal{I}_q^\nu [\text{R.L}\mathcal{D}_q^\varsigma [s(\tau)]] = \text{R.L}\mathcal{D}_q^\varsigma [\mathcal{I}_q^\nu [s(\tau)]] - \sum_{l=0}^{\mu-1} \frac{\tau^{\nu-\varsigma+l}}{\Gamma_q(\nu + l - \varsigma + 1)} \mathcal{D}_q^l s(0).$$

Lemma 2.3. *Let $\nu \in \mathbb{R}^+ \setminus \mathbb{N}$. Then, the following equality is valid*

$$\mathcal{I}_q^\nu [\text{C}\mathcal{D}_q^\nu [s(\tau)]] = s(\nu) - \sum_{l=0}^{n-1} \frac{\tau^l}{\Gamma_q(l+1)} \mathcal{D}_q^l s(0),$$

such that n is the smallest integer greater than or equal to ν .

Lemma 2.4. *For $\nu \in \mathbb{R}_+$ and $\kappa > -1$, we have*

$$\mathcal{I}_q^\nu [\tau^{(\kappa)}] = \frac{\Gamma_q(\kappa+1)}{\Gamma_q(\nu+\kappa+1)} \tau^{(\nu+\kappa)}.$$

If $\kappa = 0$, we can obtain $\mathcal{I}_q^\nu [1] = \frac{1}{\Gamma_q(\nu+1)} \tau^{(\nu)}$.

Lemma 2.5. [30] *Let G be a Banach space and $\mathbb{P} : G \rightarrow G$ be a completely continuous operator. If*

$$U = \left\{ s \in G : s = \xi \mathbb{P} s, 0 < \xi < 1 \right\},$$

is bounded, then \mathbb{P} has a fixed point in U .

In order to study the system (1.1), we need the following space

$$B_i = \left\{ s_i : s_i \in C(\Omega, \mathbb{R}) \text{ and } {}_C\mathcal{D}_q^{\alpha_i} s_i \in C(\Omega, \mathbb{R}) \right\}, \quad i = 1, 2$$

endowed with the norm

$$\|s_i\|_{B_i} = \|s_i\| + \|{}_C\mathcal{D}_q^{\alpha_i} s_i\| = \sup_{\tau \in \Omega} |s_i(\tau)| + \sup_{\tau \in \Omega} |{}_C\mathcal{D}_q^{\alpha_i} s_i(\tau)|, \quad i = 1, 2.$$

Then it is well known that $(B_i, \|\cdot\|_{B_i})$, $i = 1, 2$ is a Banach space. Obviously the product space $(B_1 \times B_2, \|(s_1, s_2)\|_{B_1 \times B_2})$ is a Banach space with norm

$$\|(s_1, s_2)\|_{B_1 \times B_2} = \|s_1\|_{B_1} + \|s_2\|_{B_2}, \quad \forall (s_1, s_2) \in B_1 \times B_2.$$

Now, we consider the Ulam-stability for the q -fractional Lane-Emden system (1.1). For $\ell_i > 0$, $i = 1, 2$ and $m : \Omega \rightarrow \mathbb{R}_+$, we give the following inequalities

$$\begin{aligned} & \left| {}_{\text{RL}}\mathcal{D}_q^{\vartheta_i} \left[{}_C\mathcal{D}_q^{\gamma_i} \left[{}_C\mathcal{D}_q^{\delta_i} + \frac{\eta_i}{\tau^{\mu_i}} \right] \right] \mathfrak{s}_i(\tau) - \left[F_i(\tau) - \theta_i \mathfrak{N}_{i1,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) \right. \right. \\ & \quad \left. \left. - \Lambda_i \mathfrak{N}_{i2,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) \right] \right| \leq \ell_i, \quad i = 1, 2, \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} & \left| {}_{\text{RL}}\mathcal{D}_q^{\vartheta_i} \left[{}_C\mathcal{D}_q^{\gamma_i} \left[{}_C\mathcal{D}_q^{\delta_i} + \frac{\eta_i}{\tau^{\mu_i}} \right] \right] \mathfrak{s}_i(\tau) - \left[F_i(\tau) - \theta_i \mathfrak{N}_{i1,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) \right. \right. \\ & \quad \left. \left. - \Lambda_i \mathfrak{N}_{i2,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) \right] \right| \leq \ell_i m(\tau), \quad i = 1, 2, \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} \mathfrak{N}_{i1,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) &= \mathfrak{N}_{i1}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), {}_C\mathcal{D}_q^{\alpha_i} \mathfrak{s}_i(\tau)), \\ \mathfrak{N}_{i2,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) &= \mathfrak{N}_{i2}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), \mathcal{I}_q^{\beta_i} \mathfrak{s}_i(\tau)), \end{aligned}$$

for $i = 1, 2$.

Definition 2.6. *The q -fractional Lane-Emden system (1.1) is stable in*

- *Ulam-Hyers sense if $\forall \ell > 0$, which $\ell = \max\{\ell_i, i = 1, 2\}$, and for each $(\mathfrak{s}_1, \mathfrak{s}_2) \in B_1 \times B_2$, where satisfied in the inequality (2.1), then there exists a solution $(\mathfrak{s}_1, \mathfrak{s}_2) \in B_1 \times B_2$ of the system (1.1) and $\exists \Sigma_{\mathfrak{N}_{i1}^*, \mathfrak{N}_{i2}^*, F_i} > 0$, $i = 1, 2$, with*

$$\|(\mathfrak{s}_1, \mathfrak{s}_2) - (\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2} \leq \Sigma_{\mathfrak{N}_{i1}^*, \mathfrak{N}_{i2}^*, F_i} \ell, \quad i = 1, 2. \quad (2.3)$$

- *Ulam-Hyers-Rassias sense, with respect to m , if for each $\ell > 0$, and for all $(\mathfrak{s}_1, \mathfrak{s}_2) \in B_1 \times B_2$, where satisfied in the inequality (2.2), then there exists a solution $(\mathfrak{s}_1, \mathfrak{s}_2) \in B_1 \times B_2$ of the system (1.1) and $\exists \Sigma_{\mathfrak{N}_{i1}^*, \mathfrak{N}_{i2}^*, F_i, m} > 0$, $i = 1, 2$, with*

$$\|(\mathfrak{s}_1, \mathfrak{s}_2) - (\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2} \leq \Sigma_{\mathfrak{N}_{i1}^*, \mathfrak{N}_{i2}^*, F_i, m} \ell m(\tau), \quad \tau \in \Omega, i = 1, 2. \quad (2.4)$$

3 Existence results for q -fractional Lane-Emden system

In this section, we will use the fixed point theory to study the above q -fractional Lane-Emden system. We give the following auxiliary result.

Lemma 3.1. *Suppose that $h_i(\mathfrak{s}) \in C(J, \mathbb{R})$, $i = 1, 2$. Then the unique solution of the fractional problem*

$$\begin{cases} {}_{\text{RL}}\mathcal{D}_q^{\vartheta_i} \left[{}_C\mathcal{D}_q^{\gamma_i} \left[{}_C\mathcal{D}_q^{\delta_i} + \frac{\eta_i}{\tau^{\mu_i}} \right] \right] \mathfrak{s}_i(\tau) = h_i(\tau), & \tau \in (0, 1], \\ {}_C\mathcal{D}_q^{\gamma_i} \left[{}_C\mathcal{D}_q^{\delta_i} + \eta_i \right] \mathfrak{s}_i(1) = 0, \quad {}_C\mathcal{D}_q^{\gamma_i} \left[{}_C\mathcal{D}_q^{\delta_i} \mathfrak{s}_i(0) \right] = 0, \quad \mathfrak{s}_i(1) = \mathcal{I}_q^{\omega_i} \mathfrak{s}_i(v_i), \end{cases} \quad (3.1)$$

for $0 < \vartheta_i, \gamma_i, \delta_i < 1$, $0 < \mu_i \leq 1$, $\omega_i, \eta_i \geq 0$, $0 < v_i < 1$ ($i = 1, 2$), is given by

$$\begin{aligned} \mathfrak{s}_i(\tau) &= \frac{1}{\Gamma_q(\vartheta_i + \gamma_i + \delta_i)} \int_0^\tau (\tau - r)^{(\vartheta_i + \gamma_i + \delta_i - 1)} h_i(r) dr \\ &\quad - \frac{1}{\Gamma_q(\delta_i)} \int_0^\tau (\tau - r)^{(\delta_i - 1)} \frac{\eta_i}{r^{\mu_i}} \mathfrak{s}_i(r) dr \\ &\quad - \frac{\tau^{\vartheta_i + \gamma_i + \delta_i - 1}}{\Gamma(\vartheta_i + \gamma_i + \delta_i)} \int_0^1 (1 - r)^{(\vartheta_i - 1)} h_i(r) dr \\ &\quad + \frac{\Gamma(\omega_i + 1)}{(1 - v_i^{\omega_i}) \Gamma_q(\vartheta_i + \gamma_i + \delta_i + \omega_i)} \int_0^{v_i} (v_i - r)^{(\vartheta_i + \gamma_i + \delta_i + \omega_i - 1)} h_i(r) dr \\ &\quad - \frac{\Gamma(\omega_i + 1)}{(1 - v_i^{\omega_i}) \Gamma_q(\delta_i + \omega_i)} \int_0^{v_i} (v_i - r)^{(\delta_i + \omega_i - 1)} \frac{\eta_i}{r^{\mu_i}} \mathfrak{s}_i(r) dr \\ &\quad + \frac{\Gamma(\omega_i + 1) (1 - v_i^{\vartheta_i + \gamma_i + \delta_i + \omega_i - 1})}{(1 - v_i^{\omega_i}) \Gamma(\vartheta_i + \gamma_i + \delta_i + \omega_i)} \int_0^1 (1 - r)^{(\vartheta_i - 1)} h_i(r) dr \\ &\quad - \frac{\Gamma(\omega_i + 1)}{(1 - v_i^{\omega_i}) \Gamma_q(\vartheta_i + \gamma_i + \delta_i + \omega_i)} \int_0^1 (1 - r)^{(\vartheta_i + \gamma_i + \delta_i + \omega_i - 1)} h_i(r) dr \\ &\quad + \frac{\Gamma(\omega_i + 1)}{(1 - v_i^{\omega_i}) \Gamma_q(\delta_i + \omega_i)} \int_0^1 (1 - r)^{(\delta_i + \omega_i - 1)} \eta_i \mathfrak{s}_i(r) dr, \end{aligned} \tag{3.2}$$

where $1 - v_i^{\omega_i} \neq 0$, $i = 1, 2$.

Proof. Applying Lemma 2.2, we get

$${}_C D_q^{\gamma_i} \left[{}_C D_q^{\delta_i} + \frac{\eta_i}{\tau^{\mu_i}} \right] \mathfrak{s}_i(\tau) = \mathcal{I}_q^{\vartheta_i} [h_i(\tau)] + c_{1,i} \tau^{\vartheta_i - 1}, \quad c_{1,i} \in \mathbb{R}, i = 1, 2. \tag{3.3}$$

Thanks to Lemma (2.3), we have

$$\left[{}_C \mathcal{D}_q^{\delta_i} + \frac{\eta_i}{\tau^{\mu_i}} \right] \mathfrak{s}_i(\tau) = \mathcal{I}_q^{\vartheta_i + \gamma_i} [h_i(\tau)] + \frac{c_{1,i} \Gamma(\vartheta_i)}{\Gamma(\vartheta_i + \gamma_i)} \tau^{\vartheta_i + \gamma_i - 1} + c_{2,i}, \tag{3.4}$$

where $c_{2,i} \in \mathbb{R}$, $i = 1, 2$. Applying the Riemann-Liouville q -fractional integral of order δ_i , $i = 1, 2$ to both sides of equation in (3.4), we get

$$\begin{aligned} \mathfrak{s}_i(\tau) &= \mathcal{I}_q^{\vartheta_i + \gamma_i + \delta_i} h_i(\tau) - \mathcal{I}_q^{\delta_i} \left[\frac{\eta_i}{\tau^{\mu_i}} \mathfrak{s}_i(\tau) \right] + c_{1,i} \frac{\Gamma(\vartheta_i) \tau^{\vartheta_i + \gamma_i + \delta_i - 1}}{\Gamma(\vartheta_i + \gamma_i + \delta_i)} \\ &\quad + c_{2,i} \frac{\tau^{\delta_i}}{\Gamma(\delta_i + 1)} + c_{3,i}, \quad c_{3,i} \in \mathbb{R}, i = 1, 2. \end{aligned} \tag{3.5}$$

Using the conditions ${}_C \mathcal{D}_q^{\gamma_i} [{}_C \mathcal{D}_q^{\delta_i} + \eta_i] \mathfrak{s}_i(1) = 0$ and ${}_C \mathcal{D}_q^{\gamma_i} [{}_C \mathcal{D}_q^{\delta_i} \mathfrak{s}_i(0)] = 0$, $i = 1, 2$, we obtain $c_{1,i} = -\mathcal{I}_q^{\vartheta_i} [h_i(1)]$ and $c_{2,i} = 0$, $i = 1, 2$. Now, applying the operator $\mathcal{I}_q^{\omega_i}$, $i = 1, 2$ to both sides of equation in (3.5), we get

$$\begin{aligned} \mathcal{I}_q^{\omega_i} [\mathfrak{s}_i(\tau)] &= \mathcal{I}_q^{\vartheta_i + \gamma_i + \delta_i + \omega_i} [h_i(\tau)] - \mathcal{I}_q^{\delta_i + \omega_i} \left[\frac{\eta_i}{\tau^{\mu_i}} \mathfrak{s}_i(\tau) \right] \\ &\quad - \frac{\Gamma(\vartheta_i) \tau^{\vartheta_i + \gamma_i + \delta_i + \omega_i - 1}}{\Gamma(\vartheta_i + \gamma_i + \delta_i + \omega_i)} \mathcal{I}_q^{\vartheta_i} [h_i(1)] + \frac{c_{3,i} \tau^{\omega_i}}{\Gamma(\omega_i + 1)}, \quad i = 1, 2. \end{aligned}$$

Using the condition $\mathfrak{s}_i(1) = \mathcal{I}_q^{\omega_i} \mathfrak{s}_i(v_i)$, $i = 1, 2$, we have

$$\begin{aligned} c_{3,i} &= \frac{\Gamma(\omega_i + 1)}{1 - v_i^{\omega_i}} \left(\mathcal{I}_q^{\vartheta_i + \gamma_i + \delta_i + \omega_i} h_i(v_i) - \mathcal{I}_q^{\delta_i + \omega_i} \left[\frac{\eta_i}{v_i^{\mu_i}} \mathfrak{s}_i(v_i) \right] \right. \\ &\quad + \frac{\Gamma(\vartheta_i) \left(1 - v_i^{\vartheta_i + \gamma_i + \delta_i + \omega_i - 1} \right)}{\Gamma(\vartheta_i + \gamma_i + \delta_i + \omega_i)} \mathcal{I}_q^{\vartheta_i} [h_i(1)] - \mathcal{I}_q^{\vartheta_i + \gamma_i + \delta_i + \omega_i} [h_i(1)] \\ &\quad \left. + \mathcal{I}_q^{\delta_i + \omega_i} [\eta_i \mathfrak{s}_i(1)] \right), \quad i = 1, 2, \end{aligned}$$

inserting the values of $c_{1,i}, c_{2,i}$ and $c_{3,i}, i = 1, 2$ in (3.5) yields the solution (3.2). \square

