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Conservation Laws, Solitary Wave Solutions, and Lie Analysis for the Nonlinear Chains of Atoms

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Abstract

Nonlinear chains of atoms(NCA) are complex systems with rich dynamics, influencing various scientific disciplines. Lie symmetry approach is considered to analyze the NCA. The Lie symmetry method is a powerful mathematical tool for analyzing and solving differential equations with symmetries, facilitating the reduction of complexity and obtaining solutions. After getting the entire vector field by using the Lie scheme, we find the optimal system of symmetries. Using the optimal system we have converted assumed PDE into nonlinear ODE. The new auxiliary scheme introduces novel approaches to complement existing techniques, enhancing accuracy and simplifying computations. Travelling wave solutions describe wave-like propagation in systems, while graphical behavior visually represents relationships and patterns in data or mathematical models. The multiplier method enables the identification of conservation laws, fundamental principles in physics that assert certain quantities remain constant over time. Understanding these concepts contributes to a deeper comprehension of nonlinear chains of atoms and their dynamics, fostering advancements in related fields.
Keywords: Nonlinear chains of atoms, Lie symmetry approach, Multiplier scheme, Conserved vectors, New auxiliary method.

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1 Introduction

The Lie symmetry analysis approach [1–8] has many applications in different fields, including physics, engineering, and mathematical modeling. It can be used to study a wide range of nonlinear PDEs, including those that are difficult to solve using other methods. Additionally, this approach provides a powerful tool for developing new theories and models that improve our understanding of complex physical systems. Overall, the

Lie symmetry analysis approach is a valuable tool for studying nonlinear PDEs and has many significant applications in different branches of science and engineering.

The Lie symmetry analysis approach [12–22] is a powerful method used in the study of nonlinear PDEs. It is based on the concept of Lie groups and Lie algebras, which are mathematical structures that describe the symmetries of a system. The Lie symmetry analysis approach involves transforming a given PDE into an equivalent system of ODEs using a Lie group transformation. This transformation is constructed from a set of symmetry generators that preserve the form of the original PDE. Once the PDE is transformed into an equivalent system of ODEs, it is possible to use various analytical and numerical methods to solve the system and obtain the solution to the original PDE. Additionally, the Lie symmetry analysis approach can be used to identify the conservation laws that govern the physical behavior of the system under study. These conservation laws provide important insights into the underlying physical mechanisms responsible for the observed behavior of the system.

Nonlinear PDEs [23–26] play a critical role in mechanical engineering by modeling complex phenomena such as stress and deformation in materials, fluid flow, and heat transfer. Unlike linear PDEs, which can be solved analytically in many cases, nonlinear PDEs require numerical or approximate methods to solve due to their complex nature. The use of nonlinear PDEs is essential in the design and optimization of mechanical systems such as turbines, engines, and aircraft. They also provide a framework for predicting the behavior of materials under different conditions, such as high temperatures, high pressure, and rapid deformation. By incorporating nonlinear PDEs into mechanical engineering models, engineers can improve the accuracy of their designs and ensure that their systems are safe, reliable, and efficient.

Nonlinear chains of atoms [27–33] have a wide range of applications in mechanical engineering, particularly in the study of materials science and solid mechanics. These models provide valuable insights into the behavior of materials at the atomic level, enabling the design of high-performance materials for various applications. Applications of nonlinear chains of atoms include the study of thermal conductivity in materials, investigating the deformation and fracture mechanisms of materials under various loading conditions, and studying the dynamics of crystals, such as the propagation of waves and the formation of defects. Nonlinear chains of atoms models are crucial in understanding the behavior of materials under extreme conditions and can inform the design of materials for high-performance applications, leading to the development of new materials with enhanced mechanical and thermal properties.

The new auxiliary method [10,11] is a recently proposed method for solving challenging nonlinear PDEs. This method involves introducing an auxiliary variable and constructing a system of coupled equations involving both the original variables and the auxiliary variable. The resulting system of equations can be solved using numerical methods to obtain the solution to the original PDE. The new auxiliary method can handle highly nonlinear PDEs that are difficult to solve using other numerical methods, such as the finite difference approach or the finite element scheme. Additionally, this method can be used to obtain exact solutions to certain types of nonlinear PDEs, reducing the computational cost required to solve some types of nonlinear PDEs. Overall, the new auxiliary method is a promising tool for solving challenging nonlinear PDEs in various fields, including physics, engineering, and mathematical modeling.

Conservation laws of nonlinear PDEs [36–39] are essential concepts that relate to the principle of conservation of physical quantities like mass, energy, and momentum. These laws are expressed in terms of PDEs and have crucial importance in various fields, including engineering, physics, and mathematical modeling. They provide a mathematical framework to predict the behavior of complex physical systems accurately and develop new theories and models to improve our understanding of the underlying physical mechanisms. Furthermore, conservation laws play a vital role in the design and analysis of physical systems and the development of numerical methods for solving challenging nonlinear PDEs, making them fundamental concepts in the study of nonlinear PDEs.

2 Formation of model

The Hamiltonian of the system is Foroutan et al. [34],

$$H = \sum_{n}\left\{ \sum_{l\neq n} \mathcal{V}(|\mathcal{U}_n - \mathcal{U}_l|) + \frac{1}{2}m\mathcal{U}_{\dot{n}}^2 \right\}, \tag{1}$$

where m is the mass of the atom, $\mathcal{V}(|\mathcal{U}_n - \mathcal{U}_l|)$ stands for nonlinear potential and dot indicates for derivative w.r.t time. We consider $l = 1 \pm n$ and the subsequent potential:

$$\mathcal{V}(h_{nl}) = \frac{1}{4}\beta_i h_{nl}^4 + \frac{1}{3}\alpha_i h_{nl}^3 + \frac{1}{2}\gamma_i h_{nl}^2, \tag{2}$$

where h_{nl} is relative displacement among $l - th$ atom and $n - th$. The index i shows the distinct interactions via the particles. We omit our focus on the first and second neighbors. From Eqs. (1) and (2) through Hamiltonian equations which are

$$\frac{\partial H}{\partial \mathcal{U}_n} = -P_{\dot{n}}, \qquad \frac{\partial H}{\partial P_{\dot{n}}} = \frac{\partial \mathcal{U}_n}{\partial \tau} = \mathcal{U}_{\dot{n}},$$

which gives us the equation of motion;

$$\begin{aligned}
\frac{d^2\mathcal{U}_n}{d\tau^2} =& \gamma_1\left(\mathcal{U}_{n+1} - 2\mathcal{U}_n + \mathcal{U}_{n-1}\right)\\
&+ \gamma_2\left(\mathcal{U}_{n+2} - 2\mathcal{U}_n + \mathcal{U}_{n-2}\right)\\
&+ \alpha_1\left\{\left(\mathcal{U}_{n+1} - \mathcal{U}_n\right)^2 - \left(\mathcal{U}_n - \mathcal{U}_{n-1}\right)^2\right\}\\
&+ \alpha_2\left\{\left(\mathcal{U}_{n+2} - \mathcal{U}_n\right)^2 - \left(\mathcal{U}_n - \mathcal{U}_{n-2}\right)^2\right\}\\
&+ \beta_1\left\{\left(\mathcal{U}_{n+1} - \mathcal{U}_n\right)^2 - \left(\mathcal{U}_n - \mathcal{U}_{n-1}\right)^2\right\}\\
&+ \beta_2\left\{\left(\mathcal{U}_{n+2} - \mathcal{U}_n\right)^2 - \left(\mathcal{U}_n - \mathcal{U}_{n-2}\right)^2\right\}.
\end{aligned} \tag{3}$$

