Stability of Geared Systems with Linear Suspension in Permanent Contact Regime

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Research Article

Keywords: Permanent contact condition, linear suspension, primary parametric resonance, combined parametric resonance

Posted Date: April 12th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-268672/v1>

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Version of Record: A version of this preprint was published at Nonlinear Dynamics on November 16th, 2021. See the published version at <https://doi.org/10.1007/s11071-021-06943-w>.

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Received: date / Accepted: date

Abstract Many experimental, analytical, and numerical works have investigated the dynamic characteristics of geared transmission systems under constant speed and lightly loaded operational conditions. In such a condition, the internal and external excitation terms can easily cause the separation of the gears in mesh, followed by free play mode and impact phases, which introduce strong nonlinearity to the system. This is while, limited works have used analytical approach to investigate the dynamic behavior of the geared systems under constant speed and heavy-load operational condition, where the governing equations of the system reduce to linear periodic time-varying system of equations and the parametric resonance is the main source of Instability. In fact, large oscillations of such a system lead into the separation of the teeth in mesh, where the permanent contact assumption is not valid anymore and the effect of the backlash, free play mode, and impact phases must be considered. Consequently, in this study, lumped parameter analysis method is used to formulate the generalized dynamic model of a one-stage spur gear pair with linear suspension. Poincare-Lindstedt method is used to determine the range of the parameters in which the primary and combined parametric resonance would not happen, where the amplitude of the vibration remains small and the permanent contact condition between the teeth in mesh is satisfied. Afterward, numerical integration along with Floquet theory is used to validate the analytical results.

Keywords Permanent contact condition · linear suspension · primary parametric resonance · combined parametric resonance

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1 Introduction

Gears are known as one of the most common mechanical part widely used in rotary machinery and power transmission systems. Many researchers have investigated the dynamic characteristics of the geared systems under influence of the internal excitation parameters like; time varying stiffness and static transmission error, as well as external excitation like; fluctuation of the load and power, to design and develop optimal, silent and reliable gear transmission systems. Originally, gears are designed to remain in permanent contact regime, where teeth in mesh do not separate during the motion. But due to the presence of clearance between the teeth in mesh, for the bad range of the parameters or in the bad operational conditions, the teeth can separate resulting in the free play mode, where the teeth are not in contact anymore. It is well known that for relatively large values of the forcing parameter the permanent contact regime is achieved, while by decreasing the value of the force, the permanent contact regime withdraws gradually, giving way to the free play mode. Low values of the forcing parameter and consequently alternating between permanent contact regime and free play mode has been one of the interesting topics in dynamic analysis of the gears. In such studies, the clearance between the teeth in mesh is expressed by a non-smooth piecewise linear function or in some works approximated with a third order polynomial [4], or by Fourier series [1]. Consequently, analytical methods, like; Melnikov method [4], harmonic balance method [1], as well as numerical [2, 7], and experimental methods [1] have been adopted to investigate the dynamic characteristics of these systems with respect to the time history chart [2], frequency content [2], frequency response [2], phase plane portrays [7], Poincare map [7], Lyapunov exponent [4] to predict stability, steady state periodic and quasiperiodic solutions [4], jump phenomena, bifurcation [4], chaotic behavior [4] and basin of attraction [8]. Many of the above-mentioned studies have neglected the effect of impact forces, when the separated teeth come back to contact, as the free play modes ends, and permanent contact regimen starts. However, there are some works specifically focusing on the effect of the impact, trying to find the parameter space that one-sided and double-sided impact phases as well as no-impact phase can happen [8]. In such a condition the gears in mesh can be considered as a non-smooth vibrio impact system which is very prone to chaotic behaviors [5].

This is while, not many attempts have been made for the analytical investigation of the gears in permanent contact regime. In this paper, without loss of the generality, it is assumed that the amplitude of the vibration is small enough such that the gears in mesh remain in permanent contact and the system does not go into the free play mode and consequently impact phases. This is a condition that can be acceptable under steady state operational conditions of the constant speed and high load [3]. Consequently, by imposing the permanent contact condition, the system's governing differential equations reduce into linear periodically time-varying system of equations. Poincare-Lindstedt method is used for the analytical calculations to study the influence of the pa-

rameters on the stability of the system and Floquet theory is used to verify the accuracy and electiveness of the analytical calculations. When using perturbation analysis, the equivalent stiffness of teeth in mesh is assumed to be small, this is while this assumption is not required in numerical analysis, and Floquet theory can be used for any small or large values of teeth stiffness. Anyhow, in this analysis, since the normalized equivalent stiffness of teeth in mesh is less than one the analytical results are acceptable. Finally, both analytical and numerical results show that there are several ranges of parametric frequency that the parametric primary or combined resonance cause unbounded response. In such a condition, significant increase in the amplitude of the oscillation cause separation of the teeth in mesh, violation of the permanent contact, and occurrence of free play mode where the nonlinear effect of the clearance between the gears in mesh and impact phases must be considered.

The rest of paper is organized as follows; In section 2, the Tribological Characteristics of the gears in mesh are described and formulated. In section 3, dynamic equations of spur gear pair in mesh with linear suspension is formulated. In section 4, Poincare-Lindstedt method is used to analyze the stability of the perturbed system while the damping and external forcing terms are dropped out. In section 5, Floquet theory is used to compute and characterize the dynamic behavior of the systems for different parametric frequencies. In section 6, the numerical results are compared with the analytical results obtained in previous sections. At the end some brief concluding remarks are presented.

2 Tribological Characteristics

The tribological characteristics of the gears in mesh contributes a great influence on the dynamic characteristics and vibrations of the geared systems which include the time varying meshing stiffness, static transmission error and backlash. In this part, more details on the nature and the mathematical expressions of the above-mentioned terms are provided.

2.1 Time Varying Gear Mesh Stiffness

The concept of equivalent tooth mesh stiffness have been very helpful in the lumped mass modelling of the gears [7]. Such that under assumption of the constant velocity operational condition, the gear stiffness coefficient can be described as a periodic function of time with fundamental frequency equal to the meshing frequency.

$$k(t) = K + k\cos(\omega t) \quad (1)$$

where K is the averaged value of the equivalent stiffness and k is the amplitude of the oscillation about K , and ω is the meshing frequency. By factoring out

the averaged value the ratio of the changing stiffness over the constant stiffness is defined as follow.

$$k(t) = K \left(1 + k_0 \cos(\omega t) \right), \quad k_0 = \frac{k}{K} \quad (2)$$

The harmonic term in Eq. (2) is the most important source of internal excitation in geared systems such that increase or decrease in the value of k_0 directly affect the amplitude of the vibration [2].

2.2 Static Transmission Error

Static transmission error, or the meshing error, is one of the most important sources of noise in the gear systems, defined as the difference between the actual and the ideal positions of the driven gear along the pressure line [3] which is mainly due to the geometrical manufacturing errors from perfect involute form and teeth deformations during the operation. Since the mean angular velocities of the gears are constant, the static transmission error can be approximated by a periodic function.

$$e(t) = E \cos(\omega t) \quad (3)$$

where E is the comprehensive error amplitude and the fundamental frequency ω is the meshing frequency [4].

2.3 Backlash Function

Backlash, looseness, or play, is usually used to represent the gear clearances, which is the gap between the gear pair along the circumference. This can be due to the manufacturing errors, wear of the tooth surface, or mounting errors. Even though, theoretically a gear pair with zero backlash is ideal but in practice some backlash is necessary. This is to avoid the jamming phenomenon, reduction of the interference and providing some space for ease of assembly, better lubrication, elastic deflection under the load, and thermal expansion. Typically, backlash is achieved by reducing the tooth thickness in manufacturing process or by increasing the distance of the center of mountings. In different works, the backlash width is approximated either by a constant value, or by a time-varying function, mostly normal distribution or fractal expression [7]. In this work, a constant value (b) is incorporated to model the gear tooth backlash width. In any case, the gear backlash function is usually represented by a non-smooth piecewise-linear, nondifferentiable and symmetric function, which is the main source of nonlinearity in the geared system, defined by the following equation.

$$f(x) = \begin{cases} x - b & +b < x \\ 0 & -b \leq x \leq +b \\ x + b & x < -b \end{cases} \quad (4)$$

where x is the dynamic transmission error, $2b$ is the total nondimensional backlash width, and b is known as half backlash. Figure 1 show that if enough load transmits through the geared system the minimum value of x remain grater than b and the system operate in permanent contact condition, where the effect of backlash do not participate in the dynamic response of the system.