In view of Lemma 3.1, we define an operator $P : B_1 \times B_2 \rightarrow B_1 \times B_2$ by

$$P(\mathfrak{s}_1, \mathfrak{s}_2)(\tau) = (P_1(\mathfrak{s}_1, \mathfrak{s}_2)(\tau), P_2(\mathfrak{s}_1, \mathfrak{s}_2)(\tau)), \quad \tau \in \Omega,$$

such that

$$\begin{aligned} P_i(\mathfrak{s}_1, \mathfrak{s}_2)(\tau) &= \int_0^\tau \frac{(\tau - r)^{(\vartheta_i + \gamma_i + \delta_i - 1)}}{\Gamma_q(\vartheta_i + \gamma_i + \delta_i)} (F_i(r) - \theta_i \mathfrak{N}_{i1,(\mathfrak{s}_1, \mathfrak{s}_2)}^*(r) - \Lambda_i \mathfrak{N}_{i2,(\mathfrak{s}_1, \mathfrak{s}_2)}^*(r)) dr \\ &\quad - \frac{1}{\Gamma_q(\delta_i)} \int_0^\tau (\tau - r)^{(\delta_i - 1)} \frac{\eta_i}{r^{\mu_i}} \mathfrak{s}_i(r) dr \\ &\quad - \int_0^1 \frac{\tau^{\vartheta_i + \gamma_i + \delta_i - 1} (1 - r)^{(\vartheta_i - 1)}}{\Gamma(\vartheta_i + \gamma_i + \delta_i)} (F_i(r) - \theta_i \mathfrak{N}_{i1,(\mathfrak{s}_1, \mathfrak{s}_2)}^*(r) - \Lambda_i \mathfrak{N}_{i2,(\mathfrak{s}_1, \mathfrak{s}_2)}^*(r)) dr \\ &\quad + \int_0^{v_i} \frac{\Gamma_q(\omega_i + 1) (v_i - r)^{(\vartheta_i + \gamma_i + \delta_i + \omega_i - 1)}}{(1 - v_i^{\omega_i}) \Gamma_q(\vartheta_i + \gamma_i + \delta_i + \omega_i)} (F_i(r) - \theta_i \mathfrak{N}_{i1,(\mathfrak{s}_1, \mathfrak{s}_2)}^*(r) - \Lambda_i \mathfrak{N}_{i2,(\mathfrak{s}_1, \mathfrak{s}_2)}^*(r)) dr \\ &\quad - \frac{\Gamma_q(\omega_i + 1)}{(1 - v_i^{\omega_i}) \Gamma_q(\delta_i + \omega_i)} \int_0^{v_i} (v_i - r)^{(\delta_i + \omega_i - 1)} \frac{\eta_i}{r^{\mu_i}} \mathfrak{s}_i(r) dr \\ &\quad + \frac{(\omega_i + 1) \left(1 - v_i^{\vartheta_i + \gamma_i + \delta_i + \omega_i - 1} \right)}{(1 - v_i^{\omega_i}) \Gamma(\vartheta_i + \gamma_i + \delta_i + \omega_i)} \int_0^1 (1 - r)^{(\vartheta_i - 1)} \\ &\quad \times (F_i(r) - \theta_i \mathfrak{N}_{i1,(\mathfrak{s}_1, \mathfrak{s}_2)}^*(r) - \Lambda_i \mathfrak{N}_{i2,(\mathfrak{s}_1, \mathfrak{s}_2)}^*(r)) dr \\ &\quad - \int_0^1 \frac{\Gamma_q(\omega_i + 1) (1 - r)^{(\vartheta_i + \gamma_i + \delta_i + \omega_i - 1)}}{(1 - v_i^{\omega_i}) \Gamma_q(\vartheta_i + \gamma_i + \delta_i + \omega_i)} (F_i(r) - \theta_i \mathfrak{N}_{i1,(\mathfrak{s}_1, \mathfrak{s}_2)}^*(r) - \Lambda_i \mathfrak{N}_{i2,(\mathfrak{s}_1, \mathfrak{s}_2)}^*(r)) dr \\ &\quad + \int_0^1 \frac{\Gamma_q(\omega_i + 1) (1 - r)^{(\delta_i + \omega_i - 1)}}{(1 - v_i^{\omega_i}) \Gamma_q(\delta_i + \omega_i)} \eta_i \mathfrak{s}_i(r) dr, \end{aligned}$$

with $v_i^{\omega_i} \neq 1$ and $\mu_i \leq 1$ ($i = 1, 2$). Also, we have

$$\mathcal{D}_q^{\alpha_i} [P_i(\mathfrak{s}_1, \mathfrak{s}_2)(\tau)] = \int_0^\tau \frac{(\tau - qr)^{(-\alpha_i)}}{\Gamma_q(1 - \alpha_i)} \mathcal{D}_q [P_i(\mathfrak{s}_1, \mathfrak{s}_2)(r)] dr, \quad i = 1, 2, \quad (3.6)$$

where

$$\begin{aligned}
& \mathcal{D}_q [P_i(\mathfrak{s}_1, \mathfrak{s}_2)(\tau)] \\
&= \int_0^\tau \frac{(\tau - r)^{(\vartheta_i + \gamma_i + \delta_i - 2)}}{\Gamma_q(\vartheta_i + \gamma_i + \delta_i - 1)} (\mathcal{F}_i(r) - \theta_i \mathfrak{N}_{i1,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(r) - \Lambda_i \mathfrak{N}_{i2,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(r)) \, dr \\
&\quad - \int_0^\tau \frac{(\tau - r)^{(\delta_i - 2)}}{\Gamma_q(\delta_i - 1)} \frac{\eta_i}{r^{\mu_i}} \mathfrak{s}_i(r) \, dr \\
&\quad - \int_0^1 \frac{[\vartheta_i + \gamma_i + \delta_i - 1]_q \tau^{\vartheta_i + \gamma_i + \delta_i - 2} (1 - r)^{(\vartheta_i - 1)}}{\Gamma(\vartheta_i + \gamma_i + \delta_i)} \\
&\quad \times (\mathcal{F}_i(r) - \theta_i \mathfrak{N}_{i1,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(r) - \Lambda_i \mathfrak{N}_{i2,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(r)) \, dr,
\end{aligned} \tag{3.7}$$

for $i = 1, 2$. we introduce the following notations

$$\begin{aligned}
\Pi_i^* &:= \frac{1}{\Gamma_q(\vartheta_i + \gamma_i + \delta_i + 1)} + \frac{1}{\vartheta_i \Gamma_q(\vartheta_i + \gamma_i + \delta_i)} \\
&\quad + \frac{\Gamma_q(\omega_i + 1) v_i^{\vartheta_i + \gamma_i + \delta_i + \omega_i}}{|1 - v_i^{\omega_i}| \Gamma_q(\vartheta_i + \gamma_i + \delta_i + \omega_i + 1)} + \frac{\Gamma_q(\omega_i + 1) |1 - v_i^{\vartheta_i + \gamma_i + \delta_i + \omega_i - 1}|}{\vartheta_i |1 - v_i^{\omega_i}| \Gamma_q(\vartheta_i + \gamma_i + \delta_i + \omega_i)} \\
&\quad + \frac{\Gamma_q(\omega_i + 1)}{|1 - v_i^{\omega_i}| \Gamma_q(\vartheta_i + \gamma_i + \delta_i + \omega_i + 1)}, \\
\Pi_i &:= \frac{\Gamma_q(1 - \mu_i)}{\Gamma_q(\delta_i - \mu_i + 1)} + \frac{\Gamma_q(\omega_i + 1) \Gamma_q(1 - \mu_i) v_i^{\delta_i + \omega_i - \mu_i}}{|1 - v_i^{\omega_i}| \Gamma_q(\delta_i + \omega_i - \mu_i + 1)} \\
&\quad + \frac{\Gamma_q(\omega_i + 1)}{|1 - v_i^{\omega_i}| \Gamma_q(\delta_i + \omega_i + 1)}, \\
\Theta_i^* &:= \frac{1}{\Gamma_q(\vartheta_i + \gamma_i + \delta_i)} + \frac{[\vartheta_i + \gamma_i + \delta_i - 1]_q}{\vartheta_i \Gamma_q(\vartheta_i + \gamma_i + \delta_i)}, \quad \Theta_i := \frac{\Gamma_q(1 - \mu_i)}{\Gamma_q(\delta_i - \mu_i)}.
\end{aligned} \tag{3.8}$$

In the following, we give the existence and uniqueness of solutions of system (1.1) by using Banach's contraction principle.

Theorem 3.2. *Let $\mathfrak{N}_{i1}, \mathfrak{N}_{i2} : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\mathcal{F}_i : \Omega \rightarrow \mathbb{R}$, $i = 1, 2$ be continuous functions. In addition we assume that*

(H₁) *There exist a constants $a_i, b_i > 0$, $i = 1, 2$ such that for all $\tau \in \Omega$ and $\mathfrak{s}_j, \dot{\mathfrak{s}}_j \in \mathbb{R}$, $j = 1, 2, 3$,*

$$\begin{aligned}
|\varphi_{i1}(\tau, \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3) - \mathfrak{N}_{i1}(\tau, \dot{\mathfrak{s}}_1, \dot{\mathfrak{s}}_2, \dot{\mathfrak{s}}_3)| &\leq a_i (|\mathfrak{s}_1 - \dot{\mathfrak{s}}_1| + |\mathfrak{s}_2 - \dot{\mathfrak{s}}_2| + |\mathfrak{s}_3 - \dot{\mathfrak{s}}_3|), \\
|\mathfrak{N}_{i2}(\tau, \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3) - \mathfrak{N}_{i2}(\tau, \dot{\mathfrak{s}}_1, \dot{\mathfrak{s}}_2, \dot{\mathfrak{s}}_3)| &\leq b_i (|\mathfrak{s}_1 - \dot{\mathfrak{s}}_1| + |\mathfrak{s}_2 - \dot{\mathfrak{s}}_2| + |\mathfrak{s}_3 - \dot{\mathfrak{s}}_3|).
\end{aligned}$$

If

$$\lambda_i^* \left(\theta_i a_i + \frac{\Lambda_i b_i (2 + \Gamma_q(\beta_i + 1))}{\Gamma_q(\beta_i + 1)} \right) < \frac{1}{2} - \lambda_i \eta_i, \quad i = 1, 2, \tag{3.9}$$

where $\lambda_i = \Pi_i + \frac{\Theta_i}{\Gamma_q(2 - \alpha_i)}$, $\lambda_i^* = \Pi_i^* + \frac{\Theta_i^*}{\Gamma_q(2 - \alpha_i)}$ and Π_i^* , Π_i , Θ_i^* and Θ_i ($i = 1, 2$) are given by (3.8). Then, the considered system (1.1) has a unique solution on Ω .

Proof. Let us define

$$\sigma \geq \max \left\{ \frac{\lambda_i^* (1 + \theta_1 + \Lambda_1) \nabla}{\frac{1}{2} - \lambda_i^* \left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (2 + \Gamma_q (\beta_1 + 1))}{\Gamma_q (\beta_1 + 1)} \right) - \lambda_i \eta_1}, \frac{\lambda_i^* (1 + \theta_2 + \Lambda_2) \Delta}{\frac{1}{2} - \lambda_i^* \left(\theta_2 a_2 + \frac{\Lambda_2 b_2 (2 + \Gamma_q (\beta_2 + 1))}{\Gamma_q (\beta_2 + 1)} \right) - \lambda_i \eta_2} \right\},$$

with $\nabla = \max \{ \nabla_j : j = 1, 2, 3 \}$ and $\Delta = \max \{ \Delta_j : j = 1, 2, 3 \}$, where ∇_j are given by:

$$\nabla_1 = \sup_{\tau \in \Omega} |\aleph_{11}(\tau, 0, 0)|, \quad \nabla_2 = \sup_{\tau \in \Omega} |\aleph_{12}(\tau, 0, 0)|, \quad \nabla_3 = \sup_{\tau \in \Omega} |F_1(\tau)|,$$

and Δ_j are given by

$$\Delta_1 = \sup_{\tau \in \Omega} |\aleph_{21}(\tau, 0, 0)|, \quad \Delta_2 = \sup_{\tau \in \Omega} |\aleph_{22}(\tau, 0, 0)|, \quad \Delta_3 = \sup_{\tau \in \Omega} |F_2(\tau)|.$$

Then we show that $PB_\sigma \subset B_\sigma$, where $B_\sigma = \{(\mathfrak{s}_1, \mathfrak{s}_2) \in B_1 \times B_2 : \|(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2} \leq \sigma\}$. Thanks to (H₁), we can write

$$\begin{aligned} |\aleph_{i1,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau)| &= |\aleph_{11}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), {}_C\mathcal{D}_q^{\alpha_i} \mathfrak{s}_i(\tau))| \\ &\leq |\aleph_{11}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), {}_C\mathcal{D}_q^{\alpha_i} \mathfrak{s}_i(\tau)) - \aleph_{11}(\tau, 0, 0)| + |\aleph_{11}(\tau, 0, 0)| \\ &\leq a_i (|\mathfrak{s}_1(\tau)| + |\mathfrak{s}_2(\tau)| + |{}_C\mathcal{D}_q^{\alpha_i} \mathfrak{s}_i(\tau)|) + \nabla_i \\ &\leq a_i (\|\mathfrak{s}_1\|_{B_1} + \|\mathfrak{s}_2\|_{B_2}) + \nabla_i \\ &\leq a_i \|(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2} + \nabla_i \leq a_i \sigma + \nabla, \quad i = 1, 2, \end{aligned} \tag{3.10}$$

$$|\aleph_{i2,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau)| = |\aleph_{12}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), \mathcal{I}_q^{\beta_i} \mathfrak{s}_i(\tau))| \tag{3.11}$$

$$\begin{aligned} &\leq |\aleph_{12}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), \mathcal{I}_q^{\beta_i} \mathfrak{s}_i(\tau)) - \aleph_{12}(\tau, 0, 0)| + |\aleph_{12}(\tau, 0, 0)| \\ &\leq b_i (|\mathfrak{s}_1(\tau)| + |\mathfrak{s}_2(\tau)| + |\mathcal{I}_q^{\beta_i} \mathfrak{s}_i(\tau)|) + \nabla_i \\ &\leq b_i \left(\|\mathfrak{s}_1\|_{B_1} + \|\mathfrak{s}_2\|_{B_2} + \frac{\|\mathfrak{s}_i\|_{B_i}}{\Gamma_q (\beta_i + 1)} \right) + \nabla_2 \\ &\leq b_i \frac{(2 + \Gamma_q (\beta_i + 1))}{\Gamma_q (\beta_i + 1)} \|\mathfrak{s}_1, \mathfrak{s}_2\|_{B_1 \times B_2} + \nabla_i \\ &\leq b_i \frac{(2 + \Gamma_q (\beta_i + 1))}{\Gamma_q (\beta_i + 1)} \sigma + \nabla, \quad i = 1, 2. \end{aligned} \tag{3.12}$$

From (3.10) and (3.11), we get

$$\begin{aligned}
\|P_1(\mathfrak{s}_1, \mathfrak{s}_2)\| &\leq \left(\frac{1}{\Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 + 1)} + \frac{1}{\vartheta_1 \Gamma_q(\vartheta_1 + \gamma_1 + \delta_1)} \right. \\
&+ \frac{\Gamma_q(\omega_i + 1) v_1^{\vartheta_1 + \gamma_1 + \delta_1 + \omega_1}}{|1 - v_1^{\omega_1}| \Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 + \omega_1 + 1)} \\
&+ \frac{\Gamma_q(\omega_1 + 1) |1 - v_1^{\vartheta_1 + \gamma_1 + \delta_1 + \omega_1 - 1}|}{\vartheta_1 |1 - v_1^{\omega_1}| \Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 + \omega_1)} \\
&+ \frac{\Gamma_q(\omega_1 + 1)}{|1 - v_1^{\omega_1}| \Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 + \omega_1 + 1)} \Big) \\
&\times \left[\left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (2 + \Gamma_q(\beta_1 + 1))}{\Gamma_q(\beta_1 + 1)} \right) \sigma + (1 + \theta_1 + \Lambda_1) \nabla \right] \\
&+ \left(\frac{\Gamma_q(1 - \mu_1)}{\Gamma_q(\delta_1 - \mu_1 + 1)} + \frac{\Gamma_q(\omega_1 + 1) \Gamma_q(1 - \mu_1) v_1^{\delta_1 + \omega_1 - \mu_1}}{|1 - v_1^{\omega_1}| \Gamma_q(\delta_1 + \omega_1 - \mu_1 + 1)} \right. \\
&\quad \left. + \frac{\Gamma_q(\omega_i + 1)}{|1 - v_1^{\omega_1}| \Gamma_q(\delta_1 + \omega_1 + 1)} \right) \eta_1 \sigma \\
&= \left[\Pi_1^* \left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (2 + \Gamma_q(\beta_1 + 1))}{\Gamma_q(\beta_1 + 1)} \right) + \Pi_1 \eta_1 \right] \sigma + \Pi_1^* (1 + \theta_1 + \Lambda_1) \nabla.
\end{aligned}$$

