On the Stability of Periodic Motions of a Two-Body System With Flexible Connection in an Elliptical Orbit

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On the Stability of Periodic Motions of a Two-Body System With Flexible Connection in an Elliptical Orbit

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Abstract This paper deals with periodic motions and their stability of a flexible connected two-body system with respect to its center of mass in a central Newtonian gravitational field on an elliptical orbit. Equations of motion are derived in a Hamiltonian form and two periodic solutions as well as the necessary conditions for their existence are acquired. By analyzing linearized equations of perturbed motions, Lyapunov instability domains and domains of stability in the first approximation are obtained. In addition, the third and fourth order resonances are investigated in linear stability domains. A constructive algorithm based on a symplectic map is used to calculate the coefficients of the normalized Hamiltonian. Then a nonlinear stability analysis for two periodic solutions is performed in the third and fourth order resonance cases as well as in the nonresonance case.

Keywords Hamiltonian system · Stability · Periodic solution · Symplectic map · Normal form · Flexible multibody system

1 Introduction

The investigation of the solar system indicated that there are simple numerical relationships between the period of rotation and revolution of many planets and

satellites [1], [2], i.e. spin-orbit resonance. The observation of Mercury in 1965 revealed that the planet Mercury is in a spin-orbit resonance of 3:2 [3], also known as Mercury-type resonance. Since then, many studies have been done to explain and analyze the 3:2 spin-orbit resonance between the rotational and orbital motions of Mercury [4, 5, 6].

As early as 1959, a second-order ordinary differential equation for planar rotations of a rigid body around its center of mass in an elliptical orbit in a central Newtonian field was derived by Beletskii [7], i.e.

$$(1 + e \cos v) \frac{d^2 \varphi}{dv^2} - 2e \sin v \frac{d\varphi}{dv} + \alpha \sin \varphi \cos \varphi = 2e \sin v. \quad (1)$$

The motion of Mercury can be described by this equation, where φ is the angle between one of the principal central axes of inertia lying in the orbital plane and the radius vector of center of mass of the system; v is the true anomaly; e is the eccentricity of the orbit. During the last few decades, the Beletskii equation has been studied and a number of articles can be found [8, 9, 10, 11, 12]. It is well known that the equation has two classical periodic solutions, one of which is $\varphi = v/2$ for $\alpha = 6e$ and the other is $\varphi = -v/2$ for $\alpha = -2e$ [8]. For the first solution, the system performs three rotations in absolute space per two revolutions of the center of mass in the orbit, also known as a Mercury-type resonance [8]. For the second solution, the system completes one rotation in absolute space during two orbital revolutions of its center of mass, that is also called the resonant rotation [9]. Furthermore, based on the Galerkin type finite-dimensional approximations, Perhysyn and Yu [10] proved the existence of the periodic solutions of the Beletskii equation. Recently, Jifeng Chu, Zaitao

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Liang, etc. [12] studied the existence of stable and unstable periodic solutions of the Beletskii equation by using a suitable version of Poincaré-Birkhoff theorem and the third order approximation method.

The stability of periodic solutions of the Beletskii equation has always been the focus of attention. According to the Beletskii equation, the attitude stability of a satellite in an elliptical orbit was investigated, and thereinto the stability of the Mercury-type resonance and the resonant rotation were particularly attractive [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. The problem on the stability of the Mercury-type resonance of a rigid body was studied by Khentov [13] in the linear statement. The same problem was performed by Markeev [14] in the rigorous nonlinear statement. By taking into account both planar and spatial perturbations, Churkina [16] investigated the stability of the Mercury-type resonance of a satellite in which the third and fourth order resonance curves were constructed and the nonlinear stability analysis of the motion on these curves were performed. In recent years, Churkina etc [17, 18] discussed the stability of the Mercury-type resonance of a rigid body that the parameters of the system lie on the boundary of the stability region of the periodic solution. A resonant rotation is another typical periodic motion of satellites in an elliptical orbit. The stability analysis of the resonant rotation was started by Khentov [19]. Whereafter, Markeev [20], Bardin etc [21, 22, 23] performed rigorous stability investigations of the resonant rotation of a rigid body satellite which center of mass moves in an elliptical orbit. The stability analysis of above resonant rotation with respect to planar perturbations were performed in [21], and intervals of stability were found for the eccentricity values close to 1. Not only that, it was also studied some special cases that the stability analysis should take into consider the terms of degrees not less than six in the expansion of the Hamiltonian of the perturbed motion. The stability problem of the resonant rotation of a symmetric rigid body was studied in [22] where both planar and spatial perturbations were considered. Furthermore, the linear and nonlinear stability analysis were studied and the results indicated that in linear approximation the spatial perturbations have no effect on the stability of the resonant rotation whereas in a nonlinear system they may lead to instability. The above mentioned problem was also analyzed in [23] and it was taken into account that the parameters lay on the boundary of the stability region of the resonant rotation, i.e. the system has a second-order resonance. Moreover, there were several investigations [12, 24, 25] about the existence and stability of periodic motions of a dumbbell satellite which consist of two point masses connected by a massless rod.

For the past few years, an approach with the combination of point mapping and cell mapping method was employed by Dayung Koh and Henryk Flshner to find periodic solutions and analyzed the global behavior of gravity gradient satellites [26] in an elliptical orbit.

During the last few decades, studies on the periodic solutions and the attitude stability of a rigid body in an elliptical orbit mainly discussed simple systems such as dynamically symmetric rigid body and dumbbell satellites etc. Although the stability of the relative equilibrium of a multibody spacecraft in a circular orbit has been studied extensively [27, 28, 29, 30, 31], to the best of authors' knowledge, there are no investigations about the attitude stability of multibody systems in an elliptical orbit at the moment. Therefore, we focus on the attitude stability of a flexible connected two-body system in an elliptical orbit. The study of the attitude motion of the above mentioned system may date back to the work of Robe and Kane [32] that investigated the linear case of the attitude dynamics of the two-body system with flexible connection. Ever since Wittenburg [33], Rimrott [34] and Liu [35, 36], etc performed the steady-state motion, stability and bifurcation of the flexible connected two-body system in a circular orbit or the no torque state. When the center of the mass of the system moves on a circular orbit, one can find relative equilibrium. However, there are periodic motions not equilibrium positions in case of elliptical orbits even for small values of the eccentricity. The aim of this paper is to study the periodic motions and their stability of a flexible connected two-body system with respect to its center of mass in a central Newtonian gravitational field on an elliptical orbit.

The remaining parts of this paper are organized as follows. In Sect. 2, the system is reduced to a two-degree-of-freedom Hamiltonian system by using a ball hinge with torsional spring as a model of flexible axis and the equations of motion of the system are formulated in the form of canonical differential equations. In Sect. 3, two particular solutions of the unperturbed motions as well as the corresponding equations of perturbed motions are obtained. In Sect. 4, the stability of the linearized equations of perturbed motions were analyzed and Lyapunov instability domains and domains of stability in the first approximation are obtained in the parameter plane of the system. In Sect. 5, an approach based on the construction of a symplectic map generated by the Hamiltonian system is used to normalize the Hamiltonian up to terms of fourth order and a complete nonlinear stability analysis is performed in Sect. 6. Finally, in Sect. 7, the conclusions of this study are given.

2 Equations of motion

Let us consider a flexible connected two-body system the center of mass of which moves on an elliptical orbit in the central Newtonian gravitational field. We assume that the attitude motion of the system does not affect the orbital motion of its center of mass on account of the linear dimensions of the system are small compared with dimensions of the elliptical orbit. Suppose that B_1 and B_2 are rigid bodies and the massless flexible axis is simplified as a spherical hinge O' with a torsional spring, as shown in Fig. 1. The orbital frame of the reference $OXYZ$ (unit vectors $\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$) is established with the center of mass O as the origin where the axis OX is directed along the radius vector of the center of mass O and the axis OZ is normal to the orbital plane and the axis OY is orthogonal to both of them. We introduce two reference frames $O_1X_1Y_1Z_1$ and $O_2X_2Y_2Z_2$ (unit vectors $\mathbf{i}_j, \mathbf{j}_j, \mathbf{k}_j, j = 1, 2$) that are rigidly connected to the rigid body B_1 and B_2 , respectively. The axis O_jZ_j ($j = 1, 2$) is parallel to the axis OZ and the axis O_1X_1 and O_2X_2 intersect at the point of the ball hinge center O' . The system allows a planar motion in the orbital plane XOY and ν denotes the true anomaly, see Fig. 2.

Let φ_j ($j = 1, 2$) is the angle between the axis O_jX_j and OX , and m_j denotes the mass of the rigid body B_j , and A_{j0}, B_{j0}, C_{j0} ($j = 1, 2$) represent the moments of inertia corresponding to the principal central axes of inertia O_jX_j, O_jY_j, O_jZ_j . The vectors from the point O to O_1 and O to O_2 are denoted as $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively; the vectors from the point O_2 to O' and O' to O_1 are denoted as $\mathbf{r}_{20}$ and $\mathbf{r}_{10}$, respectively; the vector from the point O_e to O is $\mathbf{r}$, and the distance between O_j to O is l_j ($j = 1, 2$). The stiffness of the torsional spring is assumed to be K and its damping is neglected.