In Hamiltonian's equations, $P_{\dot{n}}$ stands for generalized momentum. Assuming that the δ(inter-atom spacing) is small enough so that the continuum limit is reached, we substitute $\delta_n \to \chi$. Then

$$\mathcal{U}_{n\pm1} = \mathcal{U} \pm \delta\mathcal{U}_\chi + \frac{1}{2}\delta^2\mathcal{U}_{\chi\chi} \pm \frac{1}{6}\delta^3\mathcal{U}_{\chi\chi\chi} + \frac{1}{24}\delta^4\mathcal{U}_{\chi\chi\chi\chi} + \ldots \tag{4}$$

and

$$\mathcal{U}_{n\pm2} = \mathcal{U} \pm 2\delta\mathcal{U}_\chi + \frac{4}{2}\delta^2\mathcal{U}_{\chi\chi} \pm \frac{8}{6}\delta^3\mathcal{U}_{\chi\chi\chi} + \frac{16}{24}\delta^4\mathcal{U}_{\chi\chi\chi\chi} + \ldots, \tag{5}$$

hence, Eq.(3) can be supposed as Foroutan et al. [35]

$$\frac{\partial^2\mathcal{U}}{\partial\tau^2}=\delta_o^2\frac{\partial^2\mathcal{U}}{\partial\chi^2}+p_o\frac{\partial\mathcal{U}}{\partial\chi}\frac{\partial^2\mathcal{U}}{\partial\chi^2}+q_o\left(\frac{\partial\mathcal{U}}{\partial\chi}\right)^2\frac{\partial^2\mathcal{U}}{\partial\chi^2}+r\frac{\partial^4\mathcal{U}}{\partial\chi^4}, \tag{6}$$

with the subsequent constants;

$$\delta_o^2=\frac{\delta^2}{m}\big(\gamma_1+4\gamma_2\big),\ p_o=\frac{2\delta^3}{m}\big(\gamma_1+8\alpha_2\big),\ q_o=\frac{3\delta^4}{m}\big(\beta_1+16\beta_2\big), r=\frac{\delta^4}{12m}\big(\gamma_1+16\gamma_2\big). \tag{7}$$

Here in this paper, we will find out ...... for nonlinear Eq.(6) with the use of appropriate transformation method.

3 Preliminaries

3.1 New auxiliary approach

Assuming the general form of partial PDE is of the form:

$$\mathcal{F}(\mathcal{U},\mathcal{U}_\tau,\mathcal{U}_\chi,\mathcal{U}_{\chi\chi},...)=0, \tag{8}$$

where τ is the time part and χ is the spatial part and $\mathcal{U}=\mathcal{U}(\chi,\tau)$ is the dependent variable. We will follow the following steps.

Step 1: Suppose the new similarity variables or transformation is of the form

$$\mathcal{U}(\chi,\tau)=\mathcal{H}(\varrho),\quad \text{where}\quad \varrho=k(\chi+c\tau), \tag{9}$$

where k and c both are actual parameters for equation (8). Putting the Eq.(9) into Eq. (8) and we get the new ODE below.

$$\mathcal{P}(\mathcal{H},\mathcal{H}',\mathcal{H}'',...)=0. \tag{10}$$

Step 2: Assume the general solution for Eq.(10) is of the form

$$\mathcal{H}(\varrho)=\sum_{i=0}^{N}C_i\mathfrak{F}^{iq(\varrho)}, \tag{11}$$

in the above solution, the C_i's are constants and we will fine later and the 1^{st} ODE satisfied $q(\varrho)$.

$$q'(\varrho)=\frac{1}{\ln(\mathfrak{F})}\{\mathfrak{B}_2\mathfrak{F}^{-q(\varrho)}+\mathfrak{B}_1+\mathfrak{B}_3\mathfrak{F}^{q(\varrho)}\},\quad \mathfrak{F}>0,\quad \mathfrak{F}\neq 1. \tag{12}$$

Step 3: In this step, we will use the balancing scheme to execute the value of N. For this, we have to compare the highest order linear and nonlinear terms to find the value of N.

Step 4: Getting the coefficients of the powers of $\mathfrak{F}^{q(\varrho)}$ $(i=0,1,2,3..)$ by Eqs.(11), (12), and (8). Then collecting the same powers terms and put it equal to zero which gives us system of algebraic equations. After solving these system of euations by $Maple$.

Step 5: Finally we will get the different family of solutions for Eq.(12) of the form:
Case 1: When ${\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3 < 0$ and $\mathfrak{B}_3 \neq 0$

$$\mathfrak{F}^{q(\varrho)} = \frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{-({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{\mathfrak{B}_3} \tan\left(\frac{\sqrt{-({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{2}\varrho\right), \tag{13}$$

$$\mathfrak{F}^{q(\varrho)} = \frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{-({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{\mathfrak{B}_3} \cot\left(\frac{\sqrt{-({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{2}\varrho\right). \tag{14}$$

Case 2: When ${\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3 > 0$ and $\mathfrak{B}_3 \neq 0$

$$\mathfrak{F}^{q(\varrho)} = \frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{\mathfrak{B}_3} \tanh\left(\frac{\sqrt{({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{2}\varrho\right), \tag{15}$$

$$\mathfrak{F}^{q(\varrho)} = \frac{-\mathfrak{B}_1}{\mathfrak{B}_3} - \frac{\sqrt{({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{\mathfrak{B}_3} \coth\left(\frac{\sqrt{({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{2}\varrho\right). \tag{16}$$

Case 3: When ${\mathfrak{B}_1}^2 + \mathfrak{B}_2\mathfrak{B}_3 > 0$ and $\mathfrak{B}_3 \neq 0$ and $\mathfrak{B}_3 \neq -\mathfrak{B}_2$

$$\mathfrak{F}^{q(\varrho)} = \frac{\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{({\mathfrak{B}_1}^2 + {\mathfrak{B}_2}^2)}}{\mathfrak{B}_3} \tanh\left(\frac{\sqrt{({\mathfrak{B}_1}^2 + {\mathfrak{B}_2}^2)}}{2}\varrho\right), \tag{17}$$

$$\mathfrak{F}^{q(\varrho)} = \frac{\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{({\mathfrak{B}_1}^2 + {\mathfrak{B}_2}^2)}}{\mathfrak{B}_3} \coth\left(\frac{\sqrt{({\mathfrak{B}_1}^2 + {\mathfrak{B}_2}^2)}}{2}\varrho\right). \tag{18}$$

