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Modules in which kernels of endomorphisms are invariant under all automorphisms

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Abstract

A right R-module M is called kernel-invariant if the kernels of all endomorphisms of M are invariant under all automorphisms of M, and R is called right kernel-invariant if R_R is kernel-invariant. The classes of kernel-invariant modules and rings are simultaneous and strict generalization of two fundamental classes of modules and rings; namely, duo modules and semicommutative rings. In this paper, we show that the classes of kernel-invariant modules and rings inherits some of the important features of the aforementioned classes of modules and rings. For example, **(1)** Duo modules and uniform non-singular modules are kernel-invariant, **(2)** Endomorphism rings of kernel-invariant modules are kernel-invariant and they are abelian, **(3)** Domains are kernel-invariant which are not auto-invariant, **(4)** Semicommutative rings are kernel-invariant and the converse is true if they are clean.

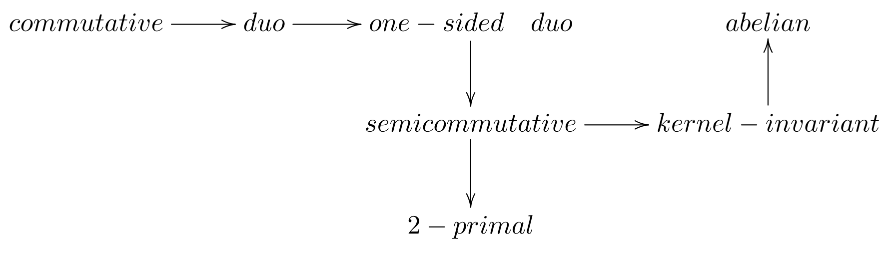
Keywords: kernel-invariant module, duo ring, semiperfect ring, semicommutative ring

1 Introduction

Throughout this paper, R denotes an associative ring with identity and all modules are assumed to be unitary. For any non-empty subsets $S \subseteq R$, $l_R(S)$ and $r_R(S)$ denote the left annihilator and the right annihilator of S in R, respectively. The Jacobson radical, the group of units, the set of all idempotent elements and singular submodule of R denoted by $J(R), U(R)$, $Id(R)$ and $Z(M)$, respectively. For a right R-module M, the endomorphism ring of M and the group automorphism of M denoted by $\text{End}(M)$ and $\text{Aut}(M)$, respectively. We write $\mathbb{Z}$ is the ring of integers.

In this paper, we introduce a new class of modules and rings, called kernel-invariant that contains properly two fundamental concepts in module theory and ring theory, namely, duo modules and semicommutative rings. More precisely, a right R-module M is called kernel-invariant if the kernels of all endomorphisms of M are invariant under all automorphisms of M, and R is called right kernel-invariant if R_R is kernel-invariant. Remark that there exists another class of modules which contains the same word "invariant": a module M is called automorphism-invariant (auto-invariant)([12]) if it is invariant under any automorphism of its injective hull. Unfortunately, there exists no a direct relation between them. Examples are provided to distinguish the class of kernel-invariant modules and rings from the aforementioned classes, and several results are established showing that this new class of modules and rings shares some of the important features of both duo modules (modules in which submodules are fully invariant) and semicommutative rings (a ring R is said to be semicommutative ([14]) if $ab = 0$ implies $aRb = 0$, for each $a, b \in R$). For example, we show that duo modules and uniform non-singular modules are kernel-invariant, and a weak duo module $M = M_1 \oplus M_2$ is kernel-invariant iff M_1 and M_2 are kernel-invariant modules in Corollary 4. Also in Section 2 of the paper, we establish a couple of decomposition theorems for kernel-invariant modules which are known to hold for both relatively projective and D3 modules.

In Section 3 of the paper, a ring R is called right kernel-invariant if R_R is a kernel-invariant right R-module. Notice that domains are kernel-invariant which are not auto-invariant and regular right self-injective rings (that are not strongly regular) are auto-invariant which are not kernel-invariant. We also show that endomorphism rings of kernel-invariant modules are kernel-invariant (Theorem 8) and they are abelian (Corollary 9). Recall that, in a semicommutative ring R, for every $x \in R$, $r_R(x)$ is an ideal of R. We obtain in Corollary 7 that R is a right kernel-invariant ring if and only if $U(R)r_R(x) = r_R(x)$ for any $x \in R$ if and only if whenever $xy = 0$ for any $x, y \in R$, then $xU(R)y = 0$. Because of this observation, we have the following chart:



We also prove that clean kernel-invariant rings are semicommutative, and a local ring is kernel-invariant iff it is semicommutative.

In Section 5, we focus on some ring extensions of kernel-invariant rings. For example, we obtain that if R is a domain, then $R[x]$ is kernel-invariant (Proposition 21), and $R[x]$ is a kernel-invariant ring iff $R[x;x^{-1}]$ is a kernel-invariant ring (Proposition 20). In this section, we also considered the amalgamated construction $A \bowtie^f K$ of rings A and B along K (an ideal of B) with respect to $f : A \to B$ which is the subring of $A \times B$. It is shown in Proposition 23 that if A and $f(A) + K$ are kernel-invariant, then so is $A \bowtie^f K$, and if $A \bowtie^f K$ is kernel-invariant, then so is A. We will study this section (and hence the paper) with an important ring extension, namely, formal matrix ring $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$, where X is an R-module and Y is an S-module. For the K-module (X, Y), it is shown that if X and Y are kernel-invariant modules, then so is (X, Y) (Proposition 24).

2 kernel-invariant modules under automorphisms

We begin with the following key observation of the paper.