Now, using (3.6) and (3.7), we can write

$$\begin{aligned}
&\left\| \mathcal{D}_q^{\alpha_1} [P_1(\mathfrak{s}_1, \mathfrak{s}_2)] \right\| \\
&\leq \frac{1}{\Gamma_q(2 - \alpha_1)} \left[\frac{1}{\Gamma_q(\vartheta_1 + \gamma_1 + \delta_1)} + \frac{[\vartheta_1 + \gamma_1 + \delta_1 - 1]_q \tau^{\vartheta_1 + \gamma_1 + \delta_1 - 2}}{\vartheta_1 \Gamma(\vartheta_1 + \gamma_1 + \delta_1)} \right] \\
&\quad \times \left[\left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (1 + \Gamma_q(\beta_1 + 1))}{\Gamma_q(\beta_1 + 1)} \right) \sigma + (1 + \theta_1 + \Lambda_1) \nabla \right] \\
&\quad + \frac{1}{\Gamma_q(2 - \alpha_1)} \frac{\eta_1 \Gamma_q(1 - \mu_1)}{\Gamma_q(\delta_1 - \mu_1)} \sigma \\
&= \frac{1}{\Gamma_q(2 - \alpha)} \left[\Theta_1^* \left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (2 + \Gamma_q(\beta_1 + 1))}{\Gamma_q(\beta_1 + 1)} \right) + \Theta_1 \eta_1 \right] \sigma \\
&\quad + \frac{\Theta_1^*}{\Gamma_q(2 - \alpha)} (1 + \theta_1 + \Lambda_1) \nabla.
\end{aligned}$$

From the definition of $\|\cdot\|_{B_1}$, we have

$$\begin{aligned}
\|P_1(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1} &\leq \left[\lambda_1^* \left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (2 + \Gamma_q(\beta_1 + 1))}{\Gamma_q(\beta_1 + 1)} \right) \right. \\
&\quad \left. + \lambda_1 \eta_1 \right] \sigma + \lambda_1^* (1 + \theta_1 + \Lambda_1) \nabla \leq \frac{\sigma}{2}.
\end{aligned}$$

In a similar manner, we can find that

$$\begin{aligned} \|P_2(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_2} &\leq \left[\lambda_2^* \left(\theta_2 a_2 + \frac{\Lambda_2 b_2 (2 + \Gamma_q (\beta_2 + 1))}{\Gamma_q (\beta_2 + 1)} \right) \right. \\ &\quad \left. + \lambda_2 \eta_2 \right] \sigma + \lambda_2^* (1 + \theta_2 + \Lambda_2) \Delta \leq \frac{\sigma}{2}. \end{aligned}$$

From these estimates, we see that $\|P(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2} \leq \sigma$. For $(\mathfrak{s}_i, \mathfrak{s}_i) \in B_\sigma$, $i = 1, 2$ and by (H₁), we can write

$$\begin{aligned} &\|P_1(\mathfrak{s}_1, \mathfrak{s}_2) - P_1(\mathfrak{s}_1, \mathfrak{s}_2)\| \\ &\leq \left[\Pi_1^* \left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (2 + \Gamma_q (\beta_1 + 1))}{\Gamma_q (\beta_1 + 1)} \right) + \Pi_1 \eta_1 \right] \|(\mathfrak{s}_1, \mathfrak{s}_2) - (\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2}. \end{aligned}$$

Also thanks to (3.6) and (3.7), we have

$$\begin{aligned} &\|\mathcal{D}_q^{\alpha_1} P_1(\mathfrak{s}_1, \mathfrak{s}_2) - \mathcal{D}_q^{\alpha_1} P_1(\mathfrak{s}_1, \mathfrak{s}_2)\| \\ &\leq \left[\frac{\Theta_1^*}{\Gamma_q (2 - \alpha_1)} \left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (2 + \Gamma_q (\beta_1 + 1))}{\Gamma_q (\beta_1 + 1)} \right) + \frac{\Theta_1 \eta_1}{\Gamma_q (2 - \alpha_1)} \right] \\ &\quad \times \|(\mathfrak{s}_1, \mathfrak{s}_2) - (\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2}. \end{aligned}$$

Hence

$$\begin{aligned} &\|P_1(\mathfrak{s}_1, \mathfrak{s}_2) - P_1(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1} \\ &= \|P_1(\mathfrak{s}_1, \mathfrak{s}_2) - P_1(\mathfrak{s}_1, \mathfrak{s}_2)\| + \|\mathcal{D}_q^{\alpha_1} P_1(\mathfrak{s}_1, \mathfrak{s}_2) - \mathcal{D}_q^{\alpha_1} P_1(\mathfrak{s}_1, \mathfrak{s}_2)\| \\ &\leq \left[\lambda_1^* \left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (2 + \Gamma_q (\beta_1 + 1))}{\Gamma_q (\beta_1 + 1)} \right) \lambda_1 \eta_1 \right] \\ &\quad \times \|(\mathfrak{s}_1, \mathfrak{s}_2) - (\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2}. \end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned} &\|P_2(\mathfrak{s}_1, \mathfrak{s}_2) - P_2(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_2} \\ &\leq \left[\lambda_2^* \left(\theta_2 a_2 + \frac{\Lambda_2 b_2 (2 + \Gamma_q (\beta_2 + 1))}{\Gamma_q (\beta_2 + 1)} \right) \lambda_2 \eta_2 \right] \\ &\quad \times \|(\mathfrak{s}_1, \mathfrak{s}_2) - (\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} &\|P(\mathfrak{s}_1, \mathfrak{s}_2) - P(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2} \\ &= \|P_1(\mathfrak{s}_1, \mathfrak{s}_2) - P_1(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1} + \|P_2(\mathfrak{s}_1, \mathfrak{s}_2) - P_2(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_2} \\ &\leq \left\{ \left[\lambda_1^* \left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (2 + \Gamma_q (\beta_1 + 1))}{\Gamma_q (\beta_1 + 1)} \right) + \lambda_1 \eta_1 \right] \right. \\ &\quad \left. + \left[\lambda_2^* \left(\theta_2 a_2 + \frac{\Lambda_2 b_2 (2 + \Gamma_q (\beta_2 + 1))}{\Gamma_q (\beta_2 + 1)} \right) + \lambda_2 \eta_2 \right] \right\} \\ &\quad \times \|(\mathfrak{s}_1, \mathfrak{s}_2) - (\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2}. \end{aligned}$$

Thanks to (3.9), we conclude that P is contractive. Hence, by the Banach contraction principle, there exists a unique fixed point which is a solution of system (1.1). \square

Now, we prove the existence of solutions of system (1.1) by using Lamma 2.5.

Theorem 3.3. *Let $F_i : \Omega \rightarrow \mathbb{R}$ and $\aleph_{i2}, \aleph_{i1} : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ for $i = 1, 2$ are continuous functions. Assume that*

(H₂) *There exist a positive constants $A_{1i}, A_{2i}, A_{3i}, i = 1, 2$ such that for any $\tau \in \Omega$ and $s_j \in \mathbb{R} (j = 1, 2, 3)$ we have*

$$|\aleph_{i1}(\tau, s_1, s_2, s_3)| \leq A_{1i}, \quad |\aleph_{i2}(\tau, s_1, s_2, s_3)| \leq A_{2i}, \quad |F_i(\tau)| \leq A_{3i}, \quad i = 1, 2.$$

If $\lambda_1\eta_1 < 1$ and $\lambda_2\eta_2 < 1$ where λ_i is given by (3.8). Then the system (1.1) has at least one solution.

Proof. The operator P is continuous in view of the continuity of functions \aleph_{i1}, \aleph_{i2} and $F_i, i = 1, 2$. We begin by showing that operator $P : B_1 \times B_2 \rightarrow B_1 \times B_2$ is completely continuous. Firstly, we demonstrate that P maps bounded sets of $B_1 \times B_2$ into bounded sets of $B_1 \times B_2$. Let us put

$$\mathbb{B}_\epsilon = \left\{ (\mathfrak{s}_1, \mathfrak{s}_2) \in B_1 \times B_2 : \|(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2} \leq \epsilon \right\}.$$

Then for all $(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathbb{B}_\epsilon$, thanks to (H₂), we have

$$\|P_1(\mathfrak{s}_1, \mathfrak{s}_2)\| \leq \Pi_1^*(A_{31} + \theta_i A_{11} + \Lambda_i A_{21}) + \Pi_1 \eta_1,$$

and by (3.6), (3.7), we get

$$\|D_q^{\alpha_1}[P_1(\mathfrak{s}_1, \mathfrak{s}_2)]\| \leq \frac{\Theta_1^*}{\Gamma_q(2 - \alpha_1)}(A_{31} + \theta_i A_{11} + \Lambda_1 A_{21}) + \frac{\Theta_1 \eta_1 \epsilon}{\Gamma_q(2 - \alpha_1)}.$$

Thus

$$\begin{aligned} \|P_1(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1} &= \|P_1(\mathfrak{s}_1, \mathfrak{s}_2)\| + \|D_q^{\alpha_1}[P_1(\mathfrak{s}_1, \mathfrak{s}_2)]\| \\ &\leq \lambda_1^*(A_{31} + \theta_1 A_{11} + \Lambda_1 A_{21}) + \lambda_1 \eta_1, \\ \|P_2(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_2} &\leq \lambda_2^*(A_{32} + \theta_2 A_{12} + \Lambda_1 A_{22}) + \lambda_2 \eta_2, \end{aligned}$$

From the above inequalities, it follows that $\|P(\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2} < \infty$. Next, we show that P is equicontinuous. Let $(\mathfrak{s}_1, \mathfrak{s}_2) \in B_1 \times B_2$ and $\tau_1, \tau_2 \in \Omega$, with $\tau_2 < \tau_1$, we have

$$\begin{aligned} &|P_i(\mathfrak{s}_1, \mathfrak{s}_2)(\tau_1) - P_i(\mathfrak{s}_1, \mathfrak{s}_2)(\tau_2)| \tag{3.13} \\ &\leq \frac{(A_{31} + \theta_1 A_{11} + \Lambda_1 A_{21})}{\Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 + 1)} \left[(\tau_1 - \tau_2)^{\vartheta_1 + \gamma_1 + \delta_1} + \left| \tau_1^{\vartheta_1 + \gamma_1 + \delta_1} - \tau_2^{\vartheta_1 + \gamma_1 + \delta_1} \right| \right] \\ &\quad + \eta_1 \epsilon \frac{\Gamma_q(1 - \mu_1)}{\Gamma_q(\delta_1 - \mu_1 + 1)} \left[(\tau_1 - \tau_2)^{\delta_1} + \left| \tau_1^{\delta_1} - \tau_2^{\delta_1} \right| \right] \\ &\quad + \frac{(A_{31} + \theta_1 A_{11} + \Lambda_1 A_{21})}{\vartheta_1 \Gamma(\vartheta_1 + \gamma_1 + \delta_1)} \left| \tau_2^{\vartheta_1 + \gamma_1 + \delta_1 - 1} - \tau_1^{\vartheta_1 + \gamma_1 + \delta_1 - 1} \right|. \end{aligned}$$

Also, by (3.6) and (3.7), we have

$$\begin{aligned}
& |D_q^{\alpha_1} P_i(\mathfrak{s}_1, \mathfrak{s}_2)(\tau_1) - D_q^{\alpha_1} P_i(\mathfrak{s}_1, \mathfrak{s}_2)(\tau_2)| \\
& \leq \frac{(A_{31} + \theta_1 A_{11} + \Lambda_1 A_{21})}{\Gamma_q(2 - \alpha_1) \Gamma_q(\vartheta_1 + \gamma_1 + \delta_1)} \left[(\tau_1 - \tau_2)^{\vartheta_1 + \gamma_1 + \delta_1 - 1} \right. \\
& \quad \left. + \left| \tau_1^{\vartheta_1 + \gamma_1 + \delta_1 - 1} - \tau_2^{\vartheta_1 + \gamma_1 + \delta_1 - 1} \right| \right] \\
& \quad + \eta_1 \epsilon \frac{\Gamma_q(1 - \mu_1)}{\Gamma_q(2 - \alpha_1) \Gamma_q(\delta_1 - \mu_1)} \left[(\tau_1 - \tau_2)^{\delta_1 - 1} + \left| \tau_1^{\delta_1 - 1} - \tau_2^{\delta_1 - 1} \right| \right] \\
& \quad + \frac{[\vartheta_1 + \gamma_1 + \delta_1 - 1]_q}{\Gamma_q(2 - \alpha_1) \Gamma(\vartheta_1 + \gamma_1 + \delta_1)} \left| \tau_1^{\vartheta_1 + \gamma_1 + \delta_1 - 2} - \tau_2^{\vartheta_1 + \gamma_1 + \delta_1 - 2} \right|.
\end{aligned} \tag{3.14}$$

Analogously, one can obtain

$$\begin{aligned}
& |P_2(\mathfrak{s}_1, \mathfrak{s}_2)(\tau_1) - P_2(\mathfrak{s}_1, \mathfrak{s}_2)(\tau_2)| \\
& \leq \frac{(A_{32} + \theta_2 A_{21} + \Lambda_2 A_{22})}{\Gamma_q(\vartheta_2 + \gamma_2 + \delta_2 + 1)} \left[(\tau_1 - \tau_2)^{\vartheta_2 + \gamma_2 + \delta_2} \right. \\
& \quad \left. + \left| \tau_1^{\vartheta_2 + \gamma_2 + \delta_2} - \tau_2^{\vartheta_2 + \gamma_2 + \delta_2} \right| \right] \\
& \quad + \eta_2 \epsilon \frac{\Gamma_q(1 - \mu_2)}{\Gamma_q(\delta_2 - \mu_2 + 1)} \left[(\tau_1 - \tau_2)^{\delta_2} + \left| \tau_1^{\delta_2} - \tau_2^{\delta_2} \right| \right] \\
& \quad + \frac{(A_{32} + \theta_2 A_{21} + \Lambda_2 A_{22})}{\vartheta_2 \Gamma(\vartheta_2 + \gamma_2 + \delta_2)} \left| \tau_1^{\vartheta_2 + \gamma_2 + \delta_2 - 1} - \tau_2^{\vartheta_2 + \gamma_2 + \delta_2 - 1} \right|,
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
& |\mathcal{D}_q^{\alpha_2} P_2(\mathfrak{s}_1, \mathfrak{s}_2)(\tau_1) - \mathcal{D}_q^{\alpha_2} P_2(\mathfrak{s}_1, \mathfrak{s}_2)(\tau_2)| \\
& \leq \frac{(A_{32} + \theta_2 A_{21} + \Lambda_2 A_{22})}{\Gamma_q(2 - \alpha_2) \Gamma_q(\vartheta_2 + \gamma_2 + \delta_2)} \left[(\tau_1 - \tau_2)^{\vartheta_2 + \gamma_2 + \delta_2 - 1} \right. \\
& \quad \left. + \left| \tau_1^{\vartheta_2 + \gamma_2 + \delta_2 - 1} - \tau_2^{\vartheta_2 + \gamma_2 + \delta_2 - 1} \right| \right] \\
& \quad + \eta_2 \epsilon \frac{\Gamma_q(1 - \mu_2)}{\Gamma_q(2 - \alpha_2) \Gamma_q(\delta_2 - \mu_2)} \left[(\tau_1 - \tau_2)^{\delta_2 - 1} + \left| \tau_1^{\delta_2 - 1} - \tau_2^{\delta_2 - 1} \right| \right] \\
& \quad + \frac{[\vartheta_2 + \gamma_2 + \delta_2 - 1]_q}{\Gamma_q(2 - \alpha_2) \Gamma(\vartheta_2 + \gamma_2 + \delta_2)} \left| \tau_1^{\vartheta_2 + \gamma_2 + \delta_2 - 2} - \tau_2^{\vartheta_2 + \gamma_2 + \delta_2 - 2} \right|.
\end{aligned} \tag{3.16}$$

