

# Haar Algorithm for The Analysis of Fractional Order Calculus Based Computation Problems Associated With Electromagnetic Waves

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## Research Article

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# Haar Algorithm for the Analysis of Fractional Order Calculus Based Computation Problems associated with Electromagnetic Waves

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**Abstract** This paper proposes a haar algorithm with the phase wise flow for three distinct cases of fractional order calculus-based electromagnetic wave machine problem. The numerical solution to these programming problems was presented in tabular and graphic form using precise, approximate analysis and haar schema for comparative analysis. Convergence research was also carried out to validate the accuracy and efficacy of the system suggested. The proposed scheme is suited for the numerical solution of the addressed type of computational problem due to its uncomplicated and easy-to-implement, professional, fastness and high convergence rate.

**Keywords** Fractional order electromagnetic waves · Fractional order derivatives · Haar scheme · Algorithm

## 1 Introduction

The fast and accurate response of the electromagnetic computational problem is becoming increasingly important in a growing number of engineering

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applications. Pace and memory are factually two significant variables in engineering. As a result, researchers are gradually inclined to measure the fractional order and to require the use of a modern computational methodology algorithm. The fractional calculus refers to the fractional order in the applied mathematics for the difference and integral equations. Current research topics in the different fields of astronomy, engineering and technology with long history in mathematics have been used in the last decades [13] - [22]. The recent benefit of this fractional computation is that it offers an extraordinary device for the representation of time, space and memory [23]-[25]. Different research activity has been noticed for the analysis of computational problem using electromagnetic tools based on fractional order calculus [43]-[47]. Due to these assets, a large number of researchers are observing that large computational electromagnetic problems (CEM) problems are having fractional-order behavior either in space or time or both. This fact elucidates that fractional-order calculus is a natural contender to offer an impressive mathematical contextual for the description of complicated CEM problems. All the materials and media which are exhibiting EM memory properties can be expressed by fractional-order calculus non-local formalism. Fractional generalization of differential forms with sundry application in physics and electromagnetic [26] has been presented in [27]-[31].

According to the literature survey done, differential and integral equation with fractional order are extensively used especially in electromagnetism. For this tenacity, we need a decisive, fast and effectual scheme for the solution of CEM problems associated with fractional-order differential equations. Nowadays, wavelets are becoming more prevalent for numerical solutions of computational problems by its venerable properties such as capacity of resolution and compact brace. In the last decade, there has been an emergent awareness in proposing numerical algorithms based on wavelet for solving CEM problems associated with partial differential equations of fraction order. The Haar scheme is straightforward and effortless to use amongst them [32]. Haar scheme has been efficaciously practiced for solving normal and partial integral/differential/integro-differential equations [33].

Hence, the first aim of this study is to fictionalize the EM wave based computational problem using different techniques suggested by researchers [34]-[38]. Further, our focus in this paper is to propose an uncomplicated, easy to implement, competent and convergent scheme to give numerical solution for the fractional order time, space and both time & space problems separately. So that researchers can apply this scheme efficiently to various types of fractional order problems based on EM Wave. The asset of this scheme is to convert the CEM problem into a series of algebraic equation which makes computation effortless as compared to NTDM [8], Power Series [9] and q-HAMT [11], etc. The obtained numerical approximations using proposed scheme are then correlated with the exact solutions as well as that of different researcher's using different techniques in the open literature.

The organization of This paper is as follows: Section 2 provides a brief description for the narrative of local fractional-order derivative / integration with the interpretation of Riemann-Liouville integral and Caputo. Section 3 encompass the EM Wave equations in the classical calculus form and fractional form using definitions of Section 2. Section 4 contains the concept and formulas related to Haar wavelet in integer and fractional form. In section 5, we have given Haar algorithm for solving CEM problems associated with EM waves in three categories. Section 6 deals with the analytical, exact and Haar solution of three different fractional-order electromagnetic waves based on computational problems each related with a given algorithm in Section 5. We have discussed the result outcome of the developed algorithm base problems in Section 7. Finally, section 8 provides a conclusion for the paper.

## 2 Narrative of Local Fractional-Order Derivative and Integration

Fractional calculus deals with the fraction order of differential / integral equations. This concept has gained wide applications in the field of engineering, science and technology, specially in signal processing, control engineering and electromagnetism. Many approaches such as Grunwald-Letnikov, Riemann-Liouville and Caputo are available to describe fraction order derivatives. Riemann-Liouville's fraction order derivative model is not fit for CEM problems as it entails the explanation of fraction order initial conditions. In an alternative manner, Caputo has been used which has the benefit of giving integer order initial conditions for fraction order differential equations. Next we present some noteworthy definitions [39]:

### 2.1 Definition 1

The most intermittent definition related to fraction order integral has been given by Riemann – Liouville in which the fractional integral operator  $J^\alpha$  ( $\alpha > 0$ ) of a function  $f(t)$ , is defined by [1], [2] as given in equations (1) to (3).

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{(\alpha-1)} f(\tau) d\tau, \alpha > 0, \alpha \in \mathfrak{R}^+, \quad (1)$$

where  $\Gamma(\cdot)$  = *well known gamma function* , and two properties related to integral operator ( $J^\alpha$ ) are given below:

#### 2.1.1 Property 1

$$J^\alpha J^\beta f(t) = J^{(\alpha+\beta)} f(t), (\alpha > 0, \beta > 0), \quad (2)$$

### 2.1.2 Property 2

$$J^\alpha t^\gamma = \left( \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} \right) \times t^{(\alpha+\gamma)}, (\gamma > -1). \quad (3)$$

### 2.2 Definition 2

Caputo [1], [2], has given the concept of fraction order derivative which is also known as Caputo Fraction order derivative. The Caputo fraction order derivative  ${}_0D_t^\alpha$  of a function  $f(t)$  is given by [1], [2] which is shown in equations (4) to (6): :

$${}_0D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{(\alpha-n+1)}} d\tau, (n-1 < \alpha \leq n, n \in N). \quad (4)$$

Two properties related to Caputo fraction order derivative are as follows:

#### 2.2.1 Property 1

$${}_0D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, 0 < \alpha < \beta+1, \beta > -1, \quad (5)$$

#### 2.2.2 Property 2

$$J^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, n-a < \alpha \leq n, n \in N. \quad (6)$$

## 3 Fractional Order Electromagnetic Waves

The Maxwell equations for the electromagnetic waves can be written as equation (7) to (12) given by [40]-[42]:

$$\tilde{\nabla} \bullet \tilde{\mathbf{D}} = 4\pi\rho(\tilde{\mathbf{r}}, t). \quad (7)$$

Putting  $\tilde{\mathbf{D}} = \epsilon\tilde{\mathbf{E}}$  in the equation above, we can get

$$\tilde{\nabla} \bullet \tilde{\mathbf{E}} = \frac{4\pi}{\epsilon}\rho(\tilde{\mathbf{r}}, t), \quad (8)$$

$$\tilde{\nabla} \bullet \tilde{\mathbf{B}} = 0, \quad (9)$$

$$\tilde{\nabla} \times \tilde{\mathbf{E}} = -\frac{1}{c} \frac{\partial \tilde{\mathbf{B}}}{\partial t}, \quad (10)$$

$$\tilde{\nabla} \times \tilde{\mathbf{H}} = \frac{4\pi}{c} \tilde{\mathbf{j}}(\tilde{\mathbf{r}}, t) + \frac{1}{c} \frac{\partial \tilde{\mathbf{D}}}{\partial t}. \quad (11)$$

( $\bullet$ ) denotes dot product and ( $\times$ ) denotes cross product. Now using  $\tilde{\mathbf{B}} = \mu\tilde{\mathbf{H}}$  in the above equation:

$$\tilde{\nabla} \times \tilde{\mathbf{B}} = \frac{4\pi\mu\tilde{\mathbf{j}}}{c}(\tilde{\mathbf{r}}, t) + \frac{\epsilon\mu}{c} \frac{\partial \tilde{\mathbf{E}}}{\partial t}, \quad (12)$$

where  $\rho(\tilde{\mathbf{r}}, t)$  is *charge density* and  $\tilde{\mathbf{j}}(\tilde{\mathbf{r}}, t)$  denotes *current density*. These two parameters are time dependent. Moreover,  $\epsilon$  and  $\mu$  are the electric constant permittivity and magnetic permeability, accordingly. These parameters are scalars in homogeneous media and tensors in anisotropic media. Here we are writing scalar potential  $\phi(x_i, t)$  and vector potential  $\tilde{\mathbf{A}}(x_i, t)$  as in equation (13) and (14), respectively:

$$\tilde{\mathbf{E}} = -\frac{1}{c} \frac{\partial A}{\partial t} - \tilde{\nabla} \phi, \quad (13)$$

$$\tilde{\mathbf{B}} = \tilde{\nabla} \times \tilde{\mathbf{A}}. \quad (14)$$

Then, considering the Lorenz gauge condition  $\partial_\mu A^\mu = 0$ , we get the following decoupled differential equations (15) and (16) for the potential:

