Multiple Hopf bifurcations of four coupled van der Pol oscillators with delay

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Multiple Hopf bifurcations of four coupled van der Pol oscillators with delay

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Abstract

In this paper, a system of four coupled van der Pol oscillators with delay is studied. Firstly, the conditions for the existence of multi-periodic solutions of the system are given. Secondly, the multi-periodic solutions of spatiotemporal patterns of the system are obtained by using symmetric bifurcation theory. The normal form of the system on the central manifold and the direction of the bifurcation periodic solution are given. Finally, numerical simulations are used to support our theoretical results .

MSC: 34K18, 35B32

Keywords: Coupled van der Pol oscillators; delay; symmetric Hopf bifurcation; periodic solution.

1 Introduction

Recently, the dynamic behavior of coupled oscillator systems has become a hot research topic. More and more research results show that the coupled oscillator system can describe the dynamic

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behavior in physics, chemistry, biology and other disciplines [1, 2].Van der Pol oscillator is an important nonlinear oscillator, and coupled van der Pol oscillators are widely used in various fields of natural science. Many scholars pay attention to the dynamic behavior caused by the interaction between coupled van der Pol oscillators, such as stability, instability and periodicity. For example, in [3], van der pol oscillators were used to describe reproduce the dynamic behavior of thecardiac muscle .in [4], the circadian rhythm of eyes has been studied by using three coupled van der Pol oscillators. In [5, 6], the dynamic behavior of van der Pol oscillator in circuit has been studied. In [7], the occurrence of explosion death has been studied by using coupled van der Pol oscillator. In [8], Batool et al. have studied the limit cycle behavior of the van der Pol model.

Since the delay in signal propagation and chemical reaction, time delay is inevitable in coupling system. In addition, time delay is ubiquitous in the coupled system and has an important impact on the dynamic behavior of the system. For example, in [9], the effect of synchronization of delayed van der Pol oscillator has been discussed. In [10, 11], some dynamic behaviors of delayed van der Pol oscillator have been verified.

Symmetric system is also one of the research hot spots. In general, symmetry describes some kind of spatial invariance of the system. In [12, 13], the Hopf bifurcation behavior of three symmetric coupled van der Pol oscillators with delay has been has been analysed. In the aspect of artificial intelligence, the movement of robots is controlled by imitating the rhythmic movement of quadruped animals when designing robots, and people often use van der Pol oscillator to design CPG(Central Pattern Generators) network [14]. For example, in [14], it is shown that CPG can be controlled by a two-mode van der Pol oscillator under certain conditions.

The van der Pol oscillator is often used in biological modeling, as follow

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \alpha(p^2 - x^2)\dot{x} - w^2 x, \end{cases}$$

where x is the output signal from oscillator. α, p and w are variable parameters which can influence the character of oscillators. Commonly, the shape of the wave is affected by parameter α, and the amplitude of an output counts on the parameter p mostly. The output frequency is mainly relying on the parameter w when the amplitude parameter p is fixed. But the alteration of parameter p can lightly change the frequency of the signal, and α also can effect the output frequency.

Let $w^2 = 1$, we study following four coupled van der Pol oscillator systems with time delay.

$$\ddot{x}_i = \alpha(p^2 - x_i^2)\dot{x}_i - x_i + a\dot{x}_i(t-\tau) + b\dot{x}_{i+1}(t-\tau) + c\dot{x}_{i+2}(t-\tau) + b\dot{x}_{i+3}(t-\tau), \tag{1.1}$$

where $i = 1, 2, 3, 4, i(\text{mod } 4)$, and a, b, c are constants, $\tau \geq 0$. System (1.1) is symmetrical, and the classical Hopf bifurcation theory cannot be used to study it, since symmetrical systems usually have multiple eigenvalues.

The content of this paper is as follows. In Section 2, the conditions for the occurrence of multiple Hopf bifurcations of the system (1.1) are given. In Section 3, the spatiotemporal patterns of the multi-periodic solutions of the system are obtained by using the results of the symmetric system in [15]. In Section 4, the normal form of the system on the central manifold is obtained by using the normal form theory in [16]. Finally, some numerical simulations are carried out to verify the theoretical results.

2 Existence of equivariant Hopf bifurcation

Let $\dot{x}_i = y_i, X_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \in R^2$ $(i = 1, 2, 3, 4, i(\text{mod } 4))$. Then system (1.1) can be rewritten as

$$\dot{X}_i = A_0 X_i(t) + AX_i(t-\tau) + BX_{i+1}(t-\tau) + CX_{i+2}(t-\tau) + BX_{i+3}(t-\tau) + g(X_i(t)), \tag{2.1}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & \alpha p^2 \end{pmatrix}, \; A = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}, \; g \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha a_1^2 b_1 \end{pmatrix}.$$

It is clear that the origin (0,0,0,0,0,0,0,0) is an equilibrium of Eq.(2.1). And the linearization of Eq.(2.1) at the origin is

$$\dot{X}_i = A_0X_i(t)+AX_i(t-\tau)+BX_{i+1}(t-\tau)+CX_{i+2}(t-\tau)+BX_{i+3}(t-\tau), i = 1, 2, 3, 4, i(\mathrm{mod}\ 4). \tag{2.2}$$

The characteristic matrix $\Delta(\lambda)$ of the system (2.2) is given by

$$\Delta(\lambda) = \lambda I_2 - L(e^{\lambda \cdot} I)$$
$$= \begin{pmatrix} \lambda I_2 - A_0 - Ae^{-\lambda\tau} & -Be^{-\lambda\tau} & -Ce^{-\lambda\tau} & -Be^{-\lambda\tau} \\ -Be^{-\lambda\tau} & \lambda I_2 - A_0 - Ae^{-\lambda\tau} & -Be^{-\lambda\tau} & -Ce^{-\lambda\tau} \\ -Ce^{-\lambda\tau} & -Be^{-\lambda\tau} & \lambda I_2 - A_0 - Ae^{-\lambda\tau} & -Be^{-\lambda\tau} \\ -Be^{-\lambda\tau} & -Ce^{-\lambda\tau} & -Be^{-\lambda\tau} & \lambda I_2 - A_0 - Ae^{-\lambda\tau} \end{pmatrix},$$

where I_2 is 2×2 identity matrix.

Let $u_r = e^{\frac{\pi}{2}ri}, f(u_r) = A_0 + Ae^{-\lambda\tau} + u_rBe^{-\lambda\tau} + u_r^2Ce^{-\lambda\tau} + u_r^3Be^{-\lambda\tau}, r = 0, 1, 2, 3$. When $r = 1, 3$, we have $\det(\lambda I_2 - f(u_1)) = \det(\lambda I_2 - f(u_3))$, and the characteristic equation of system (2.2) is

$$\begin{aligned} \det\Delta(\lambda) &= \prod_{r=0}^{3} \det(\lambda I_2 - f(u_r)) \\ &= \Delta_1\Delta_2\Delta_3^2 \\ &= 0, \end{aligned} \tag{2.3}$$

where $\Delta_1 = \lambda^2 - \lambda(\alpha p^2 + (a + 2b + c)e^{-\lambda\tau}) + 1, \Delta_2 = \lambda^2 - \lambda(\alpha p^2 + (a - 2b + c)e^{-\lambda\tau}) + 1$, $\Delta_3 = \lambda^2 - \lambda(\alpha p^2 + (a - c)e^{-\lambda\tau}) + 1$. Obviously, $\lambda = 0$ is not the root of Eq.(2.3).

We make the following assumptions:

$(H1): \alpha p^2 + (a+2b+c) < 0,$

$(H2): \alpha p^2 + (a-2b+c) < 0,$

$(H3): \alpha p^2 + (a-c) < 0,$

$(H4): \alpha^2 p^4 - (a+2b+c)^2 - 2 > 0,$

$(H5): \alpha^2 p^4 - (a-2b+c)^2 - 2 > 0,$

$(H6): \alpha^2 p^4 - (a-c)^2 < 0.$

Lemma 2.1. *If $(H1)$ and $(H4)$ are satisfied , then when $\tau \geq 0$, all roots of equation $\Delta_1 = 0$ have negative real parts.*

Proof. When $\tau = 0$, equation $\Delta_1 = 0$ becomes $\lambda^2 - \lambda(\alpha p^2 + (a+2b+c)) + 1 = 0$ and the solution is obtained as follows

$$\lambda = \frac{(\alpha p^2 + (a+2b+c)) \pm \sqrt{(\alpha p^2 + (a+2b+c))^2 - 4}}{2}.$$

By $(H1)$, the roots of equation $\Delta_1 = 0$ have negative real parts. When $\tau > 0$, let $\lambda = i\omega(\omega > 0)$ be the root of equation $\Delta_1 = 0$, then we have

$$-\omega^2 - i\omega(\alpha p^2) - (i\omega)(a+2b+c)(cos\omega\tau - isin\omega\tau) + 1 = 0.$$

Separating the real and imaginary parts, we have

$$\begin{cases} 1 - \omega^2 = \omega(a+2b+c)\sin(\omega\tau), \\ -\omega\alpha p^2 = \omega(a+2b+c)\cos(\omega\tau), \end{cases}$$

which implies

$$\omega^4 + \omega^2(\alpha^2 p^4 - (a+2b+c)^2 - 2) + 1 = 0,$$

and

$$\omega^2 = \frac{-(\alpha^2 p^4 - (a+2b+c)^2 - 2) \pm \sqrt{(\alpha^2 p^4 - (a+2b+c)^2 - 2)^2 - 4}}{2}.$$

If $(H4)$ holds, the above formula does not hold. So the lemma holds. □

A similar lemma is as follows.