Fig. 1 Backlash function

3 Generalized Model of Spur Gear Pairs in Mesh

In this section, the lumped parameter technique is used to formulate the dynamic model of a single-stage spur gear is mesh with linear suspension. As illustrated in Figs. 2 the gears in mesh are modeled as a pair of rigid disks with radiuses equal to the base circles connected by a spring-damper set along the line of the action and the transmission shafts and the supporting mounts are modeled by a set of linear spring-damper. It is assumed that only one tooth pair is in mesh at a time and all the parameters are represented in Table 1.

Fig. 2 Schematic of pinion and gear in mesh with linear suspension

Table 1 System parameters

Parameter	Description
$f(x)$	Backlash function
$e(t)$	Static transmission error
$2b$	Total backlash width
θ_p, θ_g	Torsional displacements
x_p, x_g	Vertical displacements
r_p, r_g	Base circle radius
I_p, I_g	Mass moment of inertia
$K(t)$	Time varying gear mesh stiffness
C	Constant gear mesh damping coefficient
C_p, C_g	Suspension damping coefficient
$T_p(t), T_g(t)$	Driving torque and load torque

For such a system, each disk has one rotational and one translational degree of freedom, consequently, the Newton second law is used to construct the torsional and translational differential equation of motion for each disk [2, 7].

$$\begin{aligned} M_p \ddot{x}_p + C_p \dot{x}_p + K_p x_p \\ + C(\dot{x}_p - \dot{x}_g + r_p \dot{\theta}_p - r_g \dot{\theta}_g - \dot{e}(t)) \\ + k(t)f(x_p - x_g + r_p \theta_p - r_g \theta_g - e(t)) = 0 \end{aligned} \quad (5)$$

$$\begin{aligned} M_g \ddot{x}_g + C_g \dot{x}_g + K_g x_g \\ - C(\dot{x}_p - \dot{x}_g + r_p \dot{\theta}_p - r_g \dot{\theta}_g - \dot{e}(t)) \\ - k(t)f(x_p - x_g + r_p \theta_p - r_g \theta_g - e(t)) = 0 \end{aligned} \quad (6)$$

$$\begin{aligned} I_p \ddot{\theta}_p + r_p C(\dot{x}_p - \dot{x}_g + r_p \dot{\theta}_p - r_g \dot{\theta}_g - \dot{e}(t)) \\ + r_p k(t)f(x_p - x_g + r_p \theta_p - r_g \theta_g - e(t)) = +T_p(t) \end{aligned} \quad (7)$$

$$\begin{aligned} I_g \ddot{\theta}_g - r_p C(\dot{x}_p - \dot{x}_g + r_p \dot{\theta}_p - r_g \dot{\theta}_g - \dot{e}(t)) \\ - r_g k(t)f(x_p - x_g + r_p \theta_p - r_g \theta_g - e(t)) = -T_g(t) \end{aligned} \quad (8)$$

where subscripts g and p stand for gear and pinion, respectively. The total difference between the rotation angle of the gears in mesh can be expressed by the following equation which is defined as the difference between the dynamic and static transmission errors.

$$x = x_p - x_g + r_p \theta_p - r_g \theta_g - e(t) \quad (9)$$

Besides, for considering the fluctuation of the excitation torque and the applied load, both can be decomposed into averaging and fluctuating parts [4], so Eqs. (5)-(8) can be written in the following form.

$$M_p \ddot{x}_p + C_p \dot{x}_p + K_p x_p + C\dot{x} + k(t)f(x) = 0 \quad (10)$$

$$M_g \ddot{x}_g + C_g \dot{x}_g + K_g x_g - C\dot{x} - k(t)f(x) = 0 \quad (11)$$

$$I_p \ddot{\theta}_p + r_p Cx + r_p k(t)f(x) = +\bar{T}_p + \tilde{T}_p \cos(\omega_p t) \quad (12)$$

$$I_g \ddot{\theta}_g - r_p Cx - r_g k(t)f(x) = -\bar{T}_g - \tilde{T}_g \cos(\omega_g t) \quad (13)$$

where $\bar{T}_p$ and $\bar{T}_g$ are the average torques and $\tilde{T}_p$ and $\tilde{T}_g$ are the fluctuating torques. Now by subtracting Eq. (13) from Eq. (12), the governing torsional equations of motion reduce to one equation.

$$\begin{aligned} -\ddot{x}_p + \ddot{x}_g + \ddot{x} + \frac{C}{M}\dot{x} + \frac{K}{M}(1 + k_0 \cos(\omega t))f(x) = \frac{K}{M}F \\ + \frac{K}{M}F_p \cos(\omega_p t) + \frac{K}{M}F_g \cos(\omega_g t) - \ddot{e}(t) \end{aligned} \quad (14)$$

where M is the equivalent mass representing the total inertia of the gear pair, F is the average static force transmitted through the gear pair and F_p and F_g are the fluctuating force applied on the pinion and gear, expressed by the following equations.

$$\begin{aligned} M &= \frac{I_p I_g}{I_p r_g^2 + I_g r_p^2}, \quad F = \frac{M}{K} \left(\frac{\bar{T}_p r_p}{I_p} + \frac{\bar{T}_g r_g}{I_g} \right) \\ F_p &= \frac{M}{K} \frac{\tilde{T}_p r_p}{I_p}, \quad F_g = \frac{M}{K} \frac{\tilde{T}_g r_g}{I_g} \end{aligned} \quad (15)$$

Now, by defining the following standard parameters

$$\omega_n = \sqrt{\frac{K}{M}}, \quad \zeta = \frac{C}{2\sqrt{KM}} \quad (16)$$

and using Eq. (3), one can write Eq. (14) in the following standard form

$$\begin{aligned} -\ddot{x}_p + \ddot{x}_g + \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2(1 + k_0 \sin(\omega t))f(x) &= \omega_n^2 F \\ + \omega_n^2 F_p \cos(\omega_p t) + \omega_n^2 F_g \cos(\omega_g t) + \omega^2 E \cos(\omega t) \end{aligned} \quad (17)$$

By defining the following dimensionless parameters

$$\begin{aligned} \hat{t} &= \omega_n t \\ u &= \frac{x}{b}, \quad u_p = \frac{x_p}{b}, \quad u_g = \frac{x_g}{b} \\ \Omega &= \frac{\omega}{\omega_n}, \quad \Omega_p = \frac{\omega_p}{\omega_n}, \quad \Omega_g = \frac{\omega_g}{\omega_n}, \end{aligned} \quad (18)$$

Equations (10),(11) and (17) are written in the following dimensionless form.

$$m_p \hat{u}_p + 2\zeta_p \hat{u}_p + 2\zeta \hat{u} + k_p u_p + (1 + k_0 \cos(\Omega \hat{t}))f(u) = 0 \quad (19)$$

$$m_g \hat{u}_g + 2\zeta_g \hat{u}_g - 2\zeta \hat{u} + k_g u_g - (1 + k_0 \cos(\Omega \hat{t}))f(u) = 0 \quad (20)$$

$$\begin{aligned} -\hat{u}_p + \hat{u}_g + \hat{u} + 2\zeta \hat{u} + (1 + k_0 \cos(\Omega \hat{t}))f(u) \\ = F + F_p \cos(\Omega_p \hat{t}) + F_g \cos(\Omega_g \hat{t}) + \Omega^2 E \cos(\Omega \hat{t}) \end{aligned} \quad (21)$$

where the ratio of the parameter corresponded to the suspension system over the parameter corresponded to the gear pairs in mesh are expressed as follow.

$$\begin{aligned} m_p &= \frac{M_p}{M}, \quad m_g = \frac{M_g}{M}, \\ k_p &= \frac{K_p}{K}, \quad k_g = \frac{K_g}{K} \\ \zeta_p &= \frac{C_p}{2\sqrt{KM}}, \quad \zeta_g = \frac{C_g}{2\sqrt{KM}} \end{aligned} \quad (22)$$

