Optical solitons and other solutions for Radhakrishnan–Kundu–Lakshmanan equation in birefringent fibers by an efficient computational technique

M. Bilal

Punjab University: Panjab University

Mohammad Youins (✉ profdclu@hotmail.com)

Jiangsu University

Aly Ramadan Seadawy

Cairo University

S.T.R. Rizvi

COMSATS University Islamabad

Research Article

Keywords: Optical soliton, RKL equation, Generalized exponential rational function method

Posted Date: April 12th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-285910/v1>

License: © ⓘ This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Version of Record: A version of this preprint was published at Optical and Quantum Electronics on July 22nd, 2021. See the published version at <https://doi.org/10.1007/s11082-021-03083-8>.

Optical solitons and other solutions for Radhakrishnan–Kundu–Lakshmanan equation in birefringent fibers by an efficient computational technique

M. Bilal¹, M. Younis¹, A.R. Seadawy^{2,3*}, S.T.R. Rizvi⁴

¹PUCIT, University of the Punjab, Lahore 54000, Pakistan.

² Mathematics Department, Faculty of science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia.

³ Mathematics Department, Faculty of Science, Beni-Suef University, Beni Suef, Egypt.

⁴Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore Pakistan.

Abstract

In this article, we are interested to discuss the exact optical solitons and other solutions in birefringent fibers modeled by Radhakrishnan-Kundu-Lakshmanan equation in two component form for vector solitons. We extract the solutions in the form of hyperbolic, trigonometric and exponential functions including solitary wave solutions like multiple-optical soliton, mixed complex soliton solutions. The strategy that is used to explain the dynamics of soliton is known as generalized exponential rational function method. Moreover, singular periodic wave solutions are recovered and the constraint conditions for the existence of soliton solutions are also reported. Besides, the physical action of the solution attained are recorded in terms of 3D, 2D and contour plots for distinct parameters. The achieved outcomes show that the applied computational strategy is direct, efficient, concise and can be implemented in more complex phenomena with the assistant of symbolic computations. The primary benefit of this technique is to develop a significant relationships between NLPDEs and others simple NLODEs and we have succeeded in a single move to get and organize various types of new solutions. The obtained outcomes show that the applied method is concise, direct, elementary and can be imposed in more complex phenomena with the assistant of symbolic computations

keywords: Optical soliton, RKL equation, Generalized exponential rational function method

1 Introduction

In the science and technology fields like engineering, circuit analysis, fluid mechanics, solid state physics, chemical physics, plasma physics, geochemistry, optical fiber, quantum field theory and biological sciences, NLPDEs are used as a governing model to explain the complexity of the physical phenomena. To know the behaviour of intricate physical phenomena, it is necessary to calculate the solutions of the governing NLPDEs. Generally, the solutions of the NLPDEs are categorized into three types as exact solutions, analytic solutions and numerical solutions. Finding the exact solutions of NLPDEs has the importance to discuss the stability of numerical solutions and also development of a broad range of new scholar to simplify the routine calculation. Exact solutions to NLPDEs play

*corresponding author: aly742001@yahoo.com

an important role in nonlinear science, since they can provide much physical information and more insight of the physical aspects of the problem and thus lead to further applications. Wave phenomena in dispersion, dissipation, diffusion, reaction and convection are very much important.

Furthermore, various specialists and mainstream researchers are giving more consideration to create and improve the optical transmission frameworks through optical fibers instead of birefringent fibers. However, the propagation of soliton through optical fibers governs next generation technology but there are several factors in birefringent fibers as well that are used to produce the soliton propagations. It is polarization of light that prompts bunch speed mismatch, which is at last liable for differential gathering delay and numerous other negative impacts and consequently the examination of optical soliton is one of the most intriguing and interesting zones of exploration in nonlinear optics. There are well known computational and powerful techniques to find the optical exact soliton solutions of the differential equations [1–14].

Therefore, in this study we focus on constructing the exact optical soliton solutions in different structures to the RKL equation for birefringent fibers without 4WM terms in media with Kerr-law nonlinearity which is known as the basic case of fiber nonlinearity. Most optical fibers which has been quite popular recently comply with this law nonlinearity. Also, this media indicates itself as self-phase modulation, a self-induced phase- and frequency-shift of a pulse of light when it moves along with any fiber nonlinearity. In case of birefringent fibers the pulses are polarized. Naturally, vector solitons are studied in birefringent fibers. We apply an efficient computational approach known as generalized exponential rational function method (GERFM) to find the various kinds of soliton solutions. The restraint relations are also observed during the mathematical analysis.

The RKL with Kerr law nonlinearity is given below [15–19]

$$i\Theta_t + \alpha\Theta_{xx} + \beta|\Theta|^2\Theta = i\lambda(|\Theta|^2\Theta)_x - i\delta\Theta_{xxx}, \quad (1)$$

where the dependent variable $\Theta(x, t)$ is complex-valued wave profile with two independent variables of x and t that represents spatio-temporal component, respectively. The first term characterizes temporal evolution whereas the parameters α denotes group velocity dispersion (GVD) while, the coefficients β is Kerr nonlinearity. Moreover, on right hand side the coefficients δ and λ sequentially accounts the third order dispersion (3OD) which induces soliton radiation and the effect of self-steepening to eliminate the formulation of shock waves. Thus, these compensatory effects of dispersion and nonlinearity provide the necessary balance to sustain soliton propagation.

Upon splitting the Eq. (1) for birefringent fibers into two components without 4WM [20], we arrive at:

$$i\Psi_t + \alpha_1\Psi_{xx} + (\beta_1|\Psi|^2 + \gamma_1|\varphi|^2)\Psi = i\left(\lambda_1(|\Psi|^2\Psi)_x + \theta_1(|\varphi|^2\Psi)_x\right) - i\delta_1\Psi_{xxx} \quad (2)$$

$$i\varphi_t + \alpha_2\varphi_{xx} + (\beta_2|\varphi|^2 + \gamma_2|\Psi|^2)\varphi = i\left(\lambda_2(|\varphi|^2\varphi)_x + \theta_2(|\Psi|^2\varphi)_x\right) - i\delta_2\varphi_{xxx}. \quad (3)$$

The complex valued functions $\Psi(x, t)$ and $\varphi(x, t)$ represent the wave profiles and the Eq. (2) and Eq. (3) represent the governing model for soliton transmission through birefringent fibers without 4WM. In the above coupled system, the coefficient β_j accounts for self-phase modulation (SPM) and the coefficient γ_j is the cross-phase modulation terms respectively, for $j = 1, 2$. The coefficients λ_j and θ_j correspond to self-steepening terms, while the four-wave mixing effect is discarded.

This piece of article is discussed as sequence: In section 2, the summary of the GERFM. In section 3, optical

solitons. In section 4, results and discussion and finally paper comes at conclusions in section 5.

2 The summary of GERFM

Here, we give a brief description of GERFM [21]. Let us consider a nonlinear partial differential equation (PDE)

$$\Xi(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0, \quad (4)$$

where Ξ is a polynomial in its arguments.

The essence of GERFM can be presented in the following steps

Step 1. We introduce traveling wave transformation as:

$$u(x, t) = u(\eta) \quad \text{and} \quad \eta = B(x - ct),$$

where B and c represent the amplitude component and velocity respectively. After substituting this transformation into Eq. (4), we get nonlinear ODE in the following form.

$$\Upsilon(\Lambda, \Lambda', \Lambda'', \Lambda''', \dots) = 0, \quad (5)$$

where Υ is in general a polynomial function of its arguments and $'$ denotes the derivative w.r.t η .