Case 4: When ${\mathfrak{B}_1}^2 + \mathfrak{B}_2\mathfrak{B}_3 < 0$, $\mathfrak{B}_3 \neq 0$ and $\mathfrak{B}_3 \neq -\mathfrak{B}_2$

$$\mathfrak{F}^{q(\varrho)} = \frac{\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{-({\mathfrak{B}_1}^2 + {\mathfrak{B}_2}^2)}}{\mathfrak{B}_3} \tan\left(\frac{\sqrt{-({\mathfrak{B}_1}^2 + {\mathfrak{B}_2}^2)}}{2}\varrho\right), \tag{19}$$

$$\mathfrak{F}^{q(\varrho)} = \frac{\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{-({\mathfrak{B}_1}^2 + {\mathfrak{B}_2}^2)}}{\mathfrak{B}_3} \cot\left(\frac{\sqrt{-({\mathfrak{B}_1}^2 + {\mathfrak{B}_2}^2)}}{2}\varrho\right). \tag{20}$$

Case 5: When ${\mathfrak{B}_1}^2 - {\mathfrak{B}_2}^2 < 0$ and $\mathfrak{B}_3 \neq -\mathfrak{B}_2$

$$\mathfrak{F}^{q(\varrho)} = \frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{-({\mathfrak{B}_1}^2 - {\mathfrak{B}_2}^2)}}{\mathfrak{B}_3} \tan\left(\frac{\sqrt{-({\mathfrak{B}_1}^2 - {\mathfrak{B}_2}^2)}}{2}\varrho\right), \tag{21}$$

$$\mathfrak{F}^{q(\varrho)} = \frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{-({\mathfrak{B}_1}^2 - {\mathfrak{B}_2}^2)}}{\mathfrak{B}_3} \cot\left(\frac{\sqrt{-({\mathfrak{B}_1}^2 - {\mathfrak{B}_2}^2)}}{2}\varrho\right). \tag{22}$$

Case 6: When ${\mathfrak{B}_1}^2 - {\mathfrak{B}_2}^2 > 0$ and $\mathfrak{B}_3 \neq -\mathfrak{B}_2$

$$\mathfrak{F}^{q(\varrho)} = \frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{({\mathfrak{B}_1}^2 - {\mathfrak{B}_2}^2)}}{\mathfrak{B}_3} \tanh\left(\frac{\sqrt{({\mathfrak{B}_1}^2 - {\mathfrak{B}_2}^2)}}{2}\varrho\right), \tag{23}$$

$$\mathfrak{F}^{q(\varrho)} = \frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{({\mathfrak{B}_1}^2 - {\mathfrak{B}_2}^2)}}{\mathfrak{B}_3} \coth\left(\frac{\sqrt{({\mathfrak{B}_1}^2 - {\mathfrak{B}_2}^2)}}{2}\varrho\right). \tag{24}$$

Case 7: When $\mathfrak{B}_2\mathfrak{B}_3 > 0$, $\mathfrak{B}_3 \neq 0$ and $\mathfrak{B}_1 = 0$

$$\mathfrak{F}^{q(\varrho)} = \sqrt{\frac{-\mathfrak{B}_2}{\mathfrak{B}_3}} \tanh\left(\frac{\sqrt{-\mathfrak{B}_2\mathfrak{B}_3}}{2}\varrho\right), \tag{25}$$

$$\mathfrak{F}^{q(\varrho)} = \sqrt{\frac{-\mathfrak{B}_2}{\mathfrak{B}_3}} \coth\left(\frac{\sqrt{-\mathfrak{B}_2\mathfrak{B}_3}}{2}\varrho\right). \tag{26}$$

Case 8: When $\mathfrak{B}_1 = 0$ and $\mathfrak{B}_2 = -\mathfrak{B}_3$

$$\mathfrak{F}^{q(\varrho)} = \frac{-(1+e^{2\mathfrak{B}_2\varrho}) \pm \sqrt{2(1+e^{2\mathfrak{B}_2\varrho})}}{e^{2\mathfrak{B}_2\varrho}-1}. \tag{27}$$

Case 9: When ${\mathfrak{B}_1}^2 = \mathfrak{B}_2\mathfrak{B}_3$

$$\mathfrak{F}^{q(\varrho)} = \frac{-\mathfrak{B}_2(\mathfrak{B}_1\varrho+2)}{{\mathfrak{B}_1}^2\varrho}. \tag{28}$$

Case 10: When $\mathfrak{B}_1 = k$, $\mathfrak{B}_2 = 2k$ and $\mathfrak{B}_3 = 0$

$$\mathfrak{F}^{q(\varrho)} = e^{\varrho} - 1. \tag{29}$$

Case 11: When $\mathfrak{B}_1 = k$, $\mathfrak{B}_3 = 2k$ and $\mathfrak{B}_2 = 0$

$$\mathfrak{F}^{q(\varrho)} = \frac{e^{\varrho}}{1-e^{\varrho}}. \tag{30}$$

Case 12: When $2\mathfrak{B}_1 = \mathfrak{B}_2 + \mathfrak{B}_3$

$$\mathfrak{F}^{q(\varrho)} = \frac{1+\mathfrak{B}_2 e^{\frac{1}{2}(\mathfrak{B}_2-\mathfrak{B}_3)\varrho}}{1+\mathfrak{B}_3 e^{\frac{1}{2}(\mathfrak{B}_2-\mathfrak{B}_3)\varrho}}. \tag{31}$$

Case 13: When $-2\mathfrak{B}_1 = \mathfrak{B}_2 + \mathfrak{B}_3$

$$\mathfrak{F}^{q(\varrho)} = \frac{\mathfrak{B}_2+\mathfrak{B}_2 e^{\frac{1}{2}(\mathfrak{B}_2-\mathfrak{B}_3)\varrho}}{\mathfrak{B}_3+\mathfrak{B}_3 e^{\frac{1}{2}(\mathfrak{B}_2-\mathfrak{B}_3)\varrho}}. \tag{32}$$

Case 14: When $\mathfrak{B}_2 = 0$

$$\mathfrak{F}^{q(\varrho)} = \frac{\mathfrak{B}_1 e^{\mathfrak{B}_1\varrho}}{1+\frac{\mathfrak{B}_3}{2}e^{\mathfrak{B}_1\varrho}}. \tag{33}$$

Case 15: When $\mathfrak{B}_2 = \mathfrak{B}_1 = \mathfrak{B}_3 \neq 0$

$$\mathfrak{F}^{q(\varrho)} = \frac{-(\mathfrak{B}_2\varrho+2)}{\mathfrak{B}_2\varrho}. \tag{34}$$

Case 16: When $\mathfrak{B}_2 = \mathfrak{B}_3$, $\mathfrak{B}_1 = 0$

$$\mathfrak{F}^{q(\varrho)} = \tan\left(\frac{\mathfrak{B}_2\varrho+c}{2}\right). \tag{35}$$