One can write Eqs. (19)-(21) in the following matrix form.

$$\begin{aligned}
 & \begin{bmatrix} m_p & 0 & 0 \\ 0 & m_g & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{u}_p \\ \hat{u}_g \\ \hat{u} \end{bmatrix} + 2 \begin{bmatrix} \zeta_p & 0 & \zeta \\ 0 & \zeta_g & -\zeta \\ 0 & 0 & \zeta \end{bmatrix} \begin{bmatrix} \hat{u}_p \\ \hat{u}_g \\ \hat{u} \end{bmatrix} \\
 & + \begin{bmatrix} k_p & 0 & +(1+k_0\cos(\Omega\hat{t})) \\ 0 & k_g & -(1+k_0\cos(\Omega\hat{t})) \\ 0 & 0 & +(1+k_0\cos(\Omega\hat{t})) \end{bmatrix} \begin{bmatrix} u_p \\ u_g \\ f(u) \end{bmatrix} \\
 & = \begin{bmatrix} 0 \\ 0 \\ F + F_p\cos(\Omega_p\hat{t}) + F_g\cos(\Omega_g\hat{t}) + \Omega^2 E\cos(\Omega\hat{t}) \end{bmatrix}
 \end{aligned} \tag{23}$$

with the following dimensionless backlash function

$$f(u) = \begin{cases} u - 1 & +1 < u \\ 0 & -1 \leq u \leq +1 \\ u + 1 & u < -1 \end{cases} \tag{24}$$

Equation (23) is a damped conservative system and the nonsymmetric nature of this system is due to the choice of the coordinates.

4 Analytical Calculations

The goal of this section is to use analytical methods to obtain the parameter space that the parametric resonance does not cause the instability of the system, increase in the amplitude of the oscillation, and consequently separation of the teeth in mesh. This is done under constant operational velocity and relatively high load condition. By assuming that the system is symmetric such that both driving and driven shafts have the same parameters

$$\begin{aligned}
 m &= m_p = m_g \\
 k &= k_p = k_g
 \end{aligned} \tag{25}$$

and by imposing the permanent contact condition, Eq. (23) reduces to a system of linear differential equations with periodic time-varying coefficients.

$$\begin{aligned}
& \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{u}_p \\ \hat{u}_g \\ \hat{u} \end{bmatrix} + 2 \begin{bmatrix} \zeta_p & 0 & \zeta \\ 0 & \zeta_g & -\zeta \\ 0 & 0 & \zeta \end{bmatrix} \begin{bmatrix} \hat{u}_p \\ \hat{u}_g \\ \hat{u} \end{bmatrix} \\
& + \begin{bmatrix} k & 0 & +(1+k_0\cos(\Omega\hat{t})) \\ 0 & k & -(1+k_0\cos(\Omega\hat{t})) \\ 0 & 0 & +(1+k_0\cos(\Omega\hat{t})) \end{bmatrix} \begin{bmatrix} u_p \\ u_g \\ u \end{bmatrix} \\
& = \begin{bmatrix} 0 \\ 0 \\ F + F_p\cos(\Omega_p\hat{t}) + F_g\cos(\Omega_g\hat{t}) + \Omega^2 E\cos(\Omega\hat{t}) \end{bmatrix} \\
& + \begin{bmatrix} +(1+k_0\cos(\Omega\hat{t})) \\ -(1+k_0\cos(\Omega\hat{t})) \\ +(1+k_0\cos(\Omega\hat{t})) \end{bmatrix}
\end{aligned} \tag{26}$$

To study the stability of the corresponding undamped homogenous system, the time varying terms is perturbed as follow.

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{u}_p \\ \hat{u}_g \\ \hat{u} \end{bmatrix} + \begin{bmatrix} k & 0 & +(1+\epsilon\cos(\Omega\hat{t})) \\ 0 & k & -(1+\epsilon\cos(\Omega\hat{t})) \\ 0 & 0 & +(1+\epsilon\cos(\Omega\hat{t})) \end{bmatrix} \begin{bmatrix} u_p \\ u_g \\ u \end{bmatrix} = 0 \tag{27}$$

where ϵ is a small parameter and the following transformation of variable is used to implement Poincare-Lindstedt method.

$$\tau = \Omega\hat{t}, \quad u'_p = \frac{du_p}{d\tau}, \quad u'_g = \frac{du_g}{d\tau}, \quad u' = \frac{du}{d\tau} \tag{28}$$

which results in the following system of equations.

$$\Omega^2 \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u''_p \\ u''_g \\ u'' \end{bmatrix} + \begin{bmatrix} k & 0 & +(1+\epsilon\cos\tau) \\ 0 & k & -(1+\epsilon\cos\tau) \\ 0 & 0 & +(1+\epsilon\cos\tau) \end{bmatrix} \begin{bmatrix} u_p \\ u_g \\ u \end{bmatrix} = 0 \tag{29}$$

where prime represent differentiation with respect to the new variable τ . Now by expanding the variables in the following power series

$$u_p = u_{0p} + u_{1p}\epsilon + u_{2p}\epsilon^2 + u_{3p}\epsilon^3 + \dots \tag{30}$$

$$u_g = u_{0g} + u_{1g}\epsilon + u_{2g}\epsilon^2 + u_{3g}\epsilon^3 + \dots \tag{31}$$

$$u = u_0 + u_1\epsilon + u_2\epsilon^2 + u_3\epsilon^3 + \dots \tag{32}$$

$$\Omega = \Omega_0 + \Omega_1\epsilon + \Omega_2\epsilon^2 + \Omega_3\epsilon^3 + \dots \tag{33}$$

substituting Eqs. (30)-(33) in Eq. (29), neglecting terms of $O(\epsilon^3)$, and collecting terms of the same power the following system of equations are obtained.

$$\Omega_0^2 \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u''_{0p} \\ u''_{0g} \\ u''_0 \end{bmatrix} + \begin{bmatrix} k & 0 & 1 \\ 0 & k & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{0p} \\ u_{0g} \\ u_0 \end{bmatrix} = 0 \quad (34)$$

$$\begin{aligned} & \Omega_0^2 \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u''_{1p} \\ u''_{1g} \\ u''_1 \end{bmatrix} + \begin{bmatrix} k & 0 & 1 \\ 0 & k & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{1p} \\ u_{1g} \\ u_1 \end{bmatrix} \\ & = -2\Omega_0\Omega_1 \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u''_{0p} \\ u''_{0g} \\ u''_0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} u_0 \cos \tau \end{aligned} \quad (35)$$

$$\begin{aligned} & \Omega_0^2 \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u''_{2p} \\ u''_{2g} \\ u''_2 \end{bmatrix} + \begin{bmatrix} k & 0 & 1 \\ 0 & k & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{2p} \\ u_{2g} \\ u_2 \end{bmatrix} = \\ & -(\Omega_1^2 + 2\Omega_0\Omega_2) \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u''_{0p} \\ u''_{0g} \\ u''_0 \end{bmatrix} \\ & - 2\Omega_0\Omega_1 \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u''_{1p} \\ u''_{1g} \\ u''_1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} u_1 \cos(\tau) \end{aligned} \quad (36)$$

Equation (34) is a linear homogenous system of equations with following solutions form.

$$\begin{bmatrix} u_{0p} \\ u_{0g} \\ u_0 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} e^{\lambda\tau} \quad (37)$$

By substituting Eq. (37) in Eq. (34) the corresponding eigenvalues and eigenvectors of the system are obtained as follow.

$$\lambda_1^2 = \frac{k}{m\Omega_0^2} i^2, \quad \lambda_2^2 = \frac{2k-a}{2m\Omega_0^2} i^2, \quad \lambda_3^2 = \frac{2k-b}{2m\Omega_0^2} i^2 \quad (38)$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ +2 \\ a \end{bmatrix}, \begin{bmatrix} -2 \\ +2 \\ b \end{bmatrix} \quad (39)$$

where constants a and b are a function of system parameters m and k , and i represents the imaginary unit, satisfying $i^2 = -1$.

$$a = -(m - k + 2) + \sqrt{(m + k + 2)^2 - 4km} \quad (40)$$

$$b = -(m - k + 2) - \sqrt{(m + k + 2)^2 - 4km} \quad (41)$$

For different values of k and m , the following condition is required to have real values for a and b .

$$(m + k + 2)^2 - 4km > 0 \quad (42)$$

In Fig. 3 the regions that Eq. (42) is satisfied is colored in blue, and the regions that it is not satisfied is colored in red. Since only positive values are allocated to k and m , therefore Eq. (42) is always satisfied and Eqs. (40) and (41) are real. So, the three eigenvectors represented by Eq. (39) are real for any value of k and m .

