A relaxed Kaczmarz method for fuzzy linear systems

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Abstract

A relaxed Kaczmarz method is presented for solving a class of fuzzy linear systems of equations with crisp coefficient matrix and fuzzy right-hand side. The iterative scheme is established and the convergence theorem is provided. Numerical examples show that the method is effective.

Key words: Fuzzy linear system; relaxed Kaczmarz method; iterative scheme

1 Introduction

Fuzzy linear systems (FLSs) occur in many fields, such as control problems, information, physics, statistics, engineering, economics, finance and even social sciences [12]. Thus, it is significant to study the numerical methods for solving FLSs.

A general model in [12] was proposed with embedding technique by Friedman et al. for solving a class of $n \times n$ FLSs

$$Ax = y, \quad (1.1)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is a crisp matrix, $y = [y_1, y_2, \dots, y_n]^T$ is a fuzzy vector, and $x = [x_1, x_2, \dots, x_n]^T$ is unknown. With this model, many numerical methods [1–7, 9–11, 13, 14, 16, 17, 19–22] was developed to solve FLS (1.1).

The Kaczmarz algorithm is a popular iterative projection method [23] as it is simple to implement and suitable for parallel computing. Numbers of authors studied Kaczmarz methods for solving linear systems [8, 15, 18, 23]. In this paper, a relaxed Kaczmarz algorithm is presented for fuzzy linear system (1.1).

The rest of the paper is organized as follows. Section 2 gives some basic definitions and results of FLS. In Section 3, the relaxed Kaczmarz method is established with convergence theorem. Two numerical examples in Section 4 are discussed and the conclusion is in Section 5.

2 Preliminaries

Generally, following [12], a fuzzy number is a pair of $(\underline{u}(r), \bar{u}(r))$, $0 \leq r \leq 1$, satisfying,

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- $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0, 1]$,
- $\bar{u}(r)$ is a bounded left continuous nonincreasing function over $[0, 1]$,
- $\underline{u}(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$.

The arithmetic operations of arbitrary fuzzy numbers $x = (\underline{x}(r), \bar{x}(r))$, $y = (\underline{y}(r), \bar{y}(r))$, $0 \leq r \leq 1$, and real number k , are as follows,

- (1) $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\bar{x}(r) = \bar{y}(r)$,
- (2) $x + y = (\underline{x}(r) + \underline{y}(r), \bar{x}(r) + \bar{y}(r))$, and
- (3) $kx = \begin{cases} (k\underline{x}(r), k\bar{x}(r)), & k \geq 0, \\ (k\bar{x}(r), k\underline{x}(r)), & k < 0. \end{cases}$

Definition 2.1. [12] A fuzzy number vector $X = (x_1, x_2, \dots, x_n)^T$ given by

$$x_i = (\underline{x}_i(r), \bar{x}_i(r)), \quad 1 \leq i \leq n, \quad 0 \leq r \leq 1,$$

is called a solution of the fuzzy linear system (1.1) if

$$\begin{cases} \sum_{j=1}^n a_{ij}x_j = \sum_{j=1}^n \underline{a_{ij}x_j} = \underline{y_i}, \\ \sum_{j=1}^n \overline{a_{ij}x_j} = \sum_{j=1}^n \overline{a_{ij}x_j} = \bar{y_i}. \end{cases} \quad (2.1)$$

By (2.1), Friedman et al. [12] extend FLS (1.1) to a $2n \times 2n$ crisp linear system

$$SX = Y \quad (2.2)$$

where $S = (s_{kl})$, s_{kl} are determined as follows

$$\begin{aligned} a_{ij} \geq 0 &\Rightarrow s_{ij} = a_{ij}, & s_{n+i, n+j} &= a_{ij}, \\ a_{ij} < 0 &\Rightarrow s_{i, n+j} = a_{ij}, & s_{n+i, j} &= a_{ij}, \end{aligned} \quad 1 \leq i, j \leq n,$$

and any s_{kl} which is not determined by the above items is zero, $1 \leq k, l \leq 2n$, and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix}, \quad Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{bmatrix}.$$

Further, the matrix S has the structure $\begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix}$, $A = S_1 + S_2$, and (2.2) can be rewritten as

$$\begin{cases} S_1 \underline{X} + S_2 \bar{X} = \underline{Y}, \\ S_2 \underline{X} + S_1 \bar{X} = \bar{Y}, \end{cases} \quad (2.3)$$

where

$$\underline{X} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \end{bmatrix}, \quad \bar{X} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}, \quad \underline{Y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_n \end{bmatrix}, \quad \bar{Y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_n \end{bmatrix}.$$

A theorem as follows indicates when FLS (1.1) has a unique solution.