The origin O is the center of mass of the system and therefore the radius vectors of the rigid bodies B_1 and B_2 can be denoted as

$$\mathbf{r}_j = \chi_j (\mathbf{r}_{20} + \mathbf{r}_{10}); j = 1, 2 \quad (2)$$

where

$$\mathbf{r}_{20} = l_2 \cos \varphi_2 \mathbf{i}_0 + l_2 \sin \varphi_2 \mathbf{j}_0,$$

$$\mathbf{r}_{10} = l_1 \cos \varphi_1 \mathbf{i}_0 + l_1 \sin \varphi_1 \mathbf{j}_0,$$

$$\chi_1 = -\frac{m_2}{m_1 + m_2}, \chi_2 = \frac{m_1}{m_1 + m_2}.$$

The angular velocity of the orbital reference frame is

$$\omega_0 = \dot{\nu} \mathbf{k}_0 \quad (3)$$

and the angular velocities of the rigid bodies B_1 and B_2 reference to the orbital frame can be written as

$$\omega_{j0} = \dot{\varphi}_j \mathbf{k}_0, (j = 1, 2) \quad (4)$$

in which the dot means the derivative with respect to time.

The radius vectors of the center of mass of the rigid body B_j with respect to the attractive center are represented as

$$\mathbf{r}'_j = \mathbf{r} + \mathbf{r}_j, (j = 1, 2) \quad (5)$$

in which $\mathbf{r} = r \mathbf{i}_0$. The vector $\mathbf{r}'_j$ is written in the orbital reference frame and has the form

$$\begin{aligned} \mathbf{r}'_j = & [r + \chi_j (l_1 \cos \varphi_1 + l_2 \cos \varphi_2)] \mathbf{i}_0 \\ & + \chi_j (l_1 \sin \varphi_1 + l_2 \sin \varphi_2) \mathbf{j}_0, \end{aligned} \quad (6)$$

where $j = 1, 2$. The derivative of the above equation with respect to time is obtained as

$$\begin{aligned} \dot{\mathbf{r}}'_j = & [\dot{r} - (\dot{\varphi}_1 + \dot{\nu}) \chi_j l_1 \sin \varphi_1 \\ & - (\dot{\varphi}_2 + \dot{\nu}) \chi_j l_2 \sin \varphi_2] \mathbf{i}_0 \\ & + [\dot{r} \dot{\nu} + (\dot{\varphi}_1 + \dot{\nu}) \chi_j l_1 \cos \varphi_1 \\ & + (\dot{\varphi}_2 + \dot{\nu}) \chi_j l_2 \cos \varphi_2] \mathbf{j}_0 \end{aligned} \quad (7)$$

that is the velocity of the center of mass of the rigid body B_j ($j = 1, 2$).

The kinetic energy of the flexible connected two-body system in an elliptical orbit is

$$T = \sum_{j=1}^2 \left(\frac{1}{2} m_j \dot{\mathbf{r}}'_j \dot{\mathbf{r}}'_j + \frac{1}{2} \omega_j \mathbf{J}_j \omega_j \right) \quad (8)$$

where

$$\omega_j = \omega_0 + \omega_{j0}, \mathbf{J}_j = \begin{bmatrix} A_{j0} & & \\ & B_{j0} & \\ & & C_{j0} \end{bmatrix}, j = 1, 2.$$

Substituting the Eqs. (3), (4) and (7) into the right-hand side of the Eq. (8), we obtain the expression of the kinetic energy as

$$\begin{aligned} T = & \frac{1}{2} (C_{10} + m_0 l_1^2) (\dot{\varphi}_1 + \dot{\nu})^2 \\ & + \frac{1}{2} (C_{20} + m_0 l_2^2) (\dot{\varphi}_2 + \dot{\nu})^2 \\ & + m_0 l_1 l_2 (\dot{\varphi}_1 + \dot{\nu}) (\dot{\varphi}_2 + \dot{\nu}) \cos (\varphi_1 - \varphi_2) \\ & + \frac{1}{2} (m_1 + m_2) (\dot{r}^2 + r^2 \dot{\nu}^2) \end{aligned} \quad (9)$$

in which

$$m_0 = \frac{m_1 m_2}{m_1 + m_2}. \quad (10)$$

The position vector of a point on the rigid body $B_j (j = 1, 2)$ can be expressed as

$$\mathbf{r}'_{pj} = \mathbf{r}'_j + \rho_j, \quad (11)$$

where

$$\rho_j = x_j \mathbf{i}_j + y_j \mathbf{j}_j + z_j \mathbf{k}_j, \quad (12)$$

which is the vector of the point relative to the center of mass of the rigid body $B_j (j = 1, 2)$.

Using the relation between the unit vectors $\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$ and $\mathbf{i}_j, \mathbf{j}_j, \mathbf{k}_j$, i.e.

$$\mathbf{i}_0 = \mathbf{i}_j \cos \varphi_j - \mathbf{j}_j \sin \varphi_j,$$

$$\mathbf{j}_0 = \mathbf{i}_j \sin \varphi_j + \mathbf{j}_j \cos \varphi_j,$$

$$\mathbf{k}_0 = \mathbf{k}_j,$$

where $j = 1, 2$. The Eq. (11) can be written as follows

$$\begin{aligned} \mathbf{r}'_{pj} = & r \left[\cos \varphi_j + \chi_j \left(\frac{l_1}{r} \cos \varphi_1 + \frac{l_2}{r} \cos \varphi_2 \right) \cos \varphi_j \right. \\ & + \chi_j \left(\frac{l_1}{r} \sin \varphi_1 + \frac{l_2}{r} \sin \varphi_2 \right) \sin \varphi_j + \frac{x_j}{r} \left. \right] \mathbf{i}_j \\ & - r \left[\sin \varphi_j + \chi_j \left(\frac{l_1}{r} \cos \varphi_1 + \frac{l_2}{r} \cos \varphi_2 \right) \sin \varphi_j \right. \\ & \left. - \chi_j \left(\frac{l_1}{r} \sin \varphi_1 + \frac{l_2}{r} \sin \varphi_2 \right) \cos \varphi_j - \frac{y_j}{r} \right] \mathbf{j}_j. \end{aligned} \quad (13)$$

Let the potential energy at infinitely far from the attractive center is zero and the gravitational force of a mass point on the rigid body is

$$d\mathbf{F} = -\frac{\mu dm}{r^2} \frac{\mathbf{r}}{r} \quad (14)$$

where $\mu = Gm_e$ is the coefficient of gravitation in which m_e represents the mass of attractive center and G means the gravitational constant. Therefore, by integrating the Eq. (14) on the interval $(+\infty, r)$, the gravitational potential energy of the mass point can be written

$$dU = -\frac{\mu}{r} dm. \quad (15)$$

By integrating the Eq. (15) on the whole system, we obtain the gravitational potential energy as

$$U = -\mu \int_{S_1} \frac{dm_1}{r'_{p1}} - \mu \int_{S_2} \frac{dm_2}{r'_{p2}} \quad (16)$$

where r'_{pj} represents the modulus of vector $\mathbf{r}'_{pj}$.

Substituting the Eq. (13) into the Eq. (16) and using equalities as follows

$$\begin{aligned} \int_{S_j} dm_j &= m_j; & \int_{S_j} y_j^2 dm_j &= A_{j0}; \\ \int_{S_j} x_j y_j dm_j &= 0; & \int_{S_j} x_j^2 dm_j &= B_{j0}; \\ \int_{S_j} x_j dm_j &= \int_{S_j} y_j dm_j = 0; & \int_{S_j} (x_j^2 + y_j^2) dm_j &= C_{j0}, \end{aligned}$$

the expression of the gravitational potential energy of the system can be simplified as

$$\begin{aligned} U = & \frac{3\mu}{2r^3} (A_{10} \cos^2 \varphi_1 + B_{10} \sin^2 \varphi_1 + A_{20} \cos^2 \varphi_2 \\ & + B_{20} \sin^2 \varphi_2) - \frac{2\mu m_0}{r^3} l_1 l_2 \cos(\varphi_2 - \varphi_1) \\ & + \frac{3\mu}{2r^3} m_0 (l_1 \sin \varphi_1 + l_2 \sin \varphi_2)^2 + U_0. \end{aligned} \quad (17)$$

where

$$\begin{aligned} U_0 = & -\frac{\mu}{2r^3} (A_{10} + A_{20} + B_{10} + B_{20} + C_{10} + C_{20}) \\ & - \frac{\mu}{r} (m_1 + m_2) - \frac{\mu}{r^3} m_0 (l_1^2 + l_2^2). \end{aligned}$$

It should be pointed out that the terms higher than second order about x_j/r and y_j/r are neglected in the Eq. (17), and $z_j (j = 1, 2)$ in the Eq. (12) equals to zero since we consider the planar motion of the system.