Fig. 3 Regions that Eq. (42) is satisfied

Besides, since the following equation are positive for any value of k and m .

$$\begin{aligned} 2k - a &= (m + k + 2) - \sqrt{(m + k + 2)^2 - 4km} \\ 2k - b &= (m + k + 2) + \sqrt{(m + k + 2)^2 - 4km} \end{aligned} \quad (43)$$

the three eigenvalues expressed by Eq. (38) are always complex. Therefore, for any value of the parameters, Eq. (34) has three natural frequencies.

$$\omega_{n1}^2 = \frac{k}{m\Omega_0^2}, \quad \omega_{n2}^2 = \frac{2k - a}{2m\Omega_0^2}, \quad \omega_{n3}^2 = \frac{2k - b}{2m\Omega_0^2} \quad (44)$$

such that the following relationship holds between them.

$$\omega_{n2}^2 \omega_{n3}^2 = \frac{k}{m\Omega_0^4} \quad (45)$$

$$\omega_{n1}^2 = \Omega_0^2 \omega_{n2}^2 \omega_{n3}^2 \quad (46)$$

By having the eigenvalues and eigenvectors of Eq. (34), one can write the solution of this equation in the following form.

$$\begin{aligned}
 \begin{bmatrix} u_{0p} \\ u_{0g} \\ u_0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \left(A_1 \cos(\omega_{n1}\tau) + A_2 \sin(\omega_{n1}\tau) \right) \\
 &+ \begin{bmatrix} -2 \\ +2 \\ a \end{bmatrix} \left(B_1 \cos(\omega_{n2}\tau) + B_2 \sin(\omega_{n2}\tau) \right) \\
 &+ \begin{bmatrix} -2 \\ +2 \\ b \end{bmatrix} \left(C_1 \cos(\omega_{n3}\tau) + C_2 \sin(\omega_{n3}\tau) \right)
 \end{aligned} \tag{47}$$

Substituting Eq. (47) in Eq. (35) results in the following equation.

$$\begin{aligned}
 \Omega_0^2 \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u''_{1p} \\ u''_{1g} \\ u''_1 \end{bmatrix} + \begin{bmatrix} k & 0 & 1 \\ 0 & k & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{1p} \\ u_{1g} \\ u_1 \end{bmatrix} = \\
 + 2\Omega_0\Omega_1 \begin{bmatrix} m\omega_{n1}^2 & -2m\omega_{n2}^2 & -2m\omega_{n3}^2 \\ m\omega_{n1}^2 & +2m\omega_{n2}^2 & +2m\omega_{n3}^2 \\ 0 & (a+4)\omega_{n2}^2 & (b+4)\omega_{n3}^2 \end{bmatrix} \begin{bmatrix} A_1 \cos(\omega_{n1}\tau) \\ B_1 \cos(\omega_{n2}\tau) \\ C_1 \cos(\omega_{n3}\tau) \end{bmatrix} \\
 + 2\Omega_0\Omega_1 \begin{bmatrix} m\omega_{n1}^2 & -2m\omega_{n2}^2 & -2m\omega_{n3}^2 \\ m\omega_{n1}^2 & +2m\omega_{n2}^2 & +2m\omega_{n3}^2 \\ 0 & (a+4)\omega_{n2}^2 & (b+4)\omega_{n3}^2 \end{bmatrix} \begin{bmatrix} A_2 \sin(\omega_{n1}\tau) \\ B_2 \sin(\omega_{n2}\tau) \\ C_2 \sin(\omega_{n3}\tau) \end{bmatrix} \\
 + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{a}{2} \left(B_1 \cos(\omega_{n2} - 1)\tau + B_2 \sin(\omega_{n2} - 1)\tau \right) \\
 + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{a}{2} \left(B_1 \cos(\omega_{n2} + 1)\tau + B_2 \sin(\omega_{n2} + 1)\tau \right) \\
 + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{b}{2} \left(C_1 \cos(\omega_{n3} - 1)\tau + C_2 \sin(\omega_{n3} - 1)\tau \right) \\
 + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{b}{2} \left(C_1 \cos(\omega_{n3} + 1)\tau + C_2 \sin(\omega_{n3} + 1)\tau \right)
 \end{aligned} \tag{48}$$

which contains harmonic functions with frequencies equal to

$$\omega_{n1}, \omega_{n2}, \omega_{n3}, \omega_{n2} \pm 1, \omega_{n3} \pm 1 \tag{49}$$

4.1 First Tongues

4.1.1 Primary Parametric Resonance Due to the Second Natural Frequency

In general, all the harmonic terms with frequencies equal to the first three terms in Eq. (48), the natural frequencies, are secular terms and must be removed. But for the following special case of the second natural frequency more terms become secular [9].

$$\omega_{n2} - 1 = -\omega_{n2} \rightarrow \omega_{n2} = \frac{1}{2} \quad (50)$$

Imposing this condition to Eq. (44) gives the emanating frequency of the first tongue related the second natural frequency.

$$\Omega_0^2 = \frac{2(2k - a)}{m} \quad (51)$$

Inserting Eq. (51) into Eqs. (45) and (46) provides the other two natural frequencies.

$$\omega_{n1}^2 = \frac{k}{2(2k - a)}, \quad \omega_{n2}^2 = \frac{1}{4}, \quad \omega_{n3}^2 = \frac{km}{(2k - a)^2} \quad (52)$$

By substituting Eq. (52) in Eq. (48), the following matrix form of the secular terms is obtained.

$$\begin{aligned} 2\Omega_0\Omega_1 \begin{bmatrix} m\omega_{n1}^2 & -2m\omega_{n2}^2 - \frac{a}{4\Omega_0\Omega_1} & -2m\omega_{n3}^2 \\ m\omega_{n1}^2 & 2m\omega_{n2}^2 + \frac{a}{4\Omega_0\Omega_1} & 2m\omega_{n3}^2 \\ 0 & (a+4)\omega_{n2}^2 - \frac{a}{4\Omega_0\Omega_1} & (b+4)\omega_{n3}^2 \end{bmatrix} \begin{bmatrix} A_1 \cos(\omega_{n1}\tau) \\ B_1 \cos(\omega_{n2}\tau) \\ C_1 \cos(\omega_{n3}\tau) \end{bmatrix} + \\ 2\Omega_0\Omega_1 \begin{bmatrix} m\omega_{n1}^2 & -2m\omega_{n2}^2 + \frac{a}{4\Omega_0\Omega_1} & -2m\omega_{n3}^2 \\ m\omega_{n1}^2 & 2m\omega_{n2}^2 - \frac{a}{4\Omega_0\Omega_1} & 2m\omega_{n3}^2 \\ 0 & (a+4)\omega_{n2}^2 + \frac{a}{4\Omega_0\Omega_1} & (b+4)\omega_{n3}^2 \end{bmatrix} \begin{bmatrix} A_2 \sin(\omega_{n1}\tau) \\ B_2 \sin(\omega_{n2}\tau) \\ C_2 \sin(\omega_{n3}\tau) \end{bmatrix} = 0 \end{aligned} \quad (53)$$

To remove these secular terms the determinant of each matrix in Eq. (53) must be equal to zero. These conditions provide the two first-order multipliers of Eq. (33) for the transition curves of the first tongue corresponded to the second natural frequency.

$$\Omega_1 = \pm \frac{a(b+2m+4)}{2m\Omega_0(a-b)} \quad (54)$$

In Eq. (54) the positive and negative sign are related to cosine and sine multipliers associated to the left and right transition curves, respectively. At this point, substituting the solution of Eq. (48) in Eq. (36) and removal of the secular terms provide the two second-order multipliers of Eq. (54) for the transition

curves of the first tongue corresponded to the second natural frequency. Zero-determinant condition for multipliers of both cosine and sine terms results in the following value for both left and right transition curves.

$$\Omega_2 = \frac{\Omega_0}{16m(a-b)\omega_{n1}^2} \left(b(a+2m+4)(\beta_3 + \beta_4)\omega_{n2}^2 - a(b+2m+4)(\beta_2)\omega_{n3}^2 \right) - \frac{\Omega_1^2}{2\Omega_0} \quad (55)$$

The value of β_i will be defined later in Eq. (72).