Theorem 2.2. *The matrix S is nonsingular if and only if the matrices $A = S_1 + S_2$ and $S_1 - S_2$ are both nonsingular. See [12].*

In the next section, the proposed relaxed Kaczmarz method is presented for nonsingular FLS (1.1).

3 The relaxed Kaczmarz method

For nonsingular fuzzy linear system (2.2) or (2.3), a relaxed Kaczmarz iterative scheme can be described as follows,

$$X_{k+1} = X_k + \alpha \frac{Y^{(i_k)} - S^{(i_k)} X_k}{\|S^{(i_k)}\|_2^2} \left(S^{(i_k)}\right)^T, \quad (3.1)$$

and can be implemented as the following algorithm, where $0 < \alpha < 2$ is a relaxation parameter.

RELAXED KACZMARZ ALGORITHM. *Given initial vectors $\underline{X}_0, \bar{X}_0 \in \mathbb{R}^n$, for $k = 0, 1, 2, \dots$, the following iterative scheme is taken,*

$$\begin{cases} \underline{X}_{k+1} = \underline{X}_k + \alpha \frac{(\underline{Y} - S_2 \bar{X}_k)^{(i_k)} - S_1^{(i_k)} \underline{X}_k}{\|S_1^{(i_k)}\|_2^2} \left(S_1^{(i_k)}\right)^T, \\ \bar{X}_{k+1} = \bar{X}_k + \alpha \frac{(\bar{Y} - S_2 \underline{X}_{k+1})^{(i_k)} - S_1^{(i_k)} \bar{X}_k}{\|S_1^{(i_k)}\|_2^2} \left(S_1^{(i_k)}\right)^T, \end{cases} \quad (3.2)$$

where $i_k = (k \bmod n) + 1$, $(\cdot)^{(i_k)}$ denotes the i_k th row of a matrix.

The convergence result for method (3.2) is as the following theorem.

Theorem 3.1. *Suppose that fuzzy linear system (2.2) or (2.3) is consistent. Then the iterative sequence $\left\{ \begin{bmatrix} \underline{X}_k \\ \bar{X}_k \end{bmatrix} \right\}_{k=0}^{\infty}$, generated by the relaxed Kaczmarz method (3.2) starting from an initial guess $\begin{bmatrix} \underline{X}_0 \\ \bar{X}_0 \end{bmatrix}$ with $\underline{X}_0$ and $\bar{X}_0$ in the column space of S_2 , converges to the unique solution $\begin{bmatrix} \underline{X}_* \\ \bar{X}_* \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix}^{-1} \begin{bmatrix} \underline{Y} \\ \bar{Y} \end{bmatrix}$ of (2.2). Moreover, the solution error for the iteration sequence is*

$$\|X_{k+1} - X_*\|_2^2 \leq \left(1 - \alpha(2 - \alpha) \frac{\lambda_{\min}(S^T S)}{\|S\|_F^2}\right) \|X_k - X_*\|_2^2,$$

where $\lambda_{\min}(\cdot)$ is the smallest nonzero eigenvalues of a matrix, $X_k = \begin{bmatrix} \underline{X}_k \\ \bar{X}_k \end{bmatrix}$, and $X_* = \begin{bmatrix} \underline{X}_* \\ \bar{X}_* \end{bmatrix}$.

Proof. Take $P^{(i_k)} = \left(\frac{S^{(i_k)}}{\|S^{(i_k)}\|_2} \right)^T \left(\frac{S^{(i_k)}}{\|S^{(i_k)}\|_2} \right)$, thus

$$\begin{aligned} X_{k+1} - X_* &= X_k - X_* + \alpha \frac{Y^{(i_k)} - S^{(i_k)} X_k}{\|S^{(i_k)}\|_2^2} \left(S^{(i_k)}\right)^T \\ &= X_k - X_* - \alpha \left(S^{(i_k)}\right)^T \frac{S^{(i_k)} (X_k - X_*)}{\|S^{(i_k)}\|_2^2} \\ &= \left(I - \alpha P^{(i_k)}\right) (X_k - X_*), \end{aligned}$$

