Extending a Euclidean Model of Ratio and Proportion

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Research Article

Keywords: ratio, proportion, rational number, fractions, elementary and middle school mathematics, didactical models

Posted Date: March 15th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-300613/v1>

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Extending a Euclidean Model of Ratio and Proportion

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Mar. 4, 2021

Keywords: ratio, proportion, rational number, fractions,
elementary and middle school mathematics, didactical models

DECLARATIONS

Funding: NA

Conflicts of interest/competing interests: None.

Availability of data and material: [see Code availability]

Code availability: The software, *Exploring Relative Magnitude*, mentioned in the text is being made freely available by the author through a GNU General Public License v. 3.0. It is written in Mathematica v. 12.2 and runs in the freely available Wolfram Player.

Authors' contributions: Sole author of article and of the software referred to in paper.

Abstract

This paper is written for mathematics educators and researchers engaged at the elementary and middle school levels and interested in exploring ideas and representations for introducing students to ratio and proportion and for making a smooth transition from multiplication and division by whole numbers to their counterparts with fractions. Book V of Euclid's Elements offers a scenario for deciding whether two ratios of magnitudes, embodied as a pair of line segments, are equal based on whether the ratios of magnitudes, when multiplied by the same whole numbers, n and m , each yield common products. This test of proportion can be performed using an educational software application where students are presented with a *target ratio* of commensurable magnitudes, $A:B$, and challenged to produce a *selected ratio*, $C:D$, that behaves like the target ratio under the critical conditions. The selected ratio is automatically constructed such that $C:D = m:n$, on the basis of a lattice point (n, m) chosen by the student. By adding partitive and Euclidean division to Euclid's model, five new scenarios with similar goals are proposed. Representations in the Euclidean plane, on a number line, and in the Cartesian plane provide feedback that students may use to help identify a ratio of whole numbers corresponding to the target ratio of magnitudes. The representations serve to highlight fractions as members of equivalence classes. The model remains to be investigated with teachers and students.

Comparing Ratios: Book V of the Elements and Beyond

This paper is written for mathematics educators and researchers engaged at the elementary and middle school levels and interested in exploring ideas and representations for introducing students to ratio and proportion and for making a smooth transition from multiplication and division by whole numbers to their counterparts with fractions. The ideas and representations have roots in Book V of Euclid's *Elements*, where a ratio corresponds to the relative magnitude of a pair of line segments and to the relative magnitude of a pair of whole number operators. Because ratios were limited to comparisons of entities of the same kind¹, they offer a straightforward means of expressing the relations of *less than*, *equal to*, *greater than*, and *q times as great as* through juxtaposed line segments. Although mathematical notation in Euclid's time was limited, modern notation and representational systems may prove useful in extending the model and making it suitable for mathematics education in the present-day.

In Book V of the *Elements*, *Euclid* suggests a scenario in which one multiplies line segments in pursuit of common products, that is, as two equal outcomes of multiplication. When partitive division and Euclidean division are brought into the model, additional scenarios may be developed that serve, along with the common product scenario, to compare and order ratios of magnitudes. They may also serve to identify a ratio of whole numbers corresponding to a ratio of magnitudes, provided that the magnitudes are multiples of a common unit.

We begin by describing the Euclidean scenario involving multiplication of magnitudes represented as the successive concatenation of copies of given line segments. We then propose five additional scenarios aimed at the pursuit of common quotients, the transformation of one magnitude into another (via multiplication and division by whole numbers, by fractional operators, or by fractional increments), and the deployment of Euclid's algorithm. Next, we examine how representations in the Cartesian plane and on the real line may be associated with operations on pairs of line segments and how equivalent ratios are represented in each of these systems.

The intent is to describe scenarios that may serve as resources for nurturing schemes teachers and students at different levels might deploy for reasoning about: (a) ratios, quotients, and fractions as reflections of relative magnitude; (b) multiplication and division by fractions as compositions of multiplication and division by whole numbers; (c) fractions as rational numbers, as slopes, and as constants of proportionality in functions of direct proportion; (e) representations of numbers as points on a number line.

We believe that the schemes students develop will entail perceptual structures for representing mathematical objects such as ratios, proportions and fractions as “`things’ they can describe, measure, analyze, model, and symbolize with culturally accepted words, diagrams, and signs (Abrahamson,

¹ This implies that Euclid's approach will likely be less suited for exploring ratios corresponding to intensive quantities or to linear functions where elements of the domain differ in kind from elements in the co-domain.

Dutton & Bakker, 2021, in press).” At the present stage of development, we will approach the schemes as “scenarios” to be introduced to students, leaving the sense teachers and students make of such scenarios for a future moment, when appropriate empirical data are available.

Introductory remarks

Ratio and proportion permeate the multiplicative conceptual field and entail invariants arising in diverse structures (isomorphism of measures, product of measures, and multiple proportion), symbolic representations (e.g., the number line, graphs of functions in the Cartesian plane, and algebraic notation), and situations (Vergnaud, 1983). By this standard, the model of ratio and proportion set forth in Book V of Euclid’s Elements, restricted to quantities of the same kind, is limited. For instance, it will not be directly applicable to problems entailing relations between elements of distinct “measure spaces”, that is, between elements from different quantity dimensions (such as time and distance). But there is considerable evidence that (1) students are having difficulty with fractions and rational numbers in middle school and well into high school (Carpenter, Corbitt, Kepner, Lindquist & Reys, 1980; National Mathematics Advisory Panel, 2008) and (2) their difficulties are rooted in misunderstandings about the magnitude of fractions as well as the effect of a fractional operator on the magnitude of the outcome (Siegler, Thompson, & Schneider, 2011; Siegler, Fazio, Bailey, 2013; Torbeyns, Schneider, Xin & Siegler, 2015).

Restricting our focus to comparisons of quantities of the same kind does not mean that we will be setting functions to the side, for relations between same kind of quantities may entail functions. In addition, they direct our attention towards mathematical topics related to the expansion of the concept of number, from natural numbers, through integers, to rational numbers. In this context, certain terms merit special attention.

Ratio, proportion, magnitude, and quantity are slippery terms because each has more than one distinct yet recognized meaning. Ambiguity can be minimized by restricting usage to a single meaning or by using various meanings while pointing out the intended interpretation in specific contexts. We follow both courses here.

Magnitude refers to the greatness, size, extent, or intensity of an entity, whether it be a phenomenon, a thing, a substance or a mathematical object such as a number. Entities can be ordered by their magnitudes. Collections of discrete entities can be ordered by their counts.

In ancient Greek mathematics, magnitude (μέγεθος) typically refers to some unmeasured size such as length, area, or volume, sweep of an angle, or size of a number.

There is a general agreement nowadays among scientific communities that a *quantity* is a “property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference (BIPM, IFCC, IUPAC, & ISO, 2012, p. 2).” This definition contains two distinguishable meanings: *quantity* may refer to the property itself or to measurements of the property (or counts of discrete quantities). Here we use the term to *quantity*

in the first sense. We refer to measurements or counts of a quantity as the *quantity value*. Thompson (1993) captures this distinction between quantities and quantity values:

Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them. You can think of your height, another person's height, and the amount by which one of you is taller than the other without having to know the actual values. (pp. 165–166)

There are two main kinds of *relative magnitude*. The first refers to an additive difference. In the above quotation, the “amount by which one of you is taller than the other” refers to an unmeasured additive difference. If, however, the quantity value of each person’s height is known, one value may be subtracted from the other, yielding a difference consisting of a number together with a unit of measure. The first kind of difference is a *quantity*²; the second is a *quantity value*.

Numbers by themselves, sometimes referred to as “pure numbers”, also have magnitude. And like quantity values, a pure number may be subtracted from another pure number to yield an additive difference.