Case 17: When $\mathfrak{B}_3 = 0$

$$\mathfrak{F}^{q(\varrho)} = e^{\mathfrak{B}_1\varrho} - \frac{\mathfrak{B}_2}{2\mathfrak{B}_1}. \tag{36}$$

3.2 Multiplier approach

Supposing the Eq.(8) and applying the following steps below:
1). Definig the total differential as:

$$D_i = \frac{\partial}{\partial \chi^i} + \mathcal{U}_i \frac{\partial}{\partial \mathcal{U}} + \mathcal{U}_{ij} \frac{\partial}{\partial \mathcal{U}_j} + ..., \quad i = 1, 2, 3...m, \tag{37}$$

2). Defining the Euler operator as below:

$$\frac{\delta}{\delta \mathcal{U}} = \frac{\partial}{\partial \mathcal{U}} - D_i \frac{\partial}{\partial \mathcal{U}_i} + D_{ij} \frac{\partial}{\partial \mathcal{U}_{ij}} - D_{ijk} \frac{\partial}{\partial \mathcal{U}_{ijk}} + ... \;, \tag{38}$$

3). Let us define n-tuple $\mathfrak{f} = (\mathfrak{f}^1, \mathfrak{f}^2, \mathfrak{f}^3, ..., \mathfrak{f}^m)$, $i = 1, 2, ...m$,

$$D_i \mathfrak{f}^i = 0, \tag{39}$$

eq.(39) is said to be the conservation laws and it is fulfils the all results of Eq. (8).
4). The purpose of $\Lambda(\chi, \tau, \mathcal{U})$ of the Eq. (8):

$$D_i \mathfrak{f}^i = \Lambda(\chi, \tau, \mathcal{U}) H, \tag{40}$$

for some function $\mathcal{U}(\mu^1, \mu^2, ..., \mu^m)$.
5). We will obtain the determining equations for $\Lambda(\chi, \tau, \mathcal{U})$ after calculating the derivative of $\Lambda(\chi, \tau, \mathcal{U})$ in Eq.(40):

$$\frac{\delta}{\delta \mathcal{U}} (\Lambda(\chi, \tau, \mathcal{U}) H) = 0. \tag{41}$$

Eq. (41) depends for some function $\mathcal{U}(\mu^1, \mu^2, ..., \mu^m)$. Finally, when we calculate the $\Lambda(X, t, \mathcal{U})$ with use of Eq. (41), the conservation laws can be acquired by Eq. (40).

4 Lie group analysis of Eq. (6)

Here, we are supposing the Lie approach for assumed Eq.(6). Now, suppose the one-parameter Lie group of infinitesimal transformations on $(\tau, \chi, \mathcal{U})$ given by

$$\begin{aligned} \bar{\tau} &= \tau + \varepsilon\, \zeta^1(\tau, \chi, \mathcal{U}) + O(\varepsilon^2), \\ \bar{\chi} &= \chi + \varepsilon\, \zeta^2(\tau, \chi, \mathcal{U}) + O(\varepsilon^2), \\ \bar{\mathcal{U}} &= u + \varepsilon\, \eta(\tau, \chi, \mathcal{U}) + O(\varepsilon^2), \end{aligned}$$

and $\varepsilon \ll 1$ is a Small parameter. The associated Lie algebra of infinitesimal symmetries is generated by vector fields

$$\mathfrak{X} = \zeta^1(\tau, \chi, \mathcal{U}) \partial_\tau + \zeta^2(\tau, \chi, \mathcal{U}) \partial_\chi + \eta(\tau, \chi, \mathcal{U}) \partial_{\mathcal{U}}. \tag{42}$$

Eq. (42) creates a symmetry of Eq. (6), and $\mathfrak{X}$ satisfy the Lie group conditions

$$Pr^{(4)} \mathfrak{X} \left(\frac{\partial^2 \mathcal{U}}{\partial \tau^2} = \delta_o^2 \frac{\partial^2 \mathcal{U}}{\partial \chi^2} + p_o \frac{\partial \mathcal{U}}{\partial \chi} \frac{\partial^2 \mathcal{U}}{\partial \chi^2} + q_o \left(\frac{\partial \mathcal{U}}{\partial \chi} \right)^2 \frac{\partial^2 \mathcal{U}}{\partial \chi^2} + r \frac{\partial^4 \mathcal{U}}{\partial \chi^4} \right) |_{Eq.(6)=0} = 0.$$

The $Pr^{(4)}\mathfrak{X}$ for $\mathfrak{X}$ can be written as:

$$Pr^{(4)}\mathfrak{X} = \mathfrak{X} + \eta^{\chi}\frac{\partial}{\partial \mathcal{U}_{\chi}} + \eta^{\tau\tau}\frac{\partial}{\partial \mathcal{U}_{\tau\tau}} + \eta^{\chi\chi}\frac{\partial}{\partial \mathcal{U}_{\chi\chi}} + \eta^{\chi\chi\chi\chi}\frac{\partial}{\partial \mathcal{U}_{\chi\chi\chi\chi}}, \tag{43}$$

furthermore, we have

$$\begin{cases} \eta^{\chi} = D_{\chi}(\eta) - \mathcal{U}_{\chi} D_{\chi}(\zeta^1) - \mathcal{U}_{\tau} D_{\chi}(\zeta^2), \\ \eta^{\chi\chi} = D_{\chi}(\eta^{\chi}) - \mathcal{U}_{\chi\chi} D_{\chi}(\zeta^1) - \mathcal{U}_{\tau\chi} D_{\chi}(\zeta^2), \\ \eta^{\tau} = D_{\tau}(\eta) - \mathcal{U}_{\chi} D_{t}(\zeta^1) - \mathcal{U}_{\tau} D_{\tau}(\zeta^2), \\ \eta^{\tau\tau} = D_{\tau}(\eta^{\tau}) - \mathcal{U}_{\tau\tau} D_{\tau}(\zeta^1) - \mathcal{U}_{\chi\tau} D_{t}(\zeta^2), \\ \eta^{\chi\chi\chi} = D_{\chi}(\eta^{\chi\chi}) - \mathcal{U}_{\chi\chi\chi} D_{\chi}(\zeta^1) - \mathcal{U}_{\tau\chi\chi} D_{\chi}(\zeta^2). \end{cases} \tag{44}$$

Let $(x^1, x^2) = (\chi, \tau)$, where D_i can be written as:

$$D_i = \frac{\partial}{\partial \chi^i} + \mathcal{U}_i \frac{\partial}{\partial \mathcal{U}} + \mathcal{U}_{ij}\frac{\partial}{\partial \mathcal{U}_j} + ..., \qquad i = 1, 2.$$

Substituting the values of η^i which gives us the following vectors:

$$\mathfrak{X}_1 = \frac{\partial}{\partial \chi}, \qquad \mathfrak{X}_2 = \frac{\partial}{\partial \tau}, \qquad \mathfrak{X}_3 = \frac{\partial}{\partial \mathcal{U}}, \qquad \mathfrak{X}_4 = \tau\frac{\partial}{\partial \mathcal{U}}. \tag{45}$$

We see that

$$[\mathfrak{X}_i, \mathfrak{X}_j] = 0, \quad where \;\; i, j = 1, 2, 3.$$

5 Optimal system

In this section, we observe that from the obtained vector field Eq. (45), the $\mathfrak{X} = \{\mathfrak{X}_1, \mathfrak{X}_2\}$ forms an abelian algebra. So we can use the (42) and get:

$$\begin{aligned} &\mathcal{L}_1 = <\mathfrak{X}_1>, \\ &\mathcal{L}_2 = <\mathfrak{X}_1 + k_1\mathfrak{X}_2> . \end{aligned} \tag{46}$$

5.1 Similarity reduction of Eq. (6)

Here, we will find the similarity variables and analytical results for Eq.(6).