$$\Delta^2 \phi(\tilde{\mathbf{r}}, t) - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \phi(\tilde{\mathbf{r}}, t)}{\partial t^2} = -\frac{4\pi}{\epsilon} \tilde{\rho}(\tilde{\mathbf{r}}, t), \quad (15)$$

$$\Delta^2 \tilde{\mathbf{A}}(\tilde{\mathbf{r}}, t) - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \tilde{\mathbf{A}}(\tilde{\mathbf{r}}, t)}{\partial t^2} = -\frac{4\pi}{c} \tilde{\mathbf{j}}(\tilde{\mathbf{r}}, t), \quad (16)$$

where  $\frac{\epsilon\mu}{c^2} = \frac{1}{v^2}$ . The goal of this work is to analyze the solution of fraction order CEM problems related to EM waves. So, in order to do this, we rewrite ordinary differential wave equations (8)-(10), (12) and (13)-(16) in fraction order form with respect to time using caputo definition of fraction order derivative [3], [46]-[47]. Fractional form with respect to time of equations (8)-(10), (12) are as follows from equations (17) to (24):

$$\tilde{\nabla} \bullet \tilde{\mathbf{E}} = \frac{4\pi\rho}{\epsilon}, \quad (17)$$

$$\tilde{\nabla} \bullet \tilde{\mathbf{B}} = 0, \quad (18)$$

$$\tilde{\nabla} \times \tilde{\mathbf{E}} = -\frac{1}{\sigma^{1-\gamma}} \frac{1}{c} \frac{\partial^\gamma \tilde{\mathbf{B}}}{\partial t^\gamma}, \quad (19)$$

$$\tilde{\nabla} \times \tilde{\mathbf{B}} = \frac{4\pi\mu\tilde{\mathbf{j}}}{c} + \frac{1}{\sigma^{1-\gamma}} \frac{\epsilon\mu}{c} \frac{\partial^\gamma \tilde{\mathbf{E}}}{\partial t^\gamma}. \quad (20)$$

Fractional form of equations (13)-(16) are as follows [40]-[42]:

$$\tilde{\mathbf{B}} = \tilde{\nabla} \times \tilde{\mathbf{A}}, \quad (21)$$

$$\tilde{\mathbf{E}} = -\frac{1}{\sigma^{1-\gamma}} \frac{1}{c} \frac{\partial^\gamma \tilde{A}}{\partial t^\gamma} - \tilde{\nabla} \phi, \quad (22)$$

$$\Delta \tilde{\mathbf{A}} - \frac{1}{\sigma^{2(1-\gamma)}} \frac{\epsilon\mu}{c^2} \frac{\partial^{2\gamma} \tilde{\mathbf{A}}}{\partial t^{2\gamma}} = -\frac{4\pi\mu\tilde{\mathbf{j}}}{c}, \quad (23)$$

$$\Delta \phi - \frac{1}{\sigma^{2(1-\gamma)}} \frac{\epsilon\mu}{c^2} \frac{\partial^{2\gamma} \phi}{\partial t^{2\gamma}} = -\frac{4\pi\rho}{\epsilon}, \quad (24)$$

where equations (23) and (24) are derived after applying Lorenz gauge condition. Equations (23) and (24) becomes same as Equations (15) and (16), if we put  $\gamma = 1$ . Where  $\gamma(0 < \gamma \leq 1)$  is a random parameter which tells the order of the derivative and  $\sigma$  is a new parameter telling the fraction order time factor in the system. If charge density and current density are zero, the last two equations have the following homogeneous fractional differential equations (25) and (26):

$$\Delta \tilde{\mathbf{A}} - \frac{1}{\sigma^{2(1-\gamma)}} \frac{\epsilon\mu}{c^2} \frac{\partial^{2\gamma} \tilde{\mathbf{A}}}{\partial t^{2\gamma}} = 0, \quad (25)$$

$$\Delta \phi - \frac{1}{\sigma^{2(1-\gamma)}} \frac{\epsilon\mu}{c^2} \frac{\partial^{2\gamma} \phi}{\partial t^{2\gamma}} = 0. \quad (26)$$

We are anxious in the study of the EM fields in the medium preparatory from the above two equations. We can carve the fractional equations (25) and (26) in the following manner:

$$\frac{\partial^2 Z(x, t)}{\partial x^2} - \frac{1}{\sigma^{2(1-\gamma)}} \frac{\epsilon\mu}{c^2} \frac{\partial^{2\gamma} Z(x, t)}{\partial t^{2\gamma}} = 0, \quad (27)$$

where  $Z(x, t) = \tilde{\mathbf{A}}(x, t) = \phi(x, t)$ . Equation (27) tells the fraction form regarding the time of Electromagnetic waves. Now, for fractional form with respect to space, we consider equation (27) overbearing that the spatial derivative is in fraction order and the time derivative is in integer order. Then, we have the Fraction form of electromagnetic waves with respect to space as given in equation (28):

$$\frac{1}{\sigma_x^{2(1-\delta)}} \frac{\partial^{2\delta} \tilde{Z}(x, t)}{\partial x^{2\delta}} - \frac{1}{v^2} \frac{\partial^2 \tilde{Z}(x, t)}{\partial t^2} = 0, \quad (28)$$

where  $\delta(0 < \delta \leq 1)$  is the order of differential equation which is in fraction form and  $\sigma_x$  is the parameter related to space.

#### 4 Haar Wavelet

Haar functions were introduced by Alfred Haar in 1910. It has been used since then. Haar function is the simple, fast and computationally effective one in the wavelet family which is the step function having three values 0,1 and -1 as given in equation (29) and (30). The Haar wavelet is the orthogonal family member of rectangular wave-forms whose amplitude changes from one function to another.

The Haar functions are defined in the interval  $t \in [0, 1)$ , where  $t \in [A, B]$  is the general cas, we split the time interval into  $m$  equal sub-intervals having

width  $\Delta t = \frac{B-A}{m}$  [6].

$$h_o(t) = \begin{cases} 1, & t \in [A, B] \\ 0, & \text{elsewhere,} \end{cases} \quad (29)$$

$$h_i(t) = \begin{cases} 1, & \text{for } t \in (\psi_1(i), \psi_2(i)), \\ -1, & \text{for } t \in (\psi_2(i), \psi_3(i)), \\ 0, & \text{otherwise,} \end{cases} \quad (30)$$

where  $\psi_1(i) = A + \frac{k-1}{2^j} \times (m\Delta t)$ ,  $\psi_2(i) = A + \frac{k-\frac{1}{2}}{2^j} \times (m\Delta t)$  and  $\psi_3(i) = A + \frac{k}{2^j} \times (m\Delta t)$ .

The value of index ( $i$ ) can be given by  $i = 1, 2, \dots, m$ , where  $m = 2^J$  and  $J$  is level of resolution, that is a positive integer. If we have two parameters  $j$  and  $k$  which are integer, these parameters can also be used in the disintegration of index  $i$  such that  $i = k + 2^j - 1$  where  $j = 0, 1, \dots, J$  and  $k = 1, 2, \dots, 2^j + 1$ . Any function  $y(t) \in L^2[0, 1)$  can be extended into Haar wavelets by [4] [6]-[7]

$$y(t) = c_0 h_0(t) + c_1 h_1(t) + c_2 h_2(t) + \dots \quad (31)$$

$$c_j = \int_0^t y(t) h_j(t) dt. \quad (32)$$

If  $y(t)$  is approached as piece-wise consistent in each sub-interval, the sum in Equation (31) and (32) may be sacked after  $m$  terms and subsequently the matrix in discrete form can be written as given in equation (33):

$$y \approx \sum_{i=0}^{m-1} c_i h_i(t) = \mathbf{C}_m^T \mathbf{H}_m, \quad (33)$$

where  $h_0, h_1, h_2, \dots, h_{m-1}$  are the components of Haar function in discrete form. The related collocation points are given by equation (34):

$$t_l = A + (l - \frac{1}{2})\Delta t, l = 1, 2, \dots, m-1, m. \quad (34)$$

The integration of the  $\mathbf{H}_m(\mathbf{t}) = [h_0(t), h_1(t), \dots, h_{m-1}(t)]^T$  can be approximated by [4] as given in the form of equation (35):

$$\int_0^t \mathbf{H}_m(\tau) d\tau \cong \mathbf{P} \mathbf{H}_m(\mathbf{t}), \quad (35)$$

where,  $\mathbf{P}$  is the integration matrix of square order for the Haar wavelet. For our analysis, we need general order of integration of the Haar function operational matrix. For this purpose, we have recalled fractional order  $\alpha (> 0)$  defined by I. Podlubny [5] as given in equation (36):

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0, \alpha \in \mathfrak{R}^+. \quad (36)$$