Lemma 2.2. *If* $(H2)$ *and* $(H5)$ *are satisfied, then when* $\tau \geq 0$*, all roots of equation* $\Delta_2 = 0$ *have negative real parts.*

Now consider equation

$$\Delta_3^2 = [\lambda^2 - \lambda(\alpha p^2 + (a-c)e^{-\lambda\tau}) + 1]^2 = 0.$$

When $\tau = 0$, equation $\Delta_3 = 0$ becomes $\lambda^2 - \lambda(\alpha p^2 + (a-c)) + 1 = 0$ and the solution is obtained as follows

$$\lambda = \frac{(\alpha p^2 + (a-c)) \pm \sqrt{(\alpha p^2 + (a-c))^2 - 4}}{2}.$$

Let $\lambda = i\beta(\beta > 0)$ be a root of $\Delta_3^2 = 0$. Substituting $i\beta$ into $\Delta_3^2 = 0$, we have

$$-\beta^2 - i\beta(\alpha p^2 + (a-c)e^{-i\beta\tau}) + 1 = 0.$$

The real and imaginary parts of the above equation are separated and obtained.

$$\begin{cases} 1 - \beta^2 = \beta(a-c)\sin(\beta\tau), \\ -\alpha p^2 \beta = \beta(a-c)\cos(\beta\tau). \end{cases}$$

Solving the above equation, we have

$$\beta^{\pm} = \sqrt{\frac{-(\alpha^2 p^4 - 2 - (a-c)^2) \pm \sqrt{(\alpha^2 p^4 - 2 - (a-c)^2)^2 - 4}}{2}}.$$

From the above analysis, we have the following lemma.

Lemma 2.3. *For the equation* $\Delta_3^2 = 0$*, we have the following results.*

(1) When $\tau = 0$*, all roots of the equation* $\Delta_3^2 = 0$ *have negative real parts for* $(H3)$*,*

(2) There exist $\tau_k^{\pm}$*, such that* $\Delta_3^2(\pm i\beta^{\pm}) = 0$ *holds when* $\tau = \tau_k^{\pm}(k = 0, 1, 2, \dots)$ *for* $(H6)$*.*

Where

$$\tau_k^{\pm} = \begin{cases} (1/\beta^{\pm})(\arccos(\frac{-\alpha p^2}{a-c}) + 2k\pi) & a-c < 0, \\ (1/\beta^{\pm})(2\pi - \arccos(\frac{-\alpha p^2}{a-c}) + 2k\pi) & a-c > 0, \end{cases} \quad k = 0, 1, 2 \dots.$$

Let $\lambda(\tau)=\alpha(\tau)+i\beta(\tau)$ is the root of equation (2.3), satisfying $\alpha(\tau_k^{\pm})=0$ and $\beta(\tau_k^{\pm})=\beta^{\pm}$. Taking the derivative of the equation $\Delta_3=0$ with respect to τ, we can get

$$(\frac{d\lambda}{d\tau})^{-1}=\frac{2\lambda-\alpha p^2}{-\lambda^2(a-c)e^{-\lambda\tau}}+\frac{1}{\lambda^2}-\frac{1}{\lambda}\tau.$$

When $\lambda=i\beta^{\pm}$ and $\tau=\tau_k^{\pm}$, we have

$$Re(\frac{d\lambda}{d\tau})^{-1}|_{\lambda=i\beta^{\pm},\tau=\tau_k^{\pm}}=\frac{(\beta^{\pm})^2((\beta^{\pm})^2+1)((\beta^{\pm})^2-1)}{((\beta^{\pm})^4-(\beta^{\pm})^2)^2+((\beta^{\pm})^3\alpha p^2)^2},$$

$$Re(\frac{d\lambda}{d\tau})^{-1}|_{\lambda=i\beta^{+},\tau=\tau_k^{+}}>0, Re(\frac{d\lambda}{d\tau})^{-1}|_{\lambda=i\beta^{-},\tau=\tau_k^{-}}<0, \tag{2.4}$$

which means that the transversality condition are satisfied at $\tau_k^{\pm}(k=0,1,2,\dots)$.

Let $C=C([-\tau,0],R^8)$,$z_t\in C, z_t(\theta)=z(t+\theta)$ for $-\tau\leq\theta\leq 0$, then Eq.(2.1) can be rewritten as

$$\dot{z}=Lz_t+Fz_t, \tag{2.5}$$

where

$$L\varphi=[\begin{pmatrix} A_0 & 0 & 0 & 0\\ 0 & A_0 & 0 & 0\\ 0 & 0 & A_0 & 0\\ 0 & 0 & 0 & A_0 \end{pmatrix}\varphi(0)+\begin{pmatrix} A & B & C & B\\ B & A & B & C\\ C & B & A & B\\ B & C & B & A \end{pmatrix}\varphi(-\tau)],$$

$$F\varphi=\begin{pmatrix} 0, & -\alpha\varphi_1^2\varphi_2, & 0, & -\alpha\varphi_3^2\varphi_4, & 0, & -\alpha\varphi_5^2\varphi_6, & 0, & -\alpha\varphi_7^2\varphi_8 \end{pmatrix}^T,$$

$$\varphi=(\varphi_1,\ \varphi_2,\ \varphi_3,\ \varphi_4,\ \varphi_5,\ \varphi_6,\ \varphi_7,\ \varphi_8)^T\in C.$$

Compact Lie groups can be considered as follows, the cycle group $Z_4=<\rho>$ of order 4, and the dihedral group D_4 of order 8 which is generated by Z_4 together with the flip k of order 2. D_4 acting on R^8 by

$$(\rho M)_i=M_{i+1},$$

$$(kM)_i=M_{6-i},$$

where $M_i\in R^2, i(\mathrm{mod}\,4)$. Then, we have $L(\rho\varphi)=\rho L(\varphi),\ F(\rho\varphi)=\rho F(\varphi),\ L(k\varphi)=kL(\varphi),\ F(k\varphi)=kF(\varphi)$. So we get the following lemma.

Lemma 2.4. *System (2.1) is D_4 equivariant.*

From equation (2.3), we know tha $i\beta^{\pm}$ is the dual pure imaginary root of equation (2.3) when $b \neq c$. By the lemma 2.1-2.4 and (2.4), the following theorem is obtained.

Theorem 2.1. *Suppose (H1)-(H6) is satisfied.*
(1) All roots of Eq(2.3) have negative real parts for $0 \leq \tau < \tau_0^{\pm}$ and at least a pair of roots with positive real parts for $\tau > \tau_0^{\pm}$,
(2) Zero equilibrium of system (2.1) is asymptotically stable for $0 \leq \tau < \tau_0^{\pm}$ and unstable for $\tau > \tau_0^{\pm}$,
(3) If $b \neq \pm c$, system (2.1) exhibits an equivariant Hopf bifurcation at $\tau = \tau_k^{\pm}(k = 0, 1, 2\dots)$.

3 Spatio-temporal patterns of bifurcating periodic solutions

In this section, we study the periodic solution when $\beta = \beta^+$. The case of $\beta = \beta^-$ can be considered similarly. The infinitesimal generator $A(\tau)$ of the C_0-semigroup generated by the linear system (2.2) has a pair of purely imaginary eigenvalues $\pm i\beta^+$ at the critical value $\tau_k^+(k = 0, 1, 2\dots)$, the corresponding eigenvectors can be chosen as

$$v_1(\theta) = V_1 e^{i\beta^+\theta}, v_2(\theta) = V_2 e^{i\beta^+\theta},$$

where $V_1 = \begin{pmatrix} q \\ iq \\ -q \\ -iq \end{pmatrix}$, $V_2 = \begin{pmatrix} q \\ -iq \\ -q \\ iq \end{pmatrix}$, $q = \begin{pmatrix} 1 \\ i\beta^+ \end{pmatrix}$.

Denoting the action of $\Gamma = D_4$ on R^2 by

$$\rho \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \ k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix},$$

where $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$.