Thanks to (3.13)-(3.16), we can state that

$$\|P(\mathfrak{s}_1, \mathfrak{s}_2)(\tau_1) - P(\mathfrak{s}_1, \mathfrak{s}_2)(\tau_2)\|_{B_1 \times B_2} \rightarrow 0,$$

as $\tau_1 \rightarrow \tau_2$. By using the Arzelà-Ascoli theorem, we conclude that P is a completely continuous operator. Finally, we prove that the set Φ , given by

$$\Phi = \left\{ (\mathfrak{s}_1, \mathfrak{s}_2) \in B_1 \times B_2 : (\mathfrak{s}_1, \mathfrak{s}_2) = \rho P(\mathfrak{s}_1, \mathfrak{s}_2), 0 < \rho < 1 \right\},$$

is bounded. Let $(\mathfrak{s}_1, \mathfrak{s}_2) \in \Phi$, then $(\mathfrak{s}_1, \mathfrak{s}_2) = \rho P(\mathfrak{s}_1, \mathfrak{s}_2)$, for some $0 < \rho < 1$. Hence, for $\tau \in \Omega$, we have

$$\mathfrak{s}_i(\tau) = \rho P_i(\mathfrak{s}_1, \mathfrak{s}_2), \quad i = 1, 2.$$

Thanks to (H₂), we can write

$$\|\mathfrak{s}_1\| \leq \Pi_1^*(A_{31} + \theta_1 A_{11} + \Lambda_1 A_{21}) + \Pi_1 \eta_1 \|\mathfrak{s}_1\|,$$

by (3.6) and (3.7), we get

$$\|D_q^{\alpha_1} P_1(\mathfrak{s}_1, \mathfrak{s}_2)\| \leq \frac{\Theta_1^*}{\Gamma_q(2 - \alpha_1)} (A_{31} + \theta_1 A_{11} + \Lambda_1 A_{21}) + \frac{\Theta_1}{\Gamma_q(2 - \alpha_1)} \eta_1 \|\mathfrak{s}_1\|.$$

In view of the above estimates, we get

$$\|\mathfrak{s}_1\|_{B_1} \leq \lambda_1^*(A_{31} + \theta_1 A_{11} + \Lambda_1 A_{21}) + \lambda_2 \eta_1 \|\mathfrak{s}_1\|_{B_1}. \quad (3.17)$$

In a similar manner, we can obtain

$$\|\mathfrak{s}_2\|_{B_2} \leq \lambda_2^*(A_{32} + \theta_1 A_{21} + \Lambda_1 A_{22}) + \lambda_1 \eta_2 \|\mathfrak{s}_2\|_{B_2}. \quad (3.18)$$

It follows from (3.17) and (3.18), that

$$\begin{aligned} \|\mathfrak{s}_1\|_{B_1} + \|\mathfrak{s}_2\|_{B_2} &\leq \lambda_1^*(A_{31} + \theta_1 A_{11} + \Lambda_1 A_{21}) + \lambda_2^*(A_{32} + \theta_1 A_{21} + \Lambda_1 A_{22}) \\ &\quad + \lambda_1 \eta_1 \|\mathfrak{s}_1\|_{B_1} + \lambda_2 \eta_2 \|\mathfrak{s}_2\|_{B_2}. \end{aligned}$$

Consequently

$$\|\mathfrak{s}_1, \mathfrak{s}_2\|_{B_1 \times B_2} \leq \frac{\sum_{i=1}^2 \lambda_i^*(A_{3i} + \theta_i A_{1i} + \Lambda_i A_{2i})}{\min\{1 - \lambda_2 \eta_1, 1 - \lambda_1 \eta_2\}}.$$

This shows that Φ is bounded. Thanks to Lemma 2.5, we deduce that P has at least one fixed point, which is a solution of system (1.1). \square

4 Ulam-stability of q -fractional Lane-Emden system

Theorem 4.1. *Assume that (H₁) holds. If the inequality*

$$\lambda_i^* \left(\theta_i a_i + \frac{\Lambda_i b_i (2 + \Gamma_q(\beta_i + 1))}{\Gamma_q(\beta_i + 1)} \right) + \lambda_i \eta_i < 1, \quad i = 1, 2,$$

is valid, then the q -fractional Lane Emden system (1.1) is stable in Ulam-Hyers sense.

Proof. Let us denote by $(\mathfrak{s}_1, \mathfrak{s}_2) \in B_1 \times B_2$ the unique solution of the problem

$$\left\{ \begin{array}{l} {}_{\text{R.L}}\mathcal{D}_q^{\vartheta_1} \left[{}_{\text{C}}\mathcal{D}_q^{\gamma_1} \left[{}_{\text{C}}\mathcal{D}_q^{\delta_1} + \frac{\eta_1}{\tau^{\mu_1}} \right] \right] \mathfrak{s}_1(\tau) \\ = F_1(\tau) - \theta_1 \mathfrak{N}_{11}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), {}_{\text{C}}\mathcal{D}_q^{\alpha_1} \mathfrak{s}_1(\tau)) \\ - \Lambda_1 \mathfrak{N}_{12}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), \mathcal{I}_q^{\beta_1} \mathfrak{s}_1(\tau)), \quad \tau \in (0, 1], \\ \\ {}_{\text{R.L}}\mathcal{D}_q^{\vartheta_2} \left[{}_{\text{C}}\mathcal{D}_q^{\gamma_2} \left[{}_{\text{C}}\mathcal{D}_q^{\delta_2} + \frac{\eta_2}{\tau^{\mu_2}} \right] \right] \mathfrak{s}_2(\tau) \\ = F_2(\tau) - \theta_2 \mathfrak{N}_{21}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), {}_{\text{C}}\mathcal{D}_q^{\alpha_2} \mathfrak{s}_2(\tau)) \\ - \Lambda_2 \mathfrak{N}_{22}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), \mathcal{I}_q^{\beta_2} \mathfrak{s}_2(\tau)), \quad \tau \in (0, 1], \\ \\ {}_{\text{C}}\mathcal{D}_q^{\gamma_i} \left[{}_{\text{C}}\mathcal{D}_q^{\delta_i} + \eta_i \right] \mathfrak{s}_i(1) = {}_{\text{C}}\mathcal{D}_q^{\gamma_i} \left[{}_{\text{C}}\mathcal{D}_q^{\delta_i} + \eta_i \right] \dot{\mathfrak{s}}_i(1), \\ {}_{\text{C}}\mathcal{D}_q^{\gamma_i} \left[{}_{\text{C}}\mathcal{D}_q^{\delta_i} \mathfrak{s}_i(0) \right] = {}_{\text{C}}\mathcal{D}_q^{\gamma_i} \left[{}_{\text{C}}\mathcal{D}_q^{\delta_i} \dot{\mathfrak{s}}_i(0) \right], \\ \mathfrak{s}_i(1) = \dot{\mathfrak{s}}_i(1), \quad \mathcal{I}_q^{\omega_i} \mathfrak{s}_i(v_i) = \mathcal{I}_q^{\omega_i} \dot{\mathfrak{s}}_i(v_i), \end{array} \right. \quad (4.1)$$

for $\eta_i, \theta_i, \Lambda_i > 0, 0 < \mu_i, v_i \leq 1, 0 \leq \omega_i, i = 1, 2$, and $0 < \vartheta_i, \gamma_i, \delta_i < 1, \alpha_i < \delta_i, \Omega := [0, 1]$, such that $(\dot{\mathfrak{s}}_1, \dot{\mathfrak{s}}_2) \in B_1 \times B_2$ is a solution of the inequality (2.1). From inequality (2.1), we can write

$$\begin{aligned} & \left| \dot{\mathfrak{s}}_i(\tau) - \mathcal{I}_q^{\vartheta_i + \gamma_i + \delta_i} \left[\widehat{F}_i^{\dot{\mathfrak{s}}_i}(\tau) \right] + \mathcal{I}_q^{\delta_i} \left[\frac{\eta_i}{\tau^{\mu_i}} \dot{\mathfrak{s}}_i(\tau) \right] - c_{1,i} \frac{\Gamma(\vartheta_i) \tau^{\vartheta_i + \gamma_i + \delta_i - 1}}{\Gamma(\vartheta_i + \gamma_i + \delta_i)} - c_{2,i} \frac{\tau^{\delta_i}}{\Gamma(\delta_i + 1)} - c_{3,i} \right| \\ & \leq \mathring{\ell}_i \mathcal{I}_q^{\vartheta_i + \gamma_i + \delta_i}[1], \quad i = 1, 2, \end{aligned}$$

where

$$\widehat{F}_i^{\dot{\mathfrak{s}}_i}(\tau) = F_i(\tau) - \theta_i \mathfrak{N}_{i1}(\tau, \mathfrak{s}_1(\tau), \mathfrak{s}_2(\tau), {}_{\text{C}}\mathcal{D}_q^{\alpha_1} \mathfrak{s}_1(\tau)) - \Lambda_1 \mathfrak{N}_{i2}(\tau, \mathfrak{s}_i(\tau), \mathfrak{s}_2(\tau), \mathcal{I}_q^{\beta_1} \mathfrak{s}_1(\tau)),$$

$\tau \in \Omega$. By Lemma 3.1, we get

$$|\dot{\mathfrak{s}}_1(\tau) - P_1 \dot{\mathfrak{s}}_1(\tau)| \leq \mathring{\ell}_1 \mathcal{I}_q^{\vartheta_1 + \gamma_1 + \delta_1}[1], \quad \tau \in \Omega.$$

Using Lemma 4, we obtain

$$|\dot{\mathfrak{s}}_1(\tau) - P_1 \dot{\mathfrak{s}}_1(\tau)| \leq \frac{\mathring{\ell}_1 \tau^{\vartheta_1 + \gamma_1 + \delta_1}}{\Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 + 1)}, \quad \tau \in \Omega,$$

and

$$\left| {}_{\text{R.L}}\mathcal{D}_q^{\alpha_1} \dot{\mathfrak{s}}_1(\tau) - {}_{\text{R.L}}\mathcal{D}_q^{\alpha_1} P_1 \dot{\mathfrak{s}}_1(\tau) \right| \leq \frac{\mathring{\ell}_1 \tau^{\vartheta_1 + \gamma_1 + \delta_1 - \alpha_1}}{\Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 - \alpha_1 + 1)}, \quad \tau \in \Omega,$$

which imply that

$$\|\dot{\mathfrak{s}}_1 - P_1(\dot{\mathfrak{s}}_1)\|_{B_1} \leq \frac{\mathring{\ell}_1}{\Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 + 1)} + \frac{\mathring{\ell}_1}{\Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 - \alpha_1 + 1)}. \quad (4.2)$$

On the other hand, we have

$$\begin{aligned} |\dot{s}_1(\tau) - s_1(\tau)| &= \left| \dot{s}_1(\tau) - \mathcal{I}_q^{\vartheta_1+\gamma_1+\delta_1} \left[h_1^{s_1}(\tau) \right] - \mathcal{I}_q^{\delta_1} \left[\frac{\eta_1}{\tau^{\mu_1}} s_1(\tau) \right] \right. \\ &\quad \left. + c_{1,1} \frac{\Gamma(\vartheta_1)}{\Gamma(\vartheta_1 + \gamma_1 + \delta_1)} \tau^{\vartheta_1 + \gamma_1 + \delta_1 - 1} + c_{2,1} \frac{\tau^{\delta_1}}{\Gamma(\delta_1 + 1)} + c_{3,1} \right| \\ &= |\dot{s}_1(\tau) - P_1 \dot{s}_1(\tau) + P_1 s_1(\tau) - P_1 s_1(\tau)| \\ &\leq |\dot{s}_1(\tau) - P_1 \dot{s}_1(\tau)| + |P_1 s_1(\tau) - P_1 s_1(\tau)|. \end{aligned}$$

Thanks to (4.2) and (H₁), we can write

$$\begin{aligned} \|\dot{s}_1 - s_1\|_{B_1} &\leq \frac{\overset{\circ}{\ell}_1}{\Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 + 1)} + \frac{\overset{\circ}{\ell}_1}{\Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 - \alpha_1 + 1)} \\ &\quad + \left[\lambda_i^* \left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (2 + \Gamma_q(\beta_1 + 1))}{\Gamma_q(\beta_1 + 1)} \right) + \lambda_i \eta_1 \right] \\ &\quad \times \|(\dot{s}_1, s_2) - (\dot{s}_1, s_2)\|_{B_1 \times B_2}. \end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned} \|\dot{s}_2 - s_2\|_{B_2} &\leq \frac{\overset{\circ}{\ell}_2}{\Gamma_q(\vartheta_2 + \gamma_2 + \delta_2 + 1)} + \frac{\overset{\circ}{\ell}_2}{\Gamma_q(\vartheta_2 + \gamma_2 + \delta_2 - \alpha_2 + 1)} \\ &\quad + \left[\lambda_i^* \left(\theta_2 a_2 + \frac{\Lambda_2 b_2 (2 + \Gamma_q(\beta_2 + 1))}{\Gamma_q(\beta_2 + 1)} \right) + \lambda_i \eta_2 \right] \\ &\quad \times \|(\dot{s}_1, s_2) - (\dot{s}_1, s_2)\|_{B_1 \times B_2}. \end{aligned}$$

From the above inequalities, we get

$$\begin{aligned} &\|(\dot{s}_1, s_2) - (\dot{s}_1, s_2)\|_{B_1 \times B_2} \\ &\leq \frac{\sum_{i=1}^2 \left(\frac{1}{\Gamma_q(\vartheta_i + \gamma_i + \delta_i + 1)} + \frac{1}{\Gamma_q(\vartheta_i + \gamma_i + \delta_i - \alpha_i + 1)} \right) \overset{\circ}{\ell}}{\min_{i=1,2} \{1 - F_i\}}, \end{aligned} \tag{4.3}$$

where

$$F_i := \lambda_i^* \left(\theta_i a_i + \frac{\Lambda_i b_i (2 + \Gamma_q(\beta_i + 1))}{\Gamma_q(\beta_i + 1)} \right) + \lambda_i \eta_i, \quad i = 1, 2.$$

If we put

$$\Sigma_{\aleph_{i_1}^*, \aleph_{i_2}^*, m_i} := \frac{\sum_{i=1}^2 \left(\frac{1}{\Gamma_q(\vartheta_i + \gamma_i + \delta_i + 1)} + \frac{1}{\Gamma_q(\vartheta_i + \gamma_i + \delta_i - \alpha_i + 1)} \right)}{\min_{i=1,2} \{1 - F_i\}},$$

then

$$\|(\dot{s}_1, s_2) - (\dot{s}_1, s_2)\|_{B_1 \times B_2} \leq \Sigma_{\aleph_{i_1}^*, \aleph_{i_2}^*, m_i} \overset{\circ}{\ell}.$$

Hence, the q -fractional Lane-Emden system (1.1) is stable in Ulam-Hyers sense. \square

Theorem 4.2. *If condition (H₁) is satisfied. Suppose there exists $\varrho_{i,m} > 0$ and $\partial_{i,m} > 0, i = 1, 2$ such that*

$$\mathcal{I}_q^{\vartheta_i + \gamma_i + \delta_i} [m(\tau)] \leq \varrho_{i,m} m(\tau) \text{ and } \mathcal{I}_q^{\vartheta_i + \gamma_i + \delta_i - \alpha_i} [m(\tau)] \leq \partial_{i,m} m(\tau), \quad (4.4)$$

where $i = 1, 2$, for all $\tau \in \Omega$, where $m \in C(\Omega, \mathbb{R}_+)$ is nondecreasing. Then the q -fractional Lane-Emden system (1.1) is Ulam-Hyers-Rassias stable.