Lemma 1. *The following statements are equivalent for a right R-module M with $S = \mathrm{End}(M)$ and $A = \mathrm{Aut}(M)$:*

(1) *If the kernels of all endomorphisms of M are invariant under all automorphisms of M;*
(2) *For any $\alpha \in S$, $A(\ker(\alpha)) = \ker(\alpha)$;*
(3) *For any non-empty subset I of S, $A(\ker(I)) = \ker(I)$;*
(4) *$l_S(m)A = l_S(m)$ for any $m \in M$;*
(5) *For any non-empty subset I' of M, $l_S(I')A = l_S(I')$;*
(6) *If $\alpha(m) = 0$ for any $\alpha \in S, m \in M$ then $\alpha A m = 0$.*

Proof. (1) ⇒ (2). The inclusion $A\ker(\alpha) \subseteq \ker(\alpha)$ is the statement (1). The converse inclusion always holds.
(2) ⇒ (3). Let I be a subset of S. Take any $m \in \ker(I) = \bigcap_{\alpha \in I} \ker(\alpha)$. Then $m \in \ker(\alpha)$ for all $\alpha \in I$. Since $a(m) \in \ker(\alpha)$ for all $a \in A$, we get $A\ker(I) \subseteq \ker(\alpha)$ for all $\alpha \in I$. Hence $A\ker(I) \subseteq \bigcap_{\alpha \in I} \ker(\alpha) = \ker(I)$, i.e. $\ker(I) = A\ker(I)$.
(3) ⇒ (1). The implication is obvious considering $I = \{\alpha\}$ for every $\alpha \in S$.
(1) ⇒ (4) Let $m \in M$. Take any $a \in A$ and $\alpha \in l_S(m)$. From $m \in \ker(\alpha)$ and (1), we have $a(m) \in \ker(\alpha)$. Hence $(\alpha a)(m) = 0$, i.e. $\alpha a \in l_S(m)$. It gives $l_S(m)A \subseteq l_S(m)$. The converse inclusion always holds.
(4) ⇒ (5) The implication is obvious.
(5) ⇒ (1). Let α be an endomorphism of M. By (5), $\alpha a \in l_S(m)A = l_S(m)$ for any $m \in \ker(\alpha)$ and for all $a \in A$. Then, $(\alpha a)(m) = 0$ and so $a(m) \in \ker(\alpha)$. Hence $a(\ker(\alpha)) \subseteq \ker(\alpha)$.
(4) ⇒ (6). Let α be an endomorphism of M and $m \in \ker(\alpha)$. By (4), $\alpha A \subseteq l_S(m)A = l_S(m)$, and so $\alpha A m = 0$.

(6) $\Rightarrow$ (1). Let α be an endomorphism of M. If $m \in \ker(\alpha) = 0$, then $\alpha A m = 0$ by (6). Now $a(m) \in \ker(\alpha)$ for all $a \in U$, i.e. $a \ker(\alpha) \subseteq \ker(\alpha)$. □

A module M is called *kernel-invariant* (under automorphisms) if M satisfies the equivalent conditions of Lemma 1.

Example 1. *The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ (Prüfer group) is kernel-invariant, since every nonzero proper submodule of $\mathbb{Z}_{p^\infty}$ is self-injective, we have $\alpha(K) \leq K$ for every $\alpha \in \mathrm{End}(\mathbb{Z}_{p^\infty})$ which implies $a \ker(\alpha) \subseteq \ker(\alpha)$ for all $a \in Aut(\mathbb{Z}_{p^\infty})$.*

A submodule N of a module M is called *fully invariant* if $f(N)$ is contained in N for every R-endomorphism f of M, and M is called a *(weak) duo module* provided every (direct summand) submodule of M is fully invariant.

Example 2. *Duo modules are kernel-invariant.*

A module M is called *uniform* if any two nonzero submodules of M have nonzero intersection.

Example 3. *Uniform non-singular modules are kernel-invariant. Let M be a uniform non-singular module. For any $f \in \mathrm{End}\,(M)$ with $\ker(f) \neq 0$ and $a \in \mathrm{Aut}(M)$, we have $\ker(f)$ is essential in M and so*

$$Z\left(M/\ker(f)\right) = M/\ker(f)$$

since $M/\ker(f)$ singular. Notice that $M/\ker(f) \cong \mathrm{im}(f)$. Hence

$$Z\left(M/\ker(f)\right) = Z(\mathrm{im}(f)) \quad \textit{and} \quad Z(\mathrm{im}(f)) \subseteq Z(M) \setminus \{0\}$$

(as $\mathrm{im}(f) \subseteq M$), which implies $\mathrm{im}(f) = 0$. Therefore $\ker(f) = M$, i.e., $a \ker(f) \subseteq M = \ker(f)$.

For a right R-module N, a module M is said to be *injective with respect to N* or *N-injective* if for any submodule N_1 in N, every homomorphism $N_1 \to M$ can be extended to a homomorphism $N \to M$. A module is said to be *injective* if it is injective with respect to each module. A module is said to be *quasi-injective* or *self-injective* if it is injective with respect to itself. It is well known that a module M is quasi-injective if and only if $f(M) \subseteq M$ for any endomorphism f of the injective hull of the module M, and every injective module is quasi-injective. A module M is said to be automorphism-invariant if M is invariant under any automorphism of its injective hull, i.e. for any automorphism f of $E(M)$, $f(M) \subseteq M$ where $E(M)$ denotes the injective hull of M. Some examples of the class of automorphism-invariant modules are quasi-injective modules and pseudo-injective modules (a module M is called *pseudo-injective* if every monomorphism from a submodule of M to M extends to an endomorphism of M).

Remark 1. *One may hope that the topics "kernel-invariant" and "automorphism-invariant" are related. The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ (Prüfer group) is both kernel-invariant by Example 1 and automorphism-invariant since it is quasi-injective. On the other hand, the $\mathbb{Z}$-module $\mathbb{Z}$ is kernel-invariant which is not automorphism-invariant.*

Recall that any direct summand of a (weak) duo module is also a (weak) duo module by [13, Proposition 1.3 and Proposition 1.8].