The elastic potential energy of the torsional spring can be written as

$$U_k = \frac{1}{2} K (\varphi_2 - \varphi_1)^2. \quad (18)$$

Now suppose that the rigid bodies B_1 and B_2 are identical, hence introduce the notations

$$\begin{aligned} A_{10} = A_{20} = A_0, \quad B_{10} = B_{20} = B_0, \\ C_{10} = C_{20} = C_0, \quad l_1 = l_2 = l, \\ m_1 = m_2 = 2m_0 \end{aligned} \quad (19)$$

In order to establish the equations of motions in a Hamiltonian form, the generalized coordinates are introduced as follows

$$q_1 = \frac{1}{2} (\varphi_1 + \varphi_2), \quad q_2 = \frac{1}{2} (\varphi_1 - \varphi_2). \quad (20)$$

Substituting the Eq. (20) into the Eqs. (9), (17) and (18), the kinetic energy and the gravitational potential energy of the two-body system and the elastic potential energy of the torsional spring can be expressed respectively as

$$\begin{aligned} T = & [C_0 + m_0 l^2 (1 + \cos 2q_2)] (\dot{q}_1 + \dot{q}_2)^2 \\ & + [C_0 + m_0 l^2 (1 - \cos 2q_2)] \dot{q}_2^2 \\ & + 2m_0 (\dot{r} + r^2 \dot{v}^2), \end{aligned} \quad (21)$$

$$\begin{aligned} U = & \frac{3\mu}{r^3} [A_0 (\cos^2 q_1 \cos^2 q_2 + \sin^2 q_1 \sin^2 q_2) \\ & + B_0 (\sin^2 q_1 \cos^2 q_2 + \cos^2 q_1 \sin^2 q_2)] \\ & + \frac{6\mu}{r^3} m_0 l^2 \sin^2 q_1 \cos^2 q_2 - \frac{2\mu}{r^3} m_0 l^2 \cos 2q_2 + U_0 \end{aligned} \quad (22)$$

$$U_k = 2Kq_2^2 \quad (23)$$

The generalized momentum corresponding to the generalized coordinates q_1 and q_2 can be denoted as

$$\begin{aligned} p_1 &= 2 [C_0 + m_0 l^2 (1 + \cos 2q_2)] (\dot{q}_1 + \dot{v}), \\ p_2 &= 2 [C_0 + m_0 l^2 (1 - \cos 2q_2)] \dot{q}_2 \end{aligned} \quad (24)$$

Then the Hamiltonian of the system is obtained using the formulas (21) - (24) and it can be represented as follows

$$\begin{aligned} H &= \frac{p_1^2}{4[C_0 + m_0 l^2 (1 + \cos 2q_2)]} \\ &+ \frac{p_2^2}{4[C_0 + m_0 l^2 (1 - \cos 2q_2)]} - \dot{v} p_1 \\ &+ \frac{3\mu}{r^3} [A_0 (\cos^2 q_1 \cos^2 q_2 + \sin^2 q_1 \sin^2 q_2) \\ &+ B_0 (\sin^2 q_1 \cos^2 q_2 + \cos^2 q_1 \sin^2 q_2)] \\ &+ \frac{6\mu}{r^3} m_0 l^2 \sin^2 q_1 \cos^2 q_2 \\ &- \frac{2\mu}{r^3} m_0 l^2 \cos 2q_2 + 2Kq_2^2 + H_0 \end{aligned} \quad (25)$$

where

$$\begin{aligned} H_0 &= -2m_0 (\dot{r}^2 + r^2 \dot{v}^2) - \frac{4\mu m_0}{r} \\ &- \frac{\mu}{r^3} (A_0 + B_0 + C_0) - \frac{2\mu}{r^3} m_0 l^2. \end{aligned}$$

According to the Hamiltonian (25), it is easy to write down the equation of motion of the system in canonical form

$$\dot{q}_1 = \frac{p_1}{2[C_0 + m_0 l^2 (1 + \cos 2q_2)]} - \dot{v} \quad (26a)$$

$$\dot{q}_2 = \frac{p_2}{2[C_0 + m_0 l^2 (1 - \cos 2q_2)]} \quad (26b)$$

$$\begin{aligned} \dot{p}_1 &= \frac{3\mu}{r^3} (A_0 - B_0 - m_0 l^2) \sin 2q_1 \cos 2q_2 \\ &- \frac{3\mu}{r^3} m_0 l^2 \sin 2q_1 \end{aligned} \quad (26c)$$

$$\begin{aligned} \dot{p}_2 &= - \frac{m_0 l^2 \sin 2q_2}{2[C_0 + m_0 l^2 (1 + \cos 2q_2)]^2 p_1^2} \\ &+ \frac{m_0 l^2 \sin 2q_2}{2[C_0 + m_0 l^2 (1 - \cos 2q_2)]^2 p_2^2} \\ &+ \frac{3\mu}{r^3} (A_0 - B_0 - m_0 l^2) \cos 2q_1 \sin 2q_2 \\ &- \frac{\mu}{r^3} m_0 l^2 \sin 2q_2 - 4Kq_2 \end{aligned} \quad (26d)$$

3 Particular solutions and the equations of perturbed motion

It is easy to verify the canonical equations (26a) - (26d) have the following two particular solutions.

(1) If the eccentricity of the orbit satisfies the following equality

$$e = \frac{B_0 - A_0 + 2m_0 l^2}{2(C_0 + 2m_0 l^2)}, \quad (27)$$

there is a particular solution

$$\begin{aligned} \bar{q}_1 &= \frac{v}{2}, \quad \bar{p}_1 = 3(C_0 + 2m_0 l^2), \\ \bar{q}_2 &= 0, \quad \bar{p}_2 = 0 \end{aligned} \quad (28)$$

this solution indicates that during two orbital revolutions of the center of mass of the system, the system itself completes three rotations in absolute space and the flexible rod keeps its undeformed shape, it is similar to the Mercury-type resonance.

(2) If the eccentricity of the orbit satisfies the following equality

$$e = \frac{3(A_0 - B_0 - 2m_0 l^2)}{2(C_0 + 2m_0 l^2)}, \quad (29)$$

there is another particular solution

$$\begin{aligned} \bar{q}_1 &= -\frac{v}{2}, \quad \bar{p}_1 = C_0 + 2m_0 l^2, \\ \bar{q}_2 &= 0, \quad \bar{p}_2 = 0. \end{aligned} \quad (30)$$

It means during two orbital revolutions of the center of the mass of the system, the system performs one rotation in absolute space and the flexible rod keeps its undeformed shape, it is identical with the resonant rotation.

The particular solutions (28) and (30) can be written as

$$\bar{q}_1 = q_1^*, \quad \bar{q}_2 = 0, \quad \bar{p}_1 = p_1^*, \quad \bar{p}_2 = 0 \quad (31)$$

Let us introduce perturbations $\hat{q}_j, \hat{p}_j$ ($j = 1, 2$) by means of the formulas

$$\begin{aligned} q_1 &= q_1^* + \hat{q}_1, \quad p_1 = p_1^* + C_0 \hat{p}_1, \\ q_2 &= \hat{q}_2, \quad p_2 = C_0 \hat{p}_2. \end{aligned} \quad (32)$$

Substituting the Eq. (32) into the equations of motion (26a) - (26d) with canonical form and writing them down as a series expansion as follows

$$\begin{aligned} \frac{d\hat{q}_1}{dt} &= \frac{C_0}{2(C_0 + 2m_0 l^2)} \hat{p}_1 + \frac{m_0 l^2 p_1^*}{(C_0 + 2m_0 l^2)^2} \hat{q}_2^2 \\ &+ \frac{C_0 m_0 l^2}{(C_0 + 2m_0 l^2)^2} \hat{q}_2^2 \hat{p}_1 + O_4, \end{aligned} \quad (33a)$$

$$\frac{d\hat{q}_2}{dt} = \frac{1}{2}\hat{p}_2 - \frac{m_0 l^2}{C_0} \hat{q}_2^2 \hat{p}_2 + O_4 \quad (33b)$$

$$\begin{aligned} \frac{d\hat{p}_1}{dt} = & \frac{6\mu}{r^3} \frac{(A_0 - B_0 - 2m_0 l^2) \cos 2q_1^*}{C_0} \hat{q}_1 \\ & - \frac{6\mu}{r^3} \frac{(A_0 - B_0 - 2m_0 l^2) \sin 2q_1^*}{C_0} \hat{q}_1^2 \\ & - \frac{6\mu}{r^3} \frac{(A_0 - B_0 - m_0 l^2) \sin 2q_1^*}{C_0} \hat{q}_2^2 \\ & - \frac{4\mu}{r^3} \frac{(A_0 - B_0 - 2m_0 l^2) \cos 2q_1^*}{C_0} \hat{q}_1^3 \\ & - \frac{12\mu}{r^3} \frac{(A_0 - B_0 - m_0 l^2) \cos 2q_1^*}{C_0} \hat{q}_1 \hat{q}_2^2 \\ & + O_4 \end{aligned} \quad (33c)$$