The Haar wavelet operational matrix  $P^\alpha$  for integration of the general order  $\alpha$  is given by [39] from equation (37) to equation (42):

$$\begin{aligned} \mathbf{P}^\alpha \mathbf{H}_m(\mathbf{t}) &= J^\alpha \mathbf{H}_m(\mathbf{t}) \\ &= [\mathbf{P}^\alpha \mathbf{H}_0(\mathbf{t}) + \mathbf{P}^\alpha \mathbf{H}_1(\mathbf{t}) + \mathbf{P}^\alpha \mathbf{H}_2(\mathbf{t}) + \dots + \mathbf{P}^\alpha \mathbf{H}_{m-1}(\mathbf{t})]^T \quad (37) \\ &= [\mathbf{P}\mathbf{H}_0(\mathbf{t}) + \mathbf{P}\mathbf{H}_1(\mathbf{t}) + \mathbf{P}\mathbf{H}_2(\mathbf{t}) + \dots + \mathbf{P}\mathbf{H}_{m-1}(\mathbf{t})]^T \end{aligned}$$

where

$$\mathbf{P}h_0(t) = \begin{cases} \frac{t^\alpha}{\Gamma(1+\alpha)}, & t \in [A, B], \\ 0, & \text{elsewhere,} \end{cases} \quad (38)$$

and

$$\mathbf{P}h_i(t) = \begin{cases} 0, & A \leq t < \psi_1(i), \\ \phi_1, & \psi_1(i) \leq t < \psi_2(i), \\ \phi_2, & \psi_2(i) \leq t < \psi_3(i), \\ \phi_3, & \psi_3(i) \leq t < B, \end{cases} \quad (39)$$

where

$$\phi_1 = \frac{(t - \psi_1(i))^\alpha}{\Gamma(1 + \alpha)}, \quad (40)$$

$$\phi_2 = \frac{(t - \psi_1(i))^\alpha}{\Gamma(1 + \alpha)} - 2 \frac{(t - \psi_2(i))^\alpha}{\Gamma(1 + \alpha)}, \quad (41)$$

$$\phi_3 = \frac{(t - \psi_1(i))^\alpha}{\Gamma(1 + \alpha)} - 2 \frac{(t - \psi_2(i))^\alpha}{\Gamma(1 + \alpha)} + \frac{(t - \psi_3(i))^\alpha}{\Gamma(1 + \alpha)}. \quad (42)$$

## 5 Haar Wavelet Algorithm for fractional order equation

We provided 3 algorithms in the flow chart manner. Algorithm 1 which is presented by fig. 1 is applicable when the problem has only fractional-order term with time domain. Fig. 2 is for algorithm 2 which is applicable when the problem has a fractional-order term with space domain. Finally, 3<sup>rd</sup> figure shows algorithm 3 which is valid when the problem has a fractional-order term in both domains, i.e. time and space.

### 5.1 Algorithm 1

Algorithm 1 is proposed for solving those type for computational electromagnetic problems which have fractional order term in time domain only. Flow chart related to this algorithm is given in fig 1.

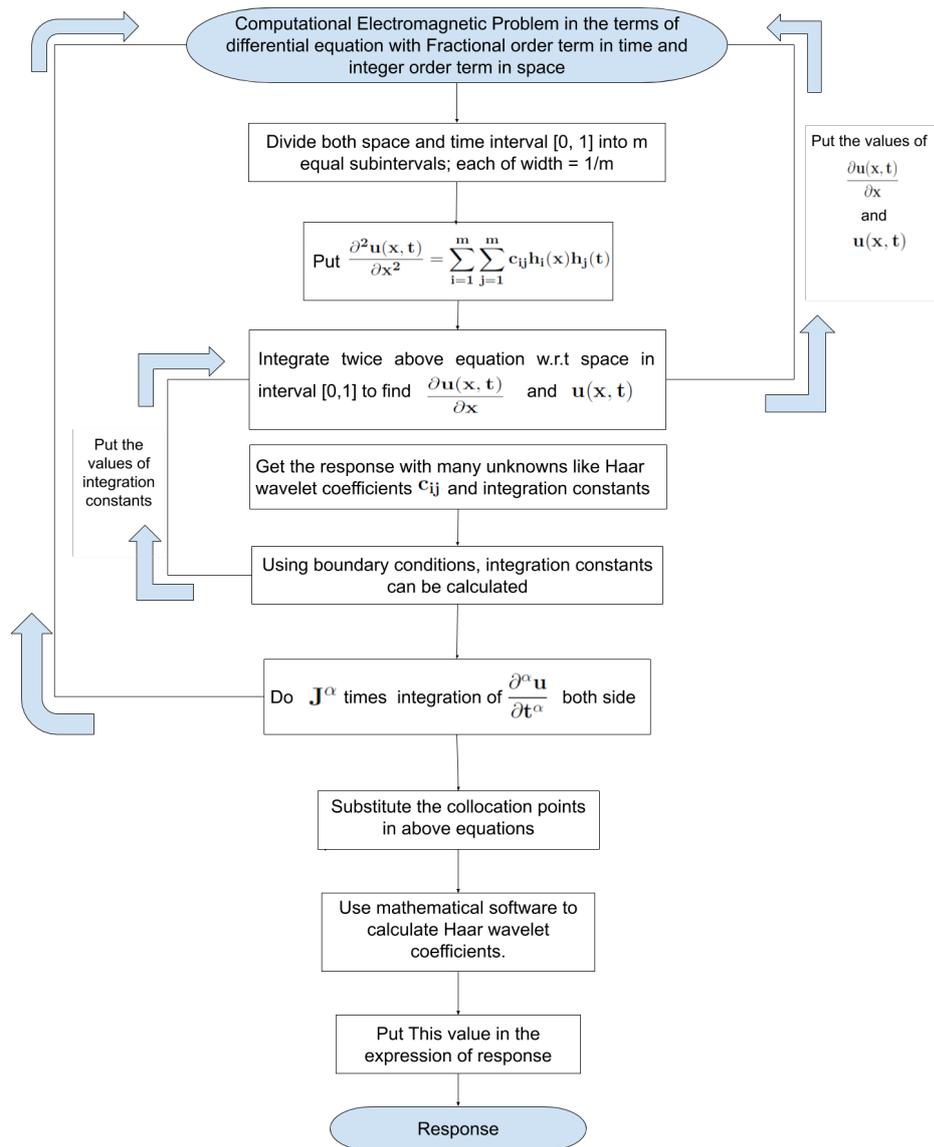


Fig. 1 Algorithm 1: When The problem has fractional order term with time domain.