A *ratio* refers to an expression of relative magnitude in a multiplicative sort sense. When the two items of interest are unmeasured quantities of the same kind, the ratio may be expressed numerically. For instance, if the height of an infant named George is one half the height of his father, Daniel, and G and H stand for their respective, *unmeasured* heights, then $G:H = 1:2$, $G = \frac{1}{2}D$, $2G = D$, $G = 0.5D$, or even $G \div D = \frac{1}{2}$. Although neither G nor D has been measured using an external unit, such as inches or centimeters, the ratio of their heights has been assigned a number.

The value 0.5 in this example might have formerly been described as “dimensionless”, but present-day convention in scientific communities would treat the present case of 0.5 as a value on the “quantity dimension 1” (BIPM, op. cit., p. 8). In other words, because 0.5 represents as ratio of two heights, it is regarded a quantity value, despite having no units and not being directly related to a quantity dimension such as length, mass, or area. Values of Mach numbers, friction values, or solid angles are, likewise, of dimension 1. Number of entities (e.g., number of children, number of correct answers) is also understood to have a quantity dimension of 1.

A *proportion* may refer to an equality of ratios of whole numbers. For instance, the equality, $2:4 = 4:6$ (or either of the variants, $\frac{2}{3} = \frac{4}{6}$, $2 \div 4 = 4 \div 6$) expresses a proportion. A proportion may also refer to an equality of quantities of the same kind. If A , B , C , and D are magnitudes represented by line segments, $A:B = C:D$, $\frac{A}{B} = \frac{C}{D}$, or $A \div B = C \div D$ represent proportions.

An *intensive quantity* entails a ratio of different kinds of quantities (Schwartz, 1996). Speed is a ratio of distance over time. Density of a substance is its mass divided by its volume. Population density refers to the number of inhabitants per unit of area. Although issues regarding intensive

² It is, more specifically, a vector quantity (in one dimension) because it involves magnitude and direction and follows the rules of vector addition.

quantities are important in mathematics and science education, there is no clear way to directly compare magnitudes of different dimensions. Consider, for example: “Which is greater: three miles, three minutes, or three kilograms?” Even so, intensive quantities often involve proportions. For instance, 3 miles: 6 minutes = 10 miles: 20 minutes, is a proportion.

For Euclid, a ratio invariably concerns the relative magnitude of quantities of the same kind.

In functions of direct proportion of the form $F(x) = \frac{n}{m}x$, the ratio of the function’s value to its input also represents a proportion. Thus, for all x in a given domain, $F(x):x = n:m$ (and $\frac{F(x)}{x} = \frac{n}{m}$ and $F(x) \div x = n \div m$). In these cases, the proportion is expressed by an equation, not simply an equality, because a variable is involved. Here $\frac{n}{m}$ is referred to as a *constant of proportionality*.

Although we begin our analysis with the comparison of ratios of fixed values (constants or constant magnitudes), when variation, equivalence, and dilations enter the scene, one shifts to equations. Such shifts signal that one has entered the domain of algebra and moved from fractions to rational numbers.

We are interested in exploring how operations involving ratios of unmeasured quantities and ratios of whole numbers can help imbue the latter with meaning. This involves understanding how quotients of whole numbers may be ordered as rational numbers on the real line and also understanding how the magnitudes of rational operators and operands determine the magnitude of the outcome of operations.

A Euclidean model of ratio and proportion

In Book V of the *Elements*, Euclid represents ratios according to the relative size of two magnitudes of the same kind represented by line segments. Two ratios of magnitudes are to be compared by operating on the magnitudes and taking note of the outcomes. In Definition 5, Euclid draws attention to the case in which a proportion holds, that is, two ratios of non-numerical magnitudes are equal:

Magnitudes are said to **be in the same ratio**, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order (Heath, 1956).

This statement can be clarified using A , B , C and D to refer to non-numerical magnitudes represented as line segments and m and n to refer to whole number multipliers. The operation of multiplying line segment A by n is to be understood as the concatenation of n instances of A , yielding the line segment, nA . The statement can be reformulated as:

If line segments A and C are multiplied by n and B and D are multiplied by m , where n and m are whole numbers, then one can compare nA with mB , and nC with mD . If the ratios $A:B$ and $C:D$ are equal, then, the outcome for the first ratio will invariably match the outcome for the second ratio. In other words, if $A:B = C:D$ then one of three outcomes will always occur: (a) $nA < mB$ and $nC < mD$; (b) $nA > mB$ and $nC > mD$; or (c) $nA = mB$ and $nC = mD$.

For ratios of commensurable magnitudes, outcomes (a) and (b) neither clearly confirm nor disconfirm that $A:B = C:D$. However, outcome (c) provides conclusive evidence that the ratios of magnitudes are equal.

A common products outcome of the form $nA = mB$ exists for every ratio of commensurable magnitudes. When a common product based on the same multipliers emerges for both ratios of magnitudes, it can be inferred that the ratios are equal. This can be summarized as follows:

Given four line segments, A , B , C and D (and assuming that line segment A is commensurable with B , and C is commensurable with D) then $A:B$ and $C:D$ are equal if and only if there exist whole numbers n and m such that, $nA = mB$ and $nC = mD$.

As mentioned above, we will restrict Euclid's Definition 5 in Book V of the *Elements* to ratios of commensurable magnitudes, given that, if two ratios of incommensurable magnitudes (think, $\sqrt{2}:1$) are equal, common products will not be attained. So, there can be no definitive evidence, based on Definition 5, that the two ratios are equal³.

A scenario based on the pursuit of common products

Figure 1 shows two pairs of line segments representing ratios of magnitudes one may compare.

Figure 1: Is $A:B = C:D$?

A ratio of magnitudes can be compared to a ratio of whole numbers. If the relative size of the magnitudes matches the relative size of the numbers, the ratios are equal. If the magnitudes are represented by line segments A and B , and the whole numbers by m and n , then, according to Euclid's definition of proportion, $nA = mB$ conveys the idea that $A:B = m:n$.

³ This does not imply that tests could not demonstrate that two ratios of incommensurable quantities could be shown to be equal. If $A:B = \sqrt{2}:1$ and $C:D = 5\sqrt{2}:5$, it could be shown that, by *alternate ratio* (see Definition 12, Book V of Euclid's *Elements*), $A:C = B:D$, which implies that $A:B = C:D$. (However, this method will not work for incommensurable ratios such as $A:B$ and $C:D$, where $A:B = \sqrt{2}:1$ and $C:D = \pi\sqrt{2}:\pi$).

Table 1 shows the outcomes of three attempts to find a common product for the two pairs of line segments (A and B ; C and D). The first attempt shows that $1A < 2B$ and $1C < 2D$. Thus, $A:B$ may equal $C:D$ but does not settle the matter. The second attempt shows that $5A > 6B$ and $5C > 6D$. Once again, the two ratios may be equal but does not imply it. In the third attempt, $3A = 5B$ and $3C = 5D$. A common product emerged for A and B and also for C and D for the multipliers $m = 5$ and $n = 3$. This allows us to conclude that $A:B = C:D$.

Table 1: Three outcomes in the common products scenario.

Attempt	Comparison of nA and mB^a	Comparison of nC and mD
#1 $n=1$ $m=2$
#2 $n=5$ $m=6$
#3 $n=3$ $m=5$

^a The scale is adjusted across attempts in order to facilitate formatting; the ratios $A:B$ and $C:D$ are invariant across such adjustments.

The outcome further establishes that each ratio of magnitudes is equal to a ratio of whole numbers; that is, $A:B = 5:3$ and $C:D = 5:3$. It is important to appreciate what this means.

The statement, $A:B = 5:3$, should not be taken to imply that A equals 5 and B equals 3. A and B are not numbers. They are the names of particular line segments. More importantly, they are to be understood as the non-numeric magnitudes of the segments that, when subjected to arithmetic operations, can be construed as two units of measure. This implies that, if we designate the lengths of line segments A and B as $1A$ and $1B$, respectively, a line segment resulting from the

concatenation of n instances of A has the value nA and a line segment resulting from the concatenation of m instances of B has the value mB .