$$\begin{aligned} \frac{d\hat{p}_2}{dt} = & - \left[\frac{m_0 l^2 p_1^{*2}}{C_0(C_0 + 2m_0 l^2)^2} \right. \\ & - \frac{6\mu}{r^3} \frac{(A_0 - B_0 - m_0 l^2)}{C_0} \cos 2q_1^* \\ & + \frac{2\mu}{r^3} \frac{m_0 l^2}{C_0} + \frac{4K}{C_0} \left. \right] \hat{q}_2 \\ & - \frac{2m_0 l^2 p_1^*}{(C_0 + 2m_0 l^2)^2} \hat{q}_2 \hat{p}_1 \\ & - \frac{12\mu}{r^3} \frac{(A_0 - B_0 - m_0 l^2) \sin 2q_1^*}{C_0} \hat{q}_1 \hat{q}_2 \\ & - \frac{C_0 m_0 l^2}{(C_0 + 2m_0 l^2)^2} \hat{q}_2 \hat{p}_1^2 + \frac{m_0 l^2}{C_0} \hat{q}_2 \hat{p}_2^2 \\ & - \frac{12\mu}{r^3} \frac{(A_0 - B_0 - m_0 l^2) \cos 2q_1^*}{C_0} \hat{q}_1^2 \hat{q}_2 \\ & + \left[\frac{2m_0 l^2 (C_0 - 4m_0 l^2) p_1^{*2}}{3C_0(C_0 + 2m_0 l^2)^3} \right. \\ & - \frac{4\mu}{r^3} \frac{(A_0 - B_0 - m_0 l^2)}{C_0} \cos 2q_1^* \\ & + \frac{4\mu}{3r^3} \frac{m_0 l^2}{C_0} \left. \right] \hat{q}_2^3 + O_4 \end{aligned} \quad (33d)$$

where O_4 represents the terms of degree greater than three in $\hat{q}_1, \hat{q}_2, \hat{p}_1$ and $\hat{p}_2$.

In the Eqs. (33a) - (33d) time t is an independent variable, we need to transform these equations of perturbed motion to equations that the true anomaly ν is the independent variable. For this purpose, the following relations are introduced [42, 43, 44]

$$\dot{\nu}^2 = \frac{\omega_0^2 (1 + e \cos \nu)^4}{(1 - e^2)^3}, \quad \frac{\mu}{r^3} = \frac{\dot{\nu}^2}{1 + e \cos \nu} \quad (34)$$

where ω_0 is the average orbital angular velocity.

In order to make the new equations of perturbed motion still have a canonical form, new perturbations are introduced by means of the following formulas

$$\begin{aligned} \hat{q}_1 &= \frac{q_1}{(1 + e \cos \nu)^2}, \quad \hat{p}_1 = \dot{\nu} p_1, \\ \hat{q}_2 &= \frac{q_2}{(1 + e \cos \nu)^2}, \quad \hat{p}_2 = \dot{\nu} p_2. \end{aligned} \quad (35)$$

Substituting the formulas (35) into the Eqs. (33a) - (33d) and considering the relations (34) the equations of perturbed motion (33a) - (33d) can be rewritten as

$$\begin{aligned} \frac{dq_1}{d\nu} = & - \frac{2e \sin \nu}{1 + e \cos \nu} q_1 + \frac{C_0 (1 + e \cos \nu)^2}{2(C_0 + 2m_0 l^2)} p_1 \\ & + \frac{\gamma^* m_0 l^2}{(C_0 + 2m_0 l^2) (1 + e \cos \nu)^2} q_2^2 \\ & + \frac{C_0 m_0 l^2}{(C_0 + 2m_0 l^2)^2 (1 + e \cos \nu)^2} q_2^2 p_1 + O_4, \end{aligned} \quad (36a)$$

$$\begin{aligned} \frac{dq_2}{d\nu} = & - \frac{2e \sin \nu}{1 + e \cos \nu} q_2 + \frac{(1 + e \cos \nu)^2}{2} p_2 \\ & - \frac{m_0 l^2}{C_0 (1 + e \cos \nu)^2} q_2^2 p_2 + O_4, \end{aligned} \quad (36b)$$

$$\begin{aligned} \frac{dp_1}{d\nu} = & - \frac{2e \sin \nu}{1 + e \cos \nu} p_1 \\ & + \frac{6(A_0 - B_0 - 2m_0 l^2)}{C_0} \frac{\cos 2q_1^*}{(1 + e \cos \nu)^3} q_1 \\ & - \frac{6(A_0 - B_0 - 2m_0 l^2)}{C_0} \frac{\sin 2q_1^*}{(1 + e \cos \nu)^5} q_1^2 \\ & - \frac{6(A_0 - B_0 - m_0 l^2)}{C_0} \frac{\sin 2q_1^*}{(1 + e \cos \nu)^5} q_2^2 \\ & - \frac{4(A_0 - B_0 - 2m_0 l^2)}{C_0} \frac{\cos 2q_1^*}{(1 + e \cos \nu)^7} q_1^3 \\ & + O_4 \end{aligned} \quad (36c)$$

$$\begin{aligned}
\frac{dp_2}{dv} = & \frac{2e \sin v}{1 + e \cos v} p_2 - \left[\frac{m_0 l^2 \gamma^{*2}}{C_0 (1 + e \cos v)^2} \right. \\
& - \frac{6 (A_0 - B_0 - m_0 l^2)}{C_0} \frac{\cos 2q_1^*}{(1 + e \cos v)^3} \\
& + \frac{2m_0 l^2}{C_0 (1 + e \cos v)^3} \\
& + \left. \frac{4K (1 - e^2)^3}{C_0 \omega_0^2 (1 + e \cos v)^6} \right] q_2 \\
& - \frac{2m_0 l^2}{(C_0 + 2m_0 l^2)^2 (1 + e \cos v)^2} q_2 p_1 \\
& - \frac{12 (A_0 - B_0 - m_0 l^2)}{C_0} \frac{\sin 2q_1^*}{(1 + e \cos v)^5} q_1 q_2 \\
& - \frac{C_0 m_0 l^2}{(C_0 + 2m_0 l^2)^2 (1 + e \cos v)^2} q_2 p_1^2 \\
& + \frac{m_0 l^2}{C_0 (1 + e \cos v)^2} q_2 p_2^2 \\
& - \frac{12 (A_0 - B_0 - m_0 l^2)}{C_0} \frac{\cos 2q_1^*}{(1 + e \cos v)^7} q_1^2 q_2 \\
& + \left[\frac{2\gamma^{*2} m_0 l^2 (C_0 - 4m_0 l^2)}{3C_0 (C_0 + 2m_0 l^2) (1 + e \cos v)^6} \right. \\
& - \frac{4 (A_0 - B_0 - m_0 l^2) \cos 2q_1^*}{C_0 (1 + e \cos v)^7} \\
& + \left. \frac{4m_0 l^2}{3C_0 (1 + e \cos v)^7} \right] q_2^3 + O_4
\end{aligned} \quad (36d)$$

in which

$$p_1^* = \gamma^* (C_0 + 2m_0 l^2) \quad (37)$$

and if the equations of perturbed motion (36a) - (36d) correspond to the particular solution (28) then $\gamma^* = 3$; if they correspond to the particular solution (30) then $\gamma^* = 1$.

Let us introduce dimensionless parameters as follows:

$$\alpha = \frac{A_0 - B_0}{C_0}, \quad \beta = \frac{C_0}{C_0 + 2m_0 l^2}, \quad k = \frac{K}{C_0 \omega_0^2}, \quad (38)$$

then the Hamiltonian corresponding to the differential equations of perturbed motion can be written

$$H = H_2 + H_3 + H_4 + \dots \quad (39)$$

where

$$\begin{aligned}
H_2 = & h_{2000} q_1^2 + h_{0200} q_2^2 + h_{1010} q_1 p_1 + h_{0101} q_2 p_2 \\
& + h_{0020} p_1^2 + h_{0002} p_2^2
\end{aligned} \quad (40)$$

$$H_3 = h_{3000} q_1^3 + h_{1200} q_1 q_2^2 + h_{0210} q_2^2 p_1 \quad (41)$$

$$\begin{aligned}
H_4 = & h_{4000} q_1^4 + h_{0400} q_2^4 + h_{2200} q_1^2 q_2^2 + h_{0220} q_2^2 p_1^2 \\
& + h_{0202} q_2^2 p_2^2
\end{aligned} \quad (42)$$

in which

$$h_{2000} = -3 \left(\alpha - \frac{1 - \beta}{\beta} \right) \frac{\cos 2q_1^*}{(1 + e \cos v)^3},$$

$$h_{1010} = h_{0101} = -\frac{2e \sin v}{1 + e \cos v}, \quad h_{0020} = \frac{\beta (1 + e \cos v)^2}{4},$$

$$\begin{aligned}
h_{0200} = & \frac{1}{2} \left[\frac{\gamma^{*2} (1 - \beta)}{2\beta (1 + e \cos v)^2} \right. \\
& - 6 \left(\alpha - \frac{1 - \beta}{2\beta} \right) \frac{\cos 2q_1^*}{(1 + e \cos v)^3} \\
& + \left. \frac{(1 - \beta)}{\beta (1 + e \cos v)^3} + \frac{4k (1 - e^2)^3}{(1 + e \cos v)^6} \right],
\end{aligned}$$

$$h_{3000} = 2 \left(\alpha - \frac{1 - \beta}{\beta} \right) \frac{\sin 2q_1^*}{(1 + e \cos v)^5},$$

$$h_{1200} = 6 \left(\alpha - \frac{1 - \beta}{2\beta} \right) \frac{\sin 2q_1^*}{(1 + e \cos v)^5},$$

$$h_{0210} = \frac{\gamma^* (1 - \beta)}{2(1 + e \cos v)^2},$$

$$h_{4000} = \left(\alpha - \frac{1 - \beta}{\beta} \right) \frac{\cos 2q_1^*}{(1 + e \cos v)^7},$$

$$h_{2200} = 6 \left(\alpha - \frac{1 - \beta}{2\beta} \right) \frac{\cos 2q_1^*}{(1 + e \cos v)^7},$$

$$h_{0220} = \frac{(1 - \beta) \beta}{4(1 + e \cos v)^2}, \quad h_{0202} = -\frac{1 - \beta}{4\beta (1 + e \cos v)^2},$$

$$\begin{aligned}
h_{0400} = & -\frac{1}{4} \left[\frac{\gamma^{*2} (1 - \beta) (3\beta - 2)}{3\beta (1 + e \cos v)^6} \right. \\
& - 4 \left(\alpha - \frac{1 - \beta}{2\beta} \right) \frac{\cos 2q_1^*}{(1 + e \cos v)^7} \\
& + \left. \frac{2(1 - \beta)}{3\beta (1 + e \cos v)^7} \right].
\end{aligned}$$