Proposition 2. *The class of kernel-invariant modules is closed under taking direct summands.*

Proof. Assume that M is a kernel-invariant right R-module and N is a direct summand of M. We write $M = N \oplus N'$ for some submodule N' of M. For any $\alpha \in \mathrm{End}(N)$ and $a \in \mathrm{Aut}(N)$, we have $\alpha \oplus 1_{N'} \in \mathrm{End}(M)$ and $a \oplus 1_{N'} \in \mathrm{Aut}(M)$. One can check that $\ker(\alpha \oplus 1'_N) = \{n + n' \in N \oplus N' \mid \alpha(n) + n' = 0\} = \ker(\alpha)$. Since M is a kernel-invariant module, we get $(a \oplus 1'_N)(\ker(\alpha \oplus 1'_N)) \subseteq \ker(\alpha \oplus 1'_N) = \ker(\alpha)$, and so $(a \oplus 1_{N'})(\ker(\alpha \oplus 1_{N'})) = (a \oplus 1_{N'})(\ker(\alpha)) = \ker(\alpha)$. We deduce that $a \ker(\alpha) \subseteq \ker(\alpha)$. □

In [13, Corollary 1.10], the authors prove that if $M = M_1 \oplus M_2$ is a weak duo module, then $\mathrm{Hom}_R(M_1, M_2) = 0$.

Proposition 3. *Let a module $M = M_1 \oplus M_2$ be a direct sum of kernel-invariant submodules M_1 and M_2. If $\mathrm{Hom}_R(M_i, M_j) = 0$ where $1 \leq i \neq j \leq 2$, then M is a kernel-invariant module.*

Proof. Let $f \in \mathrm{End}(N)$ and $a \in \mathrm{Aut}(N)$. Since $\mathrm{Hom}_R(M_i, M_j) = 0$ for $1 \leq i \neq j \leq 2$, there exist $f_1 \in \mathrm{End}(M_1)$, $f_2 \in \mathrm{End}(M_2)$ and $a_1 \in \mathrm{Aut}(M_1)$, $a_2 \in \mathrm{Aut}(M_2)$ such that

$$f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \text{ and } a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Let $(m_1, m_2) \in \ker(f)$. Then $m_1 \in \ker(f_1)$ and $m_2 \in \ker(f_2)$. By the hypothesis, we have $a_1(m_1) \in \ker(f_1)$ and $a_2(m_2) \in \ker(f_2)$. Hence

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} a_1(m_1) & 0 \\ 0 & a_2(m_2) \end{pmatrix} \in \begin{pmatrix} \ker(f_1) & 0 \\ 0 & \ker(f_2) \end{pmatrix} = \ker(f),$$

which implies $a(\ker(f)) \subseteq \ker(f)$ as desired. □

Corollary 4. *Let $M = M_1 \oplus M_2$ be a weak duo module. Then, M is a kernel-invariant module if and only if M_1 and M_2 are kernel-invariant modules.*

Given two R-modules N and M, N is called *M-projective* if for every submodule A of M, any homomorphism from N to M/A can be lifted to a homomorphism from N to M. Any two modules N and M are called *relatively projective* if N is M-projective and M is N-projective.

Lemma 5. *Let M be a kernel-invariant module with $S = \mathrm{End}(M)$ and $e^2 = e \in S$ such that $e(M)$ is projective. If $\ker(\alpha_1) \subseteq eM$ and $\ker(\alpha_2) \subseteq (1-e)R$ where α_1 and α_2 are endomorphisms of M, then $eM/\ker(\alpha_1)$ and $(1-e)M/\ker(\alpha_2)$ are relatively projective.*

Proof. Call $M_1 := eR/\ker(\alpha_1)$, $M_2 := (1-e)R/\ker(\alpha_2)$, $K := \ker(\alpha_1) \oplus \ker(\alpha_2)$. Let $\bar{L} = L/\ker(\alpha_2)$ be any submodule of M_2, where $L \leq (1-e)R$. Consider the exact sequence $M_2 \to M_2/\bar{L} \to 0$. Let $\lambda : M_1 \to M_2/\bar{L}$ be a homomorphism. Then the map $\lambda' : M_1 = eM/ker(\alpha_1) \to (1-e)M/L$ defined by $\lambda'(x + \ker(\alpha_1)) = y + L$ for all $x \in eM$ and $y \in (1-e)M$ is a homomorphism. By the projectivity of $e(M)$, λ' lifts to a homomorphism, say μ, from eM to $(1-e)M$.

Call $H := \{x + \mu(x) \mid x \in eM\}$. Then $M = H \oplus (1-e)M$ and $\delta : M \to M$ is an automorphism via $\delta(m) = em + \mu(em) + (1-e)m = m + \mu(em)$ for all $m \in M$. Since M is a kernel-invariant module, we have $\delta(K) = K$ and $\delta^{-1}(K) = K$. Clearly, $\delta' : (eM + K)/K \to (H + K)/K$ given by $\delta'(x + K) = \delta(x) + K = x + \mu(x) + K$ for all $x \in eM$ is an isomorphism. Now, if $x \in K \cap eM$, then $x + \mu(x) \in K$, which gives $\mu(x) \in K \cap (1-e)M$. Hence δ' induces a map $\bar{\mu} : eM/\ker(\alpha_1) \to (1-e)M/\ker(\alpha_2)$ given by $\bar{\mu}(x + \ker(\alpha_1)) = \mu(x) + \ker(\alpha_2)$ for all $x \in eM$. Note that $\lambda'(x + \ker(\alpha_1)) = \mu(x) + L$ for all $x \in eM$. This shows that M_1 is M_2-projective. Similarly, we can show that M_2 is M_1-projective. □

A module M is called a *D3-module* if A and B are direct summands of M with $A + B = M$, then $A \cap B$ is a direct summand of M. Besides projective and quasi-projective modules, examples of D3-modules include uniform indecomposable modules and semisimple modules.