The distinction between non-numerical magnitudes (Freudenthal, 1986), on one hand, and numbers and quantity values, on the other, receives fairly little attention in present-day K-12 mathematics education. Davydov's (1975) approach to the addition and subtraction of quantities represents a significant exception (Schmittau, 2005; Bass, 2018; Coles, 2021). In the approach, even before students are introduced to numbers, they learn to appreciate that, if A and B are strips of different lengths, such that A is longer than B (i.e., $A > B$), then there must exist a strip, C , shorter than A such that $A = B + C$. They learn to represent strip B as the part of A that remains when C is removed from A and to express this as $B = A - C$. They also learn that C can be represented as the difference between A and B and express this as $C = A - B$ ⁴.

An important premise of Davydov's approach is the idea that addition and subtraction are inherently intertwined. A statement that the result of adding two amounts⁵ is equal a third amount ($B + C = A$) implies two additional ideas related to subtraction ($A - C = B$ and $A - B = C$). In this perspective, additive relations are invariably about both addition and subtraction. This can be appreciated by thinking about additive structures as entailing a ternary relation.

A similar point can be made about multiplicative relations. To say that the result of multiplying an amount by a whole number is equal a third amount ($B \times c = A$) implies two complementary ideas regarding division ($A \div c = B$ and $A \div B = c$). So multiplicative relations of this sort are invariably about both multiplication and division⁶.

Measurement approaches to arithmetic, which maintain a sharp distinction between non-numeric magnitudes and numbers and make use of line segments or fraction strips to represent magnitudes, follow a tradition dating back to Euclid's *Elements* (Madden, 2018). They offer important ways of coordinating geometric and arithmetic thinking and representations.

Euclid's model presumes that one can work with unmeasured ratios of magnitudes and draw inferences about their relative size of the magnitudes by operating on them.

This premise is likely to evoke objections. One might object to the premise on the grounds that diagrams are inherently imprecise and subject to the limits of human perception. Pairs of line segments may have different yet perceptually indistinguishable lengths. Who is to say, for instance, whether line segments $3A$ and $5B$ in Table 1 are exactly equal in length? Could $3A$ not be greater than $5B$ by some small amount?

There is a sensible response to this objection. In a given setting, the *problem poser* (whether a teacher or a software embodiment of the model) can be informed of the ratio of whole numbers that a pair of drawn line segments is intended to represent. The problem poser can then provide precise feedback as to whether the first product segment is less than, greater than, or equal to the

⁴ This exemplifies that addition and subtraction are inherent in ternary relation. That relation can be expressed by three distinct functions. A similar point might be made about multiplication and division.

⁵ The ambiguous term "amount" is used here to stand for an unmeasured quantity, a quantity value, or a number.

⁶ The objects of multiplication and division, unlike addition and subtraction, may be of different kinds.

second product segment. The problem poser can furthermore provide an equality or inequality that removes any doubt about the outcome. Feedback of this sort appears in each of the diagrams in Table 1.

Diagrams associated with the common products scenario are to be understood as mere illustrations of assertions about relationships among quantities. The assertions are to be expressed in ways that do not require reliance on the diagrams.

Line segment scenarios for determining that two ratios are equal

Restricting Euclid's model to commensurable quantities enabled us to conceptualize the definition of proportion in Book V as referring to operations carried out in pursuit of a common product which, when found, reveals a ratio of whole numbers equal to the ratio of commensurable quantities.

We saw, in the Euclidean common products scenario, that if $A:B = m:n$, then, by Definition 5 of Book V, the product of A times n equals the product of B times m ; that is, $A \times n = B \times m$. So, a conjecture that a ratio of magnitudes equals a specified ratio of whole numbers can be tested through specified operations. If a critical outcome is obtained, the conjecture is confirmed. Otherwise, one of two possible alternative outcomes will result, implying that the ratio of magnitudes is either less than or greater than the numerical ratio. A discrepancy between the conjecture and an outcome may be exploited in preparing subsequent conjectures in pursuit of a solution.

Arithmetic operations on magnitudes are straightforward. Addition of two magnitudes is represented by joining or concatenating them. Subtraction of a smaller magnitude from a larger one can be represented as the removal of the former from the latter or a diminishing of the greater by the lesser magnitude⁷. We saw that multiplication of a magnitude A by a whole number n is represented as repeated addition, namely, the concatenation of n instances of line segment A . Partitive division of A by n is represented by the output of one part of A after A has been partitioned into n equal parts, each of which is designated as $\frac{1}{n}A$ or $\frac{A}{n}$. Euclidean division of magnitude A by B returns as quotient a whole number n while leaving a remainder magnitude less than B (possibly the null segment). Although partitive division and Euclidean division of magnitudes are discussed in the *Elements*, neither appears in the treatment of ratio and proportion in Book V. We shall, however, bring them into the extended model.

Once partitive division of magnitudes is brought into the model, additional scenarios may be devised for evaluating whether a particular ratio of whole numbers is equal to a ratio of commensurable magnitudes. Euclidean division applied to unmeasured quantities can also be integrated via the Euclidean algorithm, thereby allowing one to explore how simple continued fractions bear on issues of ratio and proportion. Such topics may be suitable for students at the secondary school level.

⁷ Subtraction of a larger magnitude from a smaller magnitude is undefined here. However, the model of subtraction can be made more general by introducing directed magnitudes, much in the way that the set of natural numbers can be expanded by introducing directed numbers.

Let us summarize the scenarios one might consider in determining the equality of two ratios. These scenarios might be introduced at different grade levels and with varying degrees of support by teachers, adjusted according to the mathematical backgrounds and needs of students. In any case, the following presentation is intended for mathematics educators, not students.

Two ratios—one of commensurable magnitudes, A and B , the other of whole numbers, m and n —are equal, that is, the proportion $A:B = m:n$ is true, if any of the following critical outcomes is reached. If any condition holds, all of the other conditions necessarily hold:

- *Common products* (Table 1): Already described.
- *Common quotients*⁸ (Table 2): division of A by m and B by n yields equal magnitudes, that is, $A \div m = B \div n$. This may be expressed as $\frac{1}{m}A = \frac{1}{n}B$.
- *Transformation of one magnitude into the other by multiplication and division by whole numbers* (Table 3): $(A \times n) \div m = (A \div m) \times n = B$ and $(B \times m) \div n = (B \div n) \times m = A$
- *Transformation of one magnitude into the other by multiplication or division by a fraction* (Table 4): $A \times \frac{n}{m} = A \div \frac{m}{n} = B$ and $B \times \frac{m}{n} = B \div \frac{n}{m} = A$. Multiplication and division by a fraction are introduced as compositions of whole number operations: for instance, $A \times \frac{n}{m} = (A \times n) \div m$.
- *Transformation of one magnitude into the other by a fractional increase or decrease* (Table 5): $A + \frac{(n-m)}{m}A = B$ and $B + \frac{(m-n)}{n}B = A$. The segment, $\frac{(n-m)}{m}A$, corresponds to an increase when $A < B$ and $m < n$; it corresponds to a decrease when $A > B$ and $m > n$. This condition offers an opening for the introduction of negative integers and directed line segments⁹.
- *Same sequence of Euclidean quotients* (Table 6): $A \langle \div \rangle B = m \langle \div \rangle n$ where $A \langle \div \rangle B$ yields the sequence of partial quotients resulting from the application of Euclid's algorithm to (A, B) ; $m \langle \div \rangle n$ returns the sequence of partial quotients resulting from the application of Euclid's algorithm to whole numbers, m and n .

Common products

This scenario is outlined in Table 1.

⁸ This condition corresponds to Euclid's notion of alternate ratios (Elements, Book V, definition 12).