Based on the Eqs. (36a) - (36d) and the particular solutions (28) and (30), if $\gamma^* = 3$, $q_1^* = \nu/2$, the unperturbed motion (37) and the Hamiltonian (39) corresponding to the particular solution (28); if $\gamma^* = 1$, $q_1^* = -\nu/2$, they corresponding to another particular solution (30). According to the Eq. (38) the relations (27) and (29) can be expressed respectively as

$$e = \frac{1 - \beta - \alpha\beta}{2} \quad (43)$$

$$e = \frac{3(\alpha\beta + \beta - 1)}{2}, \quad (44)$$

The eccentricity of an elliptical orbit should be satisfied $e \in (0, 1)$; based on the relationships of the moments of inertia of a rigid body and the Eq. (38) the dimensionless parameters α and β meet the conditions $\alpha \in (0, 1)$ and $\beta \in (0, 1)$. According to the Eqs. (43) and (44), the necessary conditions for the existence of the particular solutions (28) and (30) are given as follows:

(1) The necessary condition for the existence of the particular solution (28) is

$$0 < \alpha < 1, \quad 0 < \beta < \frac{1}{1+\alpha} \quad (45)$$

(2) The necessary condition for the existence of the particular solution (30) is

$$0 < \alpha < \frac{2}{3}, \quad \frac{1}{1+\alpha} < \beta < 1$$

or

$$\frac{2}{3} < \alpha < 1, \quad \frac{1}{1+\alpha} < \beta < \frac{5}{3(1+\alpha)}. \quad (46)$$

In a plane of dimensionless parameters α and β , the region that the particular solutions (28) and (30) exist is shown as Fig. 3. The following is a detailed stability analysis of the particular solutions (28) and (30) in their existing region.

4 Linear stability analysis

The stability analysis of the particular solutions (28) and (30) starts with the investigation of the linearized system

$$\frac{dq_j}{dv} = \frac{\partial H_2}{\partial p_j}, \quad \frac{dp_j}{dv} = -\frac{\partial H_2}{\partial q_j}, \quad (j = 1, 2) \quad (47)$$

in which the Hamiltonian H_2 is the quadratic part of the Hamiltonian (39) as shown in the Eq. (40).

The canonical variables q_1, p_1 and q_2, p_2 are decoupled in the linear system (47), therefore the two degrees of freedom linearized system (47) can be split into the following two single-degree-of-freedom linear systems

$$\frac{dq_1}{dv} = h_{1010}q_1 + 2h_{0020}p_1, \quad (48)$$

$$\frac{dp_1}{dv} = 2h_{2000}q_1 + h_{1010}p_1,$$

and

$$\frac{dq_2}{dv} = h_{0101}q_2 + 2h_{0002}p_2, \quad (49)$$

$$\frac{dp_2}{dv} = 2h_{0200}q_2 + h_{0101}p_2,$$

where the coefficients h_{1010} , h_{0020} , h_{2000} , h_{0101} , h_{0002} and h_{0200} are shown in the Eq. (40).

The question of stability for the linear system (47) can be solved by means of the analysis of roots of the corresponding characteristic equation

$$(\rho^2 - 2a_1\rho + 1) \cdot (\rho^2 - 2a_2\rho + 1) = 0 \quad (50)$$

where

$$2a_j = x_{11}^{(j)}(2\pi) + x_{22}^{(j)}(2\pi), \quad (j = 1, 2). \quad (51)$$

The functions $x_{11}^{(j)}(\nu)$, $x_{22}^{(j)}(\nu)$ are elements of the matrix $\mathbf{X}_j(\nu)$, $(j = 1, 2)$, which is the fundamental matrices of the systems (48) and (49), respectively, and satisfy the initial conditions $\mathbf{X}_j(0) = \mathbf{E}_2$, where $\mathbf{E}_2$ is the second-order identity matrix.

It is not difficult to obtain the roots of the characteristic equation (50) as follows

$$\rho_{1,2} = a_1 \pm \sqrt{a_1^2 - 1}, \quad \rho_{3,4} = a_2 \pm \sqrt{a_2^2 - 1}. \quad (52)$$

If $|a_1| > 1$ or $|a_2| > 1$ according to the Eq. (52) there is a root with the absolute value exceeding 1. In this case, the linear system (47) is unstable. Then based on Lyapunov's theorem on instability in the first approximation [45], the equilibrium position of the perturbed system with Hamiltonian (39) is unstable, it means the corresponding unperturbed motions (28) and (30) are also unstable.

If $|a_1| < 1$ and $|a_2| < 1$, then all the roots (52) are complex and the absolute values of them are equal to 1, therefore the linear system (47) is stable [45].

The linear systems (48) and (49) are integrated numerically on the interval $[0, 2\pi]$, thereby the coefficients a_1 and a_2 can be calculated. According to the above analysis, the linear stability region and the Lyapunov instability region of periodic solutions (28) and (30) are obtained in the plane that is determined by dimensionless parameters α and β . For another dimensionless parameter k related to the stiffness of torsional spring, different values are assigned to it and the linear stability region as well as Lyapunov instability region of the periodic solutions (28) and (30) are compared, as shown in Figs. 4, 5 and 6. Obviously, with the increase of parameter k the instability region of periodic solutions decrease and the corresponding linear stability region increase.

5 Normalization of the Hamiltonian

In the previous section we have already obtained the linear stability region and the Lyapunov instability region of the periodic solutions (28) and (30) by analyzing the roots of the characteristic equation of the linear

system (47). However, the stability of the linear system (47) does not mean the equilibrium position of the corresponding nonlinear system with Hamiltonian (39) is stable, i.e. the stability of the unperturbed motions (28) and (30) can not be determined. To obtain a rigorous solution of the stability problem for the periodic solutions (28) and (30), it is necessary to perform a nonlinear analysis considering the third and forth order terms in the series expansion of the Hamiltonian (39) and sometimes even higher order terms if needed. A method based on the normal form technique [38, 39, 46, 48] and KAM theory [37, 41, 47] is the most general approach for nonlinear stability study of periodic solutions of Hamiltonian systems.

The normal form of the Hamiltonian can be obtained by using an approach given in [38] which is based on the construction of a symplectic map generated by the Hamiltonian system. According to the comprehensive description of this method in [38], the symplectic map is a composite map of a linear map and a nearly identity map.

The linear map can be constructed as

$$\begin{bmatrix} q_1^{(1)} \\ q_2^{(1)} \\ p_1^{(1)} \\ p_2^{(1)} \end{bmatrix} = \mathbf{X}(2\pi) \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix} \quad (53)$$

where

$$\mathbf{X}(2\pi) = \begin{bmatrix} x_{11}^{(1)}(2\pi) & 0 & x_{12}^{(1)}(2\pi) & 0 \\ 0 & x_{11}^{(2)}(2\pi) & 0 & x_{12}^{(2)}(2\pi) \\ x_{21}^{(1)}(2\pi) & 0 & x_{22}^{(1)}(2\pi) & 0 \\ 0 & x_{21}^{(2)}(2\pi) & 0 & x_{22}^{(2)}(2\pi) \end{bmatrix},$$

$x_{ij}^{(1)}(2\pi)$ and $x_{ij}^{(2)}(2\pi)$ are the values of the fundamental matrix $\mathbf{X}_1(\nu)$ and $\mathbf{X}_2(\nu)$ at $\nu = 2\pi$, respectively.