Let N be a submodule of M. Call that *summands lift modulo* N of M if for every direct summand K/N of M/N, there exits an idempotent $e \in \text{End}(M)$ such that $K = e(M)$.

Proposition 6. *Let M be a projective kernel-invariant module and α be an endomorphism of M. If summands lift modulo* $\ker(\alpha)$*, then αM is a D3-module.*

Proof. Clearly, $\alpha(M) \cong M/\ker(\alpha)$. Call $N := \ker(\alpha)$ and $\overline{M} := M/N$. Let $\overline{A}$ and $\overline{B}$ are direct summands $\overline{M}$ such that $\overline{M} = \overline{A} + \overline{B}$. We shall show that $\bar{A} \cap \bar{B}$ is a direct summand of $\overline{M}$. Let $\overline{M} = \bar{B} \oplus \bar{B}'$. By the assumption, $\overline{B} = e(M)/N$ and $\overline{B'} = (1-e)(M)/N$ for some $e^2 = e \in \text{End}(M)$. By the proof of Lemma 5, $\overline{B'}$ is $\overline{B}$-projective. There exists a submodule $\overline{N}$ of $\overline{A}$ such that $\overline{M}$ has a decomposition $\overline{M} = \overline{N} \oplus \overline{B}$. Therefore $\overline{A} = \overline{N} \oplus (\overline{A} \cap \overline{B})$. Since $\overline{A}$ is a direct summand of $\overline{M}$, we get that $\overline{A} \cap \overline{B}$ is also a direct summand of $\overline{M}$. □

3 kernel-invariant rings

A ring R is called *right kernel-invariant* if R_R is a kernel-invariant right R-module.

Example 4. *Domains are kernel-invariant which are not automorphism-invariant.*

A ring R is called *(strongly) regular* if, for every $a \in R$, there exists $b \in R$ such that $a = aba$ (respectively, $a = a^2 b$).

Example 5. *Regular right self-injective rings (that are not strongly regular) are automorphism-invariant which are not kernel-invariant .*

The following directly follows from Lemma 1, [5, Proposition 4.3] and [9, Lemma 2.8].

Corollary 7. *The following statements are equivalent for a ring R:*

(1) *R is right kernel-invariant;*
(2) *For any $x \in R$, $U(R)r_R(x) = r_R(x)$;*
(3) *For any $x \in R$, $l_R(x)U(R) = l_R(x)$;*
(4) *For any $x \in R$, $U(R)r_R(x) = r_R(x)U(R)$;*
(5) *For any $x \in R$, $l_R(x)U(R) = U(R)l_R(x)$;*
(6) *For any non-empty subset I of R, $U(R)r_R(I) = r_R(I)$;*
(7) *For any non-empty subset I of R , $l_R(I)U(R) = l_R(I)$;*
(8) *If $xy = 0$ for any $x, y \in R$, then $xU(R)y = 0$;*
(9) *For any left ideal H and right ideal K of R, if $HK = 0$ implies $HU(R)K = 0$.*

Remark 2. *By Corollary 7, the notions of right and left kernel-invariant rings are symmetric. Hence, we will not use the terms right kernel-invariant and left kernel-invariant, and call these rings kernel-invariant rings, simply.*

Theorem 8. *Endomorphism rings of kernel-invariant modules are kernel-invariant rings.*

Proof. Let m be an element of a kernel-invariant module M with $S := \text{End}(M)$ and u be an automorphism of M. For any $f, g \in S$ with $fg = 0$, we show that $fU(S)g = 0$. Clearly, $g(m) \in \ker(f)$. Since M is a kernel-invariant module, we have $U(S)g(m) \subseteq \ker(f)$, and so $f(U(S)g)(m) = 0$ for all $m \in M$. Hence $fU(S)g = 0$, as desired. □

Proposition 9. *Every kernel-invariant ring is abelian.*

Proof. Let R be a kernel-invariant ring. For all r in R and $e = e^2 \in R$, $1 - (1 - e)re$ and $1 - er(1 - e)$ are units. Clearly, $(1 - e)[1 - (1 - e)re]e = (1 - e)re = 0$ and $e[1 - er(1 - e)](1 - e) = 0$, which imply $re = ere$ and $er = ere$. □

Corollary 10. *Endomorphism rings of kernel-invariant modules are abelian.*

A module M is called a *principally projective* (or *simply p.p.-module*) if, for any $m \in M$, $r_R(m) = eR$ where $e^2 = e \in R$ ([11]), and M is called an *S-p.p-module* if every left annihilator in $\text{End}(M)$ of any element of M is generated by an idempotent, i.e. for any $m \in M$, $l_S(m) = Se$ ([2]).