⁹ In an earlier analysis, a larger magnitude was transformed into a smaller magnitude by subtraction, that is, removal, of part of that larger line segment. This form of subtraction can be carried out without negative integers or directed line segments. However, without directed line segments different operations are required for the cases of increase and decrease. The directed line segments approach seems more promising in that it opens the way for expansion of numbers from the set of natural numbers to integers.

Common quotients

Table 2 shows two attempts to find a ratio of whole numbers equal to $A:B$ through a scenario involving division and multiplication by whole numbers. First, the ratio $2:3$ is subjected to a test. When A is divided by 2 and the result, $\frac{1}{2}A$, is multiplied by 3, the end result reveals that $\frac{3}{2}A < B$. Then, on a subsequent attempt, the student learns that $\frac{7}{4}A = B$, confirming that $A:B = 4:7$.

Table 2: Evaluation of a proportion based on a search for a common quotient.

<p>Given magnitudes A, B and whole numbers, m and n, $A:B$ is proportional to $m:n$ if $A \div m$ equals $B \div n$</p>	
<p>A person makes the conjecture that $A:B = 2:3$</p> $\frac{1}{2}A < \frac{1}{3}B$ <p>The conjecture is refuted. $A:B < 2:3$</p>	<p>$A \div 2 < B \div 3$</p>
<p>A person makes the conjecture that $A:B = 4:7$</p> $\frac{1}{4}A = \frac{1}{7}B$ <p>The conjecture is confirmed. $A:B = 4:7$</p>	<p>$A \div 4 = B \div 7$</p>

Multiplication and division by whole numbers

A proportion involving a ratio of quantities and a ratio of whole numbers may also be assessed through multiplication and division by whole numbers. Table 3 shows the results of two attempts to find a ratio of whole numbers equivalent to $A:B$. In this case, division of the antecedent quantity by m is followed by multiplication by n . While $\frac{3}{2}A$ is less than B , $\frac{7}{4}A$ equals B . This demonstrates that $A:B = 4:7$.

Table 3: Evaluation of a proportion based on division and multiplication of magnitudes by whole numbers.

<p>Given magnitudes A, B, and whole numbers m and n, $A:B = m:n$ if $(A \div m) \times n = B$</p>	
<p>A person makes the conjecture that $A:B = 2:3$</p> $(A \div 2) \times 3 = \frac{3}{2}A < B$ <p>$A:B \neq 2:3$ The conjecture is refuted.</p>
<p>A person makes the conjecture that $A:B = 4:7$</p> $(A \div 4) \times 7 = \frac{7}{4}A = B$ <p>$A:B = 4:7$ The conjecture is confirmed.</p>

Multiplication or division by a fraction

Table 4 shows the results of operations in the *multiplication (or division) by fraction scenario* used to assess the same previous conjectures. The final results show that $\frac{3}{2}A < B$ and $\frac{7}{4}A = B$. This shows, as Tables 1 to 3 did, that $A:B = 4:7$.

Table 4: Evaluation of a proportion based on multiplication or division by a fraction.

<p>Given magnitudes A, B, and whole numbers m and n,</p> <p>$A:B = m:n$ if $A \times \frac{n}{m} = B$</p>	
<p>A person makes the conjecture that $A:B = 2:3$</p> <p>$\frac{3}{2}A < B$</p> <p>The conjecture is refuted. $A:B \neq 2:3$</p>	
<p>A person makes the conjecture that $A:B = 4:7$</p> <p>$\frac{7}{4}A = B$</p> <p>The conjecture is confirmed. $A:B = 4:7$</p>	

Adding or subtracting parts

Table 5 shows how two attempts are tested under the fractional increment scenario. The conjecture that $A:B$ equals $2:3$ was evaluated by comparing $A + \frac{1}{2}A$ with B . The inequality $A + \frac{1}{2}A < B$ disconfirms the conjecture. The finding that $A + \frac{3}{4}A = B$ confirms it.

Partial quotients from Euclid's algorithm

Euclid's algorithm is normally employed as a means of producing the greatest common divisor (in the case of numbers) or the greatest common measure (in the case of quantities). We will use the algorithm with different objective in mind, namely, to determine whether the ordered list of partial quotients emerging from the algorithm is the same for two ratios.

Whereas each of the scenarios considered thus far aims at producing two equal magnitudes, the Euclidean algorithm scenario rests on a comparison of an ordered list of quotients that emerges when the algorithm is applied to both a ratio of quantities (i.e., non-numerical magnitudes) and another ratio, either of numbers or of quantities.

Table 5: Evaluation of a proportion employing a fractional increase or decrease.

<p>Given magnitudes A, B, and whole numbers m and n, $A:B = m:n$ if $A + \frac{(n-m)}{m}A = B$</p>	
<p>A person makes the conjecture that $A:B = 2:3$</p> <p>$A + \frac{(3-2)}{2}A = A + \frac{1}{2}A < B$ $A < B - \frac{1}{2}A$</p> <p>The conjecture is refuted.</p> <p>$A:B < 2:3$</p>	<p>Given magnitudes A, B, and $m=2$, $n=3$,</p>
<p>A person makes the conjecture that $A:B = 4:7$</p> <p>$A + \frac{(7-4)}{4}A = A + \frac{3}{4}A = B$ $A = B - \frac{3}{4}A$</p> <p>The conjecture is confirmed.</p> <p>$A:B = 4:7$</p> <p>Because $A/4 = B/7$ it is also possible to conclude that:</p> <p>$B + \frac{(4-7)}{7}B = B - \frac{3}{7}B = A$ $B = A + \frac{3}{7}A$</p>	<p>Given magnitudes A, B, and $m=4$, $n=7$,</p>

Table 6 shows the results obtained when Euclid's algorithm is applied to the ratio of 2:3, to a particular ratio of magnitudes $A:B$, and to the ratio of whole numbers 4:7. Although it is not necessary to use line segments to perform Euclid's algorithm on whole numbers¹⁰, the reader may find it useful for noting the parallels to cases of non-numerical magnitudes. It is found that 2:3 is associated with the quotients (0; 1, 2)¹¹. That is because 3 goes 0 times into 2, leaving a remainder of 2. That remainder, 2, goes one time into 3, leaving a remainder of 1 that goes twice into 2, leaving no remainder.

When Euclid's algorithm is applied to $A:B$, the following quotients emerge: (0; 1, 1, 3). This reveals that $A:B \neq 2:3$. If the ratios had been equal, each ratio would have generated the same ordered list of partial quotients. When Euclid's algorithm is applied to 4:7, the partial quotients, (0; 1, 1, 3), emerge. This confirms that $A:B = 4:7$.

¹⁰ There are options for representing the selected ratio using letters to represent line segments (as is the standard for target ratios).

¹¹ The leftmost digit, followed conventionally by colon, indicates how many times the consequent or second magnitude fits wholly into the antecedent or first magnitude.

Table 6: Evaluation of a proportion based on Euclid's algorithm.

<p>Given magnitudes A, B and whole numbers, m and n,</p> $A:B = m:n \text{ if } A\langle\div\rangle B \text{ equals } m\langle\div\rangle n$ <p>where $A\langle\div\rangle B$ returns the ordered list of quotients of resulting from the application of the Euclidean algorithm to magnitudes A and B, and $m\langle\div\rangle n$ signifies the respective list of quotients when Euclid's algorithm is applied to whole numbers, m and n.</p>		
<p>Given $m_1 = 2, n_1 = 3$</p> <p>quotients: (0; 1, 2)</p> <p> $A\langle\div\rangle B = (0; 1, 1, 3)$ $m_1\langle\div\rangle n_1 = (0; 1, 2)$ $A\langle\div\rangle B \neq m_1\langle\div\rangle n_1$ </p>	<p>Given magnitudes A, B</p> <p>quotients: (0; 1, 1, 3)</p>	<p>Given $m_2 = 4, n_2 = 7$</p> <p>quotients: (0; 1, 1, 3)</p> <p> $A\langle\div\rangle B = (0; 1, 1, 3)$ $c_2\langle\div\rangle d_2 = (0; 1, 1, 3)$ $A\langle\div\rangle B = c_2\langle\div\rangle d_2$ Conjecture confirmed: $A:B = 4:7$ </p>

The Euclidean algorithm may also be represented through operations involving tilings of squares (Bass, 2011). The structure and the list of quotients for $A:B$ and $4:7$ (Table 7) are the same as before (Table 6). The diagrams and quotients for $2:3$ do not match those for $A:B$. However those for $4:7$ match those for $A:B$, confirming the conjecture, $A:B = 4:7$.