The nearly identity map is written as

$$\begin{aligned} \tilde{q}_j &= q_j^{(0)} - \frac{\partial S_3}{\partial p_j^{(0)}} + \sum_{l=1}^2 \frac{\partial^2 S_3}{\partial p_j^{(0)} \partial q_l^{(0)}} \frac{\partial S_3}{\partial p_l^{(0)}} \\ &\quad - \frac{\partial S_4}{\partial p_j^{(0)}} + O_4, \\ \tilde{p}_j &= p_j^{(0)} + \frac{\partial S_3}{\partial q_j^{(0)}} - \sum_{l=1}^2 \frac{\partial^2 S_3}{\partial q_j^{(0)} \partial q_l^{(0)}} \frac{\partial S_3}{\partial p_l^{(0)}} \\ &\quad + \frac{\partial S_4}{\partial q_j^{(0)}} + O_4, \quad (j = 1, 2), \end{aligned} \quad (54)$$

where

$$\begin{aligned} S_k &= \sum_{\substack{i_1+i_2+j_1 \\ +j_2=k}} s_{i_1 i_2 j_1 j_2} \left(q_1^{(0)}\right)^{i_1} \left(q_2^{(0)}\right)^{i_2} \left(p_1^{(0)}\right)^{j_1} \left(p_2^{(0)}\right)^{j_2}, \\ s_{i_1 i_2 j_1 j_2} &= \varphi_{i_1 i_2 j_1 j_2}(2\pi), \quad (k = 3, 4), \end{aligned}$$

and functions $\varphi_{i_1 i_2 j_1 j_2}(\nu)$ satisfy the following system of ordinary differential equations:

$$\frac{d\varphi_{3000}}{d\nu} = -g_{3000} \quad (55)$$

$$\frac{d\varphi_{2010}}{d\nu} = -g_{2010} \quad (56)$$

$$\frac{d\varphi_{1200}}{d\nu} = -g_{1200} \quad (57)$$

$$\frac{d\varphi_{1101}}{d\nu} = -g_{1101} \quad (58)$$

$$\frac{d\varphi_{1020}}{d\nu} = -g_{1020} \quad (59)$$

$$\frac{d\varphi_{1002}}{d\nu} = -g_{1002} \quad (60)$$

$$\frac{d\varphi_{0210}}{d\nu} = -g_{0210} \quad (61)$$

$$\frac{d\varphi_{0111}}{d\nu} = -g_{0111} \quad (62)$$

$$\frac{d\varphi_{0030}}{d\nu} = -g_{0030} \quad (63)$$

$$\frac{d\varphi_{0012}}{d\nu} = -g_{0012} \quad (64)$$

$$\frac{d\varphi_{4000}}{d\nu} = -g_{4000} - 2g_{0210}\varphi_{3000} \quad (65)$$

$$\frac{d\varphi_{3010}}{d\nu} = -g_{3010} - 6g_{1020}\varphi_{3000} - 2g_{2010}\varphi_{2010} \quad (66)$$

$$\begin{aligned} \frac{d\varphi_{2200}}{d\nu} &= -g_{2200} - 2g_{1101}\varphi_{1200} - 3g_{0210}\varphi_{3000} \\ &\quad - g_{2010}\varphi_{1200} \end{aligned} \quad (67)$$

$$\begin{aligned} \frac{d\varphi_{2101}}{d\nu} &= -g_{2101} - 4g_{1002}\varphi_{1200} - g_{1101}\varphi_{1101} \\ &\quad - 3g_{0111}\varphi_{3000} - g_{2010}\varphi_{1101} \end{aligned} \quad (68)$$

$$\begin{aligned} \frac{d\varphi_{2020}}{dv} = & -g_{2020} - 9g_{0030}\varphi_{3000} - 4g_{1020}\varphi_{2010} \\ & - g_{2010}\varphi_{1020} \end{aligned} \quad (69)$$

$$\begin{aligned} \frac{d\varphi_{2002}}{dv} = & -g_{2002} - 2g_{1002}\varphi_{1101} - 3g_{0012}\varphi_{3000} \\ & - g_{2010}\varphi_{1002} \end{aligned} \quad (70)$$

$$\begin{aligned} \frac{d\varphi_{1210}}{dv} = & -g_{1210} - 2g_{0111}\varphi_{1200} - 2g_{1101}\varphi_{0210} \\ & - 2g_{0210}\varphi_{2010} - 2g_{1020}\varphi_{1200} \end{aligned} \quad (71)$$

$$\begin{aligned} \frac{d\varphi_{1111}}{dv} = & -g_{1111} - 4g_{0012}\varphi_{1200} - g_{0111}\varphi_{1101} \\ & - 4g_{1002}\varphi_{0210} - g_{1101}\varphi_{0111} \\ & - 2g_{0111}\varphi_{2010} - 2g_{1020}\varphi_{1101} \end{aligned} \quad (72)$$

$$\frac{d\varphi_{1030}}{dv} = -g_{1030} - 6g_{0030}\varphi_{2010} - 2g_{1020}\varphi_{1020} \quad (73)$$

$$\begin{aligned} \frac{d\varphi_{1012}}{dv} = & -g_{1012} - 2g_{0012}\varphi_{1101} - 2g_{1002}\varphi_{0111} \\ & - 2g_{0012}\varphi_{2010} - 2g_{1020}\varphi_{1002} \end{aligned} \quad (74)$$

$$\frac{d\varphi_{0400}}{dv} = -g_{0400} - g_{0210}\varphi_{1200} \quad (75)$$

$$\frac{d\varphi_{0301}}{dv} = -g_{0301} - g_{0111}\varphi_{1200} - g_{0210}\varphi_{1101} \quad (76)$$

$$\begin{aligned} \frac{d\varphi_{0220}}{dv} = & -g_{0220} - 2g_{0111}\varphi_{0210} - 3g_{0030}\varphi_{1200} \\ & - g_{0210}\varphi_{1020} \end{aligned} \quad (77)$$

$$\begin{aligned} \frac{d\varphi_{0202}}{dv} = & -g_{0202} - g_{0012}\varphi_{1200} - g_{0111}\varphi_{1101} \\ & - g_{0210}\varphi_{1002} \end{aligned} \quad (78)$$

$$\begin{aligned} \frac{d\varphi_{0121}}{dv} = & -g_{0121} - 4g_{0012}\varphi_{0210} - g_{0111}\varphi_{0111} \\ & - 3g_{0030}\varphi_{1101} - g_{0111}\varphi_{1020} \end{aligned} \quad (79)$$

$$\frac{d\varphi_{0103}}{dv} = -g_{0103} - g_{0012}\varphi_{1101} - g_{0111}\varphi_{1002} \quad (80)$$

$$\frac{d\varphi_{0040}}{dv} = -g_{0040} - 3g_{0030}\varphi_{1020} \quad (81)$$

$$\begin{aligned} \frac{d\varphi_{0022}}{dv} = & -g_{0022} - 2g_{0012}\varphi_{0111} - g_{0012}\varphi_{1020} \\ & - 3g_{0030}\varphi_{1002} \end{aligned} \quad (82)$$

$$\frac{d\varphi_{0004}}{dv} = -g_{0004} - g_{0012}\varphi_{1002} \quad (83)$$

where $g_{i_1 i_2 j_1 j_2}$ is the coefficient of function G which is a new Hamiltonian transformed from the Hamiltonian (39) using the following linear transformation

$$\begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix} = \mathbf{X}(v) \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix}, \quad (84)$$

in which $\mathbf{X}(v)$ is the fundamental matrix of the linear system (47) and the elements of the matrix $\mathbf{X}(v)$ satisfy the following ordinary differential equations

$$\begin{aligned} \frac{dx_{ij}}{dv} &= \frac{\partial H_2^{(j)}}{\partial x_{i+2,j}}, \quad \frac{dx_{i+2,j}}{dv} = -\frac{\partial H_2^{(j)}}{\partial x_{ij}}; \\ H_2^{(j)} &= H_2^*(x_{1,j}, x_{2,j}, x_{3,j}, x_{4,j}), \\ (i &= 1, 2; j = 1, 2, 3, 4). \end{aligned} \quad (85)$$

The initial conditions of the differential Eqs. (55) - (83) are $\varphi_{i_1 i_2 j_1 j_2}(0) = 0$ and $\mathbf{X}(0) = \mathbf{E}_4$ is the initial conditions of Eqs. (85), where $\mathbf{E}_4$ is a fourth order identity matrix.

The coefficients of the generating function S_k of the nearly identity map (54) are obtained by integrating the equations (55) - (83) and (85) numerically in the interval $[0, 2\pi]$.