then

$$\begin{aligned}
\|X_{k+1} - X_*\|_2^2 &= (X_k - X_*)^T \left(I - \alpha P^{(i_k)} \right)^2 (X_k - X_*) \\
&= (X_k - X_*)^T \left(I - \alpha (2 - \alpha) P^{(i_k)} \right) (X_k - X_*) \\
&= \|X_k - X_*\|_2^2 - \alpha (2 - \alpha) (X_k - X_*)^T P^{(i_k)} (X_k - X_*),
\end{aligned}$$

and

$$\begin{aligned}
(X_k - X_*)^T P^{(i_k)} (X_k - X_*) &= \frac{(X_k - X_*)^T (S^{(i_k)})^T S^{(i_k)} (X_k - X_*)}{\|S^{(i_k)}\|_2^2} \\
&= \frac{|S^{(i_k)} (X_k - X_*)|^2}{\|S^{(i_k)}\|_2^2} \geq \frac{\|S (X_k - X_*)\|_2^2}{\|S\|_F^2}.
\end{aligned}$$

As x_0 is in the column space of S , from [23], it holds that $\|S (X_k - X_*)\|_2^2 \geq \lambda_{\min}(S^T S) \|(X_k - X_*)\|_2^2$. Therefore, the following is obtained

$$\|X_{k+1} - X_*\|_2^2 \leq \left(1 - \alpha (2 - \alpha) \frac{\lambda_{\min}(S^T S)}{\|S\|_F^2} \right) \|X_k - X_*\|_2^2.$$

The proof is completed. $\square$

4 Numerical Examples

This section gives two examples to show the effectiveness of the relaxed Kaczmarz method. All implements using Matlab 7 run in a Windows 7 DELL laptop with Intel 2.80GHz CPU and 8.00GB RAM. In the numerical experiments, the initial guess is 0 and the stopping criterion is

$$\|R_k\|_2 < 10^{-6},$$

where R_k is the residual vector after k iterations, i.e., $R_k = Y - SX_k$.

In the tables, x_a and x_b mean that $SX = Y$ is solved as two numeric systems

$$S \begin{bmatrix} x_{a1} \\ x_{a2} \\ \vdots \\ x_{a,2n} \end{bmatrix} = \begin{bmatrix} y_{a1} \\ y_{a2} \\ \vdots \\ y_{a,2n} \end{bmatrix} \text{ and } S \begin{bmatrix} x_{b1} \\ x_{b2} \\ \vdots \\ x_{b,2n} \end{bmatrix} = \begin{bmatrix} y_{b1} \\ y_{b2} \\ \vdots \\ y_{b,2n} \end{bmatrix}$$

not one symbolic system

$$S \begin{bmatrix} x_{a1} + x_{b1}r \\ x_{a2} + x_{b2}r \\ \vdots \\ x_{a,2n} + x_{b,2n}r \end{bmatrix} = \begin{bmatrix} y_{a1} + y_{b1}r \\ y_{a2} + y_{b2}r \\ \vdots \\ y_{a,2n} + y_{b,2n}r \end{bmatrix}$$

in the actual calculations.

Example 4.1. Consider $n \times n$ fuzzy linear system $Ax = y$ with

$$A = \begin{bmatrix} 8 & -1 & -1 & -1 & & & & \\ -1 & 8 & -1 & -1 & \ddots & & & \\ -1 & -1 & 8 & -1 & \ddots & \ddots & & \\ -1 & -1 & -1 & 8 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots & 8 & -1 & -1 & -1 \\ & & & \ddots & \ddots & -1 & 8 & -1 & -1 \\ & & & & \ddots & -1 & -1 & 8 & -1 \\ & & & & & -1 & -1 & -1 & 8 \end{bmatrix}$$

and

$$y = \begin{bmatrix} (2+r, 2+r) \\ (2+r, 2+r) \\ \vdots \\ (2+r, 2+r) \end{bmatrix}.$$

Table 1

Iterations (IT) and Residual (RES) for Example 4.1, $\alpha = 0.8$

n	IT $_{x_a}$	RES $_{x_a}$	IT $_{x_b}$	RES $_{x_b}$
16	617	9.6319e-007	590	9.9255e-007
32	1456	9.8052e-007	1394	9.8950e-007
64	3149	9.9849e-007	3021	9.9572e-007
128	6494	9.9873e-007	6235	9.9828e-007
256	13367	9.9960e-007	12840	9.9995e-007
512	27327	9.9969e-007	26272	9.9993e-007