Step 2. Suppose that the solution of Eq. (5) can be expressed as follows

$$\Lambda(\eta) = d_0 + \sum_{k=1}^n d_k \Omega(\eta)^k + \sum_{k=1}^n f_k \Omega(\eta)^{-k}, \quad (6)$$

where

$$\Omega(\eta) = \frac{r_1 e^{s_1 \eta} + r_2 e^{s_2 \eta}}{r_3 e^{s_3 \eta} + r_4 e^{s_4 \eta}}. \quad (7)$$

The unknown coefficients d_0, d_k, f_k ($1 \leq k \leq n$) and constants r_i, s_i ($1 \leq i \leq 4$) are determined and the value of n will be evaluated by using homogeneous balance principle.

Step 3. After putting Eq. (6) into Eq. (5) we get an algebraic equation $R(\eta, e^{s_1 \eta}, e^{s_2 \eta}, e^{s_3 \eta}, e^{s_4 \eta}) = 0$. Setting each coefficient of R equal to zero, we get a system of nonlinear equations in the form of d_0, d_k, f_k ($1 \leq k \leq n$) and r_i, s_i ($1 \leq i \leq 4$) is yielded.

Step 4. on solving the above system of equations we will get the values of d_0, d_k, f_k ($1 \leq k \leq n$) and r_i, s_i ($1 \leq i \leq 4$) with the aid of any symbolic computation packages. Substituting these values in Eq. (6) we attain the soliton solutions of Eq. (4).

3 Optical solitons

For solving the above couple system of Eqs. (2)-(3), we suppose the traveling wave transformation as follows:

$$\Psi(x, t) = Q_1(\eta)e^{i\psi}, \quad (8)$$

$$\varphi(x, t) = Q_2(\eta)e^{i\psi}, \quad (9)$$

where

$$\eta = B(x - \nu t) \quad \text{and} \quad \psi = -kx + \omega t + \theta_0. \quad (10)$$

And $Q_j(\eta)$ for $(j = 1, 2)$, B , ν , θ_0 , ω and k represent the amplitude component of the soliton, velocity of soliton, phase constant, soliton wave number and soliton frequency respectively. Substituting above transformations of Eqs. (8)-(10) into Eqs. (2) and (3), we get real and imaginary parts, respectively of the form

$$B^2(\alpha_j + 3k\delta_j)Q_j'' - (\omega + \alpha_j k^2 + \delta_j k^3)Q_j + (\beta_j - k\lambda_j)Q_j^3 + \gamma_j Q_j Q_j^2 - k\theta_j Q_j Q_j^2 = 0, \quad (11)$$

$$B^3\delta_j Q_j''' - B(\nu + 2k\alpha_j + 3k^2\delta_j)Q_j' - 3B\lambda_j Q_j^2 - \theta_j B Q_j' Q_j^2 - 2\theta_j B Q_j Q_j' Q_j = 0. \quad (12)$$

In order to use the balancing rule, the important results emerged from Eqs. (11)-(12) by using of $Q_j = Q_j$ are

$$B^2(\alpha_j + 3k\delta_j)Q_j'' - (\omega + \alpha_j k^2 + \delta_j k^3)Q_j + (\beta_j - k\lambda_j + \gamma_j - k\theta_j)Q_j^3 = 0, \quad (13)$$

$$B^2\delta_j Q_j'' - (\nu + 2k\alpha_j + 3k^2\delta_j)Q_j - (\lambda_j + \theta_j)Q_j^3 = 0. \quad (14)$$

We can obtain

$$\omega = \frac{8k^3\delta_j^2 + 8k^2\alpha_j\delta_j + 3k\nu\delta_j + 2k\alpha_j^2 + \nu\alpha_j}{\delta_j}, \quad (15)$$

$$\beta_j = -\frac{2k\delta_j\theta_j + 2k\delta_j\lambda_j + \alpha_j\theta_j + \alpha_j\lambda_j + \delta_j\gamma_j}{\delta_j}, \quad (16)$$

which yields from

$$\frac{\alpha_j + 3k\delta_j}{\delta_j} = \frac{\omega + \alpha_j k^2 + \delta_j k^3}{\nu + 2k\alpha_j + 3k^2\delta_j} = -\frac{\beta_j - k\lambda_j + \gamma_j - k\theta_j}{\lambda_j + \theta_j}. \quad (17)$$

As the function Q_j satisfies Eqs. (13)-(14). We will use Eq. (13) with Eqs. (15)-(16).

By balancing the highest order derivative and nonlinear term appear in Eq. (13) we attain $n = 1$. Using $n = 1$ along with Eqs. (6)-(7), then

$$Q_j(\eta) = d_0 + d_1\Omega(\eta) + f_1\Omega(\eta)^{-1}. \quad (18)$$

Family-1: If we take $r = [-1, -1, 1, -1]$ and $s = [1, -1, 1, -1]$, then Eq. (7) changes into

$$\Omega(\eta) = -\frac{\cosh(\eta)}{\sinh(\eta)}. \quad (19)$$

Case-1:

$$d_0 = 0, \quad d_1 = 0, \quad f_1 = -\frac{\sqrt{k^3(-\delta_j) - k^2\alpha_j - \omega}}{\sqrt{-\beta_j - \gamma_j + k\theta_j + k\lambda_j}}, \quad B = \frac{\sqrt{k^3(-\delta_j) - k^2\alpha_j - \omega}}{\sqrt{2}\sqrt{\alpha_j + 3k\delta_j}}.$$

Inserting these values in Eqs. (18) and (19), then we obtain

The optical dark soliton solution as

$$\Psi(x, t) = \frac{\sqrt{k^3(-\delta_1) - k^2\alpha_1 - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k\theta_1 + k\lambda_1}} \tanh \left[\frac{\sqrt{k^3(-\delta_1) - k^2\alpha_1 - \omega}}{\sqrt{2}\sqrt{\alpha_1 + 3k\delta_1}} (x - \nu t) \right] \times e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1} t + \theta_0)}, \quad (20)$$

$$\varphi(x, t) = \frac{\sqrt{k^3(-\delta_2) - k^2\alpha_2 - \omega}}{\sqrt{-\beta_2 - \gamma_2 + k\theta_2 + k\lambda_2}} \tanh \left[\frac{\sqrt{k^3(-\delta_2) - k^2\alpha_2 - \omega}}{\sqrt{2}\sqrt{\alpha_2 + 3k\delta_2}} (x - \nu t) \right] \times e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2} t + \theta_0)}. \quad (21)$$

Here $(k^3(-\delta_j) - k^2\alpha_j - \omega)(\alpha_j + 3k\delta_j) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution. The graphical representations of the solutions are shown for different values of parameters.

Figure 1: The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (20), respectively with the parameters $k = 1, \omega = -3, \beta_1 = 3, \alpha_1 = -2.9, \gamma_1 = -4, \delta_1 = 2, \lambda_1 = 3, \theta_1 = 4, \theta_0 = -2, \nu = .8$.