5.1.1 $\mathcal{L}_1 =< \mathfrak{X}_1 >$

Using the vector $\mathcal{L}_1$, we get the new variable

$$u(\tau, \chi) = \mathcal{H}(\varrho), \quad where \;\; \varrho = \chi, \tag{47}$$

putting the (47) into Eq. (6), which gives us

$$\mathcal{U}(\tau, \chi) = m_1\tau + m_2 \tag{48}$$

where m_1 and m_2 are integration constants.

5.1.2 $\mathcal{L}_2 =< \mathfrak{X}_1 + k_1\mathfrak{X}_2$

Using the vector $\mathcal{L}_2$, we get the new variable

$$u(\tau,\chi) = \mathcal{H}(\varrho), \quad where \ \ \varrho = \chi + k_1\tau, \tag{49}$$

putting the (49) into Eq. (6), which gives us

$$6(k_1^2 - \delta_o^2)\mathcal{H}' - 3p_o(\mathcal{H}')^2 - 2q_o(\mathcal{H}')^3 - 3r\mathcal{H}''' = 0. \tag{50}$$

5.2 Application of new auxiliary method

Here, our aim is to construct the wave patterns for Eq.(6) from Eq.(50) with use of proposed technique. Using the balancing method and we obtain $N = 1$. Using the value of $N = 1$ in (11) and we have

$$\mathcal{H}(\varrho) = C_o + C_1\mathfrak{F}^{q(\varrho)}. \tag{51}$$

We have to put Eq.(51) into Eq.(50) and we get the system of equation after comparing the coefficients of $\mathfrak{F}^{q(\varrho)}$. With the use of Maple, we have solved the obtained system of equations and get the following results.

$$C_0 = V_1, \quad C_1 = -\frac{12r\mathfrak{B}_3}{p_o}, \quad k_1 = \pm\sqrt{r\mathfrak{B}_1^2 - 4r\mathfrak{B}_2\mathfrak{B}_3 + \delta_o^2}. \tag{52}$$

Using Eq.(52) into Eq.(51) and which gives us the following set of solutions.

$$\mathcal{U}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\mathfrak{F}^{q(\varrho)}, \quad where \quad \varrho = \chi \pm \sqrt{r\mathfrak{B}_1^2 - 4r\mathfrak{B}_2\mathfrak{B}_3 + \delta_o^2}\;\tau \tag{53}$$

where V_1 is orbitraryconstant.
Case:1 When ${\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathcal{U}_3 < 0$ and $\mathfrak{B}_3 \neq 0$

$$\mathcal{U}_1(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{-({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{\mathfrak{B}_3}\tan\left(\frac{\sqrt{-({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{2}\varrho\right)\right\}, \tag{54}$$

$$\mathcal{U}_2(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{-({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{\mathfrak{B}_3}\cot\left(\frac{\sqrt{-({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{2}\varrho\right)\right\}. \tag{55}$$

Case:2 When ${\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3 > 0$ and $\mathfrak{B}_3 \neq 0$

$$\mathcal{U}_3(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{\mathfrak{B}_3}\tanh\left(\frac{\sqrt{({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{2}\varrho\right)\right\}, \tag{56}$$

$$\mathcal{U}_4(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{-\mathfrak{B}_1}{\mathfrak{B}_3} - \frac{\sqrt{({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{\mathfrak{B}_3}\coth\left(\frac{\sqrt{({\mathfrak{B}_1}^2 - \mathfrak{B}_2\mathfrak{B}_3)}}{2}\varrho\right)\right\}. \tag{57}$$

Case:3 When ${\mathfrak{B}_1}^2 + \mathfrak{B}_2\mathfrak{B}_3 > 0$ and $\mathfrak{B}_3 \neq 0$ and $\mathfrak{B}_3 \neq -\mathfrak{B}_2$

$$\mathcal{U}_5(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{({\mathfrak{B}_1}^2 + {\mathfrak{B}_2}^2)}}{\mathfrak{B}_3}\tanh\left(\frac{\sqrt{({\mathfrak{B}_1}^2 + {\mathfrak{B}_2}^2)}}{2}\varrho\right)\right\}, \tag{58}$$

$$\mathcal{U}_6(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{({\mathfrak{B}_1}^2 + {\mathfrak{B}_2}^2)}}{\mathfrak{B}_3}\coth\left(\frac{\sqrt{({\mathfrak{B}_1}^2 + {\mathfrak{B}_2}^2)}}{2}\varrho\right)\right\}. \tag{59}$$

Case: 4 When $\mathfrak{B}_1{}^2 + \mathfrak{B}_2\mathfrak{B}_3 < 0$, $\mathfrak{B}_3 \neq 0$ and $\mathfrak{B}_3 \neq -\mathfrak{B}_2$

$$\mathcal{U}_7(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{-(\mathfrak{B}_1{}^2 + \mathfrak{B}_2{}^2)}}{\mathfrak{B}_3}\tan\left(\frac{\sqrt{-(\mathfrak{B}_1{}^2 + \mathfrak{B}_2{}^2)}}{2}\varrho\right)\right\}, \tag{60}$$

$$\mathcal{U}_8(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{-(\mathfrak{B}_1{}^2 + \mathfrak{B}_2{}^2)}}{\mathfrak{B}_3}\cot\left(\frac{\sqrt{-(\mathfrak{B}_1{}^2 + \mathfrak{B}_2{}^2)}}{2}\varrho\right)\right\}. \tag{61}$$

Case: 5 When $\mathfrak{B}_1{}^2 - \mathfrak{B}_2{}^2 < 0$ and $\mathfrak{B}_3 \neq -\mathfrak{B}_2$

$$\mathcal{U}_9(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{-(\mathfrak{B}_1{}^2 - \mathfrak{B}_2{}^2)}}{\mathfrak{B}_3}\tan\left(\frac{\sqrt{-(\mathfrak{B}_1{}^2 - \mathfrak{B}_2{}^2)}}{2}\varrho\right)\right\}, \tag{62}$$

$$\mathcal{U}_{10}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{-(\mathfrak{B}_1{}^2 - \mathfrak{B}_2{}^2)}}{\mathfrak{B}_3}\cot\left(\frac{\sqrt{-(\mathfrak{B}_1{}^2 - \mathfrak{B}_2{}^2)}}{2}\varrho\right)\right\}. \tag{63}$$