Table 2

Iterations (IT) and Residual (RES) for Example 4.1, $\alpha = 1$

n	IT $_{x_a}$	RES $_{x_a}$	IT $_{x_b}$	RES $_{x_b}$
16	387	9.8857e-007	371	9.9516e-007
32	937	9.8926e-007	903	9.7518e-007
64	2063	9.9769e-007	1959	9.9717e-007
128	4283	9.9940e-007	4119	9.9840e-007
256	8810	9.9827e-007	8420	9.9985e-007
512	18001	9.9982e-007	17298	9.9991e-007

Table 3

Iterations (IT) and Residual (RES) for Example 4.1, $\alpha = 1.2$

n	IT $_{x_a}$	RES $_{x_a}$	IT $_{x_b}$	RES $_{x_b}$
16	193	9.1000e-007	183	9.0818e-007
32	524	9.7139e-007	518	8.8871e-007
64	1301	9.8178e-007	1242	9.7986e-007
128	2758	9.9848e-007	2639	9.9766e-007
256	5715	9.9972e-007	5494	9.9795e-007
512	11670	9.9882e-007	11193	9.9685e-007

Example 4.2. Consider $n^2 \times n^2$ fuzzy linear system $Ax = y$ with

$$A = \begin{bmatrix} D & B^T & & & \\ B & D & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & D & B^T \\ & & & B & D \end{bmatrix},$$

where

$$B = \begin{bmatrix} 0.5 & -0.25 & & & \\ -0.25 & 0.5 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0.5 & -0.25 \\ & & & -0.25 & 0.5 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & -1 & & & \\ -1 & 5 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 5 & -1 \\ & & & -1 & 5 \end{bmatrix},$$

and

$$y = \begin{bmatrix} (1+r, 1+r) \\ (2+r, 2+r) \\ \vdots \\ (n^2+r, n^2+r) \end{bmatrix}.$$

Table 4

Iterations (IT) and Residual (RES) for Example 4.2, $\alpha = 0.98$

n	IT $_{x_a}$	RES $_{x_a}$	IT $_{x_b}$	RES $_{x_b}$
10	1772	8.9357e-007	1363	9.9337e-007
15	4248	9.9182e-007	3121	9.8859e-007
20	7948	9.8889e-007	5570	9.9014e-007
25	13014	9.9777e-007	8720	9.9602e-007
30	18828	9.9025e-007	12587	9.9366e-007
35	26732	9.9937e-007	17492	9.9987e-007

Table 5

Iterations (IT) and Residual (RES) for Example 4.2, $\alpha = 1$

n	IT $_{x_a}$	RES $_{x_a}$	IT $_{x_b}$	RES $_{x_b}$
10	1734	9.6080e-007	1283	9.3918e-007
15	4193	9.6755e-007	3023	9.9710e-007
20	7711	9.9794e-007	5543	9.9329e-007
25	12434	9.9822e-007	8704	9.9622e-007
30	18408	9.9942e-007	12563	9.9846e-007
35	25599	9.9671e-007	17283	9.9737e-007

Table 6

Iterations (IT) and Residual (RES) for Example 4.2, $\alpha = 1.01$

n	IT $_{x_a}$	RES $_{x_a}$	IT $_{x_b}$	RES $_{x_b}$
10	1713	9.8178e-007	1275	9.8845e-007
15	4135	9.9971e-007	2993	9.9549e-007
20	7707	9.9933e-007	5544	9.9592e-007
25	12546	9.9998e-007	8793	9.9290e-007
30	18441	9.9335e-007	12833	9.9414e-007
35	25672	9.9948e-007	17633	9.9540e-007

Tables 1-6 give the number of iterations (IT) and the residual of the stopping step (RES). As n increases, the method requires more iterations, and with different α the method has different convergence rate, thus, improvement should be made and the optimal parameter should be studied to change the convergence.

5 Conclusion

A relaxed Kaczmarz method is presented for solving $n \times n$ fuzzy linear system. The numerical results show that the method is effective. Further work would be improving the method and comparing with other methods, also exploring the optimal relaxation parameter.

Compliance with Ethical Standards

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Funding Supported by Key Scientific Research Project of Colleges and Universities in Henan Province (20B110012), China.

Conflict of interest The authors declare that there is no conflict of interest regarding the publication of this article.

Informed consent From all individual participants included in the study.

Authorship contributions

All authors contributed to the study conception and design.

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