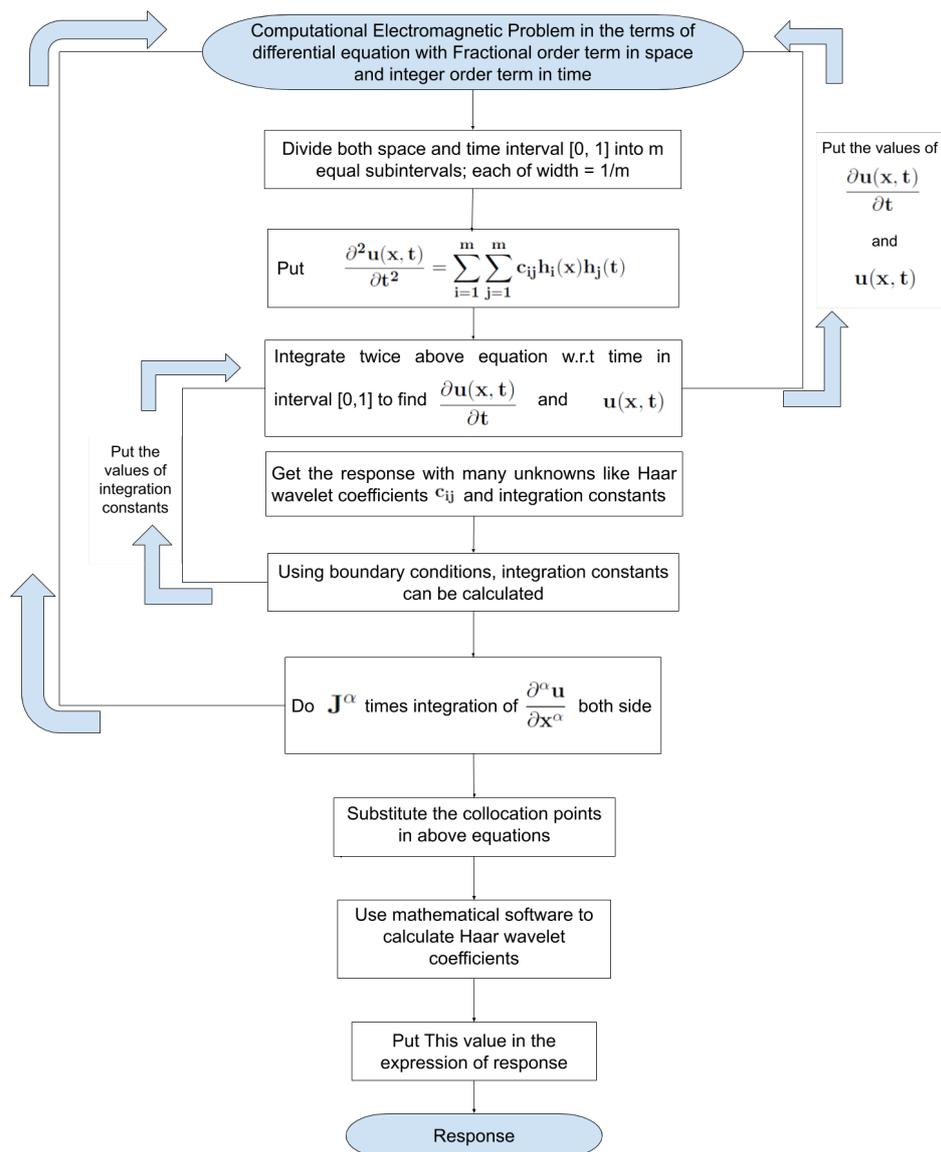
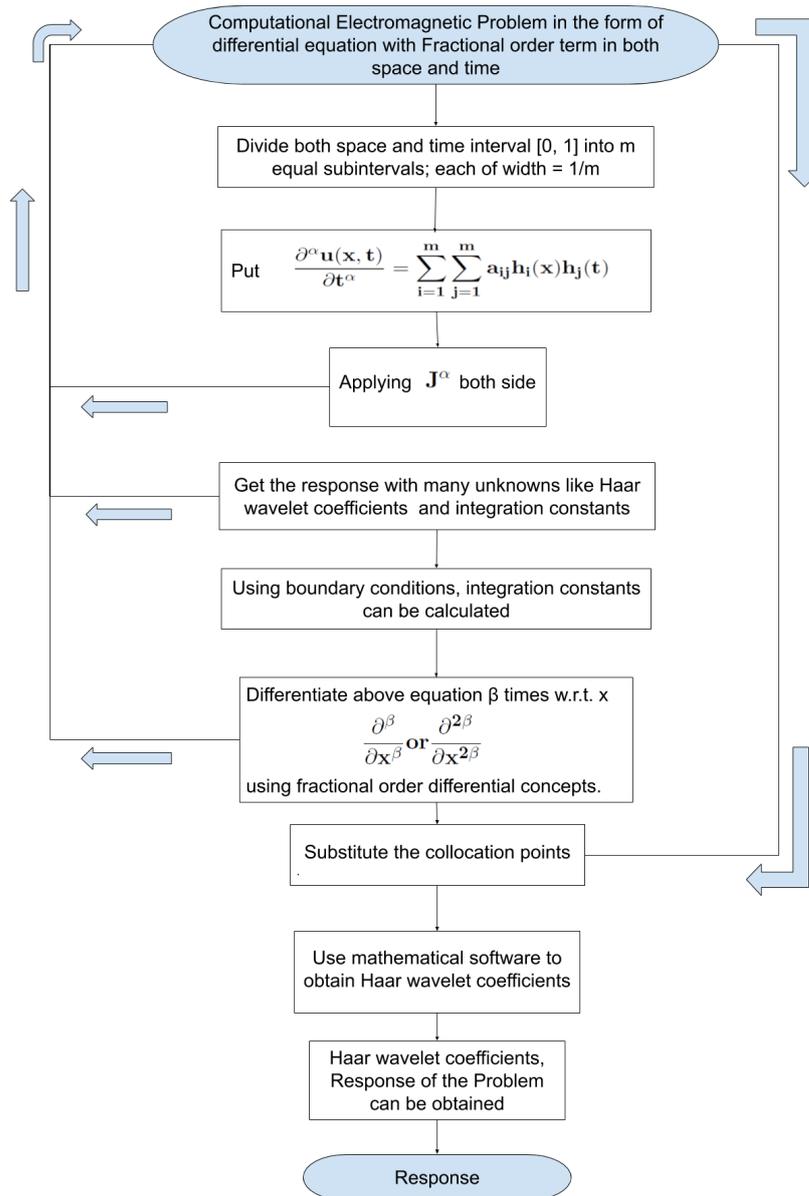


Fig. 2 Algorithm 2: When the problem has fractional order term with space domain.



**Fig. 3** Algorithm 3: The problem has fractional order term in both domains i.e. time and space.

## 5.2 Algorithm 2

Algorithm 2 is proposed for solving those type for computational electromagnetic problems which have fractional order term with space domain only. Flow chart related to this algorithm is given in Fig. 2.

## 5.3 Algorithm 3

Algorithm 3 is proposed for solving those type for computational electromagnetic problems which have fractional order term in both domains i.e. time and space. Flow chart related to this algorithm is given in Fig. 3.

## 6 Fractional order EM waves based Computational Problems and Error Estimation

We have taken three different problems to verify and check the authenticity of the proposed scheme along with error analysis. Error estimation and comparative analysis has been done after solving these problems sing approximate analytical method and proposed scheme with the help of following equation :

$$\text{Error} = \max_{n,i} |\hat{u}(x, t)_i^n - u(x, t)_i^n|. \quad (43)$$

where  $\hat{u}(x, t)_i^n$  is the Haar solution and  $u(x, t)_i^n$  is the approximate analytical solution.

### 6.1 Problem 1

Consider the 1-D fractional wave equation as  $\frac{\partial^\gamma u}{\partial t^\gamma} = \frac{x^2}{2} \frac{\partial^2 u}{\partial x^2}$  with following initial conditions given by  $\begin{cases} u(x, 0) = x \\ u_t(x, 0) = x^2 \end{cases}$ .

#### 6.1.1 Approximate Analytical Solution

The NTDM solution [cite8 of Problem 1 is given by equation (44) & (45):

$$\begin{aligned} u(x, t) &= u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \dots \\ &= x + x^2 \left[ t + \frac{t^{\gamma+1}}{\Gamma(\gamma+2)} + \frac{t^{2\gamma+1}}{\Gamma(2\gamma+2)} + \frac{t^{3\gamma+1}}{\Gamma(3\gamma+2)} + \dots \right], \end{aligned} \quad (44)$$

if  $\gamma = 2$ ,

$$u(x, t) = x + x^2 \left[ t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \dots \right]. \quad (45)$$

### 6.1.2 Exact Solution

The same problem has been solved by [8] and given the exact solution as equation (46) :

$$u(x, t) = x + x^2 \sinh t. \quad (46)$$

### 6.1.3 Haar Solution

As the given problem is in the fractional order differential term w.r.t. time and integer order term w.r.t. space, hence algorithm 1 has been followed to solve the above problem. Accordingly Haar coefficient and Haar solution has been given in equation (47) & (48) respectively:

$$\sum_{i=1}^m \sum_{j=1}^m c_{ij} = \frac{x[1 - u(1, t)]}{\sum_{i=1}^m \sum_{j=1}^m [h_i(x)h_j(t)(\mathbf{P}^2 - x\mathbf{P}^2 - \mathbf{P}^\alpha)]}, \quad (47)$$

$$u(x, t) = \frac{x[1 - u(1, t)]\mathbf{P}^2}{\sum_{i=1}^m \sum_{j=1}^m (\mathbf{P}^2 - x\mathbf{P}^2 - \mathbf{P}^\alpha)} + x \left[ u(1, t) - \frac{\sum_{i=1}^m \sum_{j=1}^m x[1 - u(1, t)][\mathbf{P}^2 h_i(x)]_{x=1}}{\sum_{i=1}^m \sum_{j=1}^m h_i(x)(\mathbf{P}^2 - x\mathbf{P}^2 - \mathbf{P}^\alpha)} \right]. \quad (48)$$

## 6.2 Problem 2

Lets consider the telegraph equation as  $\frac{\partial^\gamma v}{\partial x^\gamma} = \frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} + v$ ,  $0 < \gamma \leq 2$ ,  $0 \leq t$  which is in fractional space form. The initial and boundary conditions for the problem is given as:  $\begin{cases} v(0, t) = e^{-t} = \frac{\partial v(0, t)}{\partial x}, 0 \leq t \\ v(x, 0) = e^x, 0 < x < 1. \end{cases}$

### 6.2.1 Approximate Analytical Solution

The solution of the Problem 2 in the power series form is given by [9] in the form of equation (49) & (50):

$$v(x, t) = e^{-t} \left[ \frac{x^\gamma}{\Gamma(\gamma + 1)} + \frac{x^{\gamma+1}}{\Gamma(\gamma + 2)} + \frac{x^{2\gamma+1}}{\Gamma(2\gamma + 2)} + \dots \right], \quad (49)$$

and considering  $\gamma = 2$ , we can write

$$v(x, t) = e^{-t} \left[ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right]. \quad (50)$$