Figure 2: The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (21), respectively with the parameters $k = 1, \omega = -4, \beta_2 = 1, \alpha_2 = -2, \gamma_2 = -4.6, \delta_2 = 3, \lambda_2 = 7, \theta_2 = -3, \theta_0 = -1, \nu = .3$.

Case 2:

$$d_0 = 0, \quad d_1 = \frac{\sqrt{k^3(-\delta_j) - k^2\alpha_j - \omega}}{\sqrt{-\beta_j - \gamma_j + k\theta_j + k\lambda_j}}, \quad f_1 = 0, \quad B = \frac{\sqrt{k^3(-\delta_j) - k^2\alpha_j - \omega}}{\sqrt{2}\sqrt{\alpha_j + 3k\delta_j}}.$$

Substituting these values in Eqs. (18) and (19), we attain

The singular optical soliton solution as

$$\Psi(x, t) = -\frac{\sqrt{k^3(-\delta_1) - k^2\alpha_1 - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k\theta_1 + k\lambda_1}} \coth \left[\frac{\sqrt{k^3(-\delta_1) - k^2\alpha_1 - \omega}}{\sqrt{2}\sqrt{\alpha_1 + 3k\delta_1}} (x - \nu t) \right] \times e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1} t + \theta_0)}, \quad (22)$$

$$\varphi(x, t) = -\frac{\sqrt{k^3(-\delta_2) - k^2\alpha_2 - \omega}}{\sqrt{-\beta_2 - \gamma_2 + k\theta_2 + k\lambda_2}} \coth \left[\frac{\sqrt{k^3(-\delta_2) - k^2\alpha_2 - \omega}}{\sqrt{2}\sqrt{\alpha_2 + 3k\delta_2}}(x - \nu t) \right] \times e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2}t + \theta_0)}. \quad (23)$$

Here $(k^3(-\delta_j) - k^2\alpha_j - \omega)(\alpha_j + 3k\delta_j) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution. The graphical representations of the solutions are shown for different values of parameters.

Figure 3: The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (22), respectively with the parameters $k = 1, \omega = -6, \beta_1 = 1, \alpha_1 = -4, \gamma_1 = -2, \delta_1 = 3, \lambda_1 = 2, \theta_1 = 6, \theta_0 = -1, \nu = .7$.

Figure 4: The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (23), respectively with the parameters $k = 1, \omega = -3.6, \beta_2 = 2, \alpha_2 = -3, \gamma_2 = -5, \delta_2 = 4, \lambda_2 = 8, \theta_2 = -2, \theta_0 = -3, \nu = .3$.

Case-3:

$$d_0 = 0, \quad d_1 = \frac{\sqrt{k^2(\alpha_j + k\delta_j) + \omega}}{\sqrt{-2\beta_j - 2\gamma_j + 2k(\theta_j + \lambda_j)}}, \quad f_1 = -\frac{\sqrt{k^2(\alpha_j + k\delta_j) + \omega}}{\sqrt{-2\beta_j - 2\gamma_j + 2k(\theta_j + \lambda_j)}},$$

$$B = \frac{\sqrt{k^2(\alpha_j + k\delta_j) + \omega}}{2\sqrt{\alpha_j + 3k\delta_j}}.$$

Imposing these values in Eqs. (18) and (19), then we derive

The combined optical soliton solution as

$$\Psi(x, t) = -\frac{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega}}{\sqrt{-2\beta_1 - 2\gamma_1 + 2k(\theta_1 + \lambda_1)}} \frac{1}{\sinh \left[\frac{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega}}{2\sqrt{\alpha_1 + 3k\delta_1}} (x - \nu t) \right] \cosh \left[\frac{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega}}{2\sqrt{\alpha_1 + 3k\delta_1}} (x - \nu t) \right]} \times e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1} t + \theta_0)}, \quad (24)$$

$$\varphi(x, t) = -\frac{\sqrt{k^2(\alpha_2 + k\delta_2) + \omega}}{\sqrt{-2\beta_2 - 2\gamma_2 + 2k(\theta_2 + \lambda_2)}} \frac{1}{\sinh \left[\frac{\sqrt{k^2(\alpha_2 + k\delta_2) + \omega}}{2\sqrt{\alpha_2 + 3k\delta_2}}(x - \nu t) \right] \cosh \left[\frac{\sqrt{k^2(\alpha_2 + k\delta_2) + \omega}}{2\sqrt{\alpha_2 + 3k\delta_2}}(x - \nu t) \right]} \times e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2}t + \theta_0)}. \quad (25)$$

Here $(k^2(\alpha_j + k\delta_j) + \omega)(\alpha_j + 3k\delta_j) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution. The graphical representations of the solutions are shown for different values of parameters.

Figure 5: The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (24), respectively with the parameters $k = 2, \omega = 5, \beta_1 = .5, \alpha_1 = 1, \gamma_1 = -5, \delta_1 = 7, \lambda_1 = 3, \theta_1 = 5, \theta_0 = -3, \nu = 2$.

Figure 6: The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (25), respectively with the parameters $k = 2, \omega = 1, \beta_2 = -3, \alpha_2 = 4, \gamma_2 = 0, \delta_2 = 5, \lambda_2 = 6, \theta_2 = 7, \theta_0 = 2, \nu = 1.5$.

Family-2: If we chose $r = [-i, -i, 1, 1]$ and $s = [i, -i, i, -i]$, then Eq. (7) modify into

$$\Omega(\eta) = -\frac{\sin(\eta)}{\cos(\eta)}. \quad (26)$$

The following trigonometric and combined trigonometric traveling wave solutions are

Case-1:

$$d_0 = 0, \quad d_1 = 0, \quad f_1 = -\frac{\sqrt{k^3 \delta_j + k^2 \alpha_j + \omega}}{\sqrt{-\beta_j - \gamma_j + k \theta_j + k \lambda_j}}, \quad B = \frac{\sqrt{k^3 \delta_j + k^2 \alpha_j + \omega}}{\sqrt{2} \sqrt{\alpha_j + 3k \delta_j}}.$$

Substituting these values in Eqs. (18) and (26), we attain

$$\Psi(x, t) = \frac{\sqrt{k^3\delta_1 + k^2\alpha_1 + \omega}}{\sqrt{-\beta_1 - \gamma_1 + k\theta_1 + k\lambda_1}} \cot \left[\frac{\sqrt{k^3\delta_1 + k^2\alpha_1 + \omega}}{\sqrt{2}\sqrt{\alpha_1 + 3k\delta_1}} (x - \nu t) \right] \times e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1}t + \theta_0)}, \quad (27)$$

$$\varphi(x, t) = \frac{\sqrt{k^3\delta_2 + k^2\alpha_2 + \omega}}{\sqrt{-\beta_2 - \gamma_2 + k\theta_2 + k\lambda_2}} \cot \left[\frac{\sqrt{k^3\delta_2 + k^2\alpha_2 + \omega}}{\sqrt{2}\sqrt{\alpha_2 + 3k\delta_2}} (x - \nu t) \right] \times e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2}t + \theta_0)}. \quad (28)$$

Here $((k^3\delta_j + k^2\alpha_j) + \omega)(\alpha_j + 3k\delta_j) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution. The graphical representations of the solutions are shown for different values of parameters.

Figure 7: The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (27), respectively with the parameters $k = 5, \omega = -1, \beta_1 = -2, \alpha_1 = 1, \gamma_1 = -4, \delta_1 = 3, \lambda_1 = 2, \theta_1 = 5, \theta_0 = 3, \nu = .02$.