Lemma 3.1. *R^2 is an absolutely irreducible representation of Γ , and $\mathrm{Ker}\Delta(\tau_k^+, i\beta^+)$ is isomorphic to $R^2 \oplus R^2$.*

Proof. It can be easily proved that R^2 is an absolutely irreducible representation of Γ.

Let $\mathrm{Ker}\Delta(\tau_k^+, i\beta^+) = \{(c_1 + d_1 i)V_1 + (c_2 + d_2 i)V_2 | c_1, c_2, d_1, d_2 \in R\}$. Define a mapping J from $\mathrm{Ker}\Delta(\tau_k^+, i\beta^+)$ to R^4 by

$$J((c_1 + d_1 i)V_1 + (c_2 + d_2 i)V_2) = (c_1 + c_2, d_1 - d_2, d_1 + d_2, c_2 - c_1)^T.$$

Clearly, J is a linear isomorphism from $\mathrm{Ker}\Delta(\tau_k^+, i\beta^+)$ to R^4.

Note that

$$\begin{aligned} \rho((c_1 + d_1 i)V_1 + (c_2 + d_2 i)V_2) &= (c_1 + d_1 i)\rho(V_1) + (c_2 + d_2 i)\rho(V_2) \\ &= (c_1 + d_1 i)e^{\frac{\pi}{2}i}V_1 + (c_2 + d_2 i)e^{-\frac{\pi}{2}i}V_2 \\ &= (-d_1 + c_1 i)V_1 + (d_2 - c_2 i)V_2, \\ k((c_1 + d_1 i)V_1 + (c_2 + d_2 i)V_2) &= (c_1 + d_1 i)V_2 + (c_2 + d_2 i)V_1. \end{aligned}$$

Then

$$\begin{aligned} J(\rho((c_1 + d_1 i)V_1 + (c_2 + d_2 i)V_2)) &= J((-d_1 + c_1 i)V_1 + (d_2 - c_2 i)V_2) \\ &= (d_2 - d_1, c_1 + c_2, c_1 - c_2, d_2 + d_1)^T \\ &= \rho(J((c_1 + d_1 i)V_1 + (c_2 + d_2 i)V_2)), \end{aligned}$$

$$\begin{aligned} J(k((c_1 + d_1 i)V_1 + (c_2 + d_2 i)V_2)) &= J((c_1 + d_1 i)V_2 + (c_2 + d_2 i)V_1) \\ &= (c_2 + c_1, d_2 - d_1, d_2 + d_1, c_1 - c_2)^T \\ &= k(J((c_1 + d_1 i)V_1 + (c_2 + d_2 i)V_2)). \end{aligned}$$

This completes the proof. □

Lemma 3.2. *The generalized eigenspace*

$$U_{\pm i\beta^+} = \left\{ \sum_{j=1}^{4} y_j \varepsilon_j, j = 1,2,3,4.\ y_j \in R \right\},$$

where

$$\varepsilon_1 = \cos(\beta^+ t)\mathrm{ReV}_1 - \sin(\beta^+ \mathrm{t})\mathrm{ImV}_1,$$

$$\varepsilon_2 = \sin(\beta^+ t)\mathrm{ReV}_1 + \cos(\beta^+ \mathrm{t})\mathrm{ImV}_1,$$

$$\varepsilon_3 = \cos(\beta^+ t)\mathrm{ReV}_2 - \sin(\beta^+ \mathrm{t})\mathrm{ImV}_2,$$

$$\varepsilon_4 = \sin(\beta^+ t)\mathrm{ReV}_2 + \cos(\beta^+ \mathrm{t})\mathrm{ImV}_2.$$

It is easy to get $k(\mathrm{ReV}_1) = \mathrm{ReV}_2,\ k(\mathrm{ReV}_2) = \mathrm{ReV}_1,\ \ k(\mathrm{ImV}_1) = \mathrm{ImV}_2,\ k(\mathrm{ImV}_2) = \mathrm{ImV}_1,$ $\rho(\mathrm{ReV}_1) = -\mathrm{ImV}_1,\ \rho(\mathrm{ReV}_2) = \mathrm{ImV}_2,\ \rho(\mathrm{ImV}_1) = \mathrm{ReV}_1,\ \rho(\mathrm{ImV}_2) = -\mathrm{ReV}_2$. Thus we have the following lemma.

Lemma 3.3.

$$k\varepsilon_1 = \varepsilon_3, k\varepsilon_2 = \varepsilon_4, k\varepsilon_3 = \varepsilon_1, k\varepsilon_4 = \varepsilon_2$$

and

$$\rho\varepsilon_1 = -\varepsilon_2, \rho\varepsilon_2 = \varepsilon_1, \rho\varepsilon_3 = \varepsilon_4, \rho\varepsilon_4 = -\varepsilon_3.$$

Let $T = \frac{2\pi}{\beta^+}$ and denote P_T as the Banach space of all continuous $T-$ periodic functions x. The role of $D_4 \times S^1$ on P_T is

$$(\gamma, \theta)x(t) = \gamma x(t + \theta),\ (\gamma, \theta) \in D_4 \times S^1,\ x \in P_T,$$

where $S^1 =< e^{i\theta} >$ is the temporal.

Denoting SP_T as the subspace of P_T consisted of all T-periodic solutions of system (2.1) with $\tau = \tau_k^+$. For each subgroup $\sum \subset D_4 \times S^1$, $Fix(\sum, SP_T) = \{x \in SP_T, (\gamma, \theta)x = x,\ for\ all\ (\gamma, \theta) \in \sum\}$ is a subspace.

For the following subgroups of $D_4 \times S^1$

$$\textstyle\sum_1 = \langle (k, 1) \rangle, \quad \sum_2 = \langle (k, -1) \rangle,$$

$$\textstyle\sum_3 = \langle (\rho, e^{i\frac{T}{4}}) \rangle, \quad \sum_4 = \langle (\rho, e^{-i\frac{T}{4}}) \rangle.$$

We have the following lemma.

Lemma 3.4. $\mathrm{Fix}(\sum_j, SP_T)$ $(j = 1, 2, 3, 4.)$ *are all the two-dimensional subspaces of* SP_T. *Where*

$$\mathrm{Fix}(\textstyle\sum_1, SP_T) = \{y_1(\varepsilon_1 + \varepsilon_3) + y_2(\varepsilon_2 + \varepsilon_4)|\ y_1, y_2 \in R\},$$

$$\mathrm{Fix}(\textstyle\sum_2, SP_T) = \{y_1(\varepsilon_1 - \varepsilon_3) + y_2(\varepsilon_2 - \varepsilon_4)|\ y_1, y_2 \in R\},$$

$$\mathrm{Fix}(\textstyle\sum_3, SP_T) = \{y_1\varepsilon_3 + y_2\varepsilon_4|\ y_1, y_2 \in R\},$$

$$\mathrm{Fix}(\textstyle\sum_4, SP_T) = \{y_1\varepsilon_1 + y_2\varepsilon_2|\ y_1, y_2 \in R\}.$$

Proof. Let $x = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 + x_4\varepsilon_4$.

(i) It is necessary for us to prove $x \in \mathrm{Fix}(\sum_1, SP_T)$ if and only if $kx = x$. By the lemma 3.3, we have

$$kx = k(x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 + x_4\varepsilon_4) = x_1\varepsilon_3 + x_2\varepsilon_4 + x_3\varepsilon_1 + x_4\varepsilon_2.$$

Hence , $kx = x$ if and only if $x_1 = x_3$ and $x_2 = x_4$. This implies that $\mathrm{Fix}(\sum_1, SP_T)$ is spanned by $\varepsilon_1 + \varepsilon_3$ and $\varepsilon_2 + \varepsilon_4$.

(ii) It is enough to prove that $x \in \mathrm{Fix}(\sum_2, SP_T)$ if and only if $kx(t) = x(t + \frac{T}{2})$. It is clearly that

$$x(t + \frac{T}{2}) = x_1\varepsilon_1(t + \frac{T}{2}) + x_2\varepsilon_2(t + \frac{T}{2}) + x_3\varepsilon_3(t + \frac{T}{2}) + x_4\varepsilon_4(t + \frac{T}{2}),$$

where

$$\begin{aligned}\varepsilon_1(t + \frac{T}{2}) &= \cos(\beta^+(t + \frac{T}{2}))\mathrm{ReV}_1 - \sin(\beta^+(\mathrm{t} + \frac{\mathrm{T}}{2}))\mathrm{ImV}_1 \\ &= -\cos(\beta^+ t)\mathrm{ReV}_1 + \sin(\beta^+\mathrm{t})\mathrm{ImV}_1 \\ &= -\varepsilon_1.\end{aligned}$$

By the same token

$$\varepsilon_2(t + \frac{T}{2}) = -\varepsilon_2,\ \varepsilon_3(t + \frac{T}{2}) = -\varepsilon_3,\ \varepsilon_4(t + \frac{T}{2}) = -\varepsilon_4.$$

So

$$x(t + \frac{T}{2}) = -x_1\varepsilon_1 - x_2\varepsilon_2 - x_3\varepsilon_3 - x_4\varepsilon_4.$$

Then we have $kx(t) = x(t+\frac{T}{2})$ if and only if $-x_1 = x_3,\ x_2 = -x_4$ and $\mathrm{Fix}(\sum_2, SP_T)$ is the span of $\varepsilon_1 - \varepsilon_3$ and $\varepsilon_2 - \varepsilon_4$.