### 6.2.2 Exact Solution

The exact solution is given by [9] in equation (51):

$$v(x, t) = e^{x-t}. \quad (51)$$

### 6.2.3 Haar Solution

As the given problem 2 is in the fractional order differential term w.r.t. space and integer order term w.r.t. time, hence algorithm 2 has been used to give the solution of the above problem. Accordingly Haar coefficient and Haar solution has been given in equation (52) & (53) respectively:

$$\sum_{i=1}^m \sum_{j=1}^m d_{ij} = \frac{v(x, 1) + e^{-t}}{[\sum_{i=1}^m \sum_{j=1}^m [\mathbf{P}^2 h_i(x) h_j(t) (1 - \mathbf{P}^\alpha) + (\mathbf{P}^2 h_j(t))_{t=1} h_i(x) (\mathbf{P}^\alpha - 1 + x \mathbf{P}^\alpha) + \mathbf{P}^\alpha h_i(x) h_j(t) (\mathbf{P} - 1)]]}, \quad (52)$$

$$v(x, t) = \frac{[v(x, 1) + e^{-t}] \sum_{i=1}^m \sum_{j=1}^m [\mathbf{P}^2 h_j(t) - x (\mathbf{P}^2 h_j(t))_{t=1}]}{[\sum_{i=1}^m \sum_{j=1}^m [\mathbf{P}^2 h_j(t) (1 - \mathbf{P}^\alpha) + (\mathbf{P}^2 h_j(t))_{t=1} (\mathbf{P}^\alpha - 1 + x \mathbf{P}^\alpha) + \mathbf{P}^\alpha h_j(t) (\mathbf{P} - 1)]]}. \quad (53)$$

### 6.3 Problem 3

Now, consider the linear telegraph equation which has both time and space fractional terms as given bellow:  $\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + 2a \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + b^2 u = \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}}$ ,  $0 < \alpha \leq 1, \beta \leq 1, 0 \leq t$ . This problem subjects to the initial conditions  $\begin{cases} u(x, 0) = e^x, \\ u_t(x, 0) = -2e^x, 0 < x < 1. \end{cases}$

#### 6.3.1 Approximate Analytical Solution

Q-HAMT solution for  $a = b = 1$  and  $\alpha = 1$  is given by [11] in the form of equation (54):

$$u(x, t) = u_o(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \left(\frac{1}{n}\right)^m. \quad (54)$$

#### 6.3.2 Exact Solution

The exact solution of the above Problem 3 for  $a = b = 1$  and  $\beta = \alpha = 1$  is given by [11] in the form of equation (55):

$$u(x, t) = e^{x-2t}. \quad (55)$$

### 6.3.3 Haar Solutionn

As the given problem 3 is in the fractional form differential term with respect to both space and time, hence algorithm 3 has been used to give the solution. According to algorithm Haar coefficient and Haar solution has been given in equation (55) & (57) respectively:

$$\sum_{i=1}^m \sum_{j=1}^m a_{ij} = \frac{\frac{\partial^{2\beta} e^x}{\partial x^{2\beta}} - e^x}{[\sum_{i=1}^m \sum_{j=1}^m [h_i(x) \frac{\partial h_j(t)}{\partial t} + 2h_i(x)h_j(t) + \mathbf{P}^\alpha h_i(x)h_j(t) + \frac{\partial^{2\beta} h_i(x)}{\partial x^{2\beta}} \mathbf{P}^2 h_j(t)]]}, \quad (56)$$

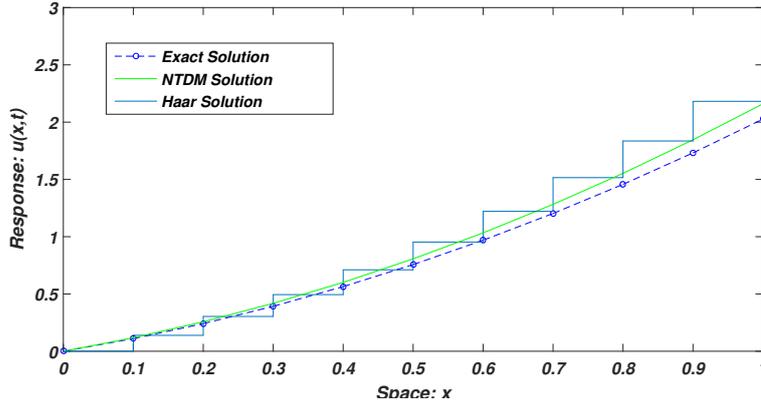
$$u(x, t) = e^x + \mathbf{P}^\alpha \sum_{i=1}^m \sum_{j=1}^m h_i(x)h_j(t)a_{ij}. \quad (57)$$

## 7 Result and Analysis

To verify the proposed algorithm, three different problems have taken, one from each algorithm. Problem 1 is based on algorithm 1. The approximate analytical solution (NTDM), Exact solution and Haar solution is given in equaions (44) - (48). The response plot for Exact solution, NTDM solution and Haar solution w.r.t. space has been given in fig. 4. As per the plot, the Haar solution is in very good compliance with the exact solution. Haar solution has staircase behaviour because it has values of 1, 0 & -1. For this problem absolute error also has been calculated for NTDM [8] and Haar method with respect to the exact solution at  $\gamma = 1$  in table 1. From this table, it is clear that the present method (Haar Method) gives error in the order of  $10^{-3}$  to  $10^{-5}$  which is less as compared to NTDM [8] at different space values. In table 2, comparison of solution for both method at different values of  $\gamma$  with exact solution has also been presented. According to table 2, Haar method gives better result as compared to NTDM [12] in good agreement to exact solution at  $\gamma = 1$ . For  $\gamma = 0.75$  and  $\gamma = 0.9$  also Haar wavelet method gives very closure result with exact solution as compared to NTDM [12].

**Table 1** Absolute Error w.r.t Exact Solution at  $\gamma = 1$

x	NTDM [8]	Present Method
0.2	5.00E-5	4.26E-5
0.4	1.20E-5	8.56E-6
0.6	2.90E-4	2.36E-4
0.8	4.90E-3	3.26E-3



**Fig. 4** Response plot of problem 1 (1-D fractional wave equation when time is in fractional order and space is in integer order)

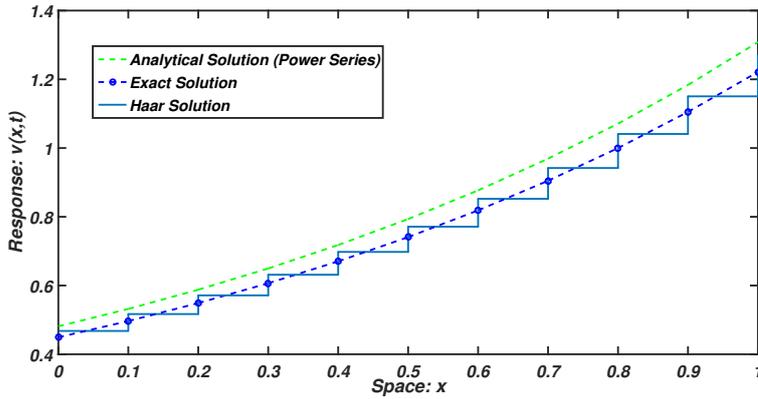
**Table 2** Comparison of solution for NTDM [12] and Haar method at different values of  $\gamma$ .

2*t	2*x	$\gamma = 0.75$		$\gamma = 0.9$		$\gamma = 1.0$		2*Exact Solution
		NTDM [12]	Haar	NTDM [12]	Haar	NTDM [12]	Haar	
3*1.0	0.3	0.3134751	0.282128	0.2677083	0.240937	0.2446431	0.244645	0.2446453
	0.6	1.2539004	1.12851	1.0708335	0.96375	0.9785727	0.978580	0.9785814
	0.9	2.8212761	2.539148	2.4093754	2.168438	2.2017886	2.201804	2.2018082
3*0.75	0.3	0.2404464	0.221211	0.2071503	0.207565	0.1905419	0.190535	0.1905300
	0.6	0.9617857	0.884843	0.8286013	0.830259	0.7621678	0.762119	0.7621200
	0.9	2.1640179	1.990896	1.8643530	1.868082	1.7148777	1.714767	1.7147700

Now, we have considered problem 2 and noticed that the computational problem is in the fractional-order differential term w.r.t. space and integer-order term w.r.t. time. Hence, we followed algorithm 2 and the response plot has been given in fig. 5. The response has been taken for  $\gamma = 0.85$ . For this problem we have also observed the absolute error at  $\gamma = 1$  for Power Series method [9] and Haar Method w.r.t. the exact solution. It is noted that Present method gives error of the order of  $10^{-3}$  to  $10^{-5}$  which is less for lower value of space as compared to Power series method as given in table 3. We have also analyzed solutions obtained at a different value of fraction order, time and space. This analysis has been conferred in tabular form in table 4, where the Haar solution is in good compliance with the exact solution at  $\gamma = 1$ . We have also given Haar solution at  $\gamma = 0.9$  and  $\gamma = 0.75$  for different time and space values.