Figure 8: The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (28), respectively with the parameters $k = 4, \omega = 3, \beta_2 = 0, \alpha_2 = 3, \gamma_2 = 0, \delta_2 = 8, \lambda_2 = 2, \theta_2 = 3, \theta_0 = 3, \nu = .3$.

Case-2:

$$d_0 = 0, \quad d_1 = -\frac{\sqrt{k^3\delta_j + k^2\alpha_j + \omega}}{\sqrt{-\beta_j - \gamma_j + k\theta_j + k\lambda_j}}, \quad f_1 = 0, \quad B = \frac{\sqrt{k^3\delta_j + k^2\alpha_j + \omega}}{\sqrt{2}\sqrt{\alpha_j + 3k\delta_j}}.$$

Replacing these values in Eqs. (18) and (26), we get

$$\Psi(x, t) = \frac{\sqrt{k^3\delta_1 + k^2\alpha_1 + \omega}}{\sqrt{-\beta_1 - \gamma_1 + k\theta_1 + k\lambda_1}} \tan \left[\frac{\sqrt{k^3\delta_1 + k^2\alpha_1 + \omega}}{\sqrt{2}\sqrt{\alpha_1 + 3k\delta_1}} (x - \nu t) \right] \times e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1} t + \theta_0)}, \quad (29)$$

$$\varphi(x, t) = \frac{\sqrt{k^3\delta_2 + k^2\alpha_2 + \omega}}{\sqrt{-\beta_2 - \gamma_2 + k\theta_2 + k\lambda_2}} \tan \left[\frac{\sqrt{k^3\delta_2 + k^2\alpha_2 + \omega}}{\sqrt{2}\sqrt{\alpha_2 + 3k\delta_2}} (x - \nu t) \right] \times e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2} t + \theta_0)}. \quad (30)$$

Here $((k^3\delta_j + k^2\alpha_j) + \omega)(\alpha_j + 3k\delta_j) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution. **Case-3:**

$$d_0 = 0, \quad d_1 = -\frac{\sqrt{k^2(\alpha_j + k\delta_j) + \omega}}{2\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad f_1 = \frac{\sqrt{k^2(\alpha_j + k\delta_j) + \omega}}{2\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad B = \frac{\sqrt{k^2(\alpha_j + k\delta_j) + \omega}}{2\sqrt{2}\sqrt{\alpha_j + 3k\delta_j}}.$$

Inserting these values in Eqs. (18) and (26), we obtained

$$\Psi(x, t) = \frac{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega}}{2\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \frac{1 - 2\cos^2 \left[\frac{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega}}{2\sqrt{2}\sqrt{\alpha_1 + 3k\delta_1}} (x - \nu t) \right]}{\sin \left[\frac{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega}}{2\sqrt{2}\sqrt{\alpha_1 + 3k\delta_1}} (x - \nu t) \right] \cos \left[\frac{\sqrt{k^2(\alpha_1 + k\delta_1) + \omega}}{2\sqrt{2}\sqrt{\alpha_1 + 3k\delta_1}} (x - \nu t) \right]} \times e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1} t + \theta_0)}, \quad (31)$$

$$\varphi(x, t) = \frac{\sqrt{k^2(\alpha_2 + k\delta_2) + \omega}}{2\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \frac{1 - 2\cos^2 \left[\frac{\sqrt{k^2(\alpha_2 + k\delta_2) + \omega}}{2\sqrt{2}\sqrt{\alpha_2 + 3k\delta_2}} (x - \nu t) \right]}{\sin \left[\frac{\sqrt{k^2(\alpha_2 + k\delta_2) + \omega}}{2\sqrt{2}\sqrt{\alpha_2 + 3k\delta_2}} (x - \nu t) \right] \cos \left[\frac{\sqrt{k^2(\alpha_2 + k\delta_2) + \omega}}{2\sqrt{2}\sqrt{\alpha_2 + 3k\delta_2}} (x - \nu t) \right]} \times e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2} t + \theta_0)}. \quad (32)$$

Here $(k^2(\alpha_j + k\delta_j) + \omega)(\alpha_j + 3k\delta_j) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution.

Family-3: If we take $r = [2, 1, 1, 1]$ and $s = [1, 0, 1, 0]$, then Eq. (7) transform into

$$\Omega(\eta) = \frac{2e^\eta + 1}{e^\eta + 1}. \quad (33)$$

Case-1:

$$d_0 = \frac{3\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad d_1 = -\frac{2\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad f_1 = 0, \quad B = \frac{\sqrt{2}\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{\alpha_j + 3k\delta_j}}.$$

Inserting these values in Eqs. (18) and (33), then

The exponential function solution can be expressed as

$$\Psi(x, t) = \left(\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \frac{1 - e^\eta}{e^\eta + 1} \right) e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1} t + \theta_0)}, \quad (34)$$

where $\eta = \frac{\sqrt{2}\sqrt{k^2(-(\alpha_1+k\delta_1))-\omega}}{\sqrt{\alpha_1+3k\delta_1}}(x - \nu t)$.

$$\varphi(x, t) = \left(\frac{\sqrt{k^2(-(\alpha_2+k\delta_2))-\omega}}{\sqrt{-\beta_2-\gamma_2+k(\theta_2+\lambda_2)}} \frac{1-e^\eta}{e^\eta+1} \right) e^{i(-kx + \frac{8k^3\delta_2^2+8k^2\alpha_2\delta_2+3k\delta_2+2k\alpha_2^2+\nu\alpha_2}{\delta_2}t + \theta_0)}, \quad (35)$$

where $\eta = \frac{\sqrt{2}\sqrt{k^2(-(\alpha_2+k\delta_2))-\omega}}{\sqrt{\alpha_2+3k\delta_2}}(x - \nu t)$.

Here $(k^2(-(\alpha_j+k\delta_j))-\omega)(-\beta_j-\gamma_j+k(\theta_j+\lambda_j)) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution. The graphical representations of the solutions are shown for different values of parameters.

Figure 9: The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (34), respectively with the parameters $k = 0, \omega = -1, \beta_1 = -2, \alpha_1 = 1, \gamma_1 = 0, \delta_1 = -3, \lambda_1 = 5, \theta_1 = 4, \theta_0 = 2, \nu = 3$.

Figure 10: The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (35), respectively with the parameters $k = 1, \omega = -4, \beta_2 = -2, \alpha_2 = 0, \gamma_2 = -5, \delta_2 = -2, \lambda_2 = 3, \theta_2 = 4, \theta_0 = 2, \nu = 1$.

Case-2:

$$d_0 = -\frac{3\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad d_1 = 0, \quad f_1 = \frac{4\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}},$$

$$B = \frac{\sqrt{2}\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{\alpha_j + 3k\delta_j}}.$$

Putting these values in Eqs. (18) and (33), then

The exponential function solution can be formulated as

$$\Psi(x, t) = \left(\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \frac{1 - 2e^\eta}{2e^\eta + 1} \right) e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1}t + \theta_0)}, \quad (36)$$

where $\eta = \frac{\sqrt{2}\sqrt{k^2(-(\alpha_1+k\delta_1))-\omega}}{\sqrt{\alpha_1+3k\delta_1}}(x-\nu t)$.

$$\varphi(x, t) = \left(\frac{\sqrt{k^2(-(\alpha_2+k\delta_2))-\omega}}{\sqrt{-\beta_2-\gamma_2+k(\theta_2+\lambda_2)}} \frac{1-2e^\eta}{2e^\eta+1} \right) e^{i(-kx + \frac{8k^3\delta_2^2+8k^2\alpha_2\delta_2+3k\delta_2+2k\alpha_2^2+\nu\alpha_2}{\delta_2}t + \theta_0)}, \quad (37)$$

where $\eta = \frac{\sqrt{2}\sqrt{k^2(-(\alpha_2+k\delta_2))-\omega}}{\sqrt{\alpha_2+3k\delta_2}}(x-\nu t)$.