(iii) It is sufficient to prove that $x \in \mathrm{Fix}(\sum_3, SP_T)$ if and only if $\rho x(t) = x(t-\frac{T}{4})$. By lemma 3.3 , we can get

$$\rho x = \rho(x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 + x_4\varepsilon_4) = -x_1\varepsilon_2 + x_2\varepsilon_1 + x_3\varepsilon_4 - x_4\varepsilon_3$$

and

$$\begin{aligned}
x(t-\frac{T}{4}) &= x_1\varepsilon_1(t-\frac{T}{4}) + x_2\varepsilon_2(t-\frac{T}{4}) + x_3\varepsilon_3(t-\frac{T}{4}) + x_4\varepsilon_4(t-\frac{T}{4}) \\
&= x_1(\cos(\beta^+(t-\frac{T}{4}))\mathrm{Re}V_1 - \sin(\beta^+(t-\frac{T}{4}))\mathrm{Im}V_1) + x_2(\sin(\beta^+(t-\frac{T}{4}))\mathrm{Re}V_1 \\
&\quad + \cos(\beta^+(t-\frac{T}{4}))\mathrm{Im}V_1) + x_3(\cos(\beta^+(t-\frac{T}{4}))\mathrm{Re}V_2 - \sin(\beta^+(t-\frac{T}{4}))\mathrm{Im}V_2) \\
&\quad + x_4(\sin(\beta^+(t-\frac{T}{4}))\mathrm{Re}V_2 + \cos(\beta^+(t-\frac{T}{4}))\mathrm{Im}V_2),
\end{aligned}$$

then

$$\cos(\beta^+(t-\frac{T}{4})) = \sin(\beta^+ t),\ \ \sin(\beta^+(t-\frac{T}{4})) = -\cos(\beta^+ t).$$

Hence

$$x(t-\frac{T}{4}) = x_1\varepsilon_2 - x_2\varepsilon_1 + x_3\varepsilon_4 - x_4\varepsilon_3.$$

It can be attained that $\rho x(t) = x(t-\frac{\pi}{4})$ if and only if $x_1 = x_2 = 0$, such that $\mathrm{Fix}(\sum_3, SP_T)$ is the span of ε_3 and ε_4.

(iv) It is adequate to prove that $x \in \mathrm{Fix}(\sum_4, SP_T)$ if and only if $\rho x(t-\frac{T}{4}) = x(t)$. Similarly, we have

$$\rho x(t-\frac{T}{4}) = \rho(x_1\varepsilon_2 - x_2\varepsilon_1 + x_3\varepsilon_4 - x_4\varepsilon_3) = x_1\varepsilon_1 + x_2\varepsilon_2 - x_3\varepsilon_3 - x_4\varepsilon_4.$$

Hence $\rho x(t-\frac{T}{4}) = x(t)$ if and only if $x_3 = x_4 = 0$, and $\mathrm{Fix}(\sum_4, SP_T)$ is the span of ε_1 and ε_2. □

By lemma 3.1, 3.2, 3.4 and theorem 4.1 of [15] , we can get the following theorem.

Theorem 3.1. *There exist three branches of small-amplitude periodic solutions with period T near $\frac{2\pi}{\beta^+}$ for equation (2.1), bifurcated from the zero equilibrium at the critical values $\tau_k^+(i = 0, 1, 2\dots)$, and they are*

(i) discrete waves: $x_i(t) = x_{i+1}(t \pm \frac{T}{4})$ for $i(mod4)$ and $t \in R$,

(ii) standing waves: $x_i(t) = x_{4+2j-i}(t + \frac{T}{2})$ for $i(mod4), j \in \{1, 2, 3, 4\}$ and $t \in R$,

(iii) mirror-reflecting solutions: $x_i(t) = x_{4+2j-i}(t)$ for $i(mod4), j \in \{1, 2, 3, 4\}$ and $t \in R$.

4 Normal form on center manifolds and bifurcation direction of bifurcated periodic solutions

In this section, we employ the algorithm and notations of [16] to derive the normal forms of system (2.1) on center manifolds and direction of bifurcation periodic solution.

Let $t \mapsto \frac{t}{\tau}$, then Eq.(2.5) can be rewritten as

$$\dot{z} = L(\tau)z_t + F(z_t, \tau), \tag{4.1}$$

where

$$L(\tau)\varphi = \tau[\begin{pmatrix} A_0 & 0 & 0 & 0 \\ 0 & A_0 & 0 & 0 \\ 0 & 0 & A_0 & 0 \\ 0 & 0 & 0 & A_0 \end{pmatrix}\varphi(0) + \begin{pmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{pmatrix}\varphi(-1)],$$

$$F(\varphi, \tau) = \tau\begin{pmatrix} 0, & -\alpha\varphi_1^2\varphi_2, & 0, & -\alpha\varphi_3^2\varphi_4, & 0, & -\alpha\varphi_5^2\varphi_6, & 0, & -\alpha\varphi_7^2\varphi_8 \end{pmatrix}^T,$$

$$\varphi = (\varphi_1,\ \varphi_2,\ \varphi_3,\ \varphi_4,\ \varphi_5,\ \varphi_6,\ \varphi_7,\ \varphi_8)^T \in C.$$

Let $\mu = \tau - \tau_k^+$, then

$$\dot{z} = L(0)z_t + G(z_t, \mu), \tag{4.2}$$

where $G(z_t, \mu) = L(\mu)z_t - L(0)z_t + F(z_t, \mu)$.

The base of the center space X at $\tau=\tau_k^+$ can be taken as $\Phi(\theta)=(\phi_1,\ \phi_2,\ \phi_3,\ \phi_4)$, where

$$\begin{aligned}&\phi_1(\theta)=e^{i\beta^+\tau_k^+\theta}V_1,\ \phi_2(\theta)=e^{-i\beta^+\tau_k^+\theta}\overline{V}_1,\\ &\phi_3(\theta)=e^{i\beta^+\tau_k^+\theta}V_2,\ \phi_4(\theta)=e^{-i\beta^+\tau_k^+\theta}\overline{V}_2,\end{aligned}\qquad -1\le\theta\le 0.$$

It is easy to check that the base for the adjoint space X^* is $\Psi(s)=(\psi_1(s),\ \psi_2(s),\ \psi_3(s),\ \psi_4(s))^T$, where

$$\begin{aligned}&\psi_1(s)=\overline{D}e^{-i\beta^+\tau_k^+s}\overline{V}_1^T,\ \psi_2(s)=De^{i\beta^+\tau_k^+s}V_1^T,\\ &\psi_3(s)=\overline{D}e^{-i\beta^+\tau_k^+s}\overline{V}_2^T,\ \psi_4(s)=De^{i\beta^+\tau_k^+s}V_2^T,\end{aligned}\qquad 0\le s\le 1,$$

and $\langle\Psi,\ \Phi\rangle=I_4$ for the adjoint bilinear form on $C^*\times C$ as

$$\langle\psi,\ \phi\rangle=\psi(0)\phi(0)-\int_{-1}^{0}\int_{0}^{\theta}\psi(\xi-\theta)d\eta(\theta,0)\phi(\xi)d\xi,$$

where

$$D=\frac{1}{4}\frac{1}{1+(\beta^+)^2-e^{i\beta^+\tau_k^+}(\beta^+)^2(a-c)}.$$

For $x\in C^4$, we have

$$\Phi(0)x=(V_1,\ \overline{V_1},\ V_2,\ \overline{V_2})x=x_1V_1+x_2\overline{V}_1+x_3V_2+x_4\overline{V}_2,$$

$$\Phi(-1)x=x_1e^{-i\beta^+\tau_k^+}V_1+x_2e^{i\beta^+\tau_k^+}\overline{V}_1+x_3e^{-i\beta^+\tau_k^+}V_2+x_4e^{i\beta^+\tau_k^+}\overline{V}_2,$$

$$\Psi(0)=(\overline{DV}_1^T,\ DV_1^T,\ \overline{DV}_2^T,\ DV_2^T)^T.$$

Define a 4×4 matrix

$$\tilde{B}=\text{diag}(i\beta^+\tau_k^+,\ \ -i\beta^+\tau_k^+,\ i\beta^+\tau_k^+,\ \ -i\beta^+\tau_k^+).$$