4.1.2 Primary Parametric Resonance Due to the Third Natural Frequency

Alternatively, for the following special case of the third natural frequency other terms in Eq. (48) become secular

$$\omega_{n3} - 1 = -\omega_{n3} \rightarrow \omega_{n3} = \frac{1}{2} \quad (56)$$

Imposing this condition to Eq. (44) gives the emanating frequency of the first tongue related the third natural frequency.

$$\Omega_0^2 = \frac{2(2k-b)}{m} \quad (57)$$

Inserting Eq. (57) into Eqs. (45) and (46) provides the other two natural frequencies.

$$\omega_{n1}^2 = \frac{k}{2(2k-b)}, \quad \omega_{n2}^2 = \frac{km}{(2k-b)^2}, \quad \omega_{n3}^2 = \frac{1}{4} \quad (58)$$

By substituting these three natural frequencies in Eq. (48), the following matrix form of the secular terms is obtained.

$$\begin{aligned} 2\Omega_0\Omega_1 \begin{bmatrix} m\omega_{n1}^2 & -2m\omega_{n2}^2 & -2m\omega_{n3}^2 - \frac{b}{4\Omega_0\Omega_1} \\ m\omega_{n1}^2 & 2m\omega_{n2}^2 & 2m\omega_{n3}^2 + \frac{b}{4\Omega_0\Omega_1} \\ 0 & (a+4)\omega_{n2}^2 & (b+4)\omega_{n3}^2 - \frac{b}{4\Omega_0\Omega_1} \end{bmatrix} \begin{bmatrix} A_1 \cos(\omega_{n1}\tau) \\ B_1 \cos(\omega_{n2}\tau) \\ C_1 \cos(\omega_{n3}\tau) \end{bmatrix} + \\ 2\Omega_0\Omega_1 \begin{bmatrix} m\omega_{n1}^2 & -2m\omega_{n2}^2 & -2m\omega_{n3}^2 + \frac{b}{4\Omega_0\Omega_1} \\ m\omega_{n1}^2 & 2m\omega_{n2}^2 & 2m\omega_{n3}^2 - \frac{b}{4\Omega_0\Omega_1} \\ 0 & (a+4)\omega_{n2}^2 & 2(b+4)\omega_{n3}^2 + \frac{b}{4\Omega_0\Omega_1} \end{bmatrix} \begin{bmatrix} A_2 \sin(\omega_{n1}\tau) \\ B_2 \sin(\omega_{n2}\tau) \\ C_2 \sin(\omega_{n3}\tau) \end{bmatrix} = 0 \end{aligned} \quad (59)$$

Similarly, the zero-determinant condition results in two first-order multipliers of Eq. (33) for the transition curves of the first tongue corresponded to the third natural frequency.

$$\Omega_1 = \mp \frac{b(a + 2m + 4)}{2m\Omega_0(a - b)} \quad (60)$$

In Eq. (60) the negative and positive sign are obtained from zero determinant condition of the cosine and sine terms which are associated to the left and right transition curves, respectively. At this point, substituting the solution of Eq. (48) in Eq. (36) and removal of the secular terms provide the two second-order multipliers of Eq. (33) for the transition curves of the first tongue corresponded to the third natural frequency. Zero determinant condition for multipliers of both cosine and sine terms results in the same value for the left and right transition curves.

$$\begin{aligned} \Omega_2 = \frac{\Omega_0}{16m(a - b)\omega_{n1}^2} & \left(b(a + 2m + 4)(\beta_4)\omega_{n2}^2 \right. \\ & \left. - a(b + 2m + 4)(\beta_1 + \beta_2)\omega_{n3}^2 \right) - \frac{\Omega_1^2}{2\Omega_0} \end{aligned} \quad (61)$$

The value of β_i will be defined later in Eq. (72).

4.1.3 Combined Parametric Resonance of Summation Type

For the following special case of the second and third natural frequency more terms in Eq. (48) become secular.

$$\begin{aligned} \omega_{n2} - 1 &= -\omega_{n3} \rightarrow \omega_{n2} + \omega_{n3} = 1 \\ \omega_{n3} - 1 &= -\omega_{n2} \rightarrow \omega_{n2} + \omega_{n3} = 1 \end{aligned} \quad (62)$$

Imposing this condition to Eq. (44) gives the emanating frequency of the first tongue related the combined parametric resonance.

$$\Omega_0^2 = \frac{(m + k + 2)}{m} + 2\sqrt{\frac{k}{m}} \quad (63)$$

Figure 4 shows the value of the natural frequencies for different normalized stiffness ratios resulted from substituting Eq. (63) in Eq. (44).

Fig. 4 Natural frequencies at combined parametric resonance, for $m = 1$

which shows that for the emanating frequency provided by Eq. (63), the summation of the second and third natural frequency is always equal to one. Imposing Eq. (62) to Eq. (48) results in the following matrix form of the secular terms.

$$\begin{aligned}
 2\Omega_0\Omega_1 \begin{bmatrix} m\omega_{n1}^2 & -2m\omega_{n2}^2 & -2m\omega_{n3}^2 & -\frac{b}{2} - \frac{a}{2} \\ m\omega_{n1}^2 & 2m\omega_{n2}^2 & 2m\omega_{n3}^2 & +\frac{b}{2} + \frac{a}{2} \\ 0 & (a+4)\omega_{n2}^2 & (b+4)\omega_{n3}^2 & -\frac{b}{2} - \frac{a}{2} \end{bmatrix} \begin{bmatrix} A_1 \cos(\omega_{n1}\tau) \\ B_1 \cos(\omega_{n2}\tau) \\ C_1 \cos(\omega_{n3}\tau) \\ C_1 \cos(\omega_{n2}\tau) \\ B_1 \cos(\omega_{n3}\tau) \end{bmatrix} + \\
 2\Omega_0\Omega_1 \begin{bmatrix} m\omega_{n1}^2 & -2m\omega_{n2}^2 & -2m\omega_{n3}^2 & +\frac{b}{2} + \frac{a}{2} \\ m\omega_{n1}^2 & 2m\omega_{n2}^2 & 2m\omega_{n3}^2 & -\frac{b}{2} - \frac{a}{2} \\ 0 & (a+4)\omega_{n2}^2 & (b+4)\omega_{n3}^2 & +\frac{b}{2} + \frac{a}{2} \end{bmatrix} \begin{bmatrix} A_2 \sin(\omega_{n1}\tau) \\ B_2 \sin(\omega_{n2}\tau) \\ C_2 \sin(\omega_{n3}\tau) \\ C_2 \sin(\omega_{n2}\tau) \\ B_2 \sin(\omega_{n3}\tau) \end{bmatrix} = 0
 \end{aligned} \tag{64}$$

4.2 Second Tongues

To find the second tongues for second and third natural frequencies and combined resonance, the general form of Eq. (48) must be solved, where the following conditions hold.

$$\omega_{n2} \neq \frac{1}{2}, \quad \omega_{n3} \neq \frac{1}{2}, \quad \omega_{n2} + \omega_{n3} \neq 1 \tag{65}$$

In such a condition, for every value of k and m the determinant of the secular terms is nonzero, and the only way to remove them is the following condition.

$$\Omega_1 = 0 \quad (66)$$

This meanse that for all the higher tongues the first order multiplier of the transition curve expressed by Eq. (33) is zero. Besides, imposing Eq. (66) to Eq. (48) results in the following nonhomoginious linear system of equations.