Case: 6 When $\mathfrak{B}_1{}^2 - \mathfrak{B}_2{}^2 > 0$ and $\mathfrak{B}_3 \neq -\mathfrak{B}_2$

$$\mathcal{U}_{11}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{(\mathfrak{B}_1{}^2 - \mathfrak{B}_2{}^2)}}{\mathfrak{B}_3}\tanh\left(\frac{\sqrt{(\mathfrak{B}_1{}^2 - \mathfrak{B}_2{}^2)}}{2}\varrho\right)\right\}, \tag{64}$$

$$\mathcal{U}_{12}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{-\mathfrak{B}_1}{\mathfrak{B}_3} + \frac{\sqrt{(\mathfrak{B}_1{}^2 - \mathfrak{B}_2{}^2)}}{\mathfrak{B}_3}\coth\left(\frac{\sqrt{(\mathfrak{B}_1{}^2 - \mathfrak{B}_2{}^2)}}{2}\varrho\right)\right\}. \tag{65}$$

Case: 7 When $\mathfrak{B}_2\mathfrak{B}_3 > 0$, $\mathfrak{B}_3 \neq 0$ and $\mathfrak{B}_1 = 0$

$$\mathcal{U}_{13}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\sqrt{\frac{-\mathfrak{B}_2}{\mathfrak{B}_3}}\tanh\left(\frac{\sqrt{-\mathfrak{B}_2\mathfrak{B}_3}}{2}\varrho\right)\right\}, \tag{66}$$

$$\mathcal{U}_{14}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\sqrt{\frac{-\mathfrak{B}_2}{\mathfrak{B}_3}}\coth\left(\frac{\sqrt{-\mathfrak{B}_2\mathfrak{B}_3}}{2}\varrho\right)\right\}. \tag{67}$$

Case: 8 When $\mathfrak{B}_1 = 0$ and $\mathfrak{B}_2 = -\mathfrak{B}_3$

$$\mathcal{U}_{15}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{-(1 + e^{2\mathfrak{B}_2\varrho}) \pm \sqrt{2(1 + e^{2\mathfrak{B}_2\varrho})}}{e^{2\mathfrak{B}_2\varrho} - 1}\right\}. \tag{68}$$

Case: 9 When $\mathfrak{B}_1{}^2 = \mathfrak{B}_2\mathfrak{B}_3$

$$\mathcal{U}_{16}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{-\mathfrak{B}_2(\mathfrak{B}_1\varrho + 2)}{\mathfrak{B}_1{}^2\varrho}\right\}. \tag{69}$$

Case: 10 When $\mathfrak{B}_1 = k$, $\mathfrak{B}_2 = 2k$ and $\mathfrak{B}_3 = 0$

$$\mathcal{U}_{17}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{e^{\varrho} - 1\right\}. \tag{70}$$

Case: 11 When $\mathfrak{B}_1 = k$, $\mathfrak{B}_3 = 2k$ and $\mathfrak{B}_2 = 0$

$$\mathcal{U}_{18}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{e^{\varrho}}{1 - e^{\varrho}}\right\}. \tag{71}$$

Case: 12 When $2\mathfrak{B}_1 = \mathfrak{B}_2 + \mathfrak{B}_3$

$$\mathcal{U}_{19}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{1+\mathfrak{B}_2 e^{\frac{1}{2}(\mathfrak{B}_2-\mathfrak{B}_3)\varrho}}{1+\mathfrak{B}_3 e^{\frac{1}{2}(\mathfrak{B}_2-\mathfrak{B}_3)\varrho}}\right\}. \tag{72}$$

Case: 13 When $-2\mathfrak{B}_1 = \mathfrak{B}_2 + \mathfrak{B}_3$

$$\mathcal{U}_{20}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{\mathfrak{B}_2+\mathfrak{B}_2 e^{\frac{1}{2}(\mathfrak{B}_2-\mathfrak{B}_3)\varrho}}{\mathfrak{B}_3+\mathfrak{B}_3 e^{\frac{1}{2}(\mathfrak{B}_2-\mathfrak{B}_3)\varrho}}\right\}. \tag{73}$$

Case: 14 When $\mathfrak{B}_2 = 0$

$$\mathcal{U}_{21}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{\mathfrak{B}_1 e^{\mathfrak{B}_1\varrho}}{1+\frac{\mathfrak{B}_3}{2}e^{\mathfrak{B}_1\varrho}}\right\}. \tag{74}$$

Case: 15 When $\mathfrak{B}_2 = \mathfrak{B}_1 = \mathfrak{B}_3 \neq 0$

$$\mathcal{U}_{22}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\frac{-(\mathfrak{B}_2\varrho+2)}{\mathfrak{B}_2\varrho}\right\}. \tag{75}$$

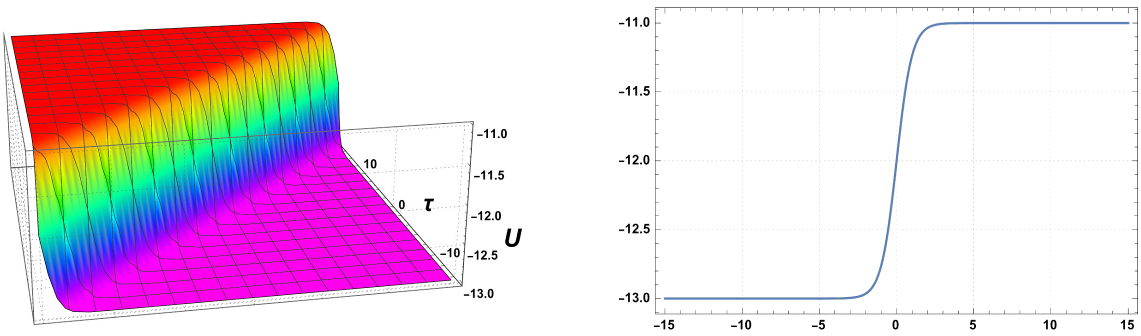
Case: 16 When $\mathfrak{B}_2 = \mathfrak{B}_3$, $\mathfrak{B}_1 = 0$

$$\mathcal{U}_{23}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{\tan\left(\frac{\mathfrak{B}_2\varrho+c}{2}\right)\right\}. \tag{76}$$

Case: 17 When $\mathfrak{B}_3 = 0$

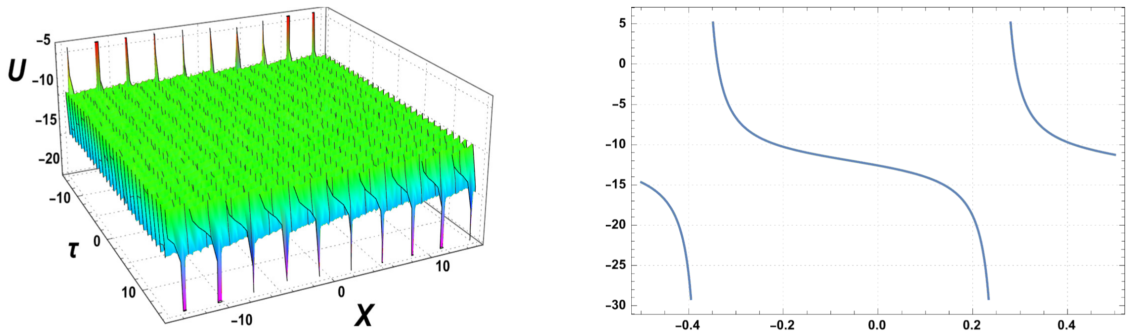
$$\mathcal{U}_{24}(\chi,\tau) = V_1 - \frac{12r\mathfrak{B}_3}{p_o}\left\{e^{\mathfrak{B}_1\varrho} - \frac{\mathfrak{B}_2}{2\mathfrak{B}_1}\right\}. \tag{77}$$

Where $\varrho = \chi \pm \sqrt{r\mathfrak{B}_1^2 - 4r\mathfrak{B}_2\mathfrak{B}_3 + \delta_o^2}\;\tau$ is given according to.