Proof. Let us denote by $(\mathfrak{s}_1, \mathfrak{s}_2) \in B_1 \times B_2$ the unique solution of 4.1. From inequality (2.2), we can write

$$\begin{aligned} & \left| \mathfrak{s}_i(\tau) - \mathcal{I}_q^{\vartheta_i + \gamma_i + \delta_i} \left[h_i^{\mathfrak{s}_i}(\tau) \right] + \mathcal{I}_q^{\delta_i} \left[\frac{\eta_i}{\tau^{\mu_i}} t_i(\tau) \right] \right. \\ & \quad \left. - c_{1,i} \frac{\Gamma(\vartheta_i) \tau^{\vartheta_i + \gamma_i + \delta_i - 1}}{\Gamma(\vartheta_i + \gamma_i + \delta_i)} - c_{2,i} \frac{\tau^{\delta_i}}{\Gamma(\delta_i + 1)} - c_{3,i} \right| \\ & \leq \mathring{\ell}_i \mathcal{I}_q^{\vartheta_i + \gamma_i + \delta_i} [m(\tau)], \quad i = 1, 2, \end{aligned}$$

where $(\mathfrak{s}_1, \mathfrak{s}_2) \in B_1 \times B_2$ is a solution of the inequality (2.2). Thanks to Lemma 3.1 and (4.4), we get

$$|\mathfrak{s}_1(\tau) - P_1 \mathfrak{s}_1(\tau)| = |\lambda_1 \mathcal{I}_q^{\vartheta_1 + \gamma_1 + \delta_1} [l(\tau)]| \leq \mathring{\ell}_1 \varrho_{1,m} m(\tau),$$

and

$$|{}_C \mathcal{D}_q^{\alpha_1} \mathfrak{s}_1(\tau) - {}_C \mathcal{D}_q^{\alpha_1} P_1 \mathfrak{s}_1(\tau)| = |\mathcal{I}_q^{\vartheta_1 + \gamma_1 + \delta_1 - \alpha_1} [m_1(\tau)]| \leq \mathring{\ell}_1 \partial_{1,m} m(\tau).$$

Also, we have

$$\begin{aligned} |\mathfrak{s}_1(\tau) - \mathfrak{s}_1(\tau)| &= \left| \mathfrak{s}_1(\tau) - \mathcal{I}_q^{\vartheta_1 + \gamma_1 + \delta_1} \left[h_1^{\mathfrak{s}_1}(\tau) \right] - \mathcal{I}_q^{\delta_1} \left[\frac{\eta_1}{\tau^{\mu_1}} \mathfrak{s}_1(\tau) \right] \right. \\ &\quad \left. + c_{1,1} \frac{\Gamma(\vartheta_1) \tau^{\vartheta_1 + \gamma_1 + \delta_1 - 1}}{\Gamma(\vartheta_1 + \gamma_1 + \delta_1)} + c_{2,1} \frac{\tau^{\delta_1}}{\Gamma(\delta_1 + 1)} + c_{3,1} \right| \\ &= |\mathfrak{s}_1(\tau) - P_1 \mathfrak{s}_1(\tau) + P_1 \mathfrak{s}_1(\tau) - P_1 \mathfrak{s}_1(\tau)| \\ &\leq |\mathfrak{s}_1(\tau) - P_1 \mathfrak{s}_1(\tau)| + |P_1 \mathfrak{s}_1(\tau) - P_1 \mathfrak{s}_1(\tau)|. \end{aligned}$$

So, by (H₁) and (4.4), we get

$$\begin{aligned} \|\mathfrak{s}_1 - \mathfrak{s}_1\|_{B_1} &\leq \mathring{\ell}_1 \varrho_{1,m} m(\tau) + \mathring{\ell}_1 \partial_{1,m} m(\tau) \\ &\quad + \left[\lambda_i^* \left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (2 + \Gamma_q(\beta_1 + 1))}{\Gamma_q(\beta_1 + 1)} \right) + \lambda_i \eta_1 \right] \\ &\quad \times \|(\mathfrak{s}_1, \mathfrak{s}_2) - (\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2}. \end{aligned}$$

Using the same arguments, we can write

$$\begin{aligned} \|\mathfrak{s}_2 - \mathfrak{s}_2\|_{B_2} &\leq \mathring{\ell}_2 \varrho_{2,m} m(\tau) + \mathring{\ell}_2 \partial_{2,m} m(\tau) \\ &\quad + \left[\lambda_i^* \left(\theta_2 a_2 + \frac{\Lambda_2 b_2 (2 + \Gamma_q(\beta_2 + 1))}{\Gamma_q(\beta_2 + 1)} \right) + \lambda_i \eta_2 \right] \\ &\quad \times \|(\mathfrak{s}_1, \mathfrak{s}_2) - (\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2}. \end{aligned}$$

Consequently, it follows that

$$\|(\mathfrak{s}_1, \mathfrak{s}_2) - (\dot{\mathfrak{s}}_1, \dot{\mathfrak{s}}_2)\|_{B_1 \times B_2} \leq \frac{\sum_{i=1}^2 (\varrho_{i,m} + \partial_{i,m})}{\min_{i=1,2} \{1 - F_i\}} \ell m(\tau).$$

If we take

$$\Sigma_{\mathfrak{N}_{i1}^*, \mathfrak{N}_{i2}^*, F_i, m} := \frac{\sum_{i=1}^2 (\varrho_{i,m} + \partial_{i,m})}{\min_{i=1,2} \{1 - F_i\}},$$

we can obtain

$$\|(\mathfrak{s}_1, \mathfrak{s}_2) - (\dot{\mathfrak{s}}_1, \dot{\mathfrak{s}}_2)\|_{B_1 \times B_2} \leq \Sigma_{\mathfrak{N}_{i1}^*, \mathfrak{N}_{i2}^*, F_i, m} \ell m(\tau), \quad \tau \in \Omega.$$

Thus, the q -fractional Lane-Emden system (1.1) is stable in Ulam-Hyers-Rassias sense. \square

5 A numerical example

We consider the following q -fractional Lane-Emden system,

$$\left\{ \begin{array}{l} {}_{\text{R.L}}\mathcal{D}_q^{1/2} \left[{}_{\text{R.L}}\mathcal{D}_q^{\ln 2/3} \left[{}_{\text{R.L}}\mathcal{D}_q^{2/5} + \frac{1}{13\tau^{0.02}} \right] \right] \mathfrak{s}_1(\tau) = \cosh(\tau^2 + 3) \\ -\frac{1}{39} \frac{e^{-\tau^2}}{e^{\tau^2} + 34\pi} \left(\cos(\mathfrak{s}_1(\tau)) + \frac{|\mathfrak{s}_2(\tau)|}{1 + |\mathfrak{s}_2(\tau)|} + \frac{1}{2} \sin^2 \left({}_{\text{R.L}}\mathcal{D}_q^{1/5} \mathfrak{s}_1(\tau) \right) \right) \\ -\frac{2}{41} \frac{1}{\sqrt{40 + \tau^2}} \left(\sin(\mathfrak{s}_1(\tau)) + \tan^{-1} \mathfrak{s}_2(\tau) + \mathcal{I}_q^{\ln 3/2} \mathfrak{s}_1(\tau) \right), \\ \\ {}_{\text{R.L}}\mathcal{D}_q^{e/7} \left[{}_{\text{R.L}}\mathcal{D}_q^{3/4} \left[{}_{\text{R.L}}\mathcal{D}_q^{\sqrt{3}/3} + \frac{2}{17\tau^{0.01}} \right] \right] \mathfrak{s}_2(\tau) = \arctan(\tau + 2) \\ -\frac{\sqrt{3}}{85} \frac{e^{-\tau^2}}{2\pi\sqrt{37 + \tau^3}} \left(\sin(2\pi\mathfrak{s}_1(\tau)) + 2\pi\mathfrak{s}_2(\tau) + \frac{2\pi \left| {}_{\text{R.L}}\mathcal{D}_q^{1/3} \mathfrak{s}_2(\tau) \right|}{2 + \left| {}_{\text{R.L}}\mathcal{D}_q^{1/3} \mathfrak{s}_2(\tau) \right|} \right) \\ -\frac{e}{100} \frac{|\mathfrak{s}_1(\tau)| + |\mathfrak{s}_2(\tau)| + \left| \mathcal{I}_q^{\sqrt{5}/2} \mathfrak{s}_2(\tau) \right|}{(3\tau^2 + 39) \left(e^{-3\tau} + |\mathfrak{s}_1(\tau)| + |\mathfrak{s}_2(\tau)| + \left| \mathcal{I}_q^{\sqrt{5}/2} \mathfrak{s}_2(\tau) \right| \right)}, \\ \\ {}_{\text{R.L}}\mathcal{D}_q^{\ln 2/3} \left[{}_{\text{R.L}}\mathcal{D}_q^{2/5} + \frac{1}{13} \right] \mathfrak{s}_1(1) = 0, \\ {}_{\text{R.L}}\mathcal{D}_q^{3/4} \left[{}_{\text{R.L}}\mathcal{D}_q^{\sqrt{3}/3} + \frac{2}{17} \right] \mathfrak{s}_2(1) = 0, \\ {}_{\text{R.L}}\mathcal{D}_q^{\ln 2/3} \left[{}_{\text{R.L}}\mathcal{D}_q^{2/5} \mathfrak{s}_1(0) \right] = 0, \quad {}_{\text{R.L}}\mathcal{D}_q^{3/4} \left[{}_{\text{R.L}}\mathcal{D}_q^{\sqrt{3}/3} \mathfrak{s}_2(0) \right] = 0, \\ \mathfrak{s}_1(1) = \mathcal{I}_q^{3/2} \mathfrak{s}_1\left(\frac{\ln 2}{5}\right), \quad \mathfrak{s}_2(1) = \mathcal{I}_q^{\frac{4}{3}} \mathfrak{s}_2\left(\frac{e}{6}\right), \end{array} \right. \quad (5.1)$$

for $\tau \in (0, 1]$, and also, consider the following inequalities

$$\left\{ \begin{array}{l} \left| \begin{array}{l} \text{R.L}\mathcal{D}_q^{1/2} \left[\text{R.L}\mathcal{D}_q^{\ln 2/3} \left[\text{R.L}\mathcal{D}_q^{2/5} + \frac{1}{13\tau^{0.02}} \right] \right] \mathfrak{s}_1(\tau) - \left[\cosh(\tau^2 + 3) \right. \\ \left. - \frac{1}{39} \mathfrak{N}_{11,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) - \frac{2}{41} \mathfrak{N}_{12,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) \right] \end{array} \right| \leq \ell_1, \\ \left| \begin{array}{l} \text{R.L}\mathcal{D}_q^{e/7} \left[\text{R.L}\mathcal{D}_q^{3/4} \left[\text{R.L}\mathcal{D}_q^{\sqrt{3}/3} + \frac{2}{17\tau^{0.01}} \right] \right] - \left[\arctan(\tau + 2) \right. \\ \left. - \frac{\sqrt{3}}{85} \mathfrak{N}_{21,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) - \frac{e}{100} \mathfrak{N}_{22,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) \right] \end{array} \right| \leq \ell_2, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \left| \begin{array}{l} \text{R.L}\mathcal{D}_q^{1/2} \left[\text{R.L}\mathcal{D}_q^{\ln 2/3} \left[\text{R.L}\mathcal{D}_q^{2/5} + \frac{1}{13\tau^{0.02}} \right] \right] \mathfrak{s}_1(\tau) - \left[\cosh(\tau^2 + 3) \right. \\ \left. - \frac{1}{39} \mathfrak{N}_{11,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) - \frac{2}{41} \mathfrak{N}_{12,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) \right] \end{array} \right| \leq \ell_1 m(\tau), \\ \left| \begin{array}{l} \text{R.L}\mathcal{D}_q^{e/7} \left[\text{R.L}\mathcal{D}_q^{3/4} \left[\text{R.L}\mathcal{D}_q^{\sqrt{3}/3} + \frac{2}{17\tau^{0.01}} \right] \right] - \left[\arctan(\tau + 2) \right. \\ \left. - \frac{\sqrt{3}}{85} \mathfrak{N}_{21,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) - \frac{e}{100} \mathfrak{N}_{22,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) \right] \end{array} \right| \leq \ell_2 m(\tau), \end{array} \right.$$

for

$$q \in \left\{ \frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10} \right\},$$

where

$$\begin{aligned} \mathfrak{N}_{11,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) &= \frac{e^{-\tau^2}}{e^{\tau^2} + 34\pi} \left(\cos \mathfrak{s}_1(\tau) + \frac{|\mathfrak{s}_2(\tau)|}{1 + |\mathfrak{s}_2(\tau)|} + \frac{1}{2} \sin^2 \left(\text{R.L}\mathcal{D}_q^{1/5} \mathfrak{s}_1(\tau) \right) \right), \\ \mathfrak{N}_{12,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) &= \frac{1}{\sqrt{40 + \tau^2}} \left(\sin \mathfrak{s}_1(\tau) + \tan^{-1} \mathfrak{s}_2(\tau) + \mathcal{I}_q^{\ln 3/2} \mathfrak{s}_1(\tau) \right), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{N}_{21,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) &= \frac{e^{-\tau^2}}{2\pi\sqrt{37 + \tau^3}} \left(\sin(2\pi\mathfrak{s}_1(\tau)) + 2\pi\mathfrak{s}_2(\tau) + \frac{2\pi \left| \text{R.L}\mathcal{D}_q^{1/3} \mathfrak{s}_2(\tau) \right|}{2 + \left| \text{R.L}\mathcal{D}_q^{1/3} \mathfrak{s}_2(\tau) \right|} \right), \\ \mathfrak{N}_{22,(\mathfrak{s}_1,\mathfrak{s}_2)}^*(\tau) &= \frac{|\mathfrak{s}_1(\tau)| + |\mathfrak{s}_2(\tau)| + \left| \mathcal{I}_q^{\sqrt{5}/2} \mathfrak{s}_2(\tau) \right|}{(3\tau^2 + 39) \left(e^{-3\tau} + |\mathfrak{s}_1(\tau)| + |\mathfrak{s}_2(\tau)| + \left| \mathcal{I}_q^{\sqrt{5}/2} \mathfrak{s}_2(\tau) \right| \right)}. \end{aligned}$$

Clearly, $\vartheta_1 = \frac{1}{2} \in (0, 1)$, $\gamma_1 = \frac{\ln 2}{3} \in (0, 1)$, $\delta_1 = \frac{2}{5} \in (0, 1)$, $\eta_1 = \frac{1}{13} > 0$, $\mu_1 = 0.02 \in (0, 1]$, $\theta_1 = \frac{1}{39} > 0$, $\Lambda_1 = \frac{2}{41} > 0$, $\alpha_1 = \frac{1}{5} < \frac{2}{5} = \delta_1$, $\beta_1 = \frac{\ln 3}{2}$, $\vartheta_2 = \frac{e}{7} \in (0, 1)$, $\gamma_2 = \frac{3}{4} \in (0, 1)$, $\delta_2 = \frac{\sqrt{3}}{3} \in (0, 1)$, $\eta_2 = \frac{2}{17} > 0$, $\mu_2 = 0.01 \in (0, 1]$, $\theta_2 = \frac{\sqrt{3}}{85} > 0$, $\Lambda_2 = \frac{e}{100} > 0$, $\alpha_2 = \frac{1}{3} < \frac{\sqrt{3}}{3} = \delta_2$,