$$\begin{aligned} \Omega_0^2 \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u''_{1p} \\ u''_{1g} \\ u''_1 \end{bmatrix} + \begin{bmatrix} k & 0 & 1 \\ 0 & k & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{1p} \\ u_{1g} \\ u_1 \end{bmatrix} = \\ + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{a}{2} \left(B_1 \cos(\omega_{n2} - 1)\tau + B_2 \sin(\omega_{n2} - 1)\tau \right) \\ + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{a}{2} \left(B_1 \cos(\omega_{n2} + 1)\tau + B_2 \sin(\omega_{n2} + 1)\tau \right) \quad (67) \\ + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{b}{2} \left(C_1 \cos(\omega_{n3} - 1)\tau + C_2 \sin(\omega_{n3} - 1)\tau \right) \\ + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{b}{2} \left(C_1 \cos(\omega_{n3} + 1)\tau + C_2 \sin(\omega_{n3} + 1)\tau \right) \end{aligned}$$

The homoginous and particular solutions of Eq. (67) are in the following form.

$$\begin{bmatrix} u_{1p} \\ u_{1g} \\ u_1 \end{bmatrix}_H = \begin{bmatrix} E \\ F \\ G \end{bmatrix} e^{\lambda\tau} \quad (68)$$

$$\begin{bmatrix} u_{1p} \\ u_{1g} \\ u_1 \end{bmatrix}_P = \begin{bmatrix} U_p \\ U_g \\ U \end{bmatrix} e^{Di\tau} \quad (69)$$

Such that i represents the imaginary unit, satisfying the equation $i^2 = -1$. Consequently, the general solution of Eq. (67) is obtained.

$$\begin{aligned}
\begin{bmatrix} u_{1p} \\ u_{1g} \\ u_1 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \left(E_1 \cos(\omega_{n1}\tau) + E_2 \sin(\omega_{n1}\tau) \right) \\
&+ \begin{bmatrix} -2 \\ 2 \\ a \end{bmatrix} \left(F_1 \cos(\omega_{n2}\tau) + F_2 \sin(\omega_{n2}\tau) \right) \\
&+ \begin{bmatrix} -2 \\ 2 \\ b \end{bmatrix} \left(G_1 \cos(\omega_{n3}\tau) + G_2 \sin(\omega_{n3}\tau) \right) \\
&+ \begin{bmatrix} \alpha_1 \\ -\alpha_1 \\ \beta_1 \end{bmatrix} \frac{a}{2} \left(B_1 \cos(\omega_{n2} - 1)\tau + B_2 \sin(\omega_{n2} - 1)\tau \right) \\
&+ \begin{bmatrix} \alpha_2 \\ -\alpha_2 \\ \beta_2 \end{bmatrix} \frac{a}{2} \left(B_1 \cos(\omega_{n2} + 1)\tau + B_2 \sin(\omega_{n2} + 1)\tau \right) \\
&+ \begin{bmatrix} \alpha_3 \\ -\alpha_3 \\ \beta_3 \end{bmatrix} \frac{b}{2} \left(C_1 \cos(\omega_{n3} - 1)\tau + C_2 \sin(\omega_{n3} - 1)\tau \right) \\
&+ \begin{bmatrix} \alpha_4 \\ -\alpha_4 \\ \beta_4 \end{bmatrix} \frac{b}{2} \left(C_1 \cos(\omega_{n3} + 1)\tau + C_2 \sin(\omega_{n3} + 1)\tau \right)
\end{aligned} \tag{70}$$

where

$$\alpha_j = \frac{\Omega_0^2 D_j^2}{(k - m\Omega_0^2 D_j^2)(1 - \Omega_0^2 D_j^2) - 2\Omega_0^2 D_j^2} \tag{71}$$

$$\beta_j = \frac{(m + 2)\Omega_0^2 D_j^2 - k}{(k - m\Omega_0^2 D_j^2)(1 - \Omega_0^2 D_j^2) - 2\Omega_0^2 D_j^2} \tag{72}$$

$j = 1, 2, 3, 4$

and

$$\begin{aligned}
D_1 &= (\omega_{n2} - 1), & D_2 &= (\omega_{n2} + 1) \\
D_3 &= (\omega_{n3} - 1), & D_4 &= (\omega_{n3} + 1)
\end{aligned} \tag{73}$$

Substituting Eqs. (47) and (70) in Eq. (36) results in

$$\begin{aligned}
\Omega_0^2 \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_{2p}'' \\ u_{2g}'' \\ u_2'' \end{bmatrix} + \begin{bmatrix} k & 0 & 1 \\ 0 & k & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{2p} \\ u_{2g} \\ u_2 \end{bmatrix} = \\
+ \begin{bmatrix} 2m\Omega_0\Omega_2\omega_{n1}^2 & -4m\Omega_0\Omega_2\omega_{n2}^2 - \frac{a}{4}(\beta_1 + \beta_2) & -4m\Omega_0\Omega_2\omega_{n3}^2 - \frac{b}{4}(\beta_3 + \beta_4) \\ 2m\Omega_0\Omega_2\omega_{n1}^2 & +4m\Omega_0\Omega_2\omega_{n2}^2 + \frac{a}{4}(\beta_1 + \beta_2) & +4m\Omega_0\Omega_2\omega_{n3}^2 + \frac{b}{4}(\beta_3 + \beta_4) \\ 0 & 2\Omega_0\Omega_2(a+4)\omega_{n2}^2 - \frac{a}{4}(\beta_1 + \beta_2) & 2\Omega_0\Omega_2(b+4)\omega_{n2}^3 - \frac{b}{4}(\beta_3 + \beta_4) \end{bmatrix} \begin{bmatrix} A_1\cos(\omega_{n1}\tau) \\ B_1\cos(\omega_{n2}\tau) \\ C_1\cos(\omega_{n3}\tau) \end{bmatrix} \\
+ \begin{bmatrix} 2m\Omega_0\Omega_2\omega_{n1}^2 & -4m\Omega_0\Omega_2\omega_{n2}^2 - \frac{a}{4}(\beta_1 + \beta_2) & -4m\Omega_0\Omega_2\omega_{n3}^2 - \frac{b}{4}(\beta_3 + \beta_4) \\ 2m\Omega_0\Omega_2\omega_{n1}^2 & +4m\Omega_0\Omega_2\omega_{n2}^2 + \frac{a}{4}(\beta_1 + \beta_2) & +4m\Omega_0\Omega_2\omega_{n3}^2 + \frac{b}{4}(\beta_3 + \beta_4) \\ 0 & 2\Omega_0\Omega_2(a+4)\omega_{n2}^2 - \frac{a}{4}(\beta_1 + \beta_2) & 2\Omega_0\Omega_2(b+4)\omega_{n2}^3 - \frac{b}{4}(\beta_3 + \beta_4) \end{bmatrix} \begin{bmatrix} A_2\sin(\omega_{n1}\tau) \\ B_2\sin(\omega_{n2}\tau) \\ C_2\sin(\omega_{n3}\tau) \end{bmatrix} \\
+ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{a}{2} \left(F_1\cos(\omega_{n2} - 1)\tau + F_2\sin(\omega_{n2} - 1)\tau \right) + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{a}{2} \left(F_1\cos(\omega_{n2} + 1)\tau + F_2\sin(\omega_{n2} + 1)\tau \right) \\
+ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{b}{2} \left(G_1\cos(\omega_{n3} - 1)\tau + G_2\sin(\omega_{n3} - 1)\tau \right) + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{b}{2} \left(G_1\cos(\omega_{n3} + 2)\tau + G_2\sin(\omega_{n3} + 2)\tau \right) \\
+ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{a}{4} \beta_1 \left(B_1\cos(\omega_{n2} - 2)\tau + B_2\sin(\omega_{n2} - 2)\tau \right) + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{a}{4} \beta_2 \left(B_1\cos(\omega_{n2} + 2)\tau + B_2\sin(\omega_{n2} + 2)\tau \right) \\
+ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{b}{4} \beta_3 \left(C_1\cos(\omega_{n3} - 2)\tau + C_2\sin(\omega_{n3} - 2)\tau \right) + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \frac{b}{4} \beta_4 \left(C_1\cos(\omega_{n3} + 2)\tau + C_2\sin(\omega_{n3} + 2)\tau \right)
\end{aligned} \tag{74}$$

which contains harmonic functions with frequencies equal to

$$\omega_{n1}, \omega_{n2}, \omega_{n3}, \omega_{n2} \pm 1, \omega_{n3} \pm 1, \omega_{n2} \pm 2, \omega_{n3} \pm 2 \tag{75}$$