Proposition 11. *Let M be a right R-module with $S = \text{End}(M)$.*

(1) *If S is a kernel-invariant ring and any cyclic submodule of M is generated by M, then M is a kernel-invariant module.*
(2) *If M is an S-p.p-module and S is a kernel-invariant ring, then M is a kernel-invariant module.*

Proof. (1) Let f be an endomorphism of M. For any $m \in \ker(f)$, there is an endomorphism g of M such that $mR = g(M)$. Then $fg(M) = f(mR) = f(m)R = 0$,

i.e. $fg = 0$. By the hypothesis, $fug = 0$ for all $u \in \text{Aut}(M)$, and so $fu(m) = 0$. Now $u(m) \in \ker(f)$. Therefore, $u \ker(f) \subseteq \ker(f)$ which yields that $\ker(f)$ is invariant under all automorphisms of M.
(2) Let α be an endomorphism of M. Clearly, $m \in \ker(\alpha)$ if and only if $\alpha \in l_S(m)$. Let $m \in \ker(\alpha)$. Since M is an S-p.p-module, there exists $e^2 = e \in S$ such that $l_S(m) = Se = l_S(1-e)$, i.e. $\alpha(1-e) = 0$. Hence S is a kernel-invariant ring. Now $\alpha u \in l_S(1-e) = 0$ for every $u \in \text{Aut}(M)$ which implies $\alpha u m = 0$. Therefore $u \ker(\alpha) \subseteq \ker(\alpha)$. □

By Example 5, regular right self-injective rings (that are not strongly regular) are not kernel-invariant.

Proposition 12. *The following conditions are equivalent for a right R-module M with $S = \text{End}(M)$:*

(1) *S is strongly regular;*
(2) *S is regular and M is kernel-invariant.*

Proof. (1) ⇒ (2) It is well-known that S is strongly regular iff S is regular and abelian (i.e. every idempotent is central). So, we shall show that M is kernel-invariant. Let α be an element of S. Then, $\ker(\alpha)$ is a direct summand of M, i.e. there is an idempotent $e \in S$ such that $\ker(\alpha) = e(M)$. Now, for every $u \in \text{Aut}(M)$, we have $u \ker(\alpha) = ue(M)$. Since S is abelian, $eu = ue$, and hence

$$u \ker(\alpha) = ue(M) = eu(M) = e(M) = \ker(\alpha),$$

as desired.
(2) ⇒ (1) By Corollary 10, S is abelian. It follows that S is strongly regular.

□

To more characterizations of End(−), we need the following some basic properties of kernel-invariant rings.

Proposition 13. *Let R and R_i $(i \in \Lambda)$ be rings, and I be an ideal of R.*

(1) *Any subring of a kernel-invariant ring is kernel-invariant.*
(2) *Let I be a reduced ideal (i.e. it does not contain nilpotent elements) of R. If R/I is a kernel-invariant ring, then R is a kernel-invariant ring.*
(3) *$\prod_{i \in \Lambda} R_i$ is kernel-invariant if and only if R_i is kernel-invariant for each $i \in \Lambda$.*
(4) *If R is kernel-invariant and $e \in Id(R)$, then the corner ring eRe is kernel-invariant.*
(5) *If R is kernel-invariant, then $J(R) r_R(x) \subseteq r_R(x)$ for each $x \in R$.*

Proof. (1) Let S be a subring of R and $x, y \in S$ such that $xy = 0$. Since $U(S) \subseteq U(R)$, we have $xU(S)y \subseteq xU(R)y = 0$, which implies S is a kernel-invariant ring.
(2) Let u be an arbitrary unit of R. Then, $u + I$ is also a unit of the ring R/I. For any $a, b \in R$ with $ab = 0$, it is easy to see that $aub \in I$. Thus $aU(R)b \subseteq I$. Moreover,

$(bIa)^2 = bIabIa = 0$ and hence $bIa = 0$, since I is a reduced ideal. On the other hand, we have

$$(aU(R)bI)^2 = aU(R)bIaU(R)bI = 0.$$

Again, since I is a reduced ideal, we get $aU(R)bI = 0$. Therefore

$$(aU(R)b)^2 = aU(R)baU(R)b \subseteq aU(R)bI = 0$$

which yields $aU(R)b = 0$.
(3) As usual, π_j denotes the canonical projection for each $i \in \Lambda$.

The necessity: Let $a_i, b_i \in R_i$ and $a_i b_i = 0$ for each $i \in \Lambda$. Then $a := (a_i)_{i\in\Lambda}$ and $b := (b_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} R_i$ for which

$$\pi_i(a) = a_i, \quad \pi_i(b) = b_i$$

and

$$\pi_j(a) = \pi_j(b) = 0$$

for all $j \neq i$. Then, $ab = 0$, hence $aU(R)b = 0$. Since $\pi_i(U(R)) = U(R_i)$, we obtain $a_i U(R_i) b_i = 0$.

The sufficiency: Let $a = (a_i)_{i\in\Lambda}, b = (b_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} R_i$ such that $ab = 0$. Then $a_i b_i = 0$ for each $i \in \Lambda$. Since each R_i is kernel-invariant, we obtain $a_i U(R_i) b_i = 0$ for all i, i.e. $aU(R)b = 0$.
(4) Suppose $exe, eye \in eRe$ and $(exe)(eye) = 0$. Since R is kernel-invariant, we have $(exe)U(R)(eye) = 0$ which implies $(exe)(eU(R)e)(eyx) = 0$. Hence $(exe)U(eRe)(eye) \subseteq (exe)(eU(R)e)(eye) = 0$.
(5) Let $j \in J(R)$, $y \in r_R(x)$ and $xy = 0$. By the hypothesis, $xU(R)y = 0$ and $x(1-j)y = 0$ since $1 - j \in U(R)$. Hence $xjy = 0$ which implies $jy \in r_R(x)$. Therefore $J(R)r_R(x) \subseteq r_R(x)$, as desired. □

A ring R is called *semicommutative* if $xy = 0$ then $xRy = 0$ where all $x, y \in R$ ([14]). Because of the definition, the right annihilator of every element of a semicommutative ring is an ideal. We also remark that semicommutative rings are abelian by [14, Lemma 2.7].