$\beta_2 = \frac{\sqrt{5}}{2}$ and $\omega_1 = \frac{3}{2}$, $\omega_2 = \frac{4}{3}$, $v_1 = \frac{\ln 2}{5} \in (0, 1)$, $v_2 = \frac{e}{6} \in (0, 1)$. Using the given data and applying (3.8), we find that in condition (H₁),

$$a_1 = \frac{1}{1 + 34\pi}, \quad a_2 = \frac{1}{\sqrt{37}}, \quad b_1 = \frac{1}{\sqrt{40}}, \quad b_2 = \frac{1}{39},$$

and

$$\begin{aligned} \Pi_1^* &= \frac{1}{\Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 + 1)} + \frac{1}{\vartheta_1 \Gamma_q(\vartheta_1 + \gamma_1 + \delta_1)} \\ &\quad + \frac{\Gamma_q(\omega_1 + 1) v_1^{\vartheta_1 + \gamma_1 + \delta_1 + \omega_1}}{|1 - v_1^{\omega_1}| \Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 + \omega_1 + 1)} \\ &\quad + \frac{\Gamma_q(\omega_1 + 1) |1 - v_1^{\vartheta_1 + \gamma_1 + \delta_1 + \omega_1 - 1}|}{\vartheta_1 |1 - v_1^{\omega_1}| \Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 + \omega_1)} \\ &\quad + \frac{\Gamma_q(\omega_1 + 1)}{|1 - v_1^{\omega_1}| \Gamma_q(\vartheta_1 + \gamma_1 + \delta_1 + \omega_1 + 1)} \\ &= \frac{1}{\Gamma_q\left(\frac{1}{2} + \frac{\ln 2}{3} + \frac{2}{5} + 1\right)} + \frac{1}{\frac{1}{2} \Gamma_q\left(\frac{1}{2} + \frac{\ln 2}{3} + \frac{2}{5}\right)} \\ &\quad + \frac{\Gamma_q\left(\frac{3}{2} + 1\right) \left(\frac{\ln 2}{5}\right)^{\frac{1}{2} + \frac{\ln 2}{3} + \frac{2}{5} + \frac{\ln 3}{2}}}{\left|1 - \left(\frac{\ln 2}{5}\right)^{3/2}\right| \Gamma_q\left(\frac{1}{2} + \frac{\ln 2}{3} + \frac{2}{5} + \frac{\ln 3}{2} + 1\right)} \\ &\quad + \frac{\Gamma_q\left(\frac{3}{2} + 1\right) \left|1 - \left(\frac{\ln 2}{5}\right)^{\frac{1}{2} + \frac{\ln 2}{3} + \frac{2}{5} + \frac{3}{2} - 1}\right|}{\frac{1}{2} \left|1 - \left(\frac{\ln 2}{5}\right)^{3/5}\right| \Gamma_q\left(\frac{1}{2} + \frac{\ln 2}{3} + \frac{2}{5} + \frac{3}{2}\right)} \\ &\quad + \frac{\Gamma_q\left(\frac{3}{2} + 1\right)}{\left|1 - \left(\frac{\ln 2}{5}\right)^{3/2}\right| \Gamma_q\left(\frac{1}{2} + \frac{\ln 2}{3} + \frac{2}{5} + \frac{3}{2} + 1\right)} = \begin{cases} 4.2162, & q = \frac{\sqrt{3}}{5}, \\ 4.1059, & q = \frac{1}{2}, \\ 3.96076, & q = \frac{9}{10}, \end{cases} \\ \Pi_2^* &= \begin{cases} 5.6215, & q = \frac{\sqrt{3}}{5}, \\ 3.3547, & q = \frac{1}{2}, \\ 4.8735, & q = \frac{9}{10}, \end{cases} \end{aligned}$$

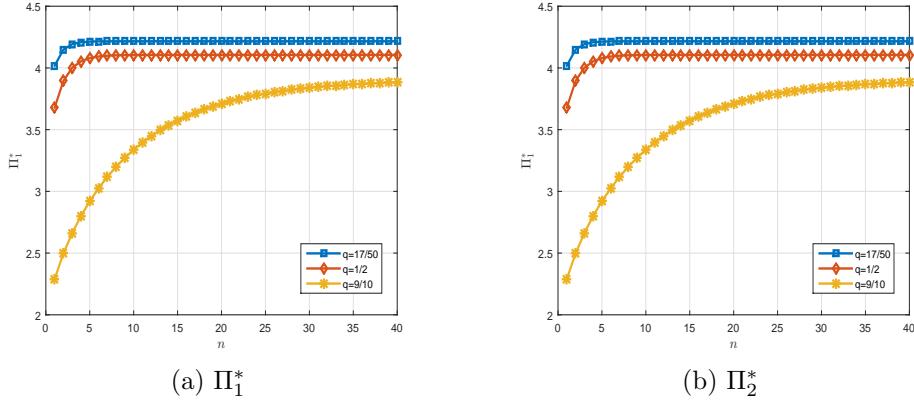
Tables 1 and 2 show the numerical results of Π_i^* and Π_i for $i = 1, 2$ of q -fractional Lane-Emden system (5.1) whenever

$$q \in \left\{ \frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10} \right\}.$$

One can see 2D plot of these variables for three cases of q in Figures 1a, 1b, and 2a, 2b respectively.

Table 1: Numerical results of Π_1^* and Π_2^* (3.18) with $q \in \{\frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10}\}$.

n	$q = \frac{\sqrt{3}}{5}$		$q = \frac{1}{2}$		$q = \frac{9}{10}$	
	Π_1^*	Π_2^*	Π_1^*	Π_2^*	Π_1^*	Π_2^*
1	4.0184	5.1728	3.6777	4.3794	2.2899	1.3636
2	4.1484	5.4664	3.8951	4.8684	2.4968	1.7479
3	4.1928	5.5678	4.0011	5.1118	2.6618	2.0789
4	4.2081	5.6029	4.0536	5.2333	2.8006	2.3691
5	4.2134	5.6151	4.0798	5.2940	2.9203	2.6260
6	4.2152	5.6193	4.0928	5.3243	3.0251	2.8547
7	4.2158	5.6207	4.0994	5.3395	3.1175	3.0590
8	4.2160	5.6212	4.1026	5.3471	3.1995	3.2420
9	4.2161	5.6214	4.1042	5.3509	3.2724	3.4060
10	4.2162	5.6215	4.1051	5.3528	3.3375	3.5533
11	4.2162	5.6215	4.1055	5.3537	3.3956	3.6856
12	4.2162	5.6215	4.1057	5.3542	3.4477	3.8045
13	4.2162	5.6215	4.1058	5.3544	3.4943	3.9115
14	4.2162	5.6215	4.1058	5.3546	3.5362	4.0077
15	4.2162	5.6215	4.1058	5.3546	3.5737	4.0943
16	4.2162	5.6215	4.1059	5.3547	3.6074	4.1722
:	:	:	:	:	:	:
31	4.2162	5.6215	4.1059	5.3547	3.8462	4.7292
32	4.2162	5.6215	4.1059	5.3547	3.8524	4.7436
33	4.2162	5.6215	4.1059	5.3547	3.8579	4.7566
:	:	:	:	:	:	:
84	4.2162	5.6215	4.1059	5.3547	3.9075	4.8733
85	4.2162	5.6215	4.1059	5.3547	3.9075	4.8733
86	4.2162	5.6215	4.1059	5.3547	3.9075	4.8734
87	4.2162	5.6215	4.1059	5.3547	3.9075	4.8734
88	4.2162	5.6215	4.1059	5.3547	3.9075	4.8735
89	4.2162	5.6215	4.1059	5.3547	3.9076	4.8735
90	4.2162	5.6215	4.1059	5.3547	3.9076	4.8735

Figure 1: Graphical representation of Π_1^* and Π_2^* of the system (5.1) for $q \in \left\{ \frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10} \right\}$.

$$\begin{aligned} \Pi_1 &= \frac{\Gamma_q(1 - \mu_1)}{\Gamma_q(\delta_1 - \mu_1 + 1)} + \frac{\Gamma_q(\omega_1 + 1) \Gamma_q(1 - \mu_1) v_1^{\delta_1 + \omega_1 - \mu_1}}{|1 - v_1^{\omega_1}| \Gamma_q(\delta_1 + \omega_1 - \mu_1 + 1)} \\ &+ \frac{\Gamma_q(\omega_1 + 1)}{|1 - v_1^{\omega_1}| \Gamma_q(\delta_1 + \omega_1 + 1)} \end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma_q(1 - 0.02)}{\Gamma_q(\frac{2}{5} - 0.02 + 1)} + \frac{\Gamma_q(\frac{3}{2} + 1) \Gamma_q(1 - 0.02) (\frac{\ln 2}{5})^{\frac{2}{5} + \frac{3}{2} - 0.02}}{\left|1 - (\frac{\ln 2}{5})^{3/2}\right| \Gamma_q(\frac{2}{5} + \frac{3}{2} - 0.02 + 1)} \\
&+ \frac{\Gamma_q(\frac{3}{2} + 1)}{\left|1 - (\frac{\ln 2}{5})^{3/2}\right| \Gamma_q(\frac{2}{5} + \frac{3}{2} + 1)} = \begin{cases} 2.0240, & q = \frac{\sqrt{3}}{5}, \\ 1.9961, & q = \frac{1}{2}, \\ 1.9372, & q = \frac{9}{10}, \end{cases}
\end{aligned}$$

Table 2: Numerical results of Π_1 and Π_2 (3.18) with $q \in \{\frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10}\}$.

n	$q = \frac{\sqrt{3}}{5}$		$q = \frac{1}{2}$		$q = \frac{9}{10}$	
	Π_1	Π_2	Π_1	Π_2	Π_1	Π_2
1	1.9385	2.5045	1.8079	2.2544	1.0382	1.0084
2	1.9951	2.5876	1.9061	2.4035	1.1812	1.1960
3	2.0141	2.6158	1.9520	2.4752	1.2900	1.3474
4	2.0205	2.6255	1.9742	2.5103	1.3770	1.4735
5	2.0228	2.6288	1.9852	2.5278	1.4486	1.5804
6	2.0235	2.6300	1.9907	2.5364	1.5088	1.6722
7	2.0238	2.6304	1.9934	2.5408	1.5599	1.7518
8	2.0239	2.6305	1.9947	2.5429	1.6039	1.8212
9	2.0239	2.6306	1.9954	2.5440	1.6420	1.8820
10	2.0239	2.6306	1.9957	2.5446	1.6751	1.9355
11	2.0240	2.6306	1.9959	2.5448	1.7041	1.9828
12	2.0240	2.6306	1.9960	2.5450	1.7296	2.0246
13	2.0240	2.6306	1.9960	2.5450	1.7521	2.0617
14	2.0240	2.6306	1.9961	2.5451	1.7719	2.0947
:	:	:	:	:	:	:
31	2.0240	2.6306	1.9961	2.5451	1.9111	2.3309
32	2.0240	2.6306	1.9961	2.5451	1.9138	2.3354
33	2.0240	2.6306	1.9961	2.5451	1.9161	2.3395
:	:	:	:	:	:	:
85	2.0240	2.6306	1.9961	2.5451	1.9371	2.3759
86	2.0240	2.6306	1.9961	2.5451	1.9371	2.3759
87	2.0240	2.6306	1.9961	2.5451	1.9371	2.3759
88	2.0240	2.6306	1.9961	2.5451	1.9371	2.3759
89	2.0240	2.6306	1.9961	2.5451	1.9371	2.3760
90	2.0240	2.6306	1.9961	2.5451	1.9372	2.3760

$$\Pi_2 = \begin{cases} 2.6306, & q = \frac{\sqrt{3}}{5}, \\ 2.5451, & q = \frac{1}{2}, \\ 2.3760, & q = \frac{9}{10}, \end{cases}$$

$$\begin{aligned}
\Theta_1^* &= \frac{1}{\Gamma_q(\vartheta_1 + \gamma_1 + \delta_1)} + \frac{[\vartheta_1 + \gamma_1 + \delta_1 - 1]_q}{\vartheta_1 \Gamma_q(\vartheta_i + \gamma_1 + \delta_1)} \\
&= \frac{1}{\Gamma_q(\frac{1}{2} + \frac{\ln 2}{3} + \frac{2}{5})} + \frac{\left[\frac{1}{2} + \frac{\ln 2}{3} + \frac{2}{5} - 1\right]_q}{\frac{1}{2} \Gamma_q(\frac{1}{2} + \frac{\ln 2}{3} + \frac{2}{5})} = \begin{cases} 1.4493, & q = \frac{\sqrt{3}}{5}, \\ 1.4088, & q = \frac{1}{2}, \\ 1.3522, & q = \frac{9}{10}, \end{cases}
\end{aligned}$$

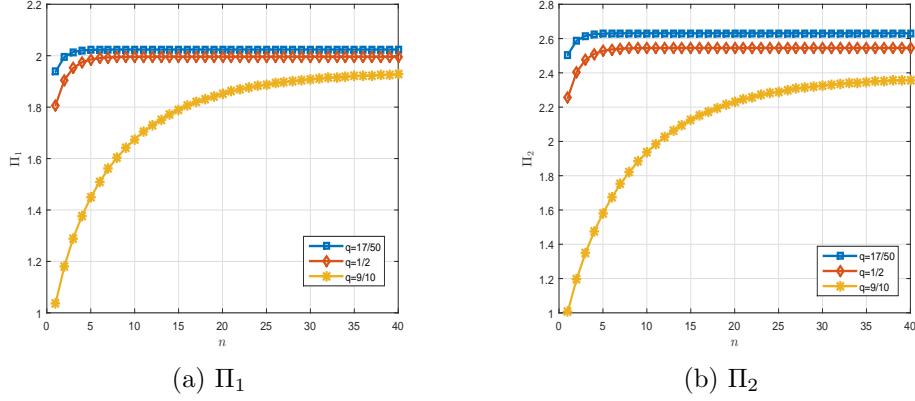


Figure 2: Graphical representation of Π_1 and Π_2 of the system (5.1) for $q \in \left\{ \frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10} \right\}$.