4.2.1 Primary Parametric Resonance Due to the Second Natural Frequency

Like before, the harmonic terms with frequencies equal to the first three terms in Eq. (74) are secular and must be removed. But for the following special case of the second natural frequency more terms in Eq. (74) become secular.

$$\omega_{n2} - 2 = -\omega_{n2} \rightarrow \omega_{n2} = 1 \tag{76}$$

Imposing This condition to Eq. (44) provides the emanating frequency of the second tongue corresponded to the second natural frequency.

$$\Omega_0^2 = \frac{2k - a}{2m} \tag{77}$$

By inserting Eq. (77) in Eqs. (45) and (46) the other two natural frequencies are obtained.

$$\omega_{n1}^2 = \frac{2k}{2k - a}, \omega_{n2}^2 = 1, \omega_{n3}^2 = \frac{4km}{(2k - a)^2} \tag{78}$$

By substituting these three natural frequencies in Eq. (74), collecting the secular terms, and imposing the zero-determinant condition to secular terms, one can find the second-order multipliers of the left and right transition curves for the second tongue corresponded to the second natural frequency.

$$\Omega_2 = \frac{\Omega_0}{16m(b-a)\omega_{n1}^2} \left(a(2m+b+4)(2\beta_1 + \beta_2)\omega_{n3}^2 - b(2m+a+4)(\beta_3 + \beta_4)\omega_{n2}^2 \right) \quad (79)$$

$$\Omega_2 = \frac{\Omega_0}{16m(b-a)\omega_{n1}^2} \left(a(2m+b+4)(\beta_2)\omega_{n3}^2 - b(2m+a+4)(\beta_3 + \beta_4)\omega_{n2}^2 \right) \quad (80)$$

4.2.2 Primary Parametric Resonance Due to the Third Natural Frequency

Alternatively, for the following special case of the third natural frequency more terms in Eq. (74) become secular.

$$\omega_{n3} - 2 = -\omega_{n3} \rightarrow \omega_{n3} = 1 \quad (81)$$

Imposing this condition to Eq. (44) provides the emanating frequency of the second tongue corresponded to the third natural frequency.

$$\Omega_0^2 = \frac{2k-b}{2m} \quad (82)$$

By inserting Eq. (82) in Eqs. (45) and (46) the other two natural frequencies are obtained.

$$\omega_{n1}^2 = \frac{2k}{2k-b}, \quad \omega_{n2}^2 = \frac{4km}{(2k-b)^2}, \quad \omega_{n3}^2 = 1 \quad (83)$$

By substituting these three natural frequencies in Eq. (74), collecting the secular terms, and imposing the zero-determinant condition to the multipliers of the cosine and sine terms, the second order multiplier of the transition curves for the second tongue corresponded to the third natural frequency are obtained.

$$\Omega_2 = \frac{\Omega_0}{16m(b-a)\omega_{n1}^2} \left(a(2m+b+4)(\beta_1 + \beta_2)\omega_{n3}^2 - b(2m+a+4)(2\beta_3 + \beta_4)\omega_{n2}^2 \right) \quad (84)$$

$$\Omega_2 = \frac{\Omega_0}{16m(b-a)\omega_{n1}^2} \left(a(2m+b+4)(\beta_1 + \beta_2)\omega_{n3}^2 - b(2m+a+4)(\beta_4)\omega_{n2}^2 \right) \quad (85)$$

4.2.3 Combined Parametric Resonance of Summation Type

For the following special case of the second and third natural frequency more terms in Eq. (74) become secular.

$$\begin{aligned}\omega_{n2} - 2 &= -\omega_{n3} \rightarrow \omega_{n2} + \omega_{n3} = 2 \\ \omega_{n3} - 2 &= -\omega_{n2} \rightarrow \omega_{n2} + \omega_{n3} = 2\end{aligned}\quad (86)$$

Imposing this condition to Eq. (44) gives the emanating frequency of the second tongue related the combined parametric resonance.

$$\Omega_0^2 = \frac{1}{4} \left(\frac{(m+k+2)}{m} + 2\sqrt{\frac{k}{m}} \right) \quad (87)$$

4.3 General Formulas

In general, the emanating frequencies and the corresponded natural frequencies related to the second natural frequency can be obtained by the following general equations.

$$\begin{aligned}\omega_{n2} - n &= -\omega_{n2} \rightarrow \omega_{n2} = \frac{n}{2} \\ \Omega_0^2 &= \frac{2(2k-a)}{n^2m} \\ \omega_{n1}^2 &= \frac{n^2k}{2(2k-a)}, \quad \omega_{n2}^2 = \frac{n^2}{4}, \quad \omega_{n3}^2 = \frac{n^2km}{(2k-a)^2} \\ n &= 1, 2, 3, \dots\end{aligned}\quad (88)$$

Similarly, the general form of the emanating frequencies and the corresponded natural frequencies related to the third natural frequency can be obtained by the following equations.

$$\begin{aligned}\omega_{n3} - n &= -\omega_{n3} \rightarrow \omega_{n3} = \frac{n}{2} \\ \Omega_0^2 &= \frac{2(2k-b)}{n^2m} \\ \omega_{n1}^2 &= \frac{n^2k}{2(2k-b)}, \quad \omega_{n2}^2 = \frac{n^2km}{(2k-b)^2}, \quad \omega_{n3}^2 = \frac{n^2}{4} \\ n &= 1, 2, 3, \dots\end{aligned}\quad (89)$$

Finally, the general form of the emanating frequencies for the combined parametric resonance can be obtained by the following equations.

$$\begin{aligned}
\omega_{n2} - n &= -\omega_{n3} \rightarrow \omega_{n2} + \omega_{n3} = n \\
\omega_{n3} - n &= -\omega_{n2} \rightarrow \omega_{n2} + \omega_{n3} = n \\
\Omega_0^2 &= \frac{1}{n^2} \left(\frac{(m+k+2)}{m} + 2\sqrt{\frac{k}{m}} \right) \\
n &= 1, 2, 3, \dots
\end{aligned} \tag{90}$$

Figure 5 demonstrate all the unstable tongues in the parametric frequency space where the red and orange tongues correspond to the second and third natural frequencies, respectively, and the green tongues correspond to the combined parametric resonance. The analytical calculations in this section clearly shows that the first natural frequency does not cause any parametric resonance.

Fig. 5 Unstable tongues corresponded to the second, third and combined natural frequencies

5 Floquet Theory

In this section it is shown how Floquet theory is used to verify the analytical results. To this end, the governing undamped homogenous differential equations of the three DOF model is written in the state space form as a system of first-order ordinary differential equations.

$$\dot{x} = A(t)x \tag{91}$$

The corresponding fundamental matrix of Eq. (91) is defined by the fundamental system of solutions where each column of this matrix is a linearly independent solution for Eq. (91).

$$X(t) = \begin{bmatrix} x_{11}(t) & x_{21}(t) & \dots & x_{n1}(t) \\ x_{12}(t) & x_{22}(t) & \dots & x_{n2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n}(t) & x_{2n}(t) & \dots & x_{nn}(t) \end{bmatrix} \quad (92)$$

Besides, Eq. (91) is a system of linear differential equations with a time periodic coefficient matrix, and T is the minimal period of coefficient matrix as it is the smallest constant positive number satisfying the following condition.

$$A(t + T) = A(t) \quad (93)$$

The main idea of the Floquet theory is that for a linear time periodic system, there is a transition matrix C that maps the states of the system from a particular time to the state of the system after one period.

$$X(t + T) = X(t)C \quad (94)$$

Such that C is a nonsingular constant matrix known as Monodromy matrix and according to the reducibility of the linear periodic systems it contains complete information about the system. For a 6×6 system, the Monodromy matrix can be constructed by integrating the equations of motion for six linearly independent initial conditions. By choosing the identity matrix as the set of linear independent initial conditions at time $t = 0$

$$X(0) = I_n \quad (95)$$

Eq. (94) simplify to the following solution space.