**Table 3** Absolute Error w.r.t Exact solution at  $\gamma = 1$ .

x	Approximate analytical solution (Power Series) [9]	Present method
0.2	3.901E-5	2.225E-5
0.4	4.760E-5	3.752E-5
0.6	5.826E-4	4.936E-4
0.8	7.251E-3	9.241E-4



**Fig. 5** Response plot of problem 2 (telegraph equation when space is in fractional order and time is in integer order )

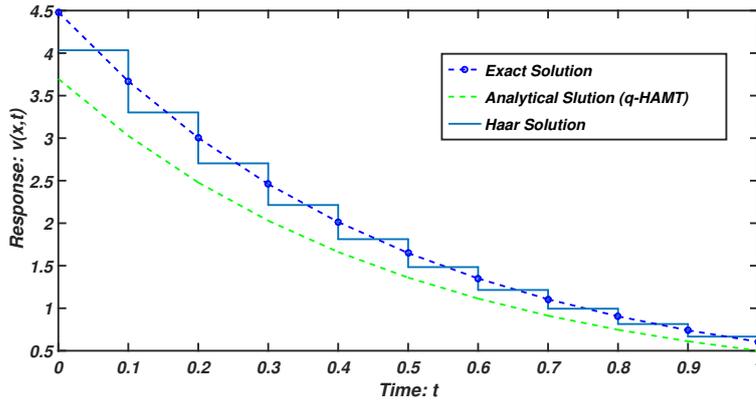
**Table 4** Analysis of solution obtained using Haar Technique at different value of  $\gamma$ , time and space.

t	x & $\gamma = 0.75$	$\gamma = 0.9$	$\gamma = 1.0$	Exact Solution	
$3*0.85$	0.2	0.510386	0.537826	0.543314	0.548802
	0.4	0.623421	0.656938	0.663642	0.670345
	0.6	0.792671	0.835287	0.843811	0.852334
$3*0.75$	0.2	0.489969	0.516311	0.52158	0.526848
	0.4	0.598444	0.630618	0.637053	0.643488
	0.6	0.760933	0.801844	0.810026	0.818208

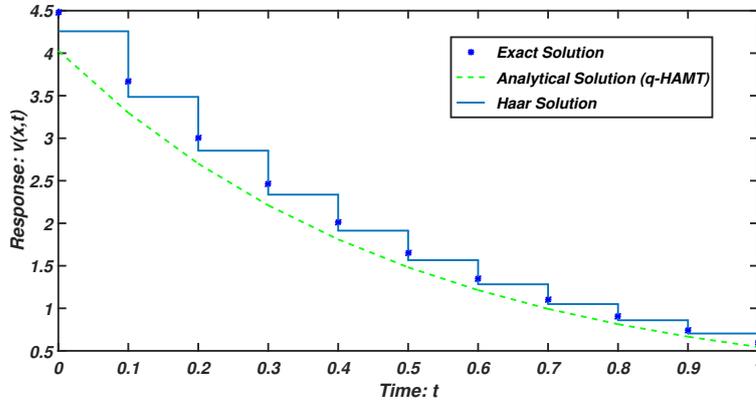
**Table 5** Absolute error for q-HAMT method [11] and haar wavelet method w.r.t exact solution at  $\gamma = 1$

x	Approximate analytical solution (q-HAMT) [11]	Present method
0.2	5.26E-6	3.01E-6
0.4	3.53E-6	1.83E-6
0.6	2.36E-5	1.23E-5
0.8	1.42E-5	8.23E-6

After the successful verification of Algorithms 1 and 2 with the help of computational Problems 1 and 2, respectively, we have considered Problem 3 which is in fractional form differential term considering both space and time. The response of this problem has been plotted in fig. 6 with respect to time at  $\alpha = 0.8$ ,  $\beta = 1$  and  $x = 1.5$ . From fig. 6, it is clear that the Haar solution approximately the same as the exact solution. We have also observed the solution for  $\alpha = 0.9$ ,  $\beta = 1$  and  $x = 1.5$  and for result agreement and found in fig. 7 that the Haar solution is good in agreement as compared to q-HAMT method [11] with respect to the exact solution. The absolute error analysis for problem 3 has been presented in table 5 at  $\alpha = 1 = \beta$ , where the present method gives comparatively less error to q-HAMT method [11] with respect to Haar Wavelet method.



**Fig. 6** Solution of time and space fractional order linear telegraph equation based computational problem at  $\alpha = 0.8, \beta = 1$  and  $x = 1.5$ .



**Fig. 7** Solution of time and space fractional order linear telegraph equation based computational problem at  $\alpha = 0.9, \beta = 1$  and  $x = 1.5$ .

## 8 Conclusion

In this work, the Haar algorithm for solving EM waves based computational problems in each category of fractional order time, space and both time & space has been Proposed in fig 1, 2 & 3 respectively. We have examined and verified the accuracy and exactness of the proposed scheme using three different computational problems of each category. All the problems have been solved using proposed scheme and compared the obtained numerical approximations with the exact solutions as well as the approximate analytical solution proposed by other researchers available in open literature like NTDM [8], Power series [9] and q-HAMT [11], etc. Their responses and absolute error have been cited in tables 1-5 to confirm the accuracy and efficiency of the proposed scheme com-

pared with numerical techniques. This scheme reveals the highest agreement with the exact solution for the targeted problems as demonstrated in figs. 4-7. Proposed Haar scheme has the ability to estimate an approximation to the residual error. From table 1-5, it is clear that present scheme gives high rate of convergence. In addition to this, the present scheme takes less computation time and space compared to other methods discussed above.

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# Figures

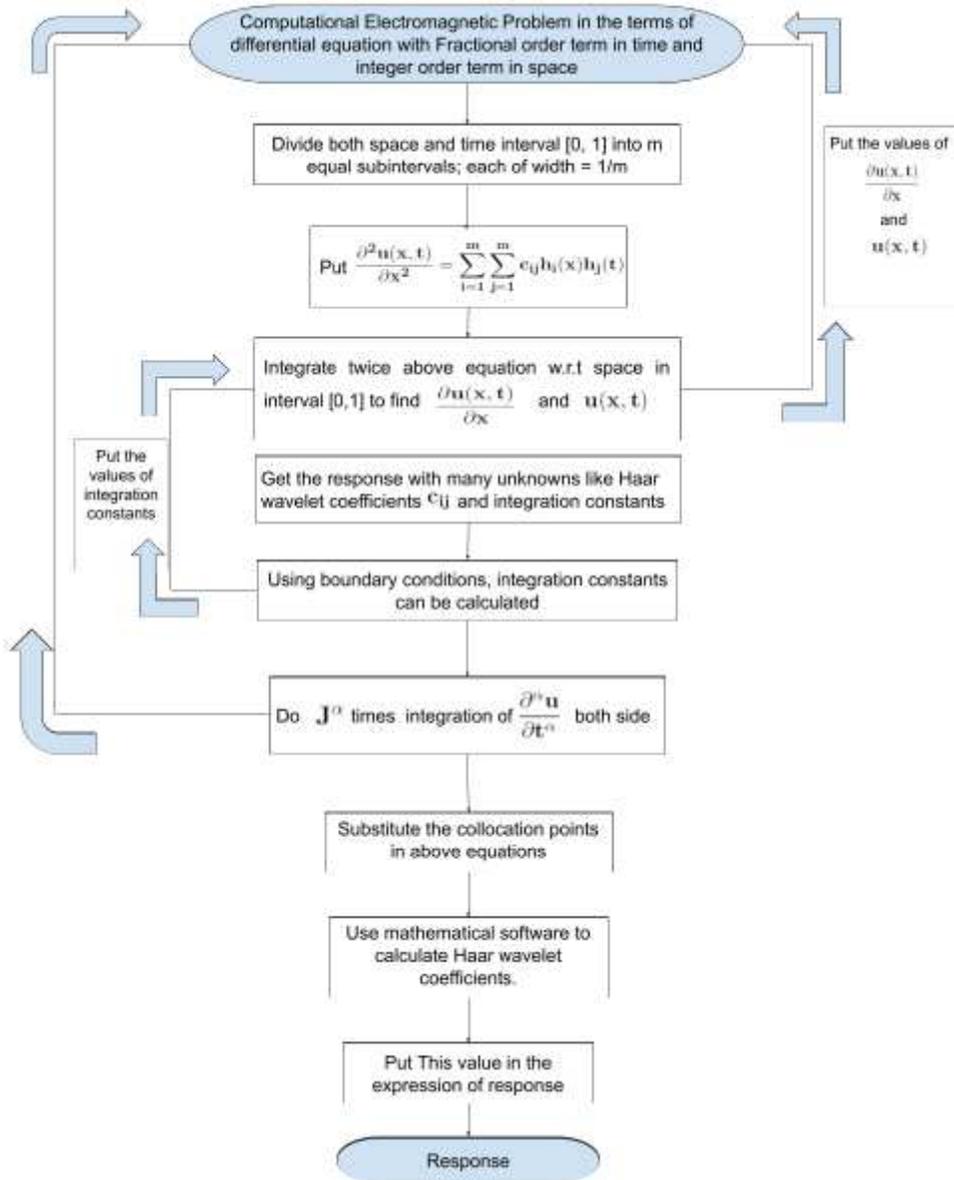


Figure 1

Algorithm 1: When The problem has fractional order term with time domain.