Here $(k^2(-(\alpha_j+k\delta_j))-\omega)(-\beta_j-\gamma_j+k(\theta_j+\lambda_j)) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution.

Family-4: If we take $r = [1-i, 1+i, -1, 1]$ and $s = [i, -i, i, -i]$, then Eq. (7) convert into

$$\Omega(\eta) = \frac{\cos(\eta) + \sin(\eta)}{\cos(\eta)}. \quad (38)$$

Case-1:

$$d_0 = \frac{\sqrt{k^2(\alpha_j+k\delta_j)+\omega}}{\sqrt{-\beta_j-\gamma_j+k(\theta_j+\lambda_j)}}, \quad d_1 = -\frac{\sqrt{k^2(\alpha_j+k\delta_j)+\omega}}{\sqrt{-\beta_j-\gamma_j+k(\theta_j+\lambda_j)}}, \quad f_1 = 0, \quad B = \frac{\sqrt{k^2(\alpha_j+k\delta_j)+\omega}}{\sqrt{2}\sqrt{\alpha_j+3k\delta_j}}.$$

Imposing these values in Eqs. (18) and (38), then

The singular periodic wave solution can be expressed as

$$\Psi(x, t) = -\frac{\sqrt{k^2(\alpha_1+k\delta_1)+\omega}}{\sqrt{-\beta_1-\gamma_1+k(\theta_1+\lambda_1)}} \tan \left[\frac{\sqrt{k^2(\alpha_1+k\delta_1)+\omega}}{\sqrt{2}\sqrt{\alpha_1+3k\delta_1}}(x-\nu t) \right] \times e^{i(-kx + \frac{8k^3\delta_1^2+8k^2\alpha_1\delta_1+3k\delta_1+2k\alpha_1^2+\nu\alpha_1}{\delta_1}t + \theta_0)}, \quad (39)$$

$$\varphi(x, t) = -\frac{\sqrt{k^2(\alpha_2+k\delta_2)+\omega}}{\sqrt{-\beta_2-\gamma_2+k(\theta_2+\lambda_2)}} \tan \left[\frac{\sqrt{k^2(\alpha_2+k\delta_2)+\omega}}{\sqrt{2}\sqrt{\alpha_2+3k\delta_2}}(x-\nu t) \right] \times e^{i(-kx + \frac{8k^3\delta_2^2+8k^2\alpha_2\delta_2+3k\delta_2+2k\alpha_2^2+\nu\alpha_2}{\delta_2}t + \theta_0)}. \quad (40)$$

Here $(k^2(\alpha_j+k\delta_j)+\omega)(\alpha_j+3k\delta_j) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution.

Case-2:

$$d_0 = \frac{\sqrt{k^2(\alpha_j+k\delta_j)+\omega}}{\sqrt{-\beta_j-\gamma_j+k(\theta_j+\lambda_j)}}, \quad d_1 = 0, \quad f_1 = -\frac{2\sqrt{k^2(\alpha_j+k\delta_j)+\omega}}{\sqrt{-\beta_j-\gamma_j+k(\theta_j+\lambda_j)}}, \quad B = \frac{\sqrt{k^3\delta_j+k^2\alpha_j+\omega}}{\sqrt{2\alpha_j+6k\delta_j}}.$$

Inserting these values in Eqs. (18) and (38), then

The combined singular periodic wave solution can be written as

$$\Psi(x, t) = \frac{\sqrt{k^2(\alpha_1+k\delta_1)+\omega}}{\sqrt{-\beta_1-\gamma_1+k(\theta_1+\lambda_1)}} \left(\frac{\sin[\frac{\sqrt{k^3\delta_1+k^2\alpha_1+\omega}}{\sqrt{2\alpha_1+6k\delta_1}}(x-\nu t)] - \cos[\frac{\sqrt{k^3\delta_1+k^2\alpha_1+\omega}}{\sqrt{2\alpha_1+6k\delta_1}}(x-\nu t)]}{\cos[\frac{\sqrt{k^3\delta_1+k^2\alpha_1+\omega}}{\sqrt{2\alpha_1+6k\delta_1}}(x-\nu t)] + \sin[\frac{\sqrt{k^3\delta_1+k^2\alpha_1+\omega}}{\sqrt{2\alpha_1+6k\delta_1}}(x-\nu t)]} \right) \times e^{i(-kx + \frac{8k^3\delta_1^2+8k^2\alpha_1\delta_1+3k\delta_1+2k\alpha_1^2+\nu\alpha_1}{\delta_1}t + \theta_0)}, \quad (41)$$

$$\varphi(x, t) = \frac{\sqrt{k^2(\alpha_2 + k\delta_2) + \omega}}{\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \left(\frac{\sin\left[\frac{\sqrt{k^3\delta_2 + k^2\alpha_2 + \omega}}{\sqrt{2\alpha_2 + 6k\delta_2}}(x - \nu t)\right] - \cos\left[\frac{\sqrt{k^3\delta_2 + k^2\alpha_2 + \omega}}{\sqrt{2\alpha_2 + 6k\delta_2}}(x - \nu t)\right]}{\cos\left[\frac{\sqrt{k^3\delta_2 + k^2\alpha_2 + \omega}}{\sqrt{2\alpha_2 + 6k\delta_2}}(x - \nu t)\right] + \sin\left[\frac{\sqrt{k^3\delta_2 + k^2\alpha_2 + \omega}}{\sqrt{2\alpha_2 + 6k\delta_2}}(x - \nu t)\right]} \right) \times e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2}t + \theta_0)}. \quad (42)$$

Here $((k^2\alpha_j + k^3\delta_j) + \omega)(2\alpha_j + 6k\delta_j) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution.

Family-5: If we take $r = [-3, -1, 1, 1]$ and $s = [1, -1, 1, -1]$, then Eq. (7) converts into

$$\Omega(\eta) = \frac{-2 \cosh(\eta) - \sinh(\eta)}{\cosh(\eta)} \quad (43)$$

Case-1:

$$d_0 = -\frac{2\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad d_1 = -\frac{\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad f_1 = 0, \\ B = \frac{\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{2}\sqrt{\alpha_j + 3k\delta_j}}.$$

Replacing these values in Eqs. (18) and (43), then

The hyperbolic function solution can be derived as

$$\Psi(x, t) = \frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \tanh \left[\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{2}\sqrt{\alpha_1 + 3k\delta_1}}(x - \nu t) \right] \times e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1}t + \theta_0)}, \quad (44)$$

$$\varphi(x, t) = \frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \tanh \left[\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{2}\sqrt{\alpha_2 + 3k\delta_2}}(x - \nu t) \right] \times e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2}t + \theta_0)}. \quad (45)$$

Here $(k^2(-(\alpha_j + k\delta_j)) - \omega)(\alpha_j + 3k\delta_j) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution.