For the purpose of simplifying the symplectic map, the following canonical univalent transformation

$$\begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix} = \mathbf{N} \begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix} \quad (86)$$

is used to introduce new variables Q_1, Q_2, P_1, P_2 . In the problem investigated in this paper, the matrix $\mathbf{N}$ can be written in the following form

$$\mathbf{N} = \begin{bmatrix} n_{11}^{(1)} & 0 & n_{12}^{(1)} & 0 \\ 0 & n_{11}^{(2)} & 0 & n_{12}^{(2)} \\ n_{21}^{(1)} & 0 & n_{22}^{(1)} & 0 \\ 0 & n_{21}^{(2)} & 0 & n_{22}^{(2)} \end{bmatrix}, \quad (87)$$

and the elements of the matrix $\mathbf{N}$ are as follows [38]:

$$\begin{aligned} n_{11}^{(j)} &= -c_j \delta_j x_{12}^{(j)}(2\pi), n_{12}^{(j)} = 0, \\ n_{21}^{(j)} &= c_j \delta_j \left[x_{11}^{(j)}(2\pi) - a_j \right], \\ n_{22}^{(j)} &= -c_j \sqrt{1 - a_j^2}, (j = 1, 2) \end{aligned}$$

where

$$\delta_j = \text{sign} \left(x_{12}^{(j)} (2\pi) \sqrt{1 - a_j^2} \right),$$

$$c_j = \left| x_{12}^{(j)} (2\pi) \sqrt{1 - a_j^2} \right|^{-\frac{1}{2}}, (j = 1, 2).$$

In terms of the new variables Q_1, Q_2, P_1, P_2 , the nearly identity map (54) can be written as the following explicit form

$$\begin{aligned} \tilde{Q}_j &= Q_j^{(0)} - \frac{\partial F_3}{\partial P_j^{(0)}} + \sum_{l=1}^2 \frac{\partial^2 F_3}{\partial P_j^{(0)} \partial Q_l^{(0)}} \cdot \frac{\partial F_3}{\partial P_l^{(0)}} \\ &\quad - \frac{\partial F_4}{\partial P_j^{(0)}} + O_4, \\ \tilde{P}_j &= P_j^{(0)} + \frac{\partial F_3}{\partial Q_j^{(0)}} - \sum_{l=1}^2 \frac{\partial^2 F_3}{\partial Q_j^{(0)} \partial Q_l^{(0)}} \cdot \frac{\partial F_3}{\partial P_l^{(0)}} \\ &\quad + \frac{\partial F_4}{\partial Q_j^{(0)}} + O_4, \end{aligned} \quad (88)$$

where $j = 1, 2$ and

$$F_3 = S_3^* \quad (89)$$

$$\begin{aligned} F_4^* &= S_4^* + \frac{1}{2} \sum_{j=1}^2 \left[(n_{1,2+j} n_{3,2+j} + n_{2,2+j} n_{4,2+j}) \left(\frac{\partial S_3^*}{\partial Q_j^{(0)}} \right)^2 \right. \\ &\quad \left. + (n_{1j} n_{3j} + n_{2j} n_{4j}) \left(\frac{\partial S_3^*}{\partial P_j^{(0)}} \right)^2 \right] \\ &\quad - \sum_{j=1}^2 \left[(n_{13} n_{3j} + n_{23} n_{4j}) \frac{\partial S_3^*}{\partial Q_1^{(0)}} \frac{\partial S_3^*}{\partial P_j^{(0)}} \right. \\ &\quad \left. + (n_{14} n_{3j} + n_{24} n_{4j}) \frac{\partial S_3^*}{\partial Q_2^{(0)}} \frac{\partial S_3^*}{\partial P_j^{(0)}} \right] \\ &\quad + (n_{13} n_{34} + n_{23} n_{44}) \frac{\partial S_3^*}{\partial Q_1^{(0)}} \frac{\partial S_3^*}{\partial Q_2^{(0)}} \\ &\quad + (n_{11} n_{32} + n_{21} n_{42}) \frac{\partial S_3^*}{\partial P_1^{(0)}} \frac{\partial S_3^*}{\partial P_2^{(0)}} \end{aligned} \quad (90)$$

$$\begin{aligned} S_k^* &= S_k \left(n_{11}^{(1)} Q_1 + n_{12}^{(1)} P_1, n_{11}^{(2)} Q_2 + n_{12}^{(2)} P_2, \right. \\ &\quad \left. n_{21}^{(1)} Q_1 + n_{22}^{(1)} P_1, n_{21}^{(2)} Q_2 + n_{22}^{(2)} P_2 \right), \end{aligned}$$

($k = 3, 4$) and n_{ij} is an element of the matrix $\mathbf{N}$.

According to the Eqs. (89), (90) and the coefficients of S_k ($k = 3, 4$), it is not difficult to calculate the coefficients of the generating functions F_3 and F_4 of the map (89). The generating functions can be expressed as

$$F_k = \sum_{i_1+i_2+j_1+j_2=k} f_{i_1 i_2 j_1 j_2} Q_1^{i_1} Q_2^{i_2} P_1^{j_1} P_2^{j_2}, \quad (91)$$

where $k = 3, 4$.

The linear canonical transformation (86) changes the linear map (53) into the normal form and the nearly identity map (54) is normalized by means of a nearly identity canonical transformation $Q_j, P_j \rightarrow \xi_j, \eta_j$:

$$P_j = \eta_j + \frac{\partial R}{\partial Q_j}, \quad \xi_j = Q_j + \frac{\partial R}{\partial \eta_j}, \quad (j = 1, 2),$$

in which the generating function R is a polynomial in Q_j, η_j ($j = 1, 2$) and the symplectic map is changed into the simplest form by choosing the coefficients of the function R (the detailed description of this method see in [38]).

After the symplectic map is normalized, it is easy to obtain the normal form of the Hamiltonian. By introducing the following polar canonical variables φ_j, r_j ($j = 1, 2$)

$$\xi_j = \sqrt{2r_j} \sin \varphi_j, \eta_j = \sqrt{2r_j} \cos \varphi_j,$$

the normal form of the Hamiltonian can be written as follows [38]:

(1) If there are no resonances up to fourth order inclusive, the normal form of the Hamiltonian can be written as

$$\begin{aligned} H &= \sigma_1 r_1 + \sigma_2 r_2 + C_{20} r_1^2 + C_{11} r_1 r_2 + C_{02} r_2^2 \\ &\quad + o \left((r_1 + r_2)^{\frac{5}{2}} \right). \end{aligned} \quad (92)$$

(2) If there is a third order resonance, the normalized Hamiltonian is

$$\begin{aligned} H &= \sigma_1 r_1 + \sigma_2 r_2 + \\ &\quad \frac{\sqrt{2}}{4\pi} r_1^{\frac{k_1}{2}} r_2^{\frac{k_2}{2}} (a_{k_1 k_2 00} \sin \gamma + b_{k_1 k_2 00} \cos \gamma) \\ &\quad + o \left((r_1 + r_2)^2 \right). \end{aligned} \quad (93)$$

(3) If there is a resonance of fourth order but there are no third order resonances, the normalized Hamiltonian as follows

$$\begin{aligned} H &= \sigma_1 r_1 + \sigma_2 r_2 + C_{20} r_1^2 + C_{11} r_1 r_2 + C_{02} r_2^2 \\ &\quad + r_1^{\frac{k_1}{2}} r_2^{\frac{k_2}{2}} (\alpha_{k_1 k_2 00} \sin \gamma + \beta_{k_1 k_2 00} \cos \gamma) \\ &\quad + o \left((r_1 + r_2)^{\frac{5}{2}} \right) \end{aligned} \quad (94)$$

where $\gamma = k_1 \varphi_1 + k_2 \varphi_2 - n\nu$.

The coefficients of the normalized Hamiltonian (92) - (94) can be expressed in terms of the coefficients of the generating functions F_3 and F_4 . The explicit formulas used to calculate the coefficients of the Hamiltonian (92) - (94) are given in [38].

6 Nonlinear stability analysis

According to the modern stability theory, it is difficult to obtain any general conditions for stability in the sense of Lyapunov for a two degrees of freedom nonautonomous Hamiltonian system. However, based on a nonlinear truncated system corresponding to the normalized Hamiltonian, there are stability theorems provide conditions on formal stability, stability for most of the initial data or stability in third approximation.

(1) Suppose there are no resonances up to fourth order inclusive, therefore the Hamiltonian can be normalized to the form (92) and the following two sufficient conditions of stability exist:

(a) If the coefficients of the normalized Hamiltonian (92) satisfy the inequality

$$C_{11}^2 - 4C_{02}C_{20} \neq 0, \quad (95)$$

then the equilibrium position of the canonical system with Hamiltonian (92) is stable for most of the initial data [37], otherwise an additional research based on terms of order higher than four in the Hamiltonian series expansion is necessary to deal with the stability problem.

(b) If the function

$$F(r_1, r_2) = C_{20}r_1^2 + C_{11}r_1r_2 + C_{02}r_2^2 \quad (96)$$

is positive (or negative) definite for $r_1 \geq 0, r_2 \geq 0$, then the equilibrium position of the canonical system with the Hamiltonian (92) is formally stable [39, 40].

In the study of stability of resonance case, i.e. k_1, k_2 satisfy $k_1\sigma_1 + k_2\sigma_2 = n$, ($n \in \mathbf{Z}$) and $|k_1| + |k_2| = 3, 4$, only $k_1k_2 \geq 0$ should be considered because other resonances cannot lead to instability of nonlinear systems [49].

(2) If there is a third order resonance and the inequality

$$a_{k_1k_200}^2 + b_{k_1k_200}^2 \neq 0 \quad (97)$$

holds, then the equilibrium position of the canonical system with Hamiltonian (93) is unstable [39].

(3) Suppose that there are no third order resonances and yet a fourth order resonance exists. If the coefficients of the normalized Hamiltonian (94) satisfy the inequality

$$|F(r_1, r_2)| > k_1^{\frac{k_1}{2}} k_2^{\frac{k_2}{2}} \sqrt{\alpha_{k_1k_200}^2 + \beta_{k_1k_200}^2}, \quad (98)$$

then the equilibrium position of the canonical system with the Hamiltonian (94) is stable in third order approximation, otherwise it is unstable [39].