Table 7: Attempts to verify two conjectures through Euclidean tilings

<p>Euclidean algorithm applied to $(2, 3)$.</p> <p>3 fits zero times into 2, leaving 2 as remainder. 2 fits one time into 3, leaving 1 as remainder. 1 fits two times into 2, leaving no remainder</p>	<p>quotients: $(0;1,2)$</p> <p>The diagram shows a rectangle with width 3 and height 2. A dashed line at width 2 divides it into a 2x2 orange square and a 1x2 green rectangle. The 1x2 green rectangle is further divided into two 1x1 green squares. The quotient sequence (0;1,2) is shown at the top.</p>
<p>Euclidean algorithm applied to (A, B).</p> <p>B fits zero times into A, leaving A as a remainder. A fits one time into B, leaving R_1 as remainder. R_1 fits one time into A, leaving R_2 as remainder. R_2 fits three times into R_1, leaving no remainder.</p>	<p>quotients: $(0;1,1,3)$</p> <p>The diagram shows a rectangle with width B and height A. A dashed line at width A divides it into an AxA orange square and an A x R1 green rectangle. The A x R1 green rectangle is further divided into an R1 x R1 green square and an A x R2 yellow rectangle. The A x R2 yellow rectangle is further divided into three R2 x R2 yellow squares. The quotient sequence (0;1,1,3) is shown at the top.</p>
<p>Euclidean algorithm applied to $(4, 7)$.</p> <p>7 fits zero times into 4, leaving 4 as a remainder. 4 fits one time into 7, leaving 3 as remainder. 3 fits one time into 4, leaving 1 as remainder. 1 fits three times into 3, leaving no remainder.</p>	<p>quotients: $(0;1,1,3)$</p> <p>The diagram shows a rectangle with width 7 and height 4. A dashed line at width 4 divides it into a 4x4 orange square and a 3x4 green rectangle. The 3x4 green rectangle is further divided into a 3x3 green square and a 1x3 yellow rectangle. The 1x3 yellow rectangle is further divided into three 1x1 yellow squares. The quotient sequence (0;1,1,3) is shown at the top.</p>

Integrating, through Software, the Euclidean Model with Representations in the Cartesian Plane and on the Real Line

Although my earlier effort to extend Euclid's model (Carraher, 1993) offered varied ways of representing fractions, it did not draw attention to fractions as members of equivalence classes of ordered pairs of whole numbers. This shortcoming is addressed in the current proposal by making explicit reference to the slopes of graphs of linear functions and to quotients of integers expressed as fractions or decimals, via points on the real line.

Ratio and proportion are represented differently in the Euclidean plane, in the Cartesian plane, (a coordinate plane), and on the real line (Table 8). In this section we suggest how the three systems of representation might be used together to explore issues of ratio and proportion. We illustrate the ideas by means of the software environment *Exploring Relative Magnitude* (Carraher, 2021), that allows representations in the three systems to be dynamically linked.

Table 8: Representation of ratios in a Euclidean line-segment model, on the real line, and in the Cartesian plane.

	Euclidean plane	The real line	Cartesian plane
Visual embodiment of a ratio as...	... a pair of parallel line segments	... a point on a number line (as well as the displayed distance from the origin to the point).	... a point in the coordinate plane along with the abscissa and ordinate; the slope of a graph of a linear function.
Resources for ordering ratios of magnitudes based on perception	Crude judgments of relative magnitude.	Simple observation of order of points on number line	Judgments of slope of lines emanating from the origin through points
Two ratios of quantities are precisely equal if...	... critical outcomes, associated with equations, emerge for each ratio when the same operators are used.	... the two points occupy same position (and have the same value) on number line.	... the two points (or two graphs) have same slope, that is, the same quotient or "rise over run".
Is there a visual depiction of arithmetic operations?	Yes. Addition, subtraction, multiplication and division on antecedent and consequent magnitudes are expressed through line segment diagrams.	No. No standard conventions exist, aside from displacements, for clearly expressing arithmetic operations.	No. But multiplicative operations on ratios may be expressed as changes in the slope of a graph of a linear function.

As noted, the principal geometric object in the extended Euclidean model is a pair of line segments represented in a plane devoid of coordinates. On the real line, a ratio is expressed as a single point or as a distance from the origin to that point. In the Cartesian plane, the ratio may be

represented as the relative size of two perpendicular line segments associated with the coordinates of a single point corresponding, respectively, to the “run” and the “rise.” The Cartesian plane is also used to represent the graph of a function corresponding to the line, $y = \frac{m}{n}x$, the slope of which corresponds to the constant of proportionality, $\frac{m}{n}$.

One compares and orders ratios differently in the three systems.

While it has been found that children as young as 5 or 6 years of age readily distinguish between ratios of magnitudes less than $\frac{1}{2}$ and ratios greater than $\frac{1}{2}$ (Spinillo and Bryant, 1991), the ordering ratios of magnitudes based solely on perceptual judgments is limited at any age. It is unclear to what extent children appreciate that a ratio of magnitudes is invariant under uniform dilations, but young children do seem to realize that, as a pair of objects approaches the observer, their relative size does not change. Even so, there is generally no reliable way to instantly ascertain, by perceptual judgment alone, whether two ratios of magnitudes are equal.

On the real line, where a ratio is expressed as a single point, positive ratios expressed as quotients diminish as they approach the origin. Proper fractions occupy the span between zero and one; improper fractions reside on positions to the right of one. Equal ratios occupy the same position on the real line.

In the Cartesian plane one may order ratios of points in the first quadrant by imagining or drawing lines from the origin through each point of interest and comparing their inclinations. The slope of the line is the quotient of the ordinate to the abscissa (“rise over run”) of any point on the line (excluding the origin). Under uniform scaling, lines with greater slopes have greater inclinations.

The pedagogical context for the software is fairly straightforward. A *problem poser* (the teacher or the software itself) covertly selects a ratio of whole numbers to underly the problem being posed. The *problem solver*, typically a student (or students), is to discover that numerical ratio. The solver formulates conjectures, receiving feedback from the poser, and presumably using the feedback to move in on a solution, which is eventually to be found by the solver and confirmed by the problem poser.

This context bears a certain similarity to activities in the *guess-my-rule* game (Davis, 1967; Carraher & Earnest, 2003; Schliemann, Carraher, & Brizuela, 2007). In guess-my-rule, the problem poser secretly chooses a function, typically a function over the integers that can be represented by a simple algebraic rule such as $n \rightarrow 2n + 3$. In this case, the problem solver must identify the function in the course of a series of trials. In each trial, the problem solver provides an integer, n . The poser then returns the value of $f(n)$. The goal for the problem solver is to come up with a description of the function that matches the function predetermined by the problem poser.