Table 3: Numerical results of Θ_1^* and Θ_2^* (3.18) with $q \in \left\{ \frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10} \right\}$.

n	$q = \frac{\sqrt{3}}{5}$		$q = \frac{1}{2}$		$q = \frac{9}{10}$	
	Θ_1^*	Θ_2^*	Θ_1^*	Θ_2^*	Θ_1^*	Θ_2^*
1	1.4120	2.9262	1.3372	2.5532	1.0586	0.9135
2	1.4370	3.1384	1.3759	2.8860	1.1205	1.1964
3	1.4451	3.2108	1.3929	3.0473	1.1619	1.4301
4	1.4478	3.2358	1.4010	3.1267	1.1922	1.6283
5	1.4488	3.2444	1.4049	3.1662	1.2158	1.7988
6	1.4491	3.2474	1.4069	3.1858	1.2347	1.9470
7	1.4492	3.2484	1.4078	3.1956	1.2503	2.0766
8	1.4492	3.2488	1.4083	3.2005	1.2632	2.1906
9	1.4493	3.2489	1.4085	3.2030	1.2742	2.2913
10	1.4493	3.2489	1.4087	3.2042	1.2835	2.3804
11	1.4493	3.2489	1.4087	3.2048	1.2916	2.4594
12	1.4493	3.2489	1.4088	3.2051	1.2985	2.5297
13	1.4493	3.2489	1.4088	3.2053	1.3046	2.5924
14	1.4493	3.2489	1.4088	3.2053	1.3099	2.6482
15	1.4493	3.2489	1.4088	3.2054	1.3146	2.6981
:	:	:	:	:	:	:
31	1.4493	3.2489	1.4088	3.2054	1.3458	3.0534
32	1.4493	3.2489	1.4088	3.2054	1.3464	3.0612
33	1.4493	3.2489	1.4088	3.2054	1.3470	3.0683
34	1.4493	3.2489	1.4088	3.2054	1.3475	3.0747
:	:	:	:	:	:	:
85	1.4493	3.2489	1.4088	3.2054	1.3522	3.1316
86	1.4493	3.2489	1.4088	3.2054	1.3522	3.1316
87	1.4493	3.2489	1.4088	3.2054	1.3522	3.1316
88	1.4493	3.2489	1.4088	3.2054	1.3522	3.1317
89	1.4493	3.2489	1.4088	3.2054	1.3522	3.1317
90	1.4493	3.2489	1.4088	3.2054	1.3522	3.1317

$$\Theta_2^* = \begin{cases} 3.2489, & q = \frac{\sqrt{3}}{5}, \\ 2.2054, & q = \frac{1}{2}, \\ 3.1317, & q = \frac{9}{10}, \end{cases}$$

Tables 3 and 4 show the numerical results of Θ_i^* and Θ_i for $i = 1, 2$ of q -fractional Lane-

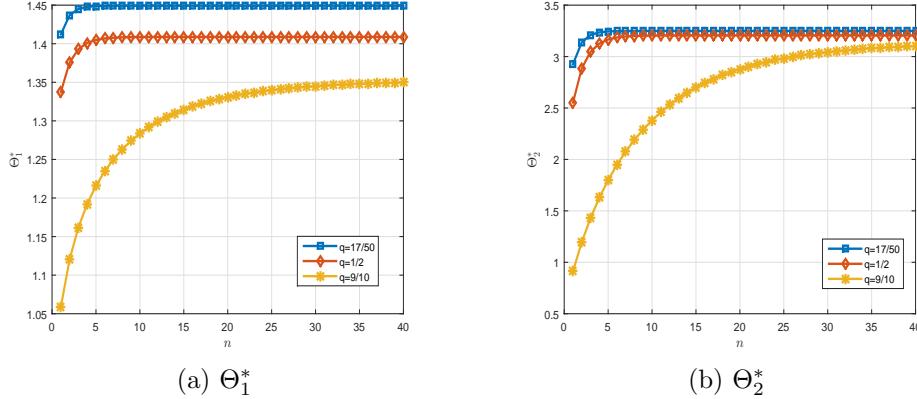


Figure 3: Graphical representation of Θ_1^* and Θ_2^* of the system (5.1) for $q \in \left\{ \frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10} \right\}$.

Emden system (5.1) whenever $q = \frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10}$. One can see 2D plot of these variables for three cases of q in Figures 3a, 3b, and 4a, 4b respectively.

$$\Theta_1 = \frac{\Gamma_q(1 - \mu_1)}{\Gamma_q(\delta_1 - \mu_1)} = \frac{\Gamma_q(1 - 0.02)}{\Gamma_q\left(\frac{2}{5} - 0.02\right)} = \begin{cases} 1.0762, & q = \frac{\sqrt{3}}{5}, \\ 1.0948, & q = \frac{1}{2}, \\ 1.1313, & q = \frac{9}{10}, \end{cases}$$

$$\Theta_2 = \frac{\Gamma_q(1 - \mu_2)}{\Gamma_q(\delta_2 - \mu_2)} = \frac{\Gamma_q(1 - 0.01)}{\Gamma_q\left(\frac{\sqrt{3}}{3} - 0.01\right)} = \begin{cases} 1.0686, & q = \frac{\sqrt{3}}{5}, \\ 1.0865, & q = \frac{1}{2}, \\ 1.1224, & q = \frac{9}{10}. \end{cases}$$

Also, from (3.9) we obtain

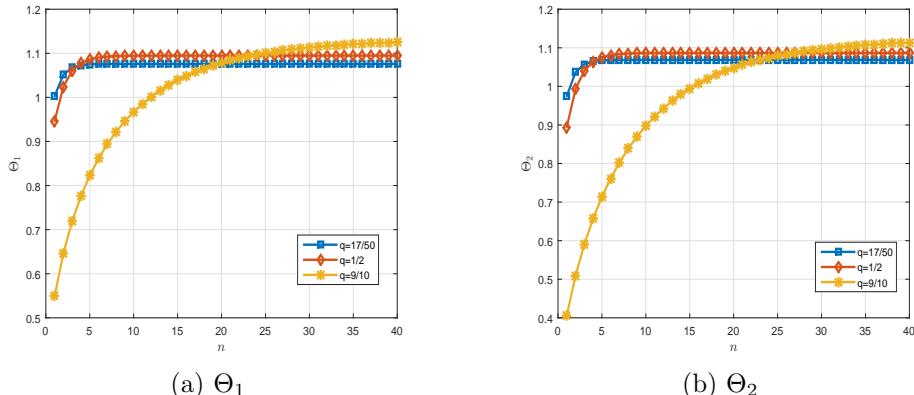
$$\lambda_1 = \Pi_1 + \frac{\Theta_1}{\Gamma_q(2 - \alpha_1)} = \Pi_1 + \frac{\Theta_1}{\Gamma_q\left(2 - \frac{1}{5}\right)} = \begin{cases} 3.1404, & q = \frac{\sqrt{3}}{5}, \\ 3.1433, & q = \frac{1}{2}, \\ 3.1468, & q = \frac{9}{10}, \end{cases}$$

$$\lambda_i^* = \Pi_i^* + \frac{\Theta_i^*}{\Gamma_q(2 - \alpha_i)} = \Pi_1^* + \frac{\Theta_1^*}{\Gamma_q\left(2 - \frac{1}{5}\right)} \begin{cases} 5.7197, & q = \frac{\sqrt{3}}{5}, \\ 5.5822, & q = \frac{1}{2}, \\ 5.3533, & q = \frac{9}{10}, \end{cases}$$

$$\lambda_1^* \left(\theta_1 a_1 + \frac{\Lambda_1 b_1 (2 + \Gamma_q(\beta_1 + 1))}{\Gamma_q(\beta_1 + 1)} \right) + \lambda_1 \eta_1 = \begin{cases} 0.3811, & q = \frac{\sqrt{3}}{5}, \\ 0.3795, & q = \frac{1}{2}, \\ 0.3769, & q = \frac{9}{10}. \end{cases} < \frac{1}{2}, \quad (5.2)$$

Table 4: Numerical results of Θ_1 and Θ_2 (3.18) with $q \in \{\frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10}\}$.

n	$q = \frac{\sqrt{3}}{5}$		$q = \frac{1}{2}$		$q = \frac{9}{10}$	
	Θ_1	Θ_2	Θ_1	Θ_2	Θ_1	Θ_2
1	1.0032	0.9752	0.9466	0.8943	0.5492	0.4067
2	1.0517	1.0368	1.0244	0.9936	0.6468	0.5096
3	1.0678	1.0576	1.0604	1.0407	0.7192	0.5908
4	1.0733	1.0648	1.0778	1.0638	0.7762	0.6575
5	1.0752	1.0673	1.0863	1.0752	0.8226	0.7136
6	1.0758	1.0681	1.0905	1.0809	0.8613	0.7615
7	1.0761	1.0684	1.0927	1.0837	0.8940	0.8028
8	1.0761	1.0685	1.0937	1.0851	0.9220	0.8387
9	1.0762	1.0685	1.0942	1.0858	0.9462	0.8700
10	1.0762	1.0686	1.0945	1.0862	0.9671	0.8976
11	1.0762	1.0686	1.0946	1.0863	0.9854	0.9218
12	1.0762	1.0686	1.0947	1.0864	1.0015	0.9433
13	1.0762	1.0686	1.0947	1.0865	1.0156	0.9623
14	1.0762	1.0686	1.0947	1.0865	1.0281	0.9791
:	:	:	:	:	:	:
31	1.0762	1.0686	1.0948	1.0865	1.1151	1.0995
32	1.0762	1.0686	1.0948	1.0865	1.1168	1.1018
33	1.0762	1.0686	1.0948	1.0865	1.1182	1.1039
34	1.0762	1.0686	1.0948	1.0865	1.1196	1.1057
:	:	:	:	:	:	:
84	1.0762	1.0686	1.0948	1.0865	1.1313	1.1223
85	1.0762	1.0686	1.0948	1.0865	1.1313	1.1223
86	1.0762	1.0686	1.0948	1.0865	1.1313	1.1224
87	1.0762	1.0686	1.0948	1.0865	1.1313	1.1224
88	1.0762	1.0686	1.0948	1.0865	1.1313	1.1224
89	1.0762	1.0686	1.0948	1.0865	1.1313	1.1224
90	1.0762	1.0686	1.0948	1.0865	1.1313	1.1224

Figure 4: Graphical representation of Θ_1 and Θ_2 of the system (5.1) for $q \in \left\{ \frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10} \right\}$.

Tables 5 and 6 show the numerical results of λ_i^* , λ_i for $i = 1, 2$, Inequalities (5.2) and (5.3) of q -fractional Lane-Emden system (5.1) whenever $q = \frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10}$. One can see 2D plot of these variables for three cases of q in Figures 5a and 5b respectively.

and

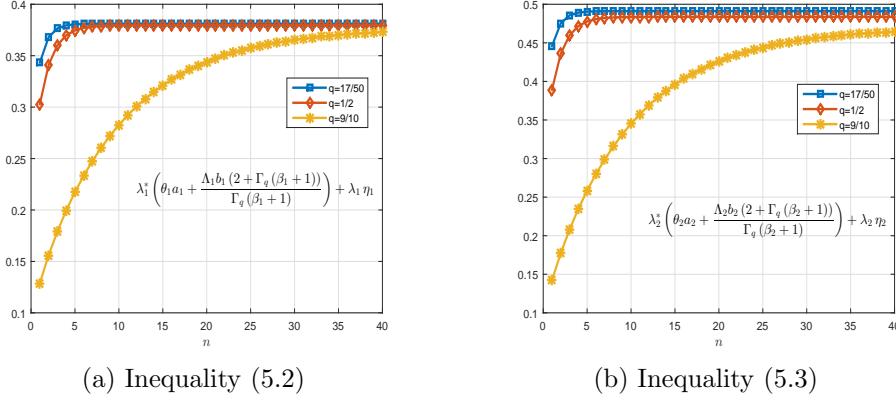


Figure 5: Graphical representation of inequality (5.2) of the system (5.1) for $q \in \left\{ \frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10} \right\}$.

Table 5: Numerical results of λ_1 , λ_1^* and inequality (5.2) with $q \in \left\{ \frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10} \right\}$.

n	$q = \frac{\sqrt{3}}{5}$			$q = \frac{1}{2}$			$q = \frac{9}{10}$		
	λ_1	λ_1^*	Ineq. (5.2)	λ_1	λ_1^*	Ineq. (5.2)	λ_1	λ_1^*	Ineq. (5.2)
1	2.8688	5.3277	0.3433	2.5830	4.7726	0.3029	1.1885	2.5798	0.1289
2	3.0463	5.5848	0.3680	2.8639	5.1815	0.3410	1.4193	2.9093	0.1560
3	3.1078	5.6730	0.3765	3.0037	5.3827	0.3602	1.6123	3.1825	0.1791
4	3.1291	5.7035	0.3795	3.0736	5.4827	0.3698	1.7785	3.4174	0.1994
5	3.1365	5.7141	0.3805	3.1085	5.5325	0.3746	1.9238	3.6226	0.2174
6	3.1391	5.7177	0.3809	3.1259	5.5574	0.3770	2.0519	3.8037	0.2334
7	3.1400	5.7190	0.3810	3.1346	5.5698	0.3783	2.1655	3.9643	0.2478
8	3.1403	5.7195	0.3811	3.1390	5.5760	0.3789	2.2665	4.1073	0.2607
9	3.1404	5.7196	0.3811	3.1412	5.5791	0.3792	2.3567	4.2349	0.2723
10	3.1404	5.7197	0.3811	3.1423	5.5807	0.3793	2.4373	4.3490	0.2827
11	3.1404	5.7197	0.3811	3.1428	5.5815	0.3794	2.5095	4.4512	0.2921
12	3.1404	5.7197	0.3811	3.1431	5.5818	0.3794	2.5741	4.5427	0.3005
13	3.1404	5.7197	0.3811	3.1432	5.5820	0.3794	2.6321	4.6247	0.3081
14	3.1404	5.7197	0.3811	3.1433	5.5821	0.3794	2.6841	4.6984	0.3150
15	3.1404	5.7197	0.3811	3.1433	5.5822	0.3795	2.7308	4.7645	0.3211
16	3.1404	5.7197	0.3811	3.1433	5.5822	0.3795	2.7727	4.8239	0.3267
:	:	:	:	:	:	:	:	:	:
30	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.0617	5.2330	0.3654
31	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.0703	5.2451	0.3666
32	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.0779	5.2559	0.3676
33	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.0848	5.2657	0.3685
34	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.0910	5.2745	0.3694
:	:	:	:	:	:	:	:	:	:
83	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.1466	5.3531	0.3769
84	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.1466	5.3531	0.3769
85	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.1466	5.3532	0.3769
86	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.1467	5.3532	0.3769
87	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.1467	5.3532	0.3769
88	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.1467	5.3533	0.3769
89	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.1467	5.3533	0.3769
90	3.1404	5.7197	0.3811	3.1434	5.5822	0.3795	3.1468	5.3533	0.3769

$$\lambda_2 = \Pi_2 + \frac{\Theta_2}{\Gamma_q(2 - \alpha_2)} = \Pi_2 + \frac{\Theta_2}{\Gamma_q(2 - \frac{1}{3})} = \begin{cases} 3.7587, & q = \frac{\sqrt{3}}{5}, \\ 3.7086, & q = \frac{1}{2}, \\ 3.6119, & q = \frac{9}{10}. \end{cases}$$

$$\lambda_2^* = \Pi_2^* + \frac{\Theta_2^*}{\Gamma_q(2 - \alpha_2)} = \Pi_2^* + \frac{\Theta_2^*}{\Gamma_q\left(2 - \frac{1}{3}\right)} = \begin{cases} 9.0514, & q = \frac{\sqrt{3}}{5}, \\ 8.7872, & q = \frac{1}{2}, \\ 8.3226, & q = \frac{9}{10}. \end{cases}$$

$$\lambda_2^* \left(\theta_2 a_2 + \frac{\Lambda_2 b_2 (2 + \Gamma_q(\beta_2 + 1))}{\Gamma_q(\beta_2 + 1)} \right) + \lambda_2 \eta_2 = \begin{cases} 0.4911, & q = \frac{\sqrt{3}}{5}, \\ 0.4837, & q = \frac{1}{2}, \\ 0.4697, & q = \frac{9}{10}. \end{cases} < \frac{1}{2}. \quad (5.3)$$

Obviously all the conditions of Theorem 3.2 are satisfied. It follows from Theorem 3.2, that

Table 6: Numerical results of λ_2 , λ_2^* and inequality (5.3) with $q \in \{\frac{\sqrt{3}}{5}, \frac{1}{2}, \frac{9}{10}\}$.