Case-2:

$$d_0 = -\frac{2\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad d_1 = 0, \quad f_1 = -\frac{3\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \\ B = \frac{\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{2}\sqrt{\alpha_j + 3k\delta_j}}.$$

Substituting these values in Eqs. (18) and (43), then

The mixed hyperbolic function solution can be formulated as

$$\Psi(x, t) = -\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \left(\frac{2 \sinh\left[\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{2\sqrt{\alpha_1 + 3k\delta_1}}}(x - \nu t)\right] + \cosh\left[\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{2\sqrt{\alpha_1 + 3k\delta_1}}}(x - \nu t)\right]}{2 \cosh\left[\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{2\sqrt{\alpha_1 + 3k\delta_1}}}(x - \nu t)\right] + \sinh\left[\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{2\sqrt{\alpha_1 + 3k\delta_1}}}(x - \nu t)\right]} \right) \times e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1}t + \theta_0)}, \quad (46)$$

$$\varphi(x, t) = -\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \left(\frac{2 \sinh\left[\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{2\sqrt{\alpha_2 + 3k\delta_2}}}(x - \nu t)\right] + \cosh\left[\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{2\sqrt{\alpha_2 + 3k\delta_2}}}(x - \nu t)\right]}{2 \cosh\left[\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{2\sqrt{\alpha_2 + 3k\delta_2}}}(x - \nu t)\right] + \sinh\left[\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{2\sqrt{\alpha_2 + 3k\delta_2}}}(x - \nu t)\right]} \right) \times e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2}t + \theta_0)}, \quad (47)$$

Here $(k^2(-(\alpha_j + k\delta_j)) - \omega)(\alpha_j + 3k\delta_j) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution.

Family-6: If we take $r = [-1, 0, 1, 1]$ and $s = [0, 1, 0, 1]$, then Eq. (7) transform into

$$\Omega(\eta) = -\frac{1}{e^\eta + 1}. \quad (48)$$

Case-1:

$$d_0 = \frac{\sqrt{k^3(-\delta_j) - k^2\alpha_j - \omega}}{\sqrt{-\beta_j - \gamma_j + k\theta_j + k\lambda_j}}, \quad d_1 = \frac{2\sqrt{k^3(-\delta_j) - k^2\alpha_j - \omega}}{\sqrt{-\beta_j - \gamma_j + k\theta_j + k\lambda_j}}, \quad f_1 = 0, \quad B = \frac{\sqrt{2}\sqrt{k^3(-\delta_j) - k^2\alpha_j - \omega}}{\sqrt{\alpha_j + 3k\delta_j}}.$$

Inserting these values in Eqs. (18) and (48), then

The exponential function solution can be shown as

$$\Psi(x, t) = \left(\frac{\sqrt{k^3(-\delta_1) - k^2\alpha_1 - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k\theta_1 + k\lambda_1}} \frac{e^\eta - 1}{1 + e^\eta} \right) e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1}t + \theta_0)}, \quad (49)$$

$$\text{where } \eta = \frac{\sqrt{2}\sqrt{k^3(-\delta_1) - k^2\alpha_1 - \omega}}{\sqrt{\alpha_1 + 3k\delta_1}}(x - \nu t).$$

$$\varphi(x, t) = \left(\frac{\sqrt{k^3(-\delta_2) - k^2\alpha_2 - \omega}}{\sqrt{-\beta_2 - \gamma_2 + k\theta_2 + k\lambda_2}} \frac{e^\eta - 1}{1 + e^\eta} \right) e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2}t + \theta_0)}, \quad (50)$$

$$\text{where } \eta = \frac{\sqrt{2}\sqrt{k^3(-\delta_2) - k^2\alpha_2 - \omega}}{\sqrt{\alpha_2 + 3k\delta_2}}(x - \nu t).$$

Here $(k^3(-\delta_j) - k^2\alpha_j - \omega)(-\beta_j - \gamma_j + k(\theta_j + \lambda_j)) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution.

Family-7: If we take $r = [-1, 1, 1, 1]$ and $s = [1, -1, 1, -1]$, then Eq. (7) transform into

$$\Omega(\eta) = -\frac{\sinh(\eta)}{\cosh(\eta)} \quad (51)$$

Case-1:

$$d_0 = 0, \quad d_1 = \frac{\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{2\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad f_1 = \frac{\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{2\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad B = \frac{\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_j - 3k\delta_j}}.$$

Inserting these values in Eqs. (18) and (51), then

The combo optical soliton solution can be transformed as

$$\Psi(x, t) = -\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{2\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \left(\frac{2 \cosh^2 \left[\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_1 - 3k\delta_1}} (x - \nu t) \right] - 1}{\cosh \left[\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_1 - 3k\delta_1}} (x - \nu t) \right] \sinh \left[\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_1 - 3k\delta_1}} (x - \nu t) \right]} \right) \times e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1}t + \theta_0)}, \quad (52)$$

$$\varphi(x, t) = -\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{2\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \left(\frac{2 \cosh^2 \left[\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_2 - 3k\delta_2}} (x - \nu t) \right] - 1}{\cosh \left[\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_2 - 3k\delta_2}} (x - \nu t) \right] \sinh \left[\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_2 - 3k\delta_2}} (x - \nu t) \right]} \right) \times e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2}t + \theta_0)}. \quad (53)$$

Here $(k^2(-(\alpha_j + k\delta_j)) - \omega)(-\alpha_j - 3k\delta_j) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution.

Case-2:

$$d_0 = 0, \quad d_1 = -\frac{\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{2\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad f_1 = -\frac{\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{2\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad B = \frac{\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_j - 3k\delta_j}}.$$

Substituting these values in Eqs. (18) and (51), then

The mixed optical soliton solution can be derived as

$$\Psi(x, t) = -\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{2\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \left(\frac{2 \sinh^2 \left[\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_1 - 3k\delta_1}} (x - \nu t) \right] + 1}{\cosh \left[\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_1 - 3k\delta_1}} (x - \nu t) \right] \sinh \left[\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_1 - 3k\delta_1}} (x - \nu t) \right]} \right) \times e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1}t + \theta_0)}, \quad (54)$$

$$\varphi(x, t) = -\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{2\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \left(\frac{2 \cosh^2 \left[\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_2 - 3k\delta_2}} (x - \nu t) \right] - 1}{\cosh \left[\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_2 - 3k\delta_2}} (x - \nu t) \right] \sinh \left[\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{2\sqrt{2}\sqrt{-\alpha_2 - 3k\delta_2}} (x - \nu t) \right]} \right) \times e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2}t + \theta_0)}. \quad (55)$$

Here $(k^2(-(\alpha_j + k\delta_j)) - \omega)(-\alpha_j - 3k\delta_j) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution. The graphical

representations of the solutions are shown for different values of parameters.

Figure 11: The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (54), respectively with the parameters $k = 8, \omega = -2, \beta_1 = -3, \alpha_1 = -6, \gamma_1 = -5, \delta_1 = -4, \lambda_1 = 4, \theta_1 = 0, \theta_0 = 1, \nu = 1.5$.