In *Exploring Relative Magnitude*, the problem poser preselects a ratio of two positive integers the terms of which are limited to a certain range (say, 1 to 15). The poser uses the integers to construct a **target ratio** of magnitudes in the form of two line segments (T and F in Figure 2). The problem solver makes a conjecture about the value of the ratio of those target line segments. He expresses this conjecture by selecting a pair of integers, m and n , that together comprise the

selected numerical ratio, $m:n$. When this is done, two new line segments appear representing the solver's **selected ratio of magnitudes ($D:C$ in Fig. 2).**

Figure 2: A target ratio of magnitudes and a selected ratio of magnitudes

The label above the selected segments, $D:C = 8:12 = 2:3$, conveys that 8:12 is equivalent to 2:3. The label at the top of the target ratio pane

$$\boxed{T:F = D:C}$$

?

poses the question of whether the target ratio equals the selected ratio. This is the issue the problem solver faces. Like the scenario envisioned in Euclid's Definition 5 of Book V of the Elements, the solver is comparing two ratios of magnitudes ($T:F$ and $D:C$). In addition, the solver is indirectly comparing a ratio of magnitudes to a ratio of whole numbers ($T:F$ and 2:3).

Embedding the extended Euclidean model in a software environment enables students and teachers to raise and assess conjectures in the context of diagrams, equations, and inequalities. A software setting may also help establish meaningful connections between the Euclidean model and representations in the Cartesian plane and on number lines.

Figure 3 displays the full screen image from which Fig. 2 was taken. At the top of the screen is a dashboard with user interface controls. In the middle part of the screen one finds the "Target Ratio" and "Selected Ratio" areas, as well as the integer lattice (a square lattice, grid lattice, or simply "grid", in the present case of a plane), titled the "Ratio Selector," on the right. A number line pane is at the bottom of the screen.

Figure 3: Full screen shot of an opening scenario in which a problem solver has conjectured that $T:F = 8:12$.

Several of the features in Fig. 3 are optionally shown. For instance, the label, $m = \frac{8}{12}n = \frac{2}{3}n$ is displayed because a “show line label” and “simplify expressions” options were set. It would have appeared simply as $m = \frac{8}{12}n$ if the simplification option had not been checked. Similarly, the label above the selected ratio, $D:C = 8:12 = 2:3$ would have appeared simply as $D:C = 8:12$ were the simplification not set. The labels, “run = 12” and “rise = 8” appear as a “rise and run” option.

Panes may be individually shown or hidden. In this way it is possible to pose problems and discuss issues from a subset of panes. For instance, a teacher or student might reduce the number of open panes to examine the relationship between the selected point in the grid and the relative size of the line segments drawn in the selected ratio pane (Fig. 4).

Figure 4: Screen displaying only two panes.

The number line in Figure 3 automatically displays ticks consistent with the consequent (the second term) of the selected numerical ratio. Given that $n = 12$, the ticks of the number line are located at multiples of $\frac{1}{12}$. When the chosen ratio is reducible, as $8:12$ is, all equivalent ratios in the grid are used to create “multiple rulers” under the number line. In the present case, rulers for thirds, sixths, ninths, twelfths and fifteenths are displayed (Figure 5) because $\frac{8}{12}$ is equivalent to $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, and $\frac{10}{15}$. The fractions are associated with the highlighted points on the line $m = \frac{8}{12}n = \frac{2}{3}n$ in Figures 3 and 5.

Figure 5: The selected ratio, $D:C$ (or $D \div C$) corresponds to each of the circled fractions under the number line (at left) and each highlighted point in the grid (at right).

This present issue merits thoughtful discussion by teachers and students, for it bears directly on the idea that there is an infinite set of fractions equivalent to $2/3$. The idea that $2/3$ is a rational number rests precisely on the notion that it may represent an arbitrary member of the equivalence class, $(\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \dots)$.

The *ratio selector* enables the problem solver to choose a ratio to match the target ratio. (Although the relative size of a ratio is fixed, the displayed width of a ratio may vary depending on the setting of a slider labelled “dilate”.) Clicking on point (n, m) signals that the solver has selected the ratio $m:n$. The software constructs the line segments D and C in the ratio of $m:n$. For the case at hand, the point $(12,8)$ was selected, corresponding to the ratio 8:12.

The grid in Figure 3 displays a blue line labelled $m = \frac{8}{12}n = \frac{2}{3}n$, on which five points have been automatically highlighted: the selected point, $(12,8)$ as well as points $(3,2)$, $(6,4)$, $(10, 6)$, and $(15,10)$. These five points are the lattice points from the displayed region that fall on the line $y = \frac{8}{12}x$ (Fig. 4). They have equivalent values of rise over run: $\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15}$. This information is also conveyed by the fact that each of these fractions falls at the same position on the number line. The target ratio and selected ratio are displayed as the blue and red points plotted on the number line. The relative position of the points shows that the target ratio, $T:F$, is less than the selected ratio, $D:C$. Because the values under the number line are chosen in accordance with the selected ratio, the value of the target ratio normally cannot be read off directly from the ticks on the number line (unless the target ratio matches a tick on the number line).

Figure 5 shows that the blue point lies somewhere between $\frac{5}{9}$ and $\frac{8}{12}$. This may prove useful. Imagine that a student proceeds to test the conjecture that $T:F$ is equal to 5:9. In this case (Fig 9) the points on the number line are closer; the vertical lines in the target ratio pane are closer. And the decimal representation of the selected quotient is $0.555555\dots$ instead of $0.666666\dots$. $\frac{5}{9}$ is less than $0.\overline{571428}$ but much closer than $\frac{8}{12}$.

There is other useful information about the target ratio in the number line pane. Above the number line, the selected ratio is displayed as a decimal quotient (Fig. 6):

● **Selected quotient:** $\frac{D}{C} = 8 \div 12 = \frac{2}{3} = 0.66666666666666666666\dots = 0.\overline{6}$

Figure 6: The selected ratio, 8:12 expressed as a quotient.

The bar above the 6 indicates that 6 is a repeating, non-terminating digit.

Had the “show target decimal” option been set, the following information about the target ratio would have also appeared (Fig. 7):

☒ show target decimal ● **Target quotient:** $\frac{T}{F} = 0.5714285714285714285\dots = 0.\overline{571428}$

Figure 7: The target ratio optionally displayed as a decimal quotient.

Students will need some time to become familiar with how the software works. Some students may be puzzled by the fact that a point (n,m) corresponds to the ratio $m:n$ rather than $n:m$. This is understandable, but the ordering of parameters is consistent with conventions for x and y axes: the point (x,y) is associated with the slope $\frac{y}{x}$, the ratio $y:x$, and the quotient $y \div x$. Even after a student has understood this, she must learn to anticipate how the size of a selected ratio depends on the location of a point in the lattice grid.

In Fig. 8 the student has activated the common product scenario and has set m equal to 8 and n equal to 12. The figure shows that, whereas $12P = 8G$ (as expected), $12T < 8F$. Although $T:F \neq 8:12$, $8:12$ appears to be close to $T:F$. This is reflected in the fact that the two dashed vertical lines associated with $12T$ and $8F$ are close (separated by a little more than $1F$). (The vertical lines are congruent in the case of the selected ratio.) Each antecedent magnitude (T and P) has been multiplied by 12; each consequent magnitude (F and G) has been multiplied by 8. A common product emerged for the selected ratio: $12P = 8G$. This is to be expected given that the software constructed P and G to satisfy the constraint that $P:G = 8:12$. No common product emerged for the target ratio: $12T < 8F$. The outcomes are consistent with the relative positions of the quotients $P \div G$ and $T \div F$ on the number line as shown in Figure 3.

Figure 8: The common product scenario is selected; the partitioned line segments and simplified expressions options are not set.

Figure 9 displays the information available after the student has posited that $9T$ equals $5F$ by virtue of selecting the ratio $5:9$. Once again, the selected ratio is close, but the target ratio, $T:F$ is slightly greater and this is reflected in the positions of the two ratios as quotients on the number line. It is also confirmed by the fact that the decimal representation of the quotient of P divided by G is slightly less than that of P divided by F .

Figure 9: The student tests the conjecture that $9T = 5F$.

Fig. 10 shows a solution. The student has found that $7T = 4F$. This implies that $T:F = 4:7$. The points corresponding to the selected and target ratio are at the same location. Furthermore, both the number line and grid indicate that the point ratio $8:14$ is equivalent to $4:7$.