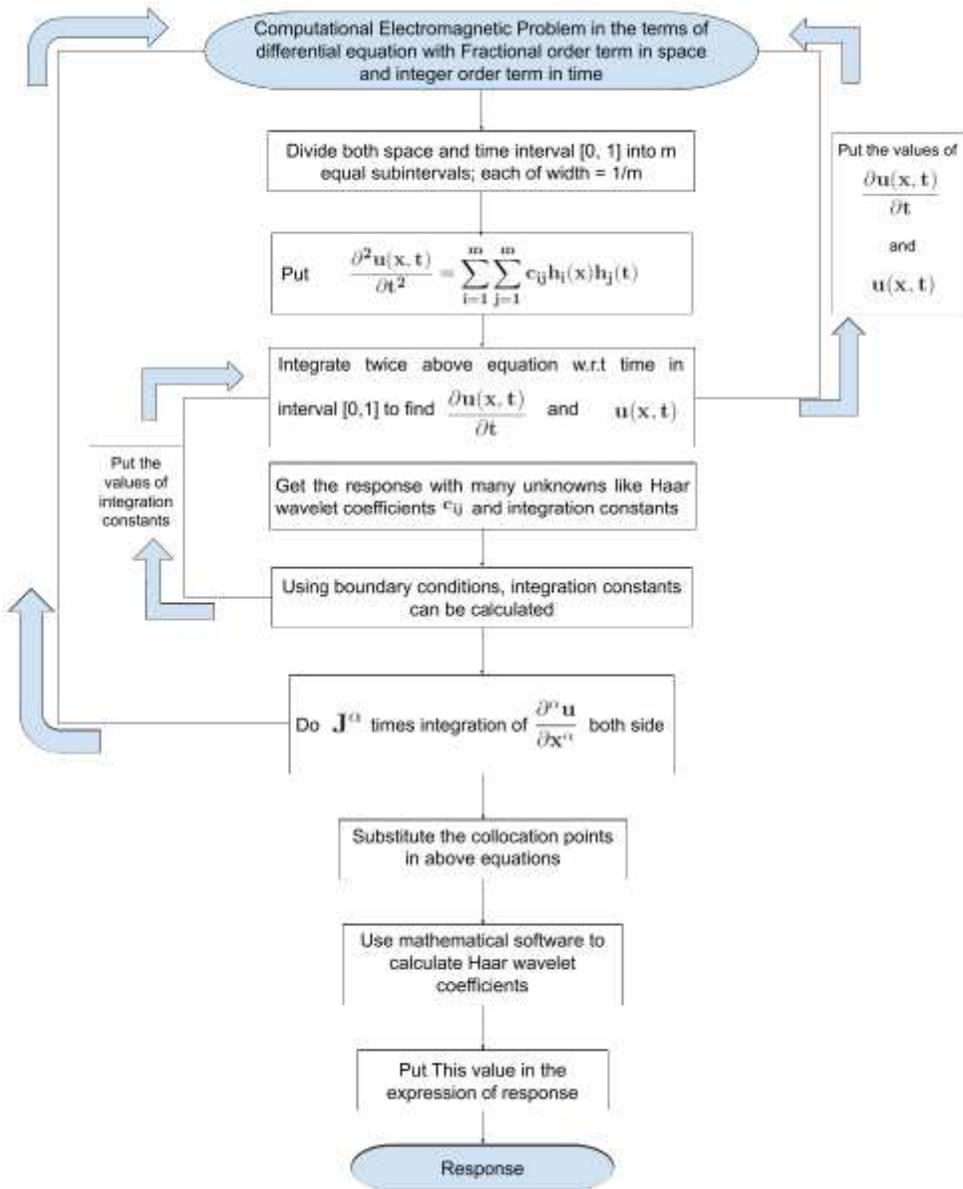
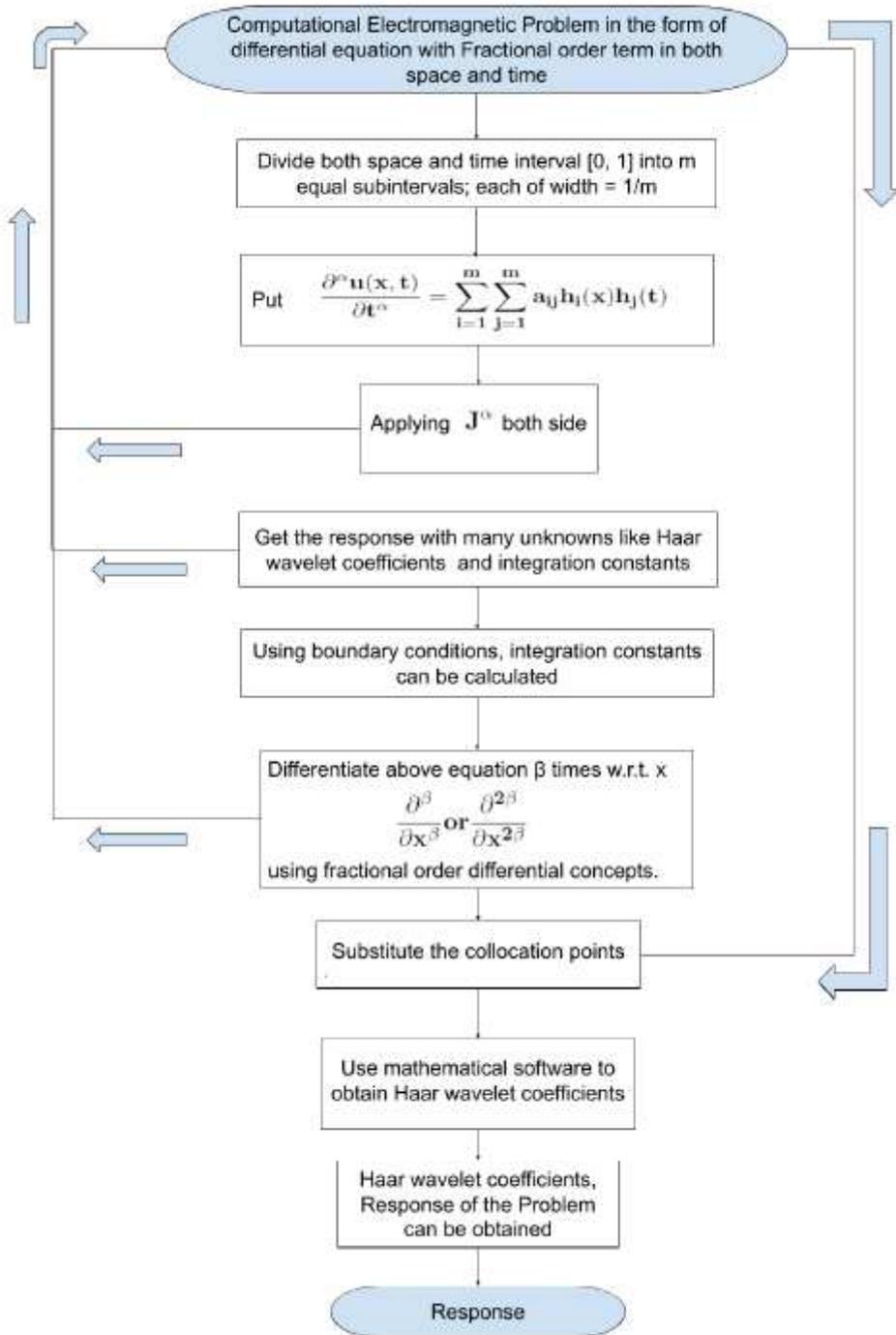


Figure 2

Algorithm 2: When the problem has fractional order term with space domain.



**Figure 3**

Algorithm 3: The problem has fractional order term in both domains i.e. time and space.

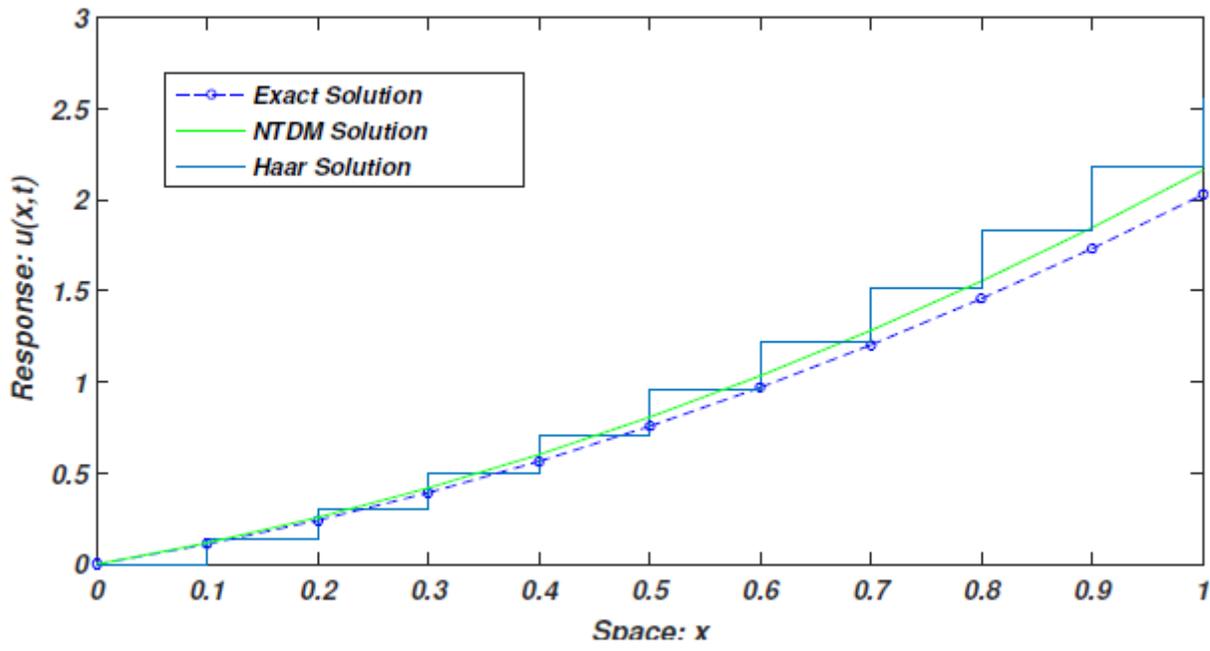


Figure 4

Response plot of problem 1 (1-D fractional wave equation when time is in fractional order and space is in integer order)