Using the decomposition of $z_t=\Phi x+y$, system (4.2) can be decomposed as

$$\begin{cases}\dot{x}=\tilde{B}x+\Psi(0)G(\Phi x+y,\mu),\\ \dot{y}=A_{Q_1}y+(I-\pi)X_0G(\Phi x+y,\mu),\end{cases}$$

where $x\in C^4,\ y\in Q_1$. We have the Taylor expansion as following

$$\dot{x}=\tilde{B}x+\sum_{j\ge 2}\frac{1}{j!}f_j^1(x,y,\mu),$$

where $f_j^1(x,y,\mu)$ are homogeneous polynomials of degree j in (x,y,μ) with coefficients in C^4. Then the normal form of (4.2) on the center manifold at the origin as for $\mu=0$ is given by

$$\dot{x}=\tilde{B}x+\frac{1}{2!}g_2^1(x,0,\mu)+\frac{1}{3!}g_3^1(x,0,\mu)+h.o.t, \tag{4.3}$$

where $g_2^1,\ g_3^1$ will be calculated in the following.

$$\begin{aligned}
\frac{1}{2!}f_2^1(x,0,\mu) &= \Psi(0)L(\mu)(\Phi x)\\
&= \Psi(0)\mu[\begin{pmatrix} A_0 & 0 & 0 & 0\\ 0 & A_0 & 0 & 0\\ 0 & 0 & A_0 & 0\\ 0 & 0 & 0 & A_0\end{pmatrix}\Phi(0)x+\begin{pmatrix} A & B & C & B\\ B & A & B & C\\ C & B & A & B\\ B & C & B & A\end{pmatrix}\Phi(-1)x]\\
&= 4\mu\begin{pmatrix} \overline{D}((m_1+n_1)x_1+(m_2+n_2)x_4)\\ D((\overline{m}_1+\overline{n}_1)x_2+(\overline{m}_2+\overline{n}_2)x_3)\\ \overline{D}((m_1+n_1)x_3+(m_2+n_2)x_2)\\ D((\overline{m}_1+\overline{n}_1)x_4+(\overline{m}_2+\overline{n}_2)x_1)\end{pmatrix},
\end{aligned} \tag{4.4}$$

where

$$m_1=\alpha p^2(\beta^+)^2+2i\beta^+,\ n_1=(\beta^+)^2(a-c)e^{-i\beta^+\tau_k^+},$$

$$m_2=-\alpha p^2(\beta^+)^2,\ n_2=-(\beta^+)^2(a-c)e^{i\beta^+\tau_k^+}.$$

Since $M_j^1p(x,\mu)=D_xp(x,\mu)\tilde{B}x-\tilde{B}p(x,\mu),\ j\geq 2$, and

$$\begin{aligned}
&\mathrm{Ker}(M_2^1)\cap\mathrm{Span}\{\mu x^q e_l : |q|=1, l=1,\ 2,\ 3,\ 4\}\\
&=\mathrm{Span}\{\mu x_1e_1,\mu x_3e_1,\mu x_2e_2,\mu x_4e_2,\mu x_1e_3,\mu x_3e_3,\mu x_2e_4,\mu x_4e_4\},
\end{aligned}$$

where $e_1,\ e_2,\ e_3,\ e_4$ is the canonical basis for C^4. We have

$$\frac{1}{2!}g_2^1(x,\ 0,\ \mu)=\frac{1}{2!}\mathrm{Proj}_{\ker(M_2^1)}f_2^1(x,\ 0,\ \mu)=4\begin{pmatrix}\overline{n}\mu x_1\\ n\mu x_2\\ \overline{n}\mu x_3\\ n\mu x_4\end{pmatrix}, \tag{4.5}$$

where $n = D(\overline{m_1} + \overline{n_1})$.

Then we need to compute

$$g_3^1(x,\ 0,\ \mu) = \mathrm{Proj}_{\mathrm{Ker}(M_3^1)} \tilde{f}_3^1(x,\ 0,\ \mu) = \mathrm{Proj}_{\mathrm{Ker}(M_3^1)} \tilde{f}_3^1(x,\ 0,\ 0) + O(\mu^2|x|),$$

with

$$\tilde{f}_3^1(x,\ 0,\ \mu) = f_3^1(x,\ 0,\ \mu) + \frac{3}{2}[(D_x f_2^1)U_2^1 - D_x U_2^1 g_2^1](x,\ 0,\ \mu) + \frac{3}{2}[(D_y f_2^1)h](x,\ 0,\ \mu),$$

where U_2^1 is the change of variables associated with the transformation from f_2^1 to g_2^1 and h is such that $M_2^2(h) = g_2^2$. It's easy to see from (4.4) and (4.5) that $f_2^1(x, 0, 0) = g_2^1(x, 0, 0) = 0$ for $\mu = 0$. Therefore, we have $\frac{1}{3!}\tilde{f}_3^1(x, 0, 0) = \frac{1}{3!} f_3^1(x, 0, 0)$ and

$$\begin{aligned}
\frac{1}{3!} f_3^1(x,\ 0,\ 0) &= \Psi(0)\tau_k^+ F(\Phi x,\ 0) \\
&= \Psi(0)\tau_k^+ \begin{pmatrix} 0 \\ -\alpha(x_1 + x_3 + x_2 + x_4)^2(x_1 + x_3 - x_2 - x_4) i\beta^+ \\ 0 \\ \alpha(x_1 - x_3 - x_2 + x_4)^2(-x_1 + x_3 - x_2 + x_4)\beta^+ \\ 0 \\ -\alpha(x_1 + x_3 + x_2 + x_4)^2(-x_1 - x_3 + x_2 + x_4) i\beta^+ \\ 0 \\ \alpha(-x_1 + x_3 + x_2 - x_4)^2(x_1 - x_3 + x_2 - x_4)\beta^+ \end{pmatrix} \\
&= 2\alpha\tau_k^+(\beta^+)^2 \begin{pmatrix} \overline{D}((x_1 + x_3 + x_2 + x_4)^2(-x_1 - x_3 + x_2 + x_4) + (x_1 - x_3 - x_2 + x_4)^2(x_1 - x_3 + x_2 - x_4)) \\ D((x_1 + x_3 + x_2 + x_4)^2(x_1 + x_3 - x_2 - x_4) + (x_1 - x_3 - x_2 + x_4)^2(x_1 - x_3 + x_2 - x_4)) \\ \overline{D}((x_1 + x_3 + x_2 + x_4)^2(-x_1 - x_3 + x_2 + x_4) + (x_1 - x_3 - x_2 + x_4)^2(-x_1 + x_3 - x_2 + x_4)) \\ D((x_1 + x_3 + x_2 + x_4)^2(x_1 + x_3 - x_2 - x_4) + (x_1 - x_3 - x_2 + x_4)^2(-x_1 + x_3 - x_2 + x_4)) \end{pmatrix},
\end{aligned}$$

$$
\begin{aligned}
&\mathrm{Ker}(M_3^1)\cap \mathrm{Span}\{\mu x^q e_l : |q|=3,\ l=1,\ 2,\ 3,\ 4\}\\
=\ &\mathrm{Span}\{x_1x_2x_3e_1,\ x_1x_3x_4e_1,\ x_1^2x_2e_1,\ x_1^2x_4e_1,\ x_3^2x_4e_1,\ x_3^2x_2e_1,\\
&x_1x_2x_4e_2,\ x_2x_3x_4e_2,\ x_2^2x_1e_2,\ x_2^2x_3e_2,\ x_4^2x_1e_2,\ x_4^2x_3e_2,\\
&x_1x_2x_3e_3,\ x_1x_3x_4e_3,\ x_1^2x_2e_3,\ x_1^2x_4e_3,\ x_3^2x_4e_3,\ x_3^2x_2e_3,\\
&x_1x_2x_4e_4,\ x_2x_3x_4e_4,\ x_2^2x_1e_4,\ x_2^2x_3e_4,\ x_4^2x_1e_4,\ x_4^2x_3e_4\},
\end{aligned}
$$

then

$$
\frac{1}{3!}g_3^1(x,\ 0,\ 0)=4\alpha\tau_k^+(\beta^+)^2\begin{pmatrix}\overline{D}(-2x_1x_3x_4-x_1^2x_2-x_3^2x_2)\\ D(-2x_2x_3x_4-x_2^2x_1-x_4^2x_1)\\ \overline{D}(-2x_1x_2x_3-x_1^2x_4-x_3^2x_4)\\ D(-2x_1x_2x_4-x_2^2x_3-x_4^2x_3)\end{pmatrix}.
$$