n	$q = \frac{\sqrt{3}}{5}$			$q = \frac{1}{2}$			$q = \frac{9}{10}$		
	λ_2	λ_2^*	Ineq. (5.3)	λ_2	λ_2^*	Ineq. (5.3)	λ_2	λ_2^*	Ineq. (5.3)
1	3.4363	7.9686	0.4460	3.0263	6.5829	0.3892	1.1495	1.6806	0.1424
2	3.6466	8.6721	0.4754	3.3668	7.6665	0.4361	1.4240	2.2830	0.1775
3	3.7198	8.9195	0.4856	3.5375	8.2224	0.4598	1.6599	2.8353	0.2079
4	3.7452	9.0057	0.4892	3.6230	8.5037	0.4717	1.8663	3.3418	0.2347
5	3.7540	9.0356	0.4905	3.6658	8.6452	0.4777	2.0484	3.8057	0.2586
6	3.7570	9.0459	0.4909	3.6872	8.7162	0.4807	2.2100	4.2298	0.2798
7	3.7581	9.0495	0.4910	3.6979	8.7517	0.4822	2.3540	4.6169	0.2989
8	3.7585	9.0508	0.4911	3.7032	8.7695	0.4830	2.4826	4.9695	0.3160
9	3.7586	9.0512	0.4911	3.7059	8.7784	0.4833	2.5976	5.2905	0.3314
10	3.7586	9.0513	0.4911	3.7073	8.7828	0.4835	2.7005	5.5821	0.3452
11	3.7587	9.0514	0.4911	3.7079	8.7850	0.4836	2.7928	5.8468	0.3576
12	3.7587	9.0514	0.4911	3.7083	8.7861	0.4837	2.8756	6.0869	0.3688
13	3.7587	9.0514	0.4911	3.7084	8.7867	0.4837	2.9499	6.3044	0.3788
14	3.7587	9.0514	0.4911	3.7085	8.7870	0.4837	3.0167	6.5014	0.3879
15	3.7587	9.0514	0.4911	3.7086	8.7871	0.4837	3.0766	6.6796	0.3960
16	3.7587	9.0514	0.4911	3.7086	8.7872	0.4837	3.1305	6.8407	0.4034
:	:	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
30	3.7587	9.0514	0.4911	3.7086	8.7872	0.4837	3.5025	7.9790	0.4545
31	3.7587	9.0514	0.4911	3.7086	8.7872	0.4837	3.5135	8.0133	0.4560
32	3.7587	9.0514	0.4911	3.7086	8.7872	0.4837	3.5234	8.0442	0.4574
33	3.7587	9.0514	0.4911	3.7086	8.7872	0.4837	3.5323	8.0720	0.4586
:	:	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
84	3.7587	9.0514	0.4911	3.7086	8.7872	0.4837	3.6119	8.3220	0.4696
85	3.7587	9.0514	0.4911	3.7086	8.7872	0.4837	3.6119	8.3221	0.4696
86	3.7587	9.0514	0.4911	3.7086	8.7872	0.4837	3.6120	8.3222	0.4696
87	3.7587	9.0514	0.4911	3.7086	8.7872	0.4837	3.6120	8.3223	0.4696
88	3.7587	9.0514	0.4911	3.7086	8.7872	0.4837	3.6120	8.3224	0.4697
89	3.7587	9.0514	0.4911	3.7086	8.7872	0.4837	3.6121	8.3225	0.4697
90	3.7587	9.0514	0.4911	3.7086	8.7872	0.4837	3.6121	8.3226	0.4697

the system (5.1) has a unique solution and is Ulam-Hyers stable with

$$\|(\mathfrak{s}_1, \mathfrak{s}_2) - (\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2} \leq \Sigma_{N_{i1}^*, N_{i2}^*, F_i} \ell, \quad \ell > 0.$$

where $\Sigma_{N_{i1}^*, N_{i2}^*, F_i} \simeq 30.576$. Let $m(\tau) = \tau^{\frac{1}{2}}$, then

$$\mathcal{I}_q^{\frac{1}{2} + \frac{\ln 2}{3} + \frac{2}{5}} [\tau^{1/2}] = \frac{\frac{1}{2} \Gamma_q\left(\frac{1}{2}\right) \tau^{7+5 \ln 2/15}}{\Gamma_q\left(\frac{36+5 \ln 2}{15}\right)} \leq \frac{\frac{1}{2} \Gamma_q\left(\frac{1}{2}\right)}{\Gamma_q\left(\frac{36+5 \ln 2}{15}\right)} \tau^{1/2} = \varrho_{1,m} m(\tau),$$

$$\mathcal{I}_q^{\frac{1}{2} + \frac{\ln 2}{3} + \frac{2}{5} - \frac{1}{5}} [\tau^{1/2}] = \frac{\frac{1}{2} \Gamma_q\left(\frac{1}{2}\right) \tau^{\frac{21+10 \ln 2}{30}}}{\Gamma_q\left(\frac{51+10 \ln 2}{30}\right)} \leq \frac{\frac{1}{2} \Gamma_q\left(\frac{1}{2}\right)}{\Gamma_q\left(\frac{51+10 \ln 2}{30}\right)} \tau^{1/2} = \varrho_{2,m} m(\tau),$$

and

$$\begin{aligned} \mathcal{I}_q^{\frac{e}{7}+\frac{3}{4}+\frac{\sqrt{3}}{3}} [\tau^{1/2}] &= \frac{\frac{1}{2}\Gamma_q\left(\frac{1}{2}\right)\tau^{\frac{24e+210+56\sqrt{3}}{168}}}{\Gamma_q\left(\frac{24e+378+56\sqrt{3}}{168}\right)} \leq \frac{\frac{1}{2}\Gamma_q\left(\frac{1}{2}\right)}{\Gamma_q\left(\frac{24e+378+56\sqrt{3}}{168}\right)}\tau^{1/2} = \partial_{1,m}m(\tau), \\ \mathcal{I}_q^{\frac{e}{7}+\frac{3}{4}+\frac{\sqrt{3}}{3}-\frac{1}{3}} [\tau^{1/2}] &= \frac{\frac{1}{2}\Gamma_q\left(\frac{1}{2}\right)\tau^{\frac{72e+462+168\sqrt{3}}{504}}}{\Gamma_q\left(\frac{72e+966+168\sqrt{3}}{504}\right)} \leq \frac{\frac{1}{2}\Gamma_q\left(\frac{1}{2}\right)}{\Gamma_q\left(\frac{72e+966+168\sqrt{3}}{504}\right)}\tau^{1/2} = \partial_{2,m}m(\tau). \end{aligned}$$

Thus condition (4.4) is satisfied with $m(\tau) = \tau^{1/2}$ and

$$\varrho_{1,m} = \frac{\frac{1}{2}\Gamma_q\left(\frac{1}{2}\right)}{\Gamma_q\left(\frac{36+5\ln 2}{15}\right)}, \quad \varrho_{2,m} = \frac{\frac{1}{2}\Gamma_q\left(\frac{1}{2}\right)}{\Gamma_q\left(\frac{51+10\ln 2}{30}\right)},$$

and

$$\partial_{1,m} = \frac{\frac{1}{2}\Gamma_q\left(\frac{1}{2}\right)}{\Gamma_q\left(\frac{24e+378+56\sqrt{3}}{168}\right)}, \quad \partial_{2,m} = \frac{\frac{1}{2}\Gamma_q\left(\frac{1}{2}\right)}{\Gamma_q\left(\frac{72e+966+168\sqrt{3}}{504}\right)}.$$

It follows from Theorem 4.2 problem (5.1) is Ulam-Hyers-Rassias stable with

$$\|(\mathfrak{s}_1, \mathfrak{s}_2) - (\mathfrak{s}_1, \mathfrak{s}_2)\|_{B_1 \times B_2} \leq \Sigma_{\aleph_{i1}^*, \aleph_{i2}^*, F_{i,m}} \ell \hat{m}(\tau),$$

here $\Sigma_{\aleph_{i1}^*, \aleph_{i2}^*, F_{i,m}} \simeq 2.4659$ and $\ell > 0$.

6 Conclusion

The sequential q -fractional Lane-Emden system involving Riemann-Liouville and Caputo type fractional q -derivatives has been investigated in this work in details. The investigation of this particular equation provides us with a powerful tool in modeling most scientific phenomena without the need to remove most parameters which have an essential role in the physical interpretation of the studied phenomena. sequential q -fractional Lane-Emden system (1.1) has been studied under some B.Cs. An example has been provided to support our results' validity and applicability in fields of physics and engineering.

Declarations

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

Funding

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Authors' contributions

MH: Actualization, methodology, formal analysis, validation, investigation, initial draft and was a major contributor in writing the manuscript. **MES:** Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft and was a major contributor in writing the manuscript. **SR:** Actualization, methodology, formal analysis, validation, investigation and initial draft. All authors read and approved the final manuscript.

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References

- [1] Abdel-Gawad, H., Aldailami, A.: On q -dynamic equations modelling and complexity. *Appl. Math. Model.* **34**(3), 697–709 (2010)
- [2] Abdeljawad, T., Samei, M.E.: Applying quantum calculus for the existence of solution of q -integro-differential equations with three criteria. *Discrete & Continuous Dynamical Systems-Series S* **14**(10), 3351–3386 (2021). DOI 10.3934/dcdss.2020440
- [3] Adibi, H., Rismani, A.: On using a modified legendre-spectral method for solving singular ivps of Lane-Emden type. *Comput. Math. Appl.* **60**, 2126–2130 (2010)
- [4] Ahmadian, A., Rezapour, S., Salahshour, S., Samei, M.E.: Solutions of sum-type singular fractional q -integro-differential equation with m -point boundary value problem using quantum calculus. *Mathematical Methods in the Applied Science* **43**(15), 8980–9004 (2020). DOI 10.1002/mma.6591
- [5] Annaby, M.H., Mansour, Z.S.: q -Fractional Calculus and Equations. Springer Heidelberg, Cambridge (2012). DOI 10.1007/978-3-642-30898-7
- [6] Bahous, Y., Dahmani, Z., Bekkouche, Z.: A two-parameter singular fractional differential equation of lane emden type. *Turkish J. Ineq.* **3**(1), 35–53 (2019)
- [7] Bekkouche, Z., Dahmani, Z., Zhang, G.: Solutions and stabilities for a $2d$ -non homogeneous Lane-Emden fractional system. *Int. J. Open Problems Compt. Math.* **11**(2), 1–14 (2018)
- [8] Benzidane, A., Dahmani, Z.: A class of nonlinear singular differential equations. *J. Interdisciplinary Mathematics* **22**(6), 991–1007 (2019)
- [9] Butt, R.I., Abdeljawad, T., Alqudah, M.A., Rehman, M.U.: Ulam stability of Caputo q -fractional delay difference equation: q -fractional gronwall inequality approach. *Journal of Inequalities and Applications* **2019**, 30 (2019)
- [10] Dobrogowska, A.: The q -deformation of the morse potential. *Appl. Math. Lett.* **26**(7), 769–773 (2013)
- [11] Emden, R.: q -Fractional Calculus and Equations. Gaskugeln, Teubner, Leipzig and Berlin (1907)
- [12] Field, C., Joshi, N., Nijhoff, F.: q -difference equations of kdv type and chazy-type second-degree difference equations. *J. Phys. A, Math. Theor.* **41**, 1–13 (2008)
- [13] Gouari, Y., Dahmani, Z., Farooq, S.E., Ahmad, F.: Fractional singular differential systems of Lane-Emden type: existence and uniqueness of solutions. *Axioms* **9**(3), 1–18 (2020)
- [14] Gouari, Y., Dahmani, Z., Sarikaya, M.Z.: A non local multi-point singular fractional integro-differential problem of Lane-Emden type. *Math. Methods Appl. Sci.* **43**(11), 6938–6949 (2020)

- [15] Hao, T., Song, S.: Solving coupled Lane-Emden equations arising in catalytic diffusion reactions by reproducing kernel hilbert space method. International Conference on Applied Science and Engineering Innovation pp. 529–536 (2015). DOI 10.2991/asei-15.2015.105
- [16] Ibrahim, R.W.: Existence of nonlinear Lane-Emden equation of fractional order. *Miskolc Mathematical Notes* **13**(1), 39–52 (2012)
- [17] Ibrahim, R.W.: Stability of a fractional differential equation. *International Journal of Mathematical, Computational, Physical and Quantum Engineering* **7**(3), 1–6 (2013)
- [18] Jarad, F., Abdeljawad, T., Baleanu, D.: Stability of q -fractional non-autonomous systems. *Nonlinear Anal. Real World Appl.* **14**(1), 780–784 (2013)
- [19] Jiang, M., Huang, R.: Existence and stability results for impulsive fractional q -difference equation. *Journal of Applied Mathematics and Physics.* **8**(7), 1413–1423 (2020)
- [20] Li, X., Han, Z., Sun, S.: Existence of positive solutions of nonlinear fractional q -difference equation with parameter. *Adv. Differ. Equ.* **2013**, 260 (2013)
- [21] Liang, S., Samei, M.E.: New approach to solutions of a class of singular fractional q -differential problem via quantum calculus. *Adv. Differ. Equ.* **2020**, 14 (2020). DOI 10.1186/s13662-019-2489-2
- [22] Mechee, S.M., Senu, N.: Numerical study of fractional differential equations of Lane-Emden type by method of collocation. *Applied Mathematics* **3**, 851–856 (2012)
- [23] Okunuga, S.A., Ehigie, J., Sofoluwe, A.B.: Treatment of Lane-Emden type equations via second derivative backward differentiation formula using boundary value technique. *Proceedings of the World Congress on Engineering IWCE*, July 4–6, , London, U.K. (2012)
- [24] Parand, K., Dehghan, M., Rezaei, A., Ghaderi, S.: An approximation algorithm for the solution of the nonlinear Lane-Emden type equations arising in astrophysics using hermite functions collocation method. *Comput. Phys. Commun.* **181**, 1096–1108 (2010)
- [25] Phuong, N.D., Etemad, S., Rezapour, S.: On two structures of the fractional q -sequential integro-differential boundary value problems. *Math. Meth. Appl. Sci.* **45**(2), 618–639 (2021)
- [26] Rajković, P.M., Marinković, S.D., Stanković, M.S.: On q -analogues of Caputo derivative and Mittag-Leffer function. *Fract. Calc. Appl. Anal.* **10**, 359–373 (2007)
- [27] Rezapour, S., Samei, M.E.: On the existence of solutions for a multi-singular pointwise defined fractional q -integro-differential equation. *Bound. Value. Probl.* **2020**, 38 (2020). DOI 10.1186/s13661-020-01342-3
- [28] Samei, M.E.: Existence of solutions for a system of singular sum fractional q -differential equations via quantum calculus. *Advances in Difference Equations* **2020**, 23 (2020). DOI 10.1186/s13662-019-2480-y
- [29] Samei, M.E., Zanganeh, H., Aydogan, S.M.: Investigation of a class of the singular fractional integro-differential quantum equations with multi-step methods. *Journal of Mathematical Extension* **17**(1), 1–545 (2021). DOI 10.30495/JME.SI.2021.2070
- [30] Smart, D.R.: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)
- [31] Tablennehas, K., Dahmani, Z., Belhamiti, M.M., Abdelnebi, A., Sarikaya, M.Z.: On a fractional problem of Lane-Emden type: Ulam type stabilities and numerical behaviors. *Adv. Differ. Equ.* **2021**, 324 (2021)
- [32] Taïeb, A., Dahmani, Z.: The high order Lane-Emden fractional differential system: Existence, uniqueness and Ulam stabilities. *Kragujevac J. Math.* **40**(2), 238–259 (2016)

- [33] Yigider, M., Tabatabaei, K., Çelik, E.: The numerical method for solving differential equations of Lane-Emden type by padé approximation. *Discrete Dyn. Nat. Soc.* **2011**, 1–9 (2011)
- [34] Yildirim, A., Ozi, T.: Solutions of singular ivps of Lane-Emden type by the variational iteration method. *Nonlinear Anal-Theor.* **70**, 2480–2484 (2009)