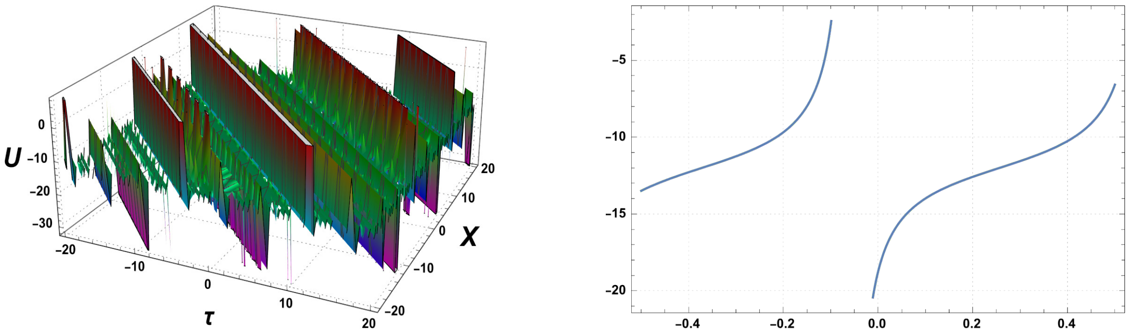
(a) 3D (b) 2D

Figure 1: Graphics of $u_1(\chi,\tau)$ for the choice of parameters $\delta_o = 1$, $r = 2$, $p_o = 3$, $\mathfrak{B}_1 = 2$, $\mathfrak{B}_2 = 1$,$\mathfrak{B}_3 = 2$, $\tau = 1$.



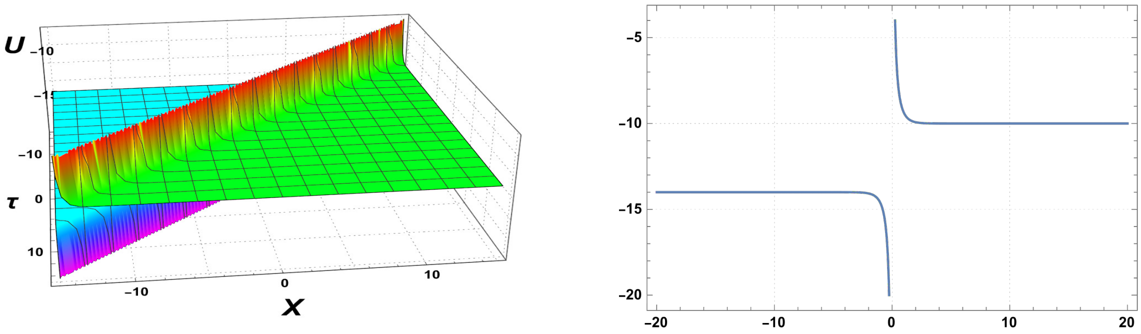
(a) 3D (b) 2D

Figure 2: Graphics of $u_2(\chi, \tau)$ for the choice of parameters $\delta_o = 2$, $r = 1$, $p_o = 1$, $\mathfrak{B}_1 = 3$, $\mathfrak{B}_2 = 2$,$\mathfrak{B}_3 = 1$, $\tau = 2$.



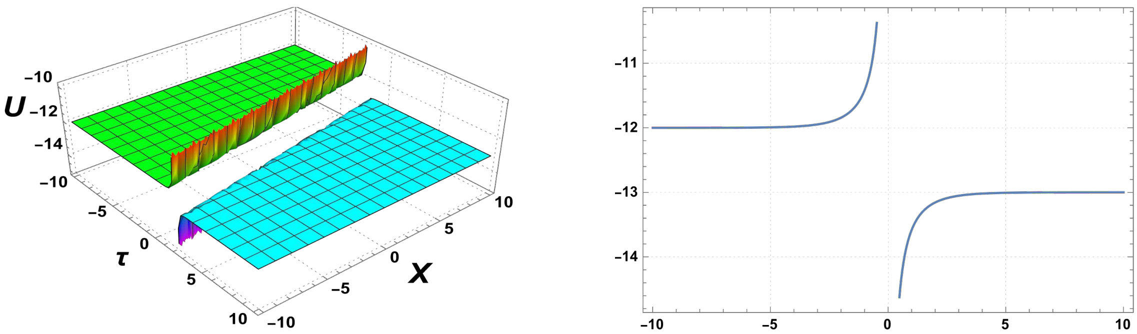
(a) 3D (b) 2D

Figure 3: Graphics of $u_3(\chi, \tau)$ for the choice of parameters $\delta_o = 1.5$, $r = 1$, $p_o = 4$, $\mathfrak{B}_1 = 4$, $\mathfrak{B}_2 = 3$,$\mathfrak{B}_3 = 5$, $\tau = 2$.



(a) 3D (b) 2D

Figure 4: Graphics of $u_4(\chi, \tau)$ for the choice of parameters $\delta_o = 5$, $r = 0.5$, $p_o = 3.5$, $\mathfrak{B}_1 = 1.5$, $\mathfrak{B}_2 = 3.5$,$\mathfrak{B}_3 = 2.5$, $\tau = 3$.



(a) 3D (b) 2D

Figure 5: Graphics of $u_{19}(\chi,\tau)$ for the choice of parameters $\delta_o = 6$, $r = 3$, $p_o = 5$, $\mathfrak{B}_1 = 1$, $\mathfrak{B}_2 = 3$,$\mathfrak{B}_3 = 5$, $\tau = 5$.