The normal form on the center manifold becomes

$$
\dot{x}=\begin{pmatrix} i\beta^+\tau_k^+ & & & \\ & -i\beta^+\tau_k^+ & & \\ & & i\beta^+\tau_k^+ & \\ & & & -i\beta^+\tau_k^+\end{pmatrix}x+4\begin{pmatrix}\overline{n}\mu x_1\\ n\mu x_2\\ \overline{n}\mu x_3\\ n\mu x_4\end{pmatrix}+\frac{1}{3!}g_3^1(x,0,0)+O(\mu^2|x|),\ (4.6)
$$

for $x\in C^4$. By the following coordinates transformation

$$
\begin{cases} x_1=w_1-iw_2,\\ x_2=w_1+iw_2,\\ x_3=w_3-iw_4,\\ x_4=w_3+iw_4.\end{cases}
$$

Let $\rho_1 = w_1^2 + w_2^2 + w_3^2 + w_4^2,\ \rho_2 = 2(w_1 w_3 + w_2 w_4)$. Eq.(4.6) becomes

$$
\begin{aligned}
\dot{w_1} &= \beta^+ \tau_k^+ w_2 + 4\mu(\mathrm{Re}(n) w_1 - \mathrm{Im}(n) w_2) - 4\alpha\tau_k^+ (\beta^+)^2 [\rho_1(\mathrm{Re}(D) w_1 - \mathrm{Im}(D) w_2) \\
&\quad + \rho_2(\mathrm{Re}(D) w_3 - \mathrm{Im}(D) w_4)] + O(\mu^2 |w| + |w|^4), \\
\dot{w_2} &= -\beta^+ \tau_k^+ w_1 + 4\mu(\mathrm{Im}(n) w_1 + \mathrm{Re}(n) w_2) - 4\alpha\tau_k^+ (\beta^+)^2 [\rho_1(\mathrm{Im}(D) w_1 + \mathrm{Re}(D) w_2) + \\
&\quad \rho_2(\mathrm{Im}(D) w_3 + \mathrm{Re}(D) w_4)] + O(\mu^2 |w| + |w|^4), \\
\dot{w_3} &= \beta^+ \tau_k^+ w_4 + 4\mu(\mathrm{Re}(n) w_3 - \mathrm{Im}(n) w_4) - 4\alpha\tau_k^+ (\beta^+)^2 [\rho_2(\mathrm{Re}(D) w_1 - \mathrm{Im}(D) w_2) \\
&\quad + \rho_1(\mathrm{Re}(D) w_3 - \mathrm{Im}(D) w_4)] + O(\mu^2 |w| + |w|^4), \\
\dot{w_4} &= -\beta^+ \tau_k^+ w_3 + 4\mu(\mathrm{Im}(n) w_3 + \mathrm{Re}(n) w_4) - 4\alpha\tau_k^+ (\beta^+)^2 [\rho_2(\mathrm{Im}(D) w_1 + \mathrm{Re}(D) w_2) \\
&\quad + \rho_1(\mathrm{Im}(D) w_3 + \mathrm{Re}(D) w_4)] + O(\mu^2 |w| + |w|^4).
\end{aligned}
\tag{4.7}
$$

Let

$$
z_1(t) = w_1(s) + i w_2(s),
$$

$$
z_2(t) = w_3(s) + i w_4(s),
$$

$$
s = \frac{t}{(1+\sigma)\beta^+ \tau_k^+},
$$

where σ is a periodic-scaling parameter, then it follows that

$$
\begin{aligned}
(1+\sigma)\dot{z_1} &= -i z_1 + \tfrac{4\mu n}{\beta^+ \tau_k^+} z_1 - 4\alpha\beta^+ D[(|z_1|^2 + |z_2|^2) z_1 + (z_1 \overline{z_2} + \overline{z_1} z_2) z_2], \\
(1+\sigma)\dot{z_2} &= -i z_2 + \tfrac{4\mu n}{\beta^+ \tau_k^+} z_2 - 4\alpha\beta^+ D[(z_1 \overline{z_2} + \overline{z_1} z_2) z_1 + (|z_1|^2 + |z_2|^2) z_2].
\end{aligned}
\tag{4.8}
$$

Let's denote $g(z,\mu)$ as follows,

$$
g(z,\mu) = \begin{pmatrix} i z_1 - \frac{4\mu n}{\beta^+ \tau_k^+} z_1 + 4\alpha\beta^+ D[(|z_1|^2 + |z_2|^2) z_1 + (z_1 \overline{z_2} + \overline{z_1} z_2) z_2] \\ i z_2 - \frac{4\mu n}{\beta^+ \tau_k^+} z_2 + 4\alpha\beta^+ D[(z_1 \overline{z_2} + \overline{z_1} z_2) z_1 + (|z_1|^2 + |z_2|^2) z_2] \end{pmatrix},
$$

then equation (4.8) can be written as

$$
(1+\sigma)\dot{z} + g(z,\mu) = 0. \tag{4.9}
$$

According to [17] (pp.296-297, Theorem 6.3 and 6.5), the bifurcations of small-amplitude periodic solution of (4.9) are comletely determined by the zero point of equation

$$
-i(1+\sigma) z + g(z,\mu) = 0, \tag{4.10}
$$

and (4.10) can be written as

$$A\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + B\begin{pmatrix} z_1^2\bar{z}_1 \\ z_2^2\bar{z}_2 \end{pmatrix} + C\begin{pmatrix} \bar{z}_1 z_2^2 \\ \bar{z}_2 z_1^2 \end{pmatrix} = 0,$$

where

$$A = A_0 + A_N(|z_1|^2 + |z_2|^2), \quad A_0 = -\frac{4\mu n}{\beta^+\tau_k^+} - i\sigma,$$

$$A_N = 8\alpha\beta^+ D, \quad B = B_0 = -4\alpha\beta^+ D, \quad C = 4\alpha\beta^+ D.$$

From the expressions of D and those above, we get the following expressions

$$\mathrm{Re}(D) = \frac{1}{4}\frac{1+(\beta^+)^2+\alpha p^2(\beta^+)^2}{(1+(\beta^+)^2+\alpha p^2(\beta^+)^2)^2+(1-(\beta^+)^2)^2(\beta^+)^2},$$

$$\mathrm{Re}(A_N + B) = 4\alpha\beta^+\mathrm{Re}(D),$$

$$\mathrm{Re}(2A_N + B + C) = 16\alpha\beta^+\mathrm{Re}(D),$$

$$\mathrm{Re}(2A_N + B - C) = 8\alpha\beta^+\mathrm{Re}(D),$$

where $\mathrm{Re}(A_N+B), \mathrm{Re}(2A_N+B+C), \mathrm{Re}(2A_N+B-C)$ are positive when $\alpha > 0$ or $\alpha < \frac{-1-(\beta^+)^2}{p^2(\beta^+)^2}$, and $\mathrm{Re}(A_N + B), \mathrm{Re}(2A_N + B + C), \mathrm{Re}(2A_N + B - C)$ are negative when $\frac{-1-(\beta^+)^2}{p^2(\beta^+)^2} < \alpha < 0$. We have the following theorem by the result of [17] (pp.383, Theorem 3.1).

Theorem 4.1. *The following statements are true.*

(i) If $\alpha > 0$ or $\alpha < \frac{-1-(\beta^+)^2}{p^2(\beta^+)^2}$, then the bifurcations are supercritical at the critical values $\tau = \tau_k^+, k = 0, 1, 2, \cdots$,

(ii) If $\frac{-1-(\beta^+)^2}{p^2(\beta^+)^2} < \alpha < 0$, then the bifurcation are subcritical at the critical values $\tau = \tau_k^+, k = 0, 1, 2, \cdots$.

5 Numerical simulations

In this section, We select parameters to simulate system (2.1). Let $\alpha = -2.5$, $p = -1, w = 1$, $a = -1, b = \frac{1}{2}, c = 2$. According to the calculation, we obtain the $\tau_0^+ = 1.2010$, $\beta^+ =$

2.1282 and $\frac{-1-(\beta^+)^2}{p^2(\beta^+)^2} = -1.2208$. On the basis of Theorem 2.1, we know the zero equilibrium is asymptotically stable when $\tau < \tau_0^+$, and it is shown in Fig.1(a) to Fig.4(a). When $\tau > \tau_0^+$, it is unstable and some periodic solutions (standing waves (Fig.1(b)-Fig.2(b)) and mirror-reflecting waves (Fig.3(b)-Fig.4(b)) appear. Figures 1(c) and 2(c) show that the standing waves of the system (2.1) bifurcated from the zero equilibrium are unstable. Figures 3(c) and 4(c) show that the mirror-reflecting waves bifurcated by the system (2.1) from the zero equilibrium are unstable. Since $\alpha < \frac{-1-(\beta^+)^2}{p^2(\beta^+)^2}$, according to the Theorem 4.1, the bifurcated periodic solutions is supercritical.