Figure 10: $T:F = 4:7$ in the common products scenario.

Options enable the teacher to control the information available to students at any moment. Young students learning about decimal numbers may find it challenging to infer a target ratio even from a simple decimal such as 0.125. (A teacher may wish to leave this option off for students who can readily produce a fraction from a decimal number.) Each of the screen panes (Fig. 3) can be minimized or maximized by clicking on the title bar above the respective pane.

So far, we have been looking at various sorts of information that can be provided in the opening scenario, before any operations have been carried out on the line segments.

Fig. 11 displays the grid with the “points visited” option set. The black points correspond to those outcomes in which the selected ratio was found to be greater than the target ratio. The yellow points, including (9,5), correspond to trials in which the selected ratio was smaller than the target ratio. Many points are plotted to highlight how the location of the points is related to the magnitude of the ratios; we expect that students guided by a teacher would normally not need to go through so many trials to arrive at a solution. The idea of registering outcomes of tests of proportion on the unit grid is suggested by Madden (2018) in his analysis of how Euclid’s approach to ratio and proportion, in Book V of the Elements, may contribute to present-day instruction about ratio and proportion.

Figure 11: The “points visited” option displays information about the outcomes of trials.

Simplified Expressions and Partitioned Line Segments

When the *simplified expressions* option is set (Fig. 12), operators are taken from the reduced ratio (2:3). Furthermore, the labels above the line segments provide information regarding both the selected ratio and the reduced selected ratio. This sort of representation is designed to encourage students to view ratios and fractions as members of an equivalence class of ordered pairs of integers. The same reasoning underlies the decision to use, by default, “multiple rulers” (Fig. 5) under the number line whenever the selected ratio is reducible.

Figure 12: The ratio 8:12 was chosen while the simplification option was set

In all scenarios, when the *partitioned line segments* option is set in addition to the *simplified expressions* option, the line segments are partitioned throughout according to the values of the simplified selected ratio (Fig. 13). Every other part is colored yellow to facilitate counting of parts. In the present case, P and T are each broken up into 2 equal parts; G and F are each broken up into 3 equal parts. This also makes it easier to notice that, whenever an “unsuccessful” ratio of whole numbers is selected, the parts of the target antecedent line segment will differ from that of parts of the target consequent. In Fig. 13, the parts of T are noticeably smaller than the parts of F . This size mismatch never occurs for the selected line segments because the selected ratio of magnitudes always matches the numerical ratio.

The *partitioned line segments* and *simplified expressions* options also apply to the line segments along the grid (Fig. 13). This is consistent with ideas and an implementation discussed by Beckmann and Kulow, 2018). The line segments P and G along the grid axes were independently dilated (in this case, multiplied by a scale factor of 1.1) to 8.8 and 13.2, respectively (their initial values were 8 and 12). The fact that P and G are broken up into 2 and 3 parts, respectively, even under dilation, provides evidence that $\frac{8.8}{13.2} = \frac{2}{3}$. For the present problem, whenever P and G are dilated along the axes, it will be possible to infer that $\frac{\text{rise}}{\text{run}} = \frac{2}{3}$.

The observant reader may have noticed in Fig. 13 that the selected ratio of magnitudes in the middle pane has been dilated (in this case, shrunken) by means of a slider control at the bottom of the central pane. Ratios are invariant under uniform dilation. Whether students appreciate this is, of course, to be determined and addressed as needed.

Figure 13: The ratio 8:12 was selected while the simplification and partitioned line segments options were set.

Might whole numbers facilitate, rather than interfere with, learning about fractional operators?

There is considerable evidence showing that, although students are first introduced to fractions in elementary school, many are still having considerable problems with fractions well into high school (Behr, Harel, Post, & Lesh, 1992). Part of the problem seems to lie in the vast gulf between representations of “fractions as pizzas” in middle school textbooks and fractions as numbers and points on the real line in mathematics proper (Wu, 2020). There is also “overwhelming evidence that a curriculum which has as its primary basis the counting of discrete objects (and hence which introduces numbers as discrete) is not an effective or rational organization (Coles, 2021, p. 1).” Children are often introduced to fractions as if they were so different from whole numbers as to be regarded as something entirely new. This is particularly apparent in the teaching and learning about multiplication and division of fractions.

Children’s experience with whole number operations leads them to generalize that “multiplication makes bigger and division makes smaller” (Bell, Swan & Taylor, 1981), a generalization that lands them in trouble when they begin to work with fractions (including decimal fractions), where multiplication by a common fraction “makes smaller” and division by a common fraction “makes bigger.” Some mathematics educators interpret this sort of discrepancy as evidence that children’s knowledge about natural numbers serves as a “conceptual barrier” (Gelman and Williams, 1998) to learning about fractions. Others have argued that fractions should be taught as fundamentally different from whole numbers, and that the children’s intuitions about how numbers are ordered need to be set aside when they work with fractions.

There is an alternative view, one that actually regards children's intuitions about the impact of whole number operations on magnitudes as potential resources for comprehending how operations with positive fractions play a role in determining the magnitude of the results. The essential idea is that multiplication by a fraction can be understood as a composition of multiplication by the numerator and division by the denominator. The numerator acts like a whole number multiplier whereas the denominator acts like a whole number divisor. Accordingly, the impact of a fractional multiplier depends on the relative size of the numerator to the denominator. When the numerator equals the denominator, the fraction neither increases nor diminishes the multiplicand; it is an identity operator. When it is greater than the denominator, the product is greater than the multiplicand. When the numerator is smaller than the denominator, the product is less than the multiplicand.

While working with *Exploring Relative Magnitude*, students can find support for such reasoning in various scenarios associated with the extended model of ratio and proportion.

It is important that students realize, early on, that division of a quantity, A , by a whole number, n , can be modeled as taking one n th of A ; that is, $A \div n = \frac{A}{n} = \frac{1}{n}A$. This is displayed in the *common quotients scenario*. In the common quotients scenario depicted in Fig. 14, a student attempts to assess whether $A:H = 3:2$ by dividing A by 3 and H by 2. The outcome showing that $A \div 3 < H \div 2$ makes clear that $A:H \neq 3:2$. In fact, it shows that $A:H < 3:2$.

The unit fractional quantities, $\frac{1}{3}A$ and $\frac{1}{2}H$, are both smaller, respectively, than A and H . Division of a quantity by a natural number greater than 1 indeed yields a unit fractional quantity that is smaller than the dividend quantity.

It is interesting to note that, when the common quotient scenario is successful (see the selected ratio pane in Fig. 14), the part from the antecedent line segment is equal to the part from the consequent line segment: $\frac{J}{3} = \frac{E}{2}$. Gridlines are drawn to highlight this fact.

Figure 14: Division by n as taking $\frac{1}{n}$ th, that is, one of n equal parts.

This stands in contrast to approaches that introduce fractions through partitioning. As Fig. 15 shows, partitioning a non-numeric magnitude into n parts can be thought of as an identity operation: $A \times \frac{n}{n} = \frac{n}{n} A = A$. It is important to establish from the outset a clear distinction between “dividing by n ” and “dividing up into n shares or parts.” This is particularly true when fractions are introduced through sharing activities.

Figure 15: Partitioning as an identity operation.

Once the operations of multiplication and division of magnitudes by whole numbers are established, they can be performed in sequence. Dividing a quantity, A , by m and multiplying by n yields a fraction: $(A \div m) \times n = \frac{n}{m} A$. Fig. 16 illustrates this for the case where $m = 3$ and $n = 2$: $(A \div 3) \times 2 = \frac{2}{3} A$.

Figure 16: Division then multiplication by whole numbers.

Fig. 17 shows the results when the integer operations are carried out in the reverse order:
 $(A \times 2) \div 3 = \frac{2}{3}A$. Because multiplication and division by integers are associative, the same result, namely, $\frac{2}{3}A$, obtains in each case.