Figure 12: The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (55), respectively with the parameters $k = 0, \omega = -4, \beta_2 = -5, \alpha_2 = -2, \gamma_2 = -2, \delta_2 = -2, \lambda_2 = 3, \theta_2 = 1, \theta_0 = 0, \nu = 1$.

Family-8: If we take $r = [1, 2, 1, 1]$ and $s = [1, 0, 1, 0]$, then Eq. (7) converts into

$$\Omega(\eta) = \frac{e^\eta + 2}{e^\eta + 1}. \quad (56)$$

Case-1:

$$d_0 = -\frac{3\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad d_1 = 0, \quad f_1 = \frac{4\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}},$$

$$B = \frac{\sqrt{2}\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{\alpha_j + 3k\delta_j}}.$$

Insert these values in Eqs. (18) and (56), then

The solitary wave solution can be written as

$$\Psi(x, t) = \left[\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \frac{e^\eta - 2}{e^\eta + 2} \right] e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1}t + \theta_0)}, \quad (57)$$

where $\eta = \frac{\sqrt{2}\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{\alpha_1 + 3k\delta_1}}(x - \nu t).$

$$\varphi(x, t) = \left[\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \frac{e^\eta - 2}{e^\eta + 2} \right] e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2}t + \theta_0)}, \quad (58)$$

where $\eta = \frac{\sqrt{2}\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{\alpha_2 + 3k\delta_2}}(x - \nu t).$

Here $(k^2(-(\alpha_j + k\delta_j)) - \omega)(-\beta_j - \gamma_j + k(\theta_j + \lambda_j)) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution.

Case-2:

$$d_0 = -\frac{3\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad d_1 = \frac{2\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{-\beta_j - \gamma_j + k(\theta_j + \lambda_j)}}, \quad f_1 = 0, \\ B = \frac{\sqrt{2}\sqrt{k^2(-(\alpha_j + k\delta_j)) - \omega}}{\sqrt{\alpha_j + 3k\delta_j}}.$$

Replacing these values in Eqs. (18) and (56), then

The solitary wave solution can be obtained as

$$\Psi(x, t) = \left(\frac{\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{-\beta_1 - \gamma_1 + k(\theta_1 + \lambda_1)}} \frac{1 - e^\eta}{1 + e^\eta} \right) e^{i(-kx + \frac{8k^3\delta_1^2 + 8k^2\alpha_1\delta_1 + 3k\delta_1 + 2k\alpha_1^2 + \nu\alpha_1}{\delta_1}t + \theta_0)}, \quad (59)$$

where $\eta = \frac{\sqrt{2}\sqrt{k^2(-(\alpha_1 + k\delta_1)) - \omega}}{\sqrt{\alpha_1 + 3k\delta_1}}(x - \nu t).$

$$\varphi = \left(\frac{\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{-\beta_2 - \gamma_2 + k(\theta_2 + \lambda_2)}} \frac{1 - e^\eta}{1 + e^\eta} \right) e^{i(-kx + \frac{8k^3\delta_2^2 + 8k^2\alpha_2\delta_2 + 3k\delta_2 + 2k\alpha_2^2 + \nu\alpha_2}{\delta_2}t + \theta_0)}, \quad (60)$$

where $\eta = \frac{\sqrt{2}\sqrt{k^2(-(\alpha_2 + k\delta_2)) - \omega}}{\sqrt{\alpha_2 + 3k\delta_2}}(x - \nu t).$

Here $(k^2(-(\alpha_j + k\delta_j)) - \omega)(-\beta_j - \gamma_j + k(\theta_j + \lambda_j)) > 0$ and $\delta_j \neq 0$ with $j = 1, 2$ for valid solution.

4 Results and discussion

The results of this paper will be beneficial for learners to analyze the most attractive applications of the RKL equation, which describes the propagation of waves without 4WM in birefringent fibers. Figures 1-12 clearly demonstrates the surfaces of the solution obtained for 3D, 2D and the contour plots, with choices of parameters for the RKL model. We can capture the behaviour of the of the acquired solution with the assistant of contour plots. In the same way, 3D figures tell us to model and demonstrate accurate physical behaviour. Through this study, we consider the exact optical soliton solutions to the nonlinear RKL model using generalized exponential rational function approach. The authors proposed different analytic approach in newly issued article and reported some fascinating findings. We can grasp from all the graphs that the GERFM is very efficient and more precise

in evaluating the equation under consideration.

5 Conclusions

This paper extracted the dynamics of optical solitons in the RKL equation without 4WM in birefringent fibers. The solutions are achieved in the shape of exponential, trigonometric and hyperbolic functions as well as exact optical soliton solutions by the mechanism of GERFM under different constraint conditions which provide the guaranty and validity of solutions are also listed. In addition, we attained the singular periodic wave solutions. The model must be extended to govern DWDM networks so that parallel communication of soliton dynamics can be addressed. These new families of solutions are shown the power, effectiveness and fruitfulness of this method. This article shelters the application of optical fibers. Also, these fresh solutions have many applications in physics and other branches of physical sciences.

References

- [1] B. Younas, M. Younis, Chirped solitons in optical monomode fibres modelled with Chen–Lee–Liu equation, *Pramana – J. Phys.* 94, 3 (2020). <https://doi.org/10.1007/s12043-019-1872-6>
- [2] A.R. Seadawy, M. Arshad, D. Lu, Dispersive optical solitary wave solutions of strain wave equation in micro-structured solids and its applications, *Physica A: Statistical Mechanics and its Applications*, 540 (2020) 123122.
- [3] M. Younis, M. Bilal, S.Rehman, U. Younas, S.T.R. Rizvi, Investigation of optical solitons in birefringent polarization preserving fibers with four-wave mixing effect. *Int. J. Mod. Phys. B*, (2020) 2050113.
- [4] G.D. Donne, M.B. Hubert, A.R. Seadawy, T. Etienne, G. Betchewe, S.Y. Doka, Chirped soliton solutions of Fokas–Lenells equation with perturbation terms and the effect of spatio-temporal dispersion in the modulational instability analysis, *The European Physical Journal Plus*, 135 (2) (2020) 212.
- [5] M. Younis, U. Younas, S.U. Rehman, M. Bilal, A. Waheed, Optical bright-dark and Gaussian soliton with third order dispersion, *Optik* 134 (2017) 233–238.
- [6] H.U. Rehman, M.S. Saleem, M. Zubair, S. Jafar, I. Latif, Optical solitons with Biswas-Arshed model using mapping method, *Optik* 194 (2019) 163091.
- [7] M. Iqbal, A.R. Seadawy, D. Lu, X. Xianwe, Construction of a weakly nonlinear dispersion solitary wave solution for the Zakharov–Kuznetsov-modified equal width dynamical equation, *Indian Journal of Physics*, (2019) 1-10.
- [8] M.S. Osman, K.U. Tariq, A. Bekir, A. Elmoasry, N.S. Elazab, M. Younis, Investigation of soliton solutions with different wave structures to the $(2+1)$ -dimensional Heisenberg ferromagnetic spin chain equation, *Communications in Theoretical Physics* 72 (3) (2020) 035002.
- [9] M. Younis, T.A. Sulaiman, M. Bilal, S.U. Rehman, U. Younas, Modulation instability analysis, optical and other solutions to the modified nonlinear Schrodinger equation, *Communications in Theoretical Physics*, 72 (2020) 065001 (12pp).