$$C = X(T) \quad (96)$$

Consequently, numerical methods can be used to integrate Eq. (91) with initial conditions expressed by Eq. (95) for one period of oscillation. Therefore, the following Monodromy matrix is constructed out of the six solution vectors.

$$C = \begin{bmatrix} x_{11}(T) & x_{21}(T) & x_{31}(T) & x_{41}(T) & x_{51}(T) & x_{61}(T) \\ x_{12}(T) & x_{22}(T) & x_{32}(T) & x_{42}(T) & x_{52}(T) & x_{62}(T) \\ x_{13}(T) & x_{23}(T) & x_{33}(T) & x_{43}(T) & x_{53}(T) & x_{63}(T) \\ x_{14}(T) & x_{24}(T) & x_{34}(T) & x_{44}(T) & x_{54}(T) & x_{64}(T) \\ x_{15}(T) & x_{25}(T) & x_{35}(T) & x_{45}(T) & x_{55}(T) & x_{65}(T) \\ x_{16}(T) & x_{26}(T) & x_{36}(T) & x_{46}(T) & x_{56}(T) & x_{66}(T) \end{bmatrix} \quad (97)$$

According to Floquet Theory, Eq. (91) is stable if all the eigenvalues of the Monodromy matrix, known as Floquet Multipliers, lie on the unit circle [6]. The Monodromy matrix for this system has six eigenvalues, which are demonstrate in Figs 6-9. Figures 6 represent the real, imaginary, and absolute values of all the eigenvalues for different values of the parametric frequency, and Figs 7-9 represent the root locus of the eigenvalues on the unit circle only for the unstable cases.

Fig. 6 Floquet Multipliers

Fig. 7 Root locus of unstable cases corresponded to the second natural frequency

Fig. 8 Root locus of unstable cases corresponded to the third natural frequency

Fig. 9 Root locus of unstable cases corresponded to the combined resonance

Figure 6 show that two of the eigenvalues are always on the unit circle and do not cause instability. These two eigenvalues correspond to the first natural frequency of the system. This is while, for some parameters the other four eigenvalues leave the unit circle. Figures 7 and 8 show that there are cases that only two of these eigenvalues are real and one of them fall outside of the unit circle. These cases of instability correspond to the second and third natural frequencies. Figure 9 show that there are cases that none of the eigenvalues are real and two of them fall outside the unit circle. These cases of instability correspond to the combined parametric frequency.

6 Simulation

In this section, the fourth-fifth order Runge-Kutta numerical integration algorithm is used to examine the accuracy of the analytical results.

6.1 Stability Chart

Figures 10-15 show the first two unstable tongues corresponded to the second and third natural frequencies and combined parametric resonance where the stiffness of the suspension is increased gradually. In these plots the colored area is obtained by Floquet theory such that the red areas correspond to the second natural frequency, the orange areas correspond to the third natural frequency, and the green regions correspond to parametric combination

resonances. The black solid lines are plotted by using the analytical formula obtained by Poincare-Lindstedt method.

Fig. 10 Stability chart of three DOF model for $k = 0.5$

Fig. 11 Stability chart of three DOF model for $k = 1$

Fig. 12 Stability chart of three DOF model for $k = 5$

Fig. 13 Stability chart of three DOF model for $k = 10$

Fig. 14 Stability chart of three DOF model for $k = 15$

Fig. 15 Stability chart of three DOF model for $k = 20$

Figures 10-15 show a good agreement between the analytical and numerical calculations for both models. Such that by increase in the value of the stiffness, all the unstable tongues shift toward higher frequencies while the width of the tongues corresponded to the second natural frequency increase and the width of the tongues corresponded to the third natural frequency and combined parametric frequency decrease. To have a better sense about the location and width of the unstable tongues, Eqs. (51), (57), (60) are used to plot the value of the emanating frequencies of the first tongues in Fig. 16. Eqs. (54) and (60) are used to plot the slope of the second and third natural frequencies in Fig. 17.

Fig. 16 Emanating frequency of the first tongues

Fig. 17 Slope of the first tongues

Figures 16 and 17 show that by choosing relatively high values for the stiffness of the suspension, the emanating frequency and the slop of the first unstable tongue related to the second natural frequency both merge to constant values. This is while the emanating frequency and the slop of the first unstable tongue related to the third natural frequency and combined parametric frequency merge to infinity and zero respectively. This means that for high values of the stiffness for the suspension, when the mountings can be assumed rigid, the unstable tongues corresponded to the third natural frequency and combined parametric resonance vanish.

6.2 Time response

Figures 18-21 demonstrate the effect of changing the average transmitting force on the time response of the system. Figure 18 show that for low values of the transmitting force, the system enter the dead zoon area, so the permanent contact condition is violated, and the linear model is not valid. In such a condition the effect of the backlash and impact phase must be considered. This is while, Fig 19 show that for the larger values of the transmitting force the time response of system shifts up and remains outside of the dead zoon area. In such a condition, the stability of the system can be determined by the stability chart demonstrated in Fig. 15. Figure 20 show that by applying larger transmitting force, the response of the system gets more distant from the dead zone area which make system more robust toward any kind of external disturbances. Figure 21 show that even for large values of the transmitting force if the system parameters is not design properly the parametric resonance cause increase in the amplitude of the vibration, entering into the dead zone area, and violation of the permanent contact regime.

Fig. 18 Time response for $\zeta = 0.05, k = 20, k_0 = 0.3, F = 1, \Omega = 1.5$

Fig. 19 Time response for $\zeta = 0.05, k = 20, k_0 = 0.3, F = 2, \Omega = 1.5$

Fig. 20 Time response for $\zeta = 0.05, k = 20, k_0 = 0.3, F = 3, \Omega = 1.5$

Fig. 21 Time response for $\zeta = 0.05, k = 20, k_0 = 0.3, F = 3, \Omega = 1.9$

7 Conclusion

The outline of this work is to investigate the effect of the deformation of the mountings on the dynamic behavior of the geared transmission systems. This is done with the assumption that the system operates under constant angular velocity and relatively large loads. In such a condition for small amplitude parametric oscillation the gear pair in mesh remain in permanent contact regime which is the desired operational condition for every geared system. This is while for large amplitude parametric oscillation, the separation of teeth, free play mode, and impact phase happens, which results is noise, excessive vibrations, and chaotic behaviors. For every linear periodic time varying system the large amplitude oscillation is mainly associated to the parametric resonance, therefore Poincare-Lindstedt method and Floquet theory is used to find the parameter space that primary and combined parametric resonance do not happen and the response of the system remain bounded. Consequently, numerical analysis show that the model with linear suspension has a total of six Floquet multipliers. Two of these multipliers remain inside the unit circle for every parametric frequency and do not cause instability which correspond to the first natural frequency obtained by analytical calculation. According to the numerical results, the instability of the system is related to the other four Floquet multipliers which correspond to the second and third natural frequencies, obtained by the analytical calculation. There are frequencies that two of these multipliers are real and one of them falls outside the unit circle, which is associated to the parametric primary resonances. This is while for some other frequencies, all four multipliers are complex when two of them fall outside the unit circle which is associated to the parametric combination resonances of summation type. Finally, the results show for large values of stiffness, the primary resonance tongue corresponded to the second natural frequency expand and become the main source of instability. This is while, the primary resonance tongue corresponded to the third natural frequency and combined parametric resonance shift toward higher frequencies and get narrower. However, they still exist and for high operational frequencies can cause large amplitude oscillation and separation of the tooth in mesh.

8 Declarations

Funding: this study is not funded by any organization or company.

Conflicts of interest/Competing interests: the author declare that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Availability of data and material: Not applicable.

Code availability: Not applicable.

Authors' contributions (optional)

Ethics approval: Not applicable.

Consent to participate: the Author warrants that the Work has not been published before in any form except as a preprint, that the Work is not being concurrently submitted to and is not under consideration by another publisher, that the persons listed above are listed in the proper order and that no author entitled to credit has been omitted.

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