Figure 17: Multiplication then division by whole numbers.

Fig. 18 illustrates the scenario for multiplication of a quantity by a fraction. This may seem insignificant when represented algebraically: $A \times \frac{2}{3} = \frac{2}{3}A$. However, something remarkable emerges when one examines the diagrams in Figures 16 -18: the result is $\frac{2}{3}A$ in each case! Dividing by m and then multiplying the result by n has the same effect as multiplying by $\frac{n}{m}$. Multiplying by n and then dividing by m also is equivalent to multiplying by $\frac{n}{m}$.

Multiplication (or division) by a fraction can be understood as a composition of whole number multiplication and division. There need be no conflict between one's intuitions about whole numbers and fractions when fractions are understood this way. This topic is ripe for research.

There is no need to discuss division by a fraction as a separate case. Division of a quantity by a fraction of the form $\frac{n}{m}$ can simply be defined as multiplication by the inverse, $\frac{m}{n}$. In the model, a single scenario suffices for both cases.

Figure 18: Multiplying by a fraction (or dividing by its inverse).

Similarity in the Cartesian Plane

This paper has provided an introduction to a model that aims to unite ideas and representations related to teaching and learning about ratio and proportion, fractions, rational number, and linear functions.

In the software, *Exploring Relative Magnitude*, right triangles or rectangles may be displayed (Fig. 19), in addition to line segments, to allow students to explore issues regarding the geometric similarity of triangles or rectangles. There is compelling evidence that students in grade 3 may benefit significantly from discussions about proportionality in such geometric contexts (Lehrer, Strom & Confrey, 2002). It is likely that activities off the computer such as comparing triangles made in paper may be helpful.

Figure 19: Right triangles descending from the line $m = \frac{4}{6}n$ are similar.

Concluding Remarks

The model under discussion rests on the notion that one can find a ratio of whole numbers equal to a ratio of magnitudes if certain critical outcomes emerge from various operations on the magnitudes.

The diagrams shown in this paper should not be regarded as self-evident. They are intended to invite and provide structure for discussion among students and teachers about relationships among various mathematical objects.

In the real world, one can never be certain that two magnitudes are commensurable or incommensurable. Likewise, one can never be certain that two magnitudes are exactly equal. This uncertainty stems from measurement error and the limits of human perception.

A virtual environment can bypass such limitations by working with idealized, exact ratios and providing unambiguous information in the form of equalities and inequalities regarding the outcomes of operations on the ratios. Furthermore, it can restrict the set of candidate ratios to those expressible through a manageable range of whole numbers.

Euclid's definition of proportion, given in Book V of the *Elements*, is highly regarded for having provided Greeks of antiquity a means of representing ratios of incommensurable magnitudes without requiring the invention of irrational numbers. It also proves to be surprisingly relevant for applications to commensurable magnitudes and rational numbers where it can offer a test of whether a particular ratio of magnitudes equals a particular ratio of whole numbers. The test will be either "merely suggestive" or "definitive" depending on whether one assumes a real world or idealized perspective. In the idealized perspective, line segments A and B are proportional to whole numbers m and n if $nA = mB$. Such a perspective can be supported in a virtual world where one works with exact ratios and unambiguous notation.

When operations of partitive division and Euclidean division on magnitudes are united with multiplication, additional tests of proportion emerge. Each scenario offers a means for determining whether a given ratio of magnitudes equals a selected ratio of whole numbers. And each scenario is associated with notational variants for expressing proportions. Key variants, each a counterpart to $A:B = m:n$, are listed in Table 9.

Table 9: Variants of $A:B = m:n$

scenario	critical outcome confirming that $A:B = m:n$
common products	$A \times n = B \times m$
common quotients	$A \div m = B \div n$
whole number operators	$(A \times n) \div m = B$
fractional operators	$A \times \frac{n}{m} = B$
fractional increment	$A + \frac{(n-m)}{m}A = B$
Euclidean algorithm	$A = q_1B + R_1$ $B = q_2R_1 + R_2$ $R_1 = q_3R_2 + R_3$ $\dots$ $R_{n-2} = q_nR_{n-1}$ $C = q_1D + W_1$ $D = q_2W_1 + W_2$ $W_1 = q_3W + W_3$ $\dots$ $W_{n-2} = q_nW_{n-1}$

The principal systems of representation appearing in the model included: (1) line segment diagrams; (2) unit grid representations; (3) number line representations and (4) arithmetic notation. The systems of representation offer various means of expressing the equivalence of ratios (Table 10).

Table 10: Equivalence in various systems of representation.

Representation	Illustrations of equivalence
Line segment diagrams	<ul style="list-style-type: none"> if an ordered pair of magnitudes (A,B) yields a critical outcome (Error! Reference source not found.) for whole numbers m and n, $A:B = m:n$.
Unit grid	<ul style="list-style-type: none"> the “rise over run” is invariant for points on the same graph of the line $y = \frac{m}{n}x$, where m and n are whole numbers. all right triangles under the graph are similar.
Number line	<ul style="list-style-type: none"> equivalent fractions (e.g. $\frac{6}{8}, \frac{3}{4}$) occupy the same position on a number line
Arithmetic notation	<ul style="list-style-type: none"> reducing a fraction (e.g. from $\frac{6}{8}$ to $\frac{3}{4}$) does not change its value all equivalent ordered pairs of whole numbers yield the same decimal quotient ($6 \div 8 = 3 \div 4 = 0.75$)

Equivalence is of paramount importance. Ultimately it underlies the shift from whole numbers to positive rational numbers, where fractions are regarded as members of equivalence classes.

A model built on ratios of unmeasured magnitudes offers some advantages over models in which fractions are given construed as the number of highlighted parts among some total number of parts. This is true whether the model entails discrete quantities or continuous quantities that are associated, from the outset, with a given number of units. In principle, working with quantities that are not measured in conventional units may provide students with special opportunities for reasoning about relative magnitude.

Representations in the Cartesian plane are expected to be helpful for students to regard literals, such as m and n , not merely as constants but more generally, as variables. They can begin to view expressions such as $\frac{m}{n} = \frac{4}{7}$ not simply as equalities but as equations. And they can learn to view an inequality such as $\frac{m}{n} < \frac{2}{3}$ as referring to a region in the plane delineated by the graph of a linear function, and where each point in the region corresponds to an ordered pair of numbers satisfying the inequality.

It is too early to know whether the model will make a contribution to mathematics education. Any potential benefits will depend on future developments and investigations. And, given the ambitious nature of the model, it would seem that a substantial investment of time and energy will be required for teachers to become familiar with the model and capable of using it productively in their classrooms. I would be grateful to hear from educators interested in working together on this.

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Figures

Figure 1

Is $A:B = C:D$?

Figure 2

A target ratio of magnitudes and a selected ratio of magnitudes

Figure 3

Full screen shot of an opening scenario in which a problem solver has conjectured that $T:F = 8:12$.

Figure 4

Screen displaying only two panes.

Figure 5

The selected ratio, $D:C$ (or $D \div C$) corresponds to each of the circled fractions under the number line (at left) and each highlighted point in the grid (at right).

Figure 9

The student tests the conjecture that $9T = 5F$.

Figure 10

$T:F = 4:7$ in the common products scenario.

Figure 11

The “points visited” option displays information about the outcomes of trials.

Figure 12

The ratio 8:12 was chosen while the simplification option was set

Figure 13

The ratio 8:12 was selected while the simplification and partitioned line segments options were set.

Figure 14

Division by n as taking $1/n$ th, that is, one of n equal parts.

Figure 15

Partitioning as an identity operation.

Figure 16

Division then multiplication by whole numbers.

Figure 17

Multiplication then division by whole numbers.

Figure 18

Multiplying by a fraction (or dividing by its inverse).

Figure 19

Right triangles descending from the line $m = \frac{4}{6} n$ are similar.