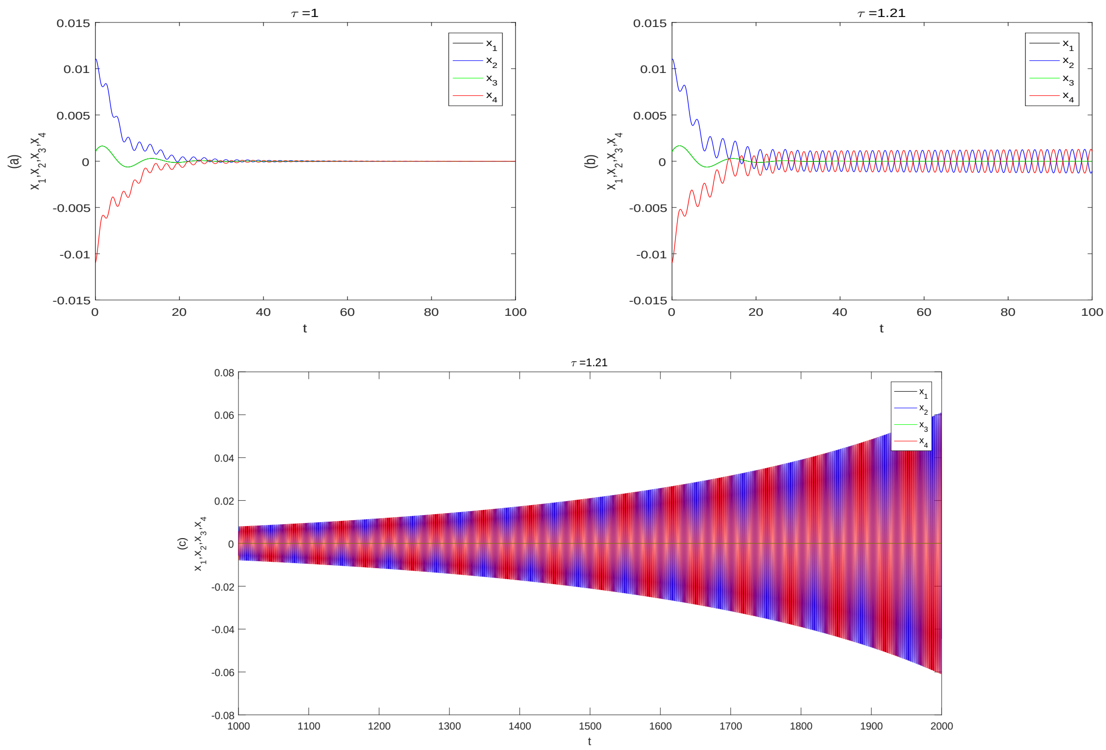

Figure 1: Trajectories $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ of system (2.1) with the initial values $x_1(0) = 0.001, y_1(0) = 0.001, x_2(0) = 0.011, y_2(0) = 0.001, x_3(0) = 0.001, y_3(0) = 0.001, x_4(0) = -0.011, y_4(0) = 0.001$ at $\tau = 1$ (a) and $\tau = 1.21$ ((b) and (c)). (a) represents that the zero equilibrium is asymptotically stable at $\tau = 1 < \tau_0^+ = 1.2010$. (b) represents that periodic solution (standing wave) is generated at $\tau = 1.21 > \tau_0^+ = 1.2010$, indicating that the zero equilibrium is unstable. (c) is a long-term behavior at $\tau = 1.21 > \tau_0^+ = 1.2010$, representing that the standing wave is unstable.

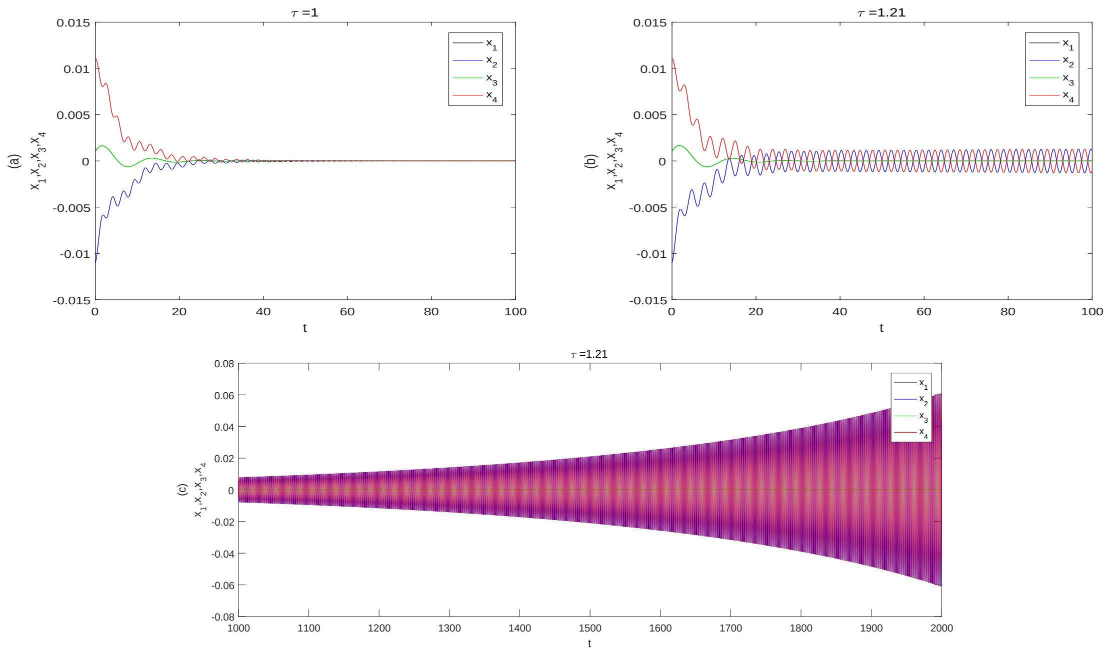

Figure 2: Trajectories $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ of system (2.1) with the initial values $x_1(0) = 0.001, y_1(0) = 0.001, x_2(0) = -0.011, y_2(0) = 0.001, x_3(0) = 0.001, y_3(0) = 0.001, x_4(0) = 0.011, y_4(0) = 0.001$ at $\tau = 1$ (a) and $\tau = 1.21$ ((b) and (c)). (a) represents that the zero equilibrium is asymptotically stable at $\tau = 1 < \tau_0^+ = 1.2010$. (b) represents that periodic solution (standing wave) is generated at $\tau = 1.21 > \tau_0^+ = 1.2010$, indicating that the zero equilibrium is unstable. (c) is a long-term behavior at $\tau = 1.21 > \tau_0^+ = 1.2010$, representing that the standing wave is unstable.

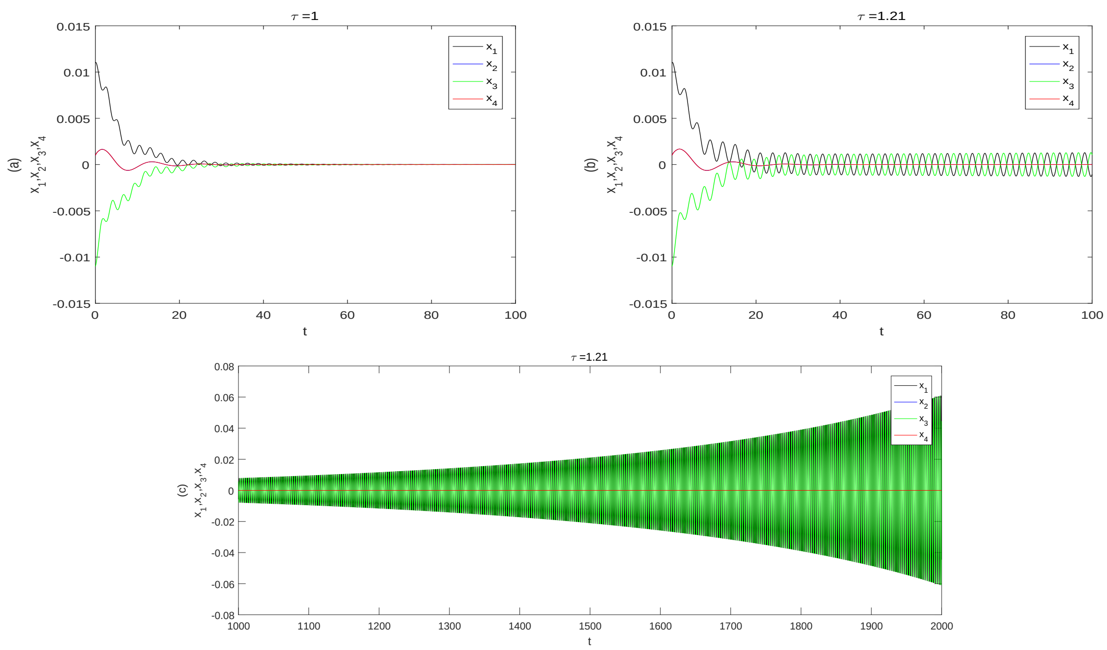

Figure 3: Trajectories $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ of system (2.1) with the initial values $x_1(0) = 0.011, y_1(0) = 0.001, x_2(0) = 0.001, y_2(0) = 0.001, x_3(0) = -0.011, y_3(0) = 0.001, x_4(0) = 0.001, y_4(0) = 0.001$ at $\tau = 1$ (a) and $\tau = 1.21$((b) and (c)). (a) represents that the zero equilibrium is asymptotically stable at $\tau = 1 < \tau_0^+ = 1.2010$. (b) represents that periodic solution (mirror-reflecting wave) is generated at $\tau = 1.21 > \tau_0^+ = 1.2010$, indicating that the zero equilibrium is unstable. (c) is a long-term behavior at $\tau = 1.21 > \tau_0^+ = 1.2010$, representing that the mirror-reflecting wave is unstable.