In the following stability analysis of the nonlinear Hamiltonian system, we consider the case that the dimensionless parameter k corresponding to the stiffness of the torsional spring is finite, i.e. $k = 0.2$.

Figs. 7 and 8 show the third and fourth order resonances in the linear stable region of the particular solutions (28) and (30), respectively. The special points in third order resonances represent that they meet the conditions of the third order resonances, i.e. $k_1\sigma_1 + k_2\sigma_2 = n$, ($n \in \mathbf{Z}$) and $|k_1| + |k_2| = 3$, but the coefficients $a_{k_1k_200}$ and $b_{k_1k_200}$ of the Hamiltonian (93) all equal to zero and there are no fourth order resonances, therefore these points belong to the nonresonance case. The special points in fourth order resonances mean the case that the conditions of the fourth order resonances are satisfied, i.e. $k_1\sigma_1 + k_2\sigma_2 = n$, ($n \in \mathbf{Z}$) and $|k_1| + |k_2| = 4$, but the coefficients $\alpha_{k_1k_200}$ and $\beta_{k_1k_200}$ of the fourth order resonance terms are vanished, hence it can also be regarded as the nonresonance case.

In the linear stable domain of particular solutions (28) and (30), Figs. 9 and 10 show the existence region of the nonresonance case, third order, fourth order resonance and the special points. When the dimensionless parameters α and β are located in the region of no resonances up to fourth order, the stability problem can be solved by using the sufficient conditions introduced in Eqs. (95) and (96). On the other hand, if the dimensionless parameters α and β in the resonance domain, it is easy to solve the stability of the third and fourth order resonances respectively by means of the sufficient conditions determined by inequalities (97) and (98). If the dimensionless parameters in the region of special points, however, it is necessary to consider the terms of order higher than four in the Hamiltonian series expansion and introduce an additional method for solving this stability problem.

Results of stability analysis for the nonresonance case are given in the plane determined by dimensionless parameters α and β , as shown in Figs. 11 and 12, as well as the regions of stable for most initial data and formal stability are obtained. As previously mentioned, the special points in third and fourth order resonances are deemed as the nonresonance case because the normalized Hamiltonian in this case is identical with the normalized Hamiltonian in the nonresonance case.

In accordance with the stability condition expressed in Eq. (97), it is obvious that if there is a third order resonance and the coefficients of the third order resonance terms are not vanish, the equilibrium position of the equations of perturbed motion is Lyapunov unstable. Figs. 13 and 14 show the results of stability analysis for the third order resonance case. For a two-degree

freedom system, there are generally four kinds of resonance relations in the third order resonance case [38], i.e. $3\sigma_1 = n$, $3\sigma_2 = n$, $\sigma_1 + 2\sigma_2 = n$, $2\sigma_1 + \sigma_2 = n$, however, in the problem considered in this paper there is only one resonance relation, $\sigma_1 + 2\sigma_2 = n$, for particular solution (28) and two resonance relation, $3\sigma_1 = n$ and $\sigma_1 + 2\sigma_2 = n$, for particular solution (30). Therefore, in the third order resonance case both the particular solutions (28) and (30) are unstable in the sense of Lyapunov, as shown in Figs. 13 and 14.

It is more complicated than a third order resonance case when there is a fourth order resonance but no third order resonances. In the fourth order resonance case, the following five relations are possible [38], i.e. $4\sigma_1 = n$, $4\sigma_2 = n$, $2\sigma_1 + 2\sigma_2 = n$, $\sigma_1 + 3\sigma_2 = n$, $3\sigma_1 + \sigma_2 = n$.

The results of stability analysis for the case of resonances $2\sigma_1 + 2\sigma_2 = n$, $3\sigma_1 + \sigma_2 = n$, $4\sigma_2 = n$, $\sigma_1 + 3\sigma_2 = n$ and $4\sigma_1 = n$, are shown in Figs. 15 - 23, respectively. And according to the stability conditions mentioned in (98), the domain of stable in third-order approximation and the Lyapunov instability regions of the particular solutions (28) and (30) are presented in the plane determined by the dimensionless parameters α and β . It is worth mentioning that there is no resonance $4\sigma_1 = n$ for the particular solution (28).

7 Conclusions

In this paper, we investigated the periodic solutions and the stability of the attitude motion of a flexible connected two-body system which center of mass moves on an elliptical orbit in a central Newtonian gravitational field. It was indicated that the attitude motion of the system discussed in this paper exist two classical periodic solutions, i.e. the Mercury-type resonance and the resonant rotation. By introducing dimensionless parameters, the necessary conditions of the existence of the above mentioned periodic solutions were obtained in the dimensionless parameters plane. The linear stability analysis indicated that with the enhancement of the stiffness of the torsional spring the linear stability region of the periodic solutions increase. Furthermore, the normalized Hamiltonian corresponding to the perturbed motion was obtained as well as the rigorous nonlinear stability analysis was performed in the linear stable domain. The nonlinear stability analysis showed that the unperturbed periodic motion is unstable if there is a third order resonance, otherwise if there is a fourth order resonance the unstable region and the region of stable for third order approximation appear in the linear stable domain. For the nonresonance case it was derived that the unperturbed periodic motions are stable for most of the initial data with an exception

of several special points. Moreover, formally stable regions for the periodic solutions were also derived in the nonresonance case. For the special points in nonresonance case, it is necessary to introduce a method based on terms of order higher than four in the Hamiltonian series expansion to resolve the stability problem.

Conflict of interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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Fig. 1 Schematic diagram of a flexible connected two-body system

Fig. 2 The trajectory of the center of mass of a flexible connected two-body system

Fig. 3 The region for the existence of periodic solutions, where the region I is the existence region of solution (28) and region II is the existence region of solution (30)

Fig. 4 Results of a linear stability analysis in the plane of dimensionless parameters α and β , where another dimensionless parameter $k = 0$ and region I and region II is the linear stable region of solution (28) and (30); region III and IV is the unstable region of solution (28) and (30), respectively.

Fig. 5 Results of a linear stability analysis in the plane of dimensionless parameters α and β , where another dimensionless parameter k is a finite value equal to 0.2 and region I and region II is the linear stable region of solution (28) and (30); region III and IV is the unstable region of solution (28) and (30), respectively.

Fig. 7 The third and fourth order resonances in the linear stability region of the particular solution (28) where I is the third order resonances, II is the fourth order resonances, III is the special points in third order resonances and IV is the special points of fourth order resonances

Fig. 6 Results of a linear stability analysis in the plane of dimensionless parameters α and β , where another dimensionless parameter $k = \infty$ and region I and region II is the linear stable region of solution (28) and (30); region III and IV is the unstable region of solution (28) and (30), respectively.

Fig. 8 The third and fourth order resonances in the linear stability region of the particular solution (30) where I is the third order resonances, II is the fourth order resonances, III is the special points in third order resonances and IV is the special points of fourth order resonances

Fig. 9 The third order, fourth order resonance and nonresonance region of the particular solution (28) where in region I no resonances up to fourth order, II is the third order resonances, III is the fourth order resonances and IV is the special points

Fig. 11 Results of stability analysis of the particular solution (28) for nonresonance case in which I is the region of stable for most initial data and II is the region of formally stable.

Fig. 10 The third order, fourth order resonance and nonresonance region of the particular solution (30) where in region I no resonances up to fourth order, II is the third order resonances, III is the fourth order resonances and IV is the special points

Fig. 12 Results of stability analysis of the particular solution (30) for nonresonance case in which I is the region of stable for most initial data and II is the region of formally stable.

Fig. 13 Results of stability analysis of the particular solution (28) in third order resonance

Fig. 15 Results of stability analysis of the particular solution (28), in the case of resonance $2\sigma_1 + 2\sigma_2 = n$, in which I is unstable points and II is stable in third order approximation

Fig. 14 Results of stability analysis of the particular solution (30) in third order resonance

Fig. 16 Results of stability analysis of the particular solution (30), in the case of resonance $2\sigma_1 + 2\sigma_2 = n$, in which I is unstable points and II is stable points in third order approximation

Fig. 17 Results of stability analysis of the particular solution (28), in the case of resonance $3\sigma_1 + \sigma_2 = n$, in which I is unstable points and II is stable points in third order approximation

Fig. 19 Results of stability analysis of the particular solution (28), in the case of resonance $4\sigma_2 = n$, in which I is unstable points and II is stable points in third order approximation

Fig. 18 Results of stability analysis of the particular solution (30), in the case of resonance $3\sigma_1 + \sigma_2 = n$, in which I is unstable points and II is stable points in third order approximation

Fig. 20 Results of stability analysis of the particular solution (30), in the case of resonance $4\sigma_2 = n$, in which I is unstable points and II is stable points in third order approximation

Fig. 21 Results of stability analysis of the particular solution (28), in the case of resonance $\sigma_1 + 3\sigma_2 = n$, in which I is unstable points and II is stable points in third order approximation

Fig. 23 Results of stability analysis of the particular solution (30), in the case of resonance $4\sigma_1 = n$, in which I is unstable points and II is stable points in third order approximation

Fig. 22 Results of stability analysis of the particular solution (30), in the case of resonance $\sigma_1 + 3\sigma_2 = n$, in which I is unstable points and II is stable points in third order approximation

Figures

Figure 1

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