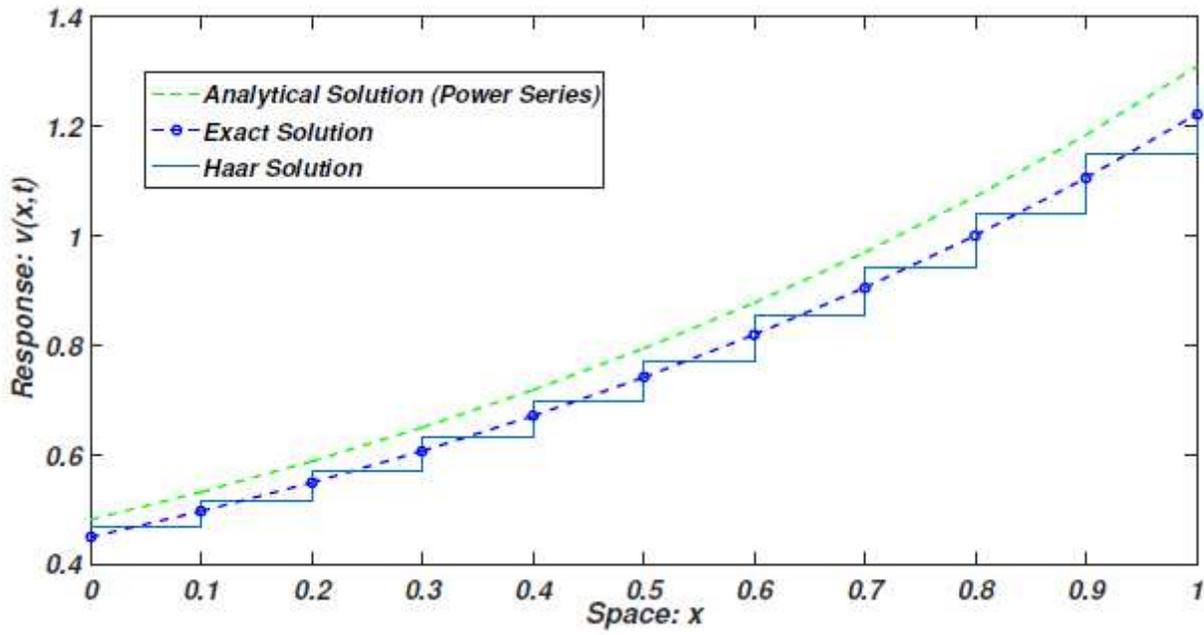


Figure 5

Response plot of problem 2 (telegraph equation when space is in fractional order and time is in integer order)

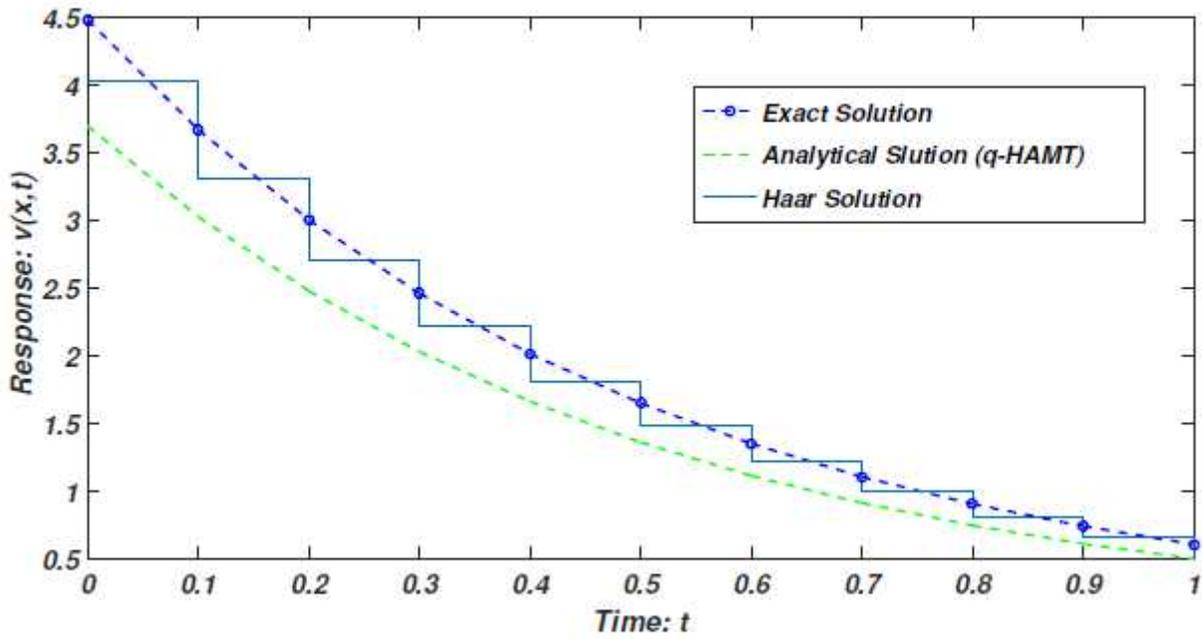


Figure 6

Solution of time and space fractional order linear telegraph equation based computational problem at  $\alpha = 0:8$ ;  $\beta = 1$  and  $x = 1:5$ .

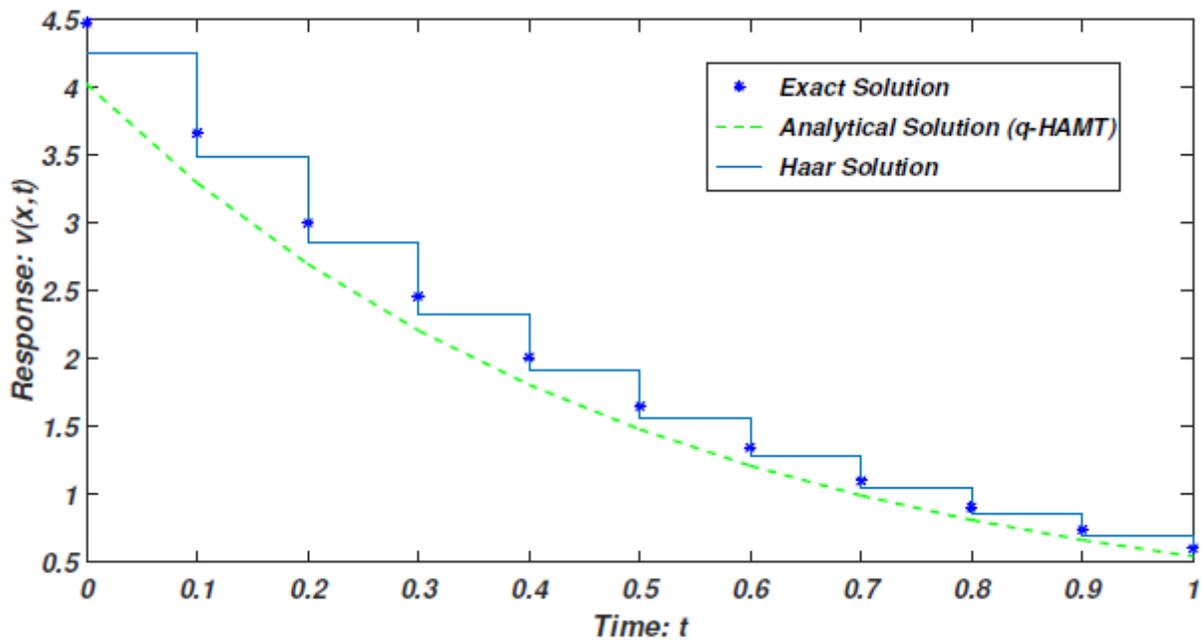


Figure 7

Solution of time and space fractional order linear telegraph equation based computational problem at  $\beta = 0:9$ ;  $\alpha = 1$  and  $x = 1:5$ .