Proposition 14. *Let M be a kernel-invariant module.*

1. *If* $\mathrm{End}(M)$ *is a local ring, then* $\mathrm{End}(M)$ *is a semicommutative ring.*
2. $\mathrm{End}(M)$ *is a semiperfect ring if and only if* $\mathrm{End}(M)$ *is a finite product of local semicommutative rings.*

Proof. By Theorem 8, $S = \mathrm{End}(M)$ is a kernel-invariant ring.
(1) For $\alpha, \beta \in \mathrm{End}(M)$ with $\alpha\beta = 0$, we have $\alpha U(S)\beta = 0$ by Corollary 7. Since $\mathrm{End}(M)$ is local ring, we have either $f \in U(S)$ or $1 - f \in U(S)$ for every $f \in \mathrm{End}(M)$. Then $\alpha f \beta = 0$ which implies $\alpha S \beta = 0$. Thus, S is a semicomumative ring.
(2) Assume that $\mathrm{End}(M)$ is a semiperfect ring. By Corollary 10, $S = \mathrm{End}(M)$ is abelian. Since $S = \mathrm{End}(M)$ is a semiperfect ring, we have a decomposition $S = e_1 S \oplus e_2 S \oplus ... \oplus e_n S$ where $\{e_1, e_2, \dots, e_n\}$ is a set of local orthogonal idempotents. As S is abelian, we obtain $e_i S = S e_i = e_i S e_i$, so that each $e_i S$ is local semicommutative.

The converse is obvious. □

If X is a subset of a ring R, we denote by $\langle X\rangle$ the subgroup of the abelian group $(R, +, -, 0)$ generated by the set X.

Proposition 15. *Let $R = \langle U(R) \cup Id(R) \cup J(R)\rangle$. Then $R = \langle U(R) \cup Id(R)\rangle$ and R is a kernel-invariant ring if and only if R is semicommutative.*

Proof. If $j \in J(R)$, then $j - 1 \in U(R)$, hence $j \in \langle U(R) \cup Id(R)\rangle$. It proves that $J(R) \subseteq \langle U(R) \cup Id(R)\rangle$.

Let R be a kernel-invariant ring and $x \in R$. It is enough to prove that $r_R(x)$ is a two-sided ideal. Take $a \in r_R(x)$ and $y \in R$. Then $y = u_1 + u_2 + ... + u_i + e_1 + e_2 + ... + e_j$ for each $u_i \in U(R)$ and $e_j \in Id(R)$. Since R is a kernel-invariant ring and $xa = 0$, we obtain $xU(R)a = 0$ which implies $u_i a \in r_R(x)$. Furthermore, as R is abelian, $e_j a = ae_j$ and $ae_j \in r_R(x)$, so $e_j a \in r_R(x)$. Hence $ya \in r_R(x)$, or $r_R(x)$ is a left ideal, which gives $r_R(x)$ is a two-sided ideal for each $x \in R$.

The reverse implication is obvious. □

Corollary 16. *A local ring is kernel-invariant if and only if it is semicommutative.*

A ring R is called *clean* if, for any $r \in R$, there is an idempotent e of R such that $r - e \in U(R)$ [7].

Corollary 17. *A semicommutative ring is kernel-invariant. The converse is true if it is clean.*

Proof. Let R be a semicommutative ring and $xy = 0$ where $x, y \in R$. Then $xU(R)y \subseteq xRy$ implies $xU(R)y = 0$, i.e. R is kernel-invariant.

For the converse, if the ring R is clean then $R = \langle U(R) \cup Id(R)\rangle$. By Proposition 15, R is semicommutative. □

Corollary 18. *A semiperfect ring is kernel-invariant if and only if it is semicommutative.*

Proof. Assume that R is a kernel-invariant ring. Then R has a decomposition $R = e_1R \oplus e_2R \oplus ... \oplus e_nR$ where $e_1, e_2, ..., e_n$ is a finite orthogonal set of local idempotents. Since R is abelian by Corollary 10, each e_iR is kernel-invariant by Proposition 13. Hence each e_iR local semicommutative, i.e. R is semicommutative.

The converse implication is obvious. □

4 Some extensions of kernel-invariant rings

A ring R is called *Armendariz* if $f(x)g(x) = 0$ where $f(x) = \Sigma_{i=1}^{t} a_i x^i, g(x) = \Sigma_{j=1}^{n} b_j x^j \in R[x]$ implies $a_i b_j = 0$ for every i and j ([3]).

Proposition 19. *Let R be a prime ring. Then $R[x]/(x^n)$ is Armendariz if and only if $R[x]/(x^n)$ is kernel-invariant .*

Proof. Let R be a prime ring.
($:\Rightarrow$) By [1, Corollary 2.16], $R[x]/(x^n)$ is Armendariz if and only if $R[x]/(x^n)$ is semi-commutative. Hence $R[x]/(x^n)$ is kernel-invariant by Corollary 17.
($\Leftarrow:$) By [9, Theorem 2.16] and [10, Corollary 1.5], we shall prove that R is kernel-invariant. Let $a, b \in R$ satisfy $ab = 0$. Consider an arbitrary unit $u(x) = u + (x^n)$ of $R[x]/(x^n)$ ring. For $f(x) = a + (x^n)$, $g(x) = b + (x^n) \in R[x]/(x^n)$, we have $f(x)g(x) = ab + (x^n) = 0 + (x^n) = 0_{R[x]/(x^n)}$. Therefore, $f(x)U(R[x]/(x^n))g(x) = 0$ which implies $aub = 0$. □

Proposition 20. *$R[x]$ is a kernel-invariant ring if and only if $R[x; x^{-1}]$ is a kernel-invariant ring.*

Proof. ($:\Rightarrow$) Let $\Omega := \{1; x; x^2; ...\}$. Then $R[x, x^{-1}] = \Omega^{-1}R[x]$. For every $u \in \Omega^{-1}R[x]$ we get $u = x^{-k}u_1$, where $u_1 \in U(R[x])$ and $x^k \in \Omega$. Let $f := x^{-i}f_1$ and $g := x^{-j}g_1$ be elements of $\Omega^{-1}R[x]$, where $f_1, g_1 \in R[x]$. If $fg = 0$, then $f_1g_1 = 0$. Hence $f_1u_1g_1 = 0$ which implies $x^{-i}f_1x^kux^{-j}g_1 = 0$, i.e. $fug = 0$. Thus, $R[x; x^{-1}]$ is a kernel-invariant ring.
($\Leftarrow:$) This implication follows from by Proposition 13 since $R[x] \subseteq R[x; x^{-1}]$. □

Recall that domains are kernel-invariant (see Example 4).