6 Graphics and discussion

Graphical representation of obtained solutions is discussed here in this section. By using the new auxiliary method we have constructed the analytical behaviour of considered model in the form of trigonometric functions, hyperbolic trigonometric functions, exponential, and algebraic type results. The graph of the tangent function is periodic with a period of Π and has vertical asymptotes at odd multiples of $\frac{Pi}{2}$. As χ approaches these vertical asymptotes, the tangent function approaches positive or negative infinity depending on the direction of approach. The graph of the cotangent function is also periodic with a period of Π and has horizontal asymptotes at even multiples of Π. As χ approaches these horizontal asymptotes, the cotangent function approaches zero. We have plotted the behaviour of some obtained results. Fig.(1) shows the graphical behavior of $u_1(\chi,\tau)$ for the choice of parameters $\delta_o = 1$, $r = 2$, $p_o = 3$, $\mathfrak{B}_1 = 2$, $\mathfrak{B}_2 = 1$,$\mathfrak{B}_3 = 2$, $\tau = 1$. Fig. (2) represent the behaviour of $u_2(\chi,\tau)$ for the choice of parameters $\delta_o = 2$, $r = 1$, $p_o = 1$, $\mathfrak{B}_1 = 3$, $\mathfrak{B}_2 = 2$,$\mathfrak{B}_3 = 1$, $\tau = 2$. Fig. (3) shows the Graphics of $u_3(\chi,\tau)$ for the choice of parameters $\delta_o = 1.5$, $r = 1$, $p_o = 4$, $\mathfrak{B}_1 = 4$, $\mathfrak{B}_2 = 3$,$\mathfrak{B}_3 = 5$, $\tau = 2$. Fig. (4) shows the Graphics of $u_4(\chi,\tau)$ for the choice of parameters $\delta_o = 5$, $r = 0.5$, $p_o = 3.5$, $\mathfrak{B}_1 = 1.5$, $\mathfrak{B}_2 = 3.5$,$\mathfrak{B}_3 = 2.5$, $\tau = 3$. Fig. (5) represent the behaviour of $u_{19}(\chi,\tau)$ for the choice of parameters $\delta_o = 6$, $r = 3$, $p_o = 5$, $\mathfrak{B}_1 = 1$, $\mathfrak{B}_2 = 3$,$\mathfrak{B}_3 = 5$, $\tau = 5$.

7 Conservation laws

In this portion, we will construct the conservation laws by multiplier approach for Eq.(6). We obtain the determinant equation for $\Lambda(X,t,u)$ by Eq. (41).

$$\frac{\delta}{\delta\mathcal{U}}\left[\Lambda\left(\frac{\partial^2\mathcal{U}}{\partial\tau^2} - \delta_o^2\frac{\partial^2\mathcal{U}}{\partial\chi^2} - p_o\frac{\partial\mathcal{U}}{\partial\chi}\frac{\partial^2\mathcal{U}}{\partial\chi^2} - q_o\left(\frac{\partial\mathcal{U}}{\partial\chi}\right)^2\frac{\partial^2\mathcal{U}}{\partial\chi^2} - r\frac{\partial^4\mathcal{U}}{\partial\chi^4}\right)\right] = 0. \tag{78}$$

Using Eq.(38), we can write the Euler operator is of the form

$$\frac{\delta}{\delta\mathcal{U}} = \frac{\partial}{\partial\mathcal{U}} - D_\tau\frac{\partial}{\partial\mathcal{U}_\tau} - D_\chi\frac{\partial}{\partial\mathcal{U}_\chi} + D_\tau^2\frac{\partial}{\partial\mathcal{U}_{\tau\tau}} + D_\chi^2\frac{\partial}{\partial\mathcal{U}_{\chi\chi}} + D_\chi D_\tau\frac{\partial}{\partial\mathcal{U}_{\tau\chi}} - ..., \tag{79}$$

defining the total derivative operators D_τ and D_χ from Eq. (37).

$$\begin{aligned}
D_\tau &= \frac{\partial}{\partial \tau} + \mathcal{U}_\tau \frac{\partial}{\partial \mathcal{U}} + \mathcal{U}_{\tau\tau} \frac{\partial}{\partial \mathcal{U}_\tau} + \mathcal{U}_{\tau\chi} \frac{\partial}{\partial \mathcal{U}_\chi} ...,\\
D_\chi &= \frac{\partial}{\partial \chi} + \mathcal{U}_\chi \frac{\partial}{\partial \mathcal{U}} + \mathcal{U}_{\chi\chi} \frac{\partial}{\partial \mathcal{U}_\chi} + \mathcal{U}_{\tau\chi} \frac{\partial}{\partial \mathcal{U}_\tau} ...,
\end{aligned} \tag{80}$$

computing Eq. (78) and we obtain the following multipliers and conservation laws as:

$$\Lambda = C_1 \tau + C_2, \tag{81}$$

by Eqs. (40) and (81), The following conservation laws are found.

$$\begin{aligned}
\mathcal{T}^\tau =& \mathcal{U}_\tau (C_1 \tau + C_2) - C_1 \mathcal{U},\\
\mathcal{T}^\chi =& (C_1 \tau + C_2) \left(\frac{-1}{3} \mathcal{U}_\chi^3 q_o - \frac{1}{2} \mathcal{U}_\chi^2 p_o - \mathcal{U}_\chi \delta_o^2 - r \mathcal{U}_{\chi\chi\chi} \right).
\end{aligned} \tag{82}$$

Using Eq. (82) and we get the following two cases of conservation laws.
Case1 : For $C_1 = 1,\ C_2 = 0$, then $\Lambda_1 = \tau$, we get the following fluxes:

$$\begin{aligned}
\mathcal{T}_1^\tau &= \tau \mathcal{U}_\tau - \mathcal{U},\\
\mathcal{T}_1^\chi &= \frac{-1}{3} \mathcal{U}_\chi^3 \tau q_o - \frac{1}{2} \mathcal{U}_\chi^2 \tau p_o - \mathcal{U}_\chi \tau \delta_o^2 - r \mathcal{U}_{\chi\chi\chi} \tau.
\end{aligned} \tag{83}$$

Case2 : For $C_1 = 0,\ C_2 = 1$, then $\Lambda_1 = 1$, we get the following fluxes:

$$\begin{aligned}
\mathcal{T}_2^\tau &= \mathcal{U}_\tau,\\
\mathcal{T}_2^\chi &= \frac{-1}{3} \mathcal{U}_\chi^3 q_o - \frac{1}{2} \mathcal{U}_\chi^2 p_o - \mathcal{U}_\chi \delta_o^2 - r \mathcal{U}_{\chi\chi\chi}.
\end{aligned} \tag{84}$$

8 Conclusion

In this research, nonlinear chains of atoms(NCA) are studied. NCA are complex systems that exhibit rich dynamics and phenomena, making their study crucial in various scientific fields. The Lie symmetry method provides a powerful mathematical tool for analyzing and solving differential equations with symmetries, reducing complexity and obtaining exact or approximate solutions. The introduction of a new auxiliary scheme enhances existing techniques, offering additional insights, improved accuracy, or simplified computations. Travelling wave solutions describe wave-like behavior propagating through systems, while graphical behavior provides visual representations of relationships and patterns in data or mathematical models. The multiplier method allows for the identification of conservation laws, which are fundamental principles in physics that state certain quantities remain constant over time. Understanding conservation laws and utilizing mathematical techniques such as the Lie symmetry method, travelling wave solutions, and graphical analysis contributes to a deeper understanding of nonlinear chains of atoms and their dynamics.

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Data availability statements: All data that support the findings of this study are included within the article (and any supplementary files).

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