- [10] D. Lu, A.R. Seadawy, I. Ahmed, Peregrine-like rational solitons and their interaction with kink wave for the resonance nonlinear Schrödinger equation with Kerr law of nonlinearity, *Modern Physics Letters B*, 33 (24) (2019) 1950292.
- [11] A.R. Seadawy, A. Abdullah, Nonlinear complex physical models: optical soliton solutions of the complex Hirota dynamical model, *Indian Journal of Physics*, (2019) 1-10.
- [12] A.R. Seadawy, M. Arshad, D. Lu, Modulation stability analysis and solitary wave solutions of nonlinear higher-order Schrödinger dynamical equation with second-order spatiotemporal dispersion, *Indian Journal of Physics*, 93 (8) (2019) 1041-1049.
- [13] M. Iqbal, A.R. Seadawy, O.H. Khalil, D. Lu, Propagation of long internal waves in density stratified ocean for the $(2+1)$ -dimensional nonlinear Nizhnik-Novikov-Vesselov dynamical equation, *Results in Physics* 16 (2020) 102838.
- [14] M. Iqbal, A.R. Seadawy, D. Lu, X. Xia, Construction of bright–dark solitons and ion-acoustic solitary wave solutions of dynamical system of nonlinear wave propagation, *Modern Physics Letters A*, 34 (37) (2019) 1950309.
- [15] A. Biswas, Y. Yildirim, E. Yasar, M. F. Mahmood, A. S. Alshomrani, Q. Zhou, S. P. Moshokoa, M. Belic, Optical soliton perturbation for Radhakrishnan–Kundu–Lakshmanan equation with a couple of integration schemes, *Optik*. 163 (2018) 126–136.
- [16] N.A. Kudryashov, D.V. Safonova, A. Biswas, Painleve analysis and a solution to the traveling wave reduction of the Radhakrishnan–Kundu–Lakshmanan equation, *Regul Chaotic Dyn.* 24 (2019) 607-614.
- [17] H.U. Rehman, M.S. Saleem, A.M. Sultan, M. Iftikhar, Comments on "Dynamics of optical solitons with Radhakrishnan–Kundu–Lakshmanan model via two reliable integration schemes, *Optik* 178 (2019) 557–566.
- [18] T.A. Sulaiman, H. Bulut, G. Yel, S.S. Atas, Optical solitons to the fractional perturbed Radhakrishnan–Kundu–Lakshmanan model, *Opt. Quant. Electron.* 50 (2018) 372. <https://doi.org/10.1007/s11082-018-1641-7>.
- [19] B. Sturdevant, D.A. Lott, A. Biswas, Topological 1-soliton solution of the generalized Radhakrishnan, Kundu, Lakshmanan equation with nonlinear dispersion, *Mod. Phys. Lett. B.* 24 (2010) 1825–1831. <https://doi.org/10.1142/S0217984910024109>.
- [20] Y. Yildirim, A. Biswas, M. Ekici, H. Triki, O. Gonzalez-Gaxiolah, A.K Alzahranic, M.R. Belici, Optical solitons in birefringent fibers for Radhakrishnan–Kundu–Lakshmanan equation with five prolific integration norms, *Optik* 208 (2020) 164550.
- [21] B. Ghanbari, M. Inc, A.i Yusuf, D. Baleanu, New solitary wave solutions and stability analysis of the Benney-Luke and the Phi-4 equations in mathematical physics, *AIMS Mathematics*, 4(6) (2019) 1523–1539.

Figures

Figure 1

The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (20), respectively with the parameters $k = 1$, $\omega = -3$, $\beta_1 = 3$, $\alpha_1 = -2.9$, $\gamma_1 = -4$, $\delta_1 = 2$, $\lambda_1 = 3$, $\theta_1 = 4$, $\theta_0 = -2$, $v = .8$.

Figure 2

The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (21), respectively with the parameters $k = 1$, $\omega = -4$, $\beta_2 = 1$, $\alpha_2 = -2$, $\gamma_2 = -4.6$, $\delta_2 = 3$, $\lambda_2 = 7$, $\theta_2 = -3$, $\theta_0 = -1$, $v = .3$.

Figure 3

The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (22), respectively with the parameters $k = 1$, $\omega = -6$, $\beta_1 = 1$, $\alpha_1 = -4$, $\gamma_1 = -2$, $\delta_1 = 3$, $\lambda_1 = 2$, $\theta_1 = 6$, $\theta_0 = -1$, $v = .7$.

Figure 4

The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (23), respectively with the parameters $k = 1$, $\omega = -3.6$, $\beta_2 = 2$, $\alpha_2 = -3$, $\gamma_2 = -5$, $\delta_2 = 4$, $\lambda_2 = 8$, $\theta_2 = -2$, $\theta_0 = -3$, $v = .3$.

Figure 5

The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (24), respectively with the parameters $k = 2$, $\omega = 5$, $\beta_1 = .5$, $\alpha_1 = 1$, $\gamma_1 = -5$, $\delta_1 = 7$, $\lambda_1 = 3$, $\theta_1 = 5$, $\theta_0 = -3$, $v = 2$.

Figure 6

The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (25), respectively with the parameters $k = 2$, $\omega = 1$, $\beta_2 = -3$, $\alpha_2 = 4$, $\gamma_2 = 0$, $\delta_2 = 5$, $\lambda_2 = 6$, $\theta_2 = 7$, $\theta_0 = 2$, $v = 1.5$.

Figure 7

The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (27), respectively with the parameters $k = 5$, $\omega = -1$, $\beta_1 = -2$, $\alpha_1 = 1$, $\gamma_1 = -4$, $\delta_1 = 3$, $\lambda_1 = 2$, $\theta_1 = 5$, $\theta_0 = 3$, $v = .02$.

Figure 8

The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (28), respectively with the parameters $k = 4$, $\omega = 3$, $\beta_2 = 0$, $\alpha_2 = 3$, $\gamma_2 = 0$, $\delta_2 = 8$, $\lambda_2 = 2$, $\theta_2 = 3$, $\theta_0 = 3$, $v = .3$.

Figure 9

The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (34), respectively with the parameters $k = 0$, $\omega = -1$, $\beta_1 = -2$, $\alpha_1 = 1$, $\gamma_1 = 0$, $\delta_1 = -3$, $\lambda_1 = 5$, $\theta_1 = 4$, $\theta_0 = 2$, $v = 3$.

Figure 10

The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (35), respectively with the parameters $k = 1$, $\omega = -4$, $\beta_2 = -2$, $\alpha_2 = 0$, $\gamma_2 = -5$, $\delta_2 = -2$, $\lambda_2 = 3$, $\theta_2 = 4$, $\theta_0 = 2$, $v = 1$.

Figure 11

The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (54), respectively with the parameters $k = 8$, $\omega = -2$, $\beta_1 = -3$, $\alpha_1 = -6$, $\gamma_1 = -5$, $\delta_1 = -4$, $\lambda_1 = 4$, $\theta_1 = 0$, $\theta_0 = 1$, $v = 1.5$.

Figure 12

The (a), (b) and (c) show the 3D, 2D and contour physical behaviour of solution (55), respectively with the parameters $k = 0$, $\omega = -4$, $\beta_2 = -5$, $\alpha_2 = -2$, $\gamma_2 = -2$, $\delta_2 = -2$, $\lambda_2 = 3$, $\theta_2 = 1$, $\theta_0 = 0$, $v = 1$.