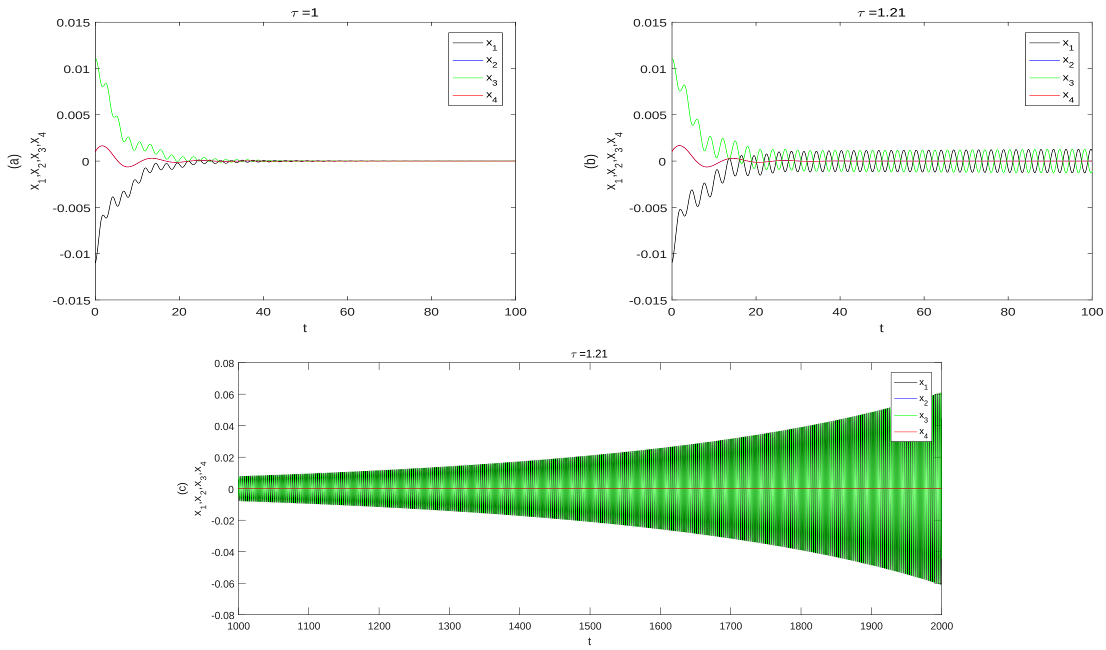

Figure 4: Trajectories $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ of system (2.1) with the initial values $x_1(0) = -0.011, y_1(0) = 0.001, x_2(0) = 0.001, y_2(0) = 0.001, x_3(0) = 0.011, y_3(0) = 0.001, x_4(0) = 0.001, y_4(0) = 0.001$ at $\tau = 1$ (a) and $\tau = 1.21$((b) and (c)). (a) represents that the zero equilibrium is asymptotically stable at $\tau = 1 < \tau_0^+ = 1.2010$. (b) represents that periodic solution (mirror-reflecting wave) is generated at $\tau = 1.21 > \tau_0^+ = 1.2010$, indicating that the zero equilibrium is unstable. (c) is a long-term behavior at $\tau = 1.21 > \tau_0^+ = 1.2010$, representing that the mirror-reflecting wave is unstable.

6 Conclusions

In this paper, four coupled van der Pol oscillators system (1.1) is discussed. The system is symmetrical and D_4 equivariant. By analyzing the characteristic equation of the system, the conditions and critical value $\tau_k^{\pm}(k = 0, 1, 2 \dots)$ of losing stability of the zero equilibrium are obtained. At the critical value $\tau_k^{\pm}(k = 0, 1, 2 \dots)$, the system exhibits asynchronous periodic solutions: discrete waves, mirror-emitted waves and standing waves. The normal form of the system on the central manifold is obtained, and the bifurcation direction is studied. Finally, the theoretical results are supported by using numerical simulation. In addition, standing waves and mirror-reflecting waves are unstable by numerical simulation. Unfortunately, the discrete wave of the system has not been simulated.

Declarations

Competing interests The author declares that they have no competing interests.

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Data Availability Data sharing is not applicable to this article, as no datasets were generated or analyzed during the current study.

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References

[1] Mathias. S. Heltberg, Yuanxu Jiang, Yingying Fan, et al. [2023] "Coupled oscillator cooperativity as a control mechanism in chronobiology", *cell systems*, **14(5)**, pp.382-391.e5.

[2] Mariamo Mussa Juane, David García-Selfa , Alberto P. Mu uzuri. [2020] "Turing instability in nonlinear chemical oscillators coupled via an active medium",*Chaos, Solitons* & *Fractals*, **133**, 109603.

[3] A. Acosta , R.Gallo , P. García , D.Peluffo-Ordóez. [2023] "Positive invariant regions for a modified Van Der Pol equation modeling heart action", *Applied Mathematics and Computation*, **442(1)**, 127732.

[4] Rastogi, P. Rastogi, V. [2019] "Synchronization dynamics and numerical simulation of three coupled oscillators through extended Poincaré-Cartan invariants", *AIP Conference Proceedings*, **2061(1)**, 020017.

[5] J.V.Ngamsa Tegnitsap , H.B.Fotsin , E.B.Megam Ngouonkadi. [2021] "Magnetic coupling based control of a chaotic circuit: Case of the van der Pol oscillator coupled to a linear circuit", *Chaos, Solitons & Fractals*, **152**, 111319.

[6] Yanli Wang , Xianghong Li , Yongjun Shen .[2023] "Vibration reduction mechanism of Van der Pol oscillator under low-frequency forced excitation by means of nonlinear energy sink", *International Journal of Non-Linear Mechanics*, **152**,104389.

[7] Zhao, N. Sun, Z. Yang, X. Xu, W. [2018] "Explosive death of conjugate coupled Van der Pol oscillators on networks", *Physical Review E***97(6)**, 062203.

[8] Batool, A. Hanif, A. Hamayun, M.T. Ali, S.M.N. [2018] "Control design for the compensation of limit cycles in Van der Pol oscillator", *Proceedings - 2017 13th International Conference on Emerging Technologies, ICET2017*, **2018-January**, pp.1-6.

[9] Henrici, A. Neukom, M. [2016] "Synchronization of van der Pol oscillators with delayed coupling", *ICMC 2016 - 42nd International Computer Music Conference, Proceedings*, pp.212-217.

[10] Mohamed Elfouly. [2023] "Hopf bifurcation and chaotic motion for Van der Pol model as two-delay differential equation in basal ganglia disorder", *Brain Stimulation* , **16(1)**, pp.305.

[11] Xindong Ma , Yue Yu , Lifeng Wang. [2021] "Complex mixed-mode vibration types triggered by the pitchfork bifurcation delay in a driven van der Pol-Duffing oscillator", *Applied Mathematics and Computation*, **411**, 126522.

[12] Zhang, C.R. Zheng, B.D. Wang, L.C. [2011] "Multiple Hopf bifurcation of three coupled van der Pol oscillators with delay", *Applied Mathematics and Computation*, **217(17)**, pp.7155-7166.

[13] Song, Y.L. Xu, J. Zhang, T.H. [2011] "Bifurcation, amplitude death and oscillation patterns in a system of three coupled van der Pol oscillators with diffusively delayed velocity coupling", *Chaos*, **21(2)**, 02311

[14] Krakhovskaya, N. Astakhov, S. [2018] "Forced Synchronization of Central Pattern Generator of the Van der Pol Oscillator with an Additional Feedback Loop", *DCNAIR*, **8589215**, pp.87-89.

[15] Wu, J. [1998] "Symmetric functional equations and neural networks with memory", *T.Am.Math.Soc*, **350(12)**, pp.4799-4838.

[16] Faria, T. Magalháes, L.T. [1995] "Normal forms for retarded functional differential equation with parameters and applications to Hopf bifurcation", *J. Differ. Equations*, **122(2)**, pp.181-200.

[17] Golubitsky, M. Stewart, I. Schaeffer,D. [1988] "Singularities and Groups in Bifurcation Theory", *Springer-Verlag, New York.*