Proposition 21. *If R is a domain, then $R[x]$ is kernel-invariant .*

Proof. If R is domain then $U(R[x]) = U(R)$ by [4, Lemma 2.4] and R is an Armendariz ring as well as kernel-invariant. If $f(x)g(x) = 0$ where $f(x) = \sum a_ix^i \in R[x]$ and $g(x) = \sum b_jx^j \in R[x]$, then $a_jb_j = 0$ for all $a_i, b_j \in R$. Hence $a_jub_j = 0$, where $u \in U(R)$, which gives $fU(R[x])g = 0$. □

Next, we study on finite kernel-invariant rings. A ring R is called a *minimal non-commutative kernel-invariant* ring if R has the smallest order $|R|$ among the non-commutative kernel-invariant rings. The field with p^n elements is denoted by $GP(p^n)$ for a prime number p and natural number n.

Theorem 22. *If R is a minimal non-commutative kernel-invariant ring, then R is a local ring with $|R| = 16$ and R is isomorphic to one of the following rings R_1, R_2, R_3, R_4 and R_5.*

(1) $R_1 = \mathbb{Z}_2\langle x, y\rangle/I$, *where I is an ideal of $\mathbb{Z}_2\langle x, y\rangle$ generated by $x^3, y^3, yx, x^2 - xy, y^2 - xy$.*
(2) $R_2 = \mathbb{Z}_4\langle x, y\rangle/I$, *where I is an ideal of $\mathbb{Z}_4\langle x, y\rangle$ generated by $x^3, y^3, yx, x^2 - xy, x^2 - 2, y^2 - 2, 2x, 2y$.*
(3) $R_3 = \left\{ \begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix} \;\middle|\; a, b \in GF(2^2) \right\}$.
(4) $R_4 = \mathbb{Z}_2\langle x, y\rangle/I$, *where I is an ideal of $\mathbb{Z}_2\langle x, y\rangle$ generated by $x^3, y^2, yx, x^2 - xy$*
(5) $R_5 = \mathbb{Z}_4\langle x, y\rangle/I$, *where I is an ideal of $\mathbb{Z}_4\langle x, y\rangle$ generated by $x^3, y^2, yx, x^2 - xy, x^2 - 2, 2x, 2y$.*

Proof. Let R be a minimal non-commutative kernel-invariant ring. By [16, Example 2], R_1 is a noncommutative duo ring with 16 elements. From the minimality of $|R|$, it infers that $|R| \leq 16$. We can write $|R| = p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$ with p_i prime numbers and $m_j \in \mathbb{N}$. Then, we have a decomposition $R = R_1 \oplus R_2 \oplus \cdots \oplus R_k$ (ideal direct sum), where each ideal R_i is of order $p_i^{m_i}$ and has a unity. If each $m_i < 3$, then each R_i is commutative, a contradiction. There exists a $m_i \geq 3$. Suppose that $m_i = 3$ and $p_i = 2$. Then $R_i \cong \begin{pmatrix} GF(2) & GF(2) \\ 0 & GF(2) \end{pmatrix}$ again by [6, Proposition]. But $T := \begin{pmatrix} GF(2) & GF(2) \\ 0 & GF(2) \end{pmatrix}$ is not kernel-invariant (since $r_T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & GF(2) \end{pmatrix}$ is not invariant under the unit $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of T), a contradiction. It follows that either $m_i > 3$ or $p_i > 2$. We deduce that $|R| \geq 16$, and so $|R| = 16$. As R is an artinian ring, we have a decomposition $R = e_1 R \oplus e_2 R \oplus \cdots \oplus e_n R$ where $\{e_1, e_2, \dots, e_n\}$ is a finite orthogonal set of local idempotents. But R is abelian by Proposition 9, and so each $e_i R$ is an ideal of R. By Theorem 8, we have $e_i R = e_i R e_i$ a local kernel-invariant ring. Since R is a minimal non-commutative kernel-invariant ring, we must have $n = 1$ and $R = e_1 R = e_1 R e_1$. They imply that R is a local ring. This yields that R is isomorphic to an R_i ($i = 1, 2, 3, 4, 5$) by [15, Theorem 8]. □

Remark 3. *The ring R_5 is a kernel-invariant ring which is not duo.*

Let A and B be two rings with identity, K be an ideal of B and $f : A \to B$ be a ring homomorphism. We consider the subring of $A \times B$ defined by:

$$A \bowtie^f K := \{(a, f(a) + k) \mid a \in A, k \in K\}$$

which is called *the amalgamated construction* of A with B along K with respect to f.

Proposition 23. *Let A and B be a pair of ring, $f : A \to B$ be a ring homomorphism and K be a proper ideal of B. Then the following hold:*

(1) *If A and $f(A) + K$ are kernel-invariant, then so is $A \bowtie^f K$.*
(2) *If $A \bowtie^f K$ is kernel-invariant, then so is A.*
(3) *If f is injective, A is reduced and K is a nil ideal, then $A \bowtie^f K$ is kernel-invariant if and only if $f(A) + K$ is so.*

Proof. (1) If A and $f(A) + K$ are kernel-invariant rings, then $A \times (f(A) + K)$ is kernel-invariant by Proposition 13 (3). Since $A \bowtie^f K$ is a subring of ring $A \times (f(A) + K)$, it is kernel-invariant by Proposition 13 (1).
(2) Since A is isomorphic to a subring of the kernel-invariant ring $A \bowtie^f K$, it is kernel-invariant by Proposition 13 (1).
(3) Let $x \in f(A) \cap K$. Since K is a nil ideal, there exists a natural number n such that $x^n = 0$. Let $x = f(a)$ for some $a \in A$. Then $x^n = (f(a))^n = 0$ implies $f(a) = 0$. Hence $f(A) \cap K = 0$. By [5, Lemma 2.5], $A \bowtie^f K \cong f(A) + K$. □

Let $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a formal matrix ring, X be an R-module and Y be an S-module. Consider R-homomorphisms $f : Y \otimes {}_S N \to X$ given by $f(y \otimes n) = yn$ where $y \in Y, n \in N$ and $g : X \otimes {}_R M \to Y$ given by $g(x \otimes m) = xm$, where $x \in X, m \in M$, such that $(yn)m = y(nm)$ and $(xm)n = x(mn)$. Then (X, Y) is a K-module.

Consider two R-modules X, A and two S-modules Y, B. Then the K-module $W = (A, B)$ is a submodule of the K-module (X, Y) if and only if $A \leq X$, $B \leq Y$, $BN \leq A$ and $AM \leq B$.

Proposition 24. *Let X be an R-module and Y be an S-module. If X and Y are kernel-invariant modules, then so is $V = (X, Y)$.*

Proof. Let $u \in \mathrm{Aut}(X)$ and $v \in \mathrm{Aut}(Y)$ such that $u(yn) = v(y)n$ and $v(xm) = u(x)m$. Let $\alpha \in \mathrm{End}(X); \beta \in \mathrm{End}(Y)$ such that $\alpha(yn) = \beta(y)n, \beta(xm) = \alpha(x)m$. It is easy to see that $(\alpha, \beta) \in \mathrm{End}(V)$ and $(u, v) \in \mathrm{Aut}(V)$. We shall show that $(u(\ker(\alpha)), v(\ker(\beta)) \subseteq (\ker(\alpha), \ker(\beta))$. For any $x \in u(ker(\alpha))$, we have $x \in \ker(\alpha)$ by the hypothesis. For any $m \in M$, $\beta(xm) = \alpha(x)m = 0$, implies $xm \in \ker(\beta)$. Hence $u(\ker \alpha)M \subseteq \ker(\beta)$. Similarly, for any $y \in v(\ker(\beta))$, we have $y \in \ker(\beta)$ by the hypothesis. For any $n \in N$, $\alpha(yn) = \beta(y)n = 0$ implies $yn \in \ker(\alpha)$. Hence $v(\ker(\beta))N \subseteq \ker(\alpha)$, as desired. □

Now we form the group of row vectors $(X, \mathrm{Hom}_R(N, X))$, where we consider the group $\mathrm{Hom}_R(N, X)$ as an S-module in a standard way. In fact, we have a K-module with homomorphisms of the module multiplication

$$\begin{aligned} g : \mathrm{Hom}_R(N, X) \otimes N &\to X \\ \eta \otimes n &\mapsto \eta(n) \\ f : X \otimes M &\to \mathrm{Hom}_R(N, X) \\ x \otimes m &\mapsto \mu \end{aligned}$$

where $\mu(m) = x(mn)$, $m \in M$, $n \in N$, $x \in X$ and $\eta \in \mathrm{Hom}_R(N, X)$.

We will write $H(X)$ instead of $\mathrm{Hom}_R(M, X)$ and $H(Y)$ instead of $\mathrm{Hom}_S(N, Y)$.

Let X be an R-module, (A, B) be a K-module and $\alpha : A \to B$ be an R-homomorphism. Then there exists a unique map $\beta : B \to H(X)$ such that $\beta(b)(n) = \alpha(bn)$, $b \in B$ and $n \in N$ by ([8, Lemma 3.1.3]). We also notice that β is an S-homomorphism and $(\alpha; \beta)$ is a K-homomorphism.

Lemma 25. *Let f be an epimorphism, X be a R-module, $(X, H(X))$ be a K-module and $i : X \to X$ be an isomorphism. Then there exists an isomorphism $j : H(X) \to H(X)$ such that $j(\eta)(n) = i(\eta(n))$ and i is unique.*

Proof. By [8, Lemma 3.1.3],

$$\begin{array}{ccc} X \otimes {}_R M & \xrightarrow{f} & H(X) \\ {\scriptstyle i\otimes 1_M}\downarrow & & \downarrow{\scriptstyle j} \\ X \otimes {}_R M & \xrightarrow{f} & H(X) \end{array}$$

$$\begin{array}{ccc} X \otimes {}_R M & \xrightarrow{f} & H(X) \\ {\scriptstyle i^{-1}\otimes 1_M}\downarrow & & \downarrow{\scriptstyle j'} \\ X \otimes {}_R M & \xrightarrow{f} & H(X) \end{array}$$

the diagrams are commutative. Hence,

$$j'f = f(i^{-1} \otimes 1_M)$$

and

$$jj'f = jf\left(i^{-1} \otimes 1_M\right) = f\left(i \otimes 1_M\right)\left(i^{-1} \otimes 1_M\right) = f\left(ii^{-1} \otimes 1_M 1_M\right) = f\left(1_M \otimes 1_M\right)$$

which imply $jj' = 1$. □

Proposition 26. *Let $(X, H(X))$ be a K-module. If $(X, H(X))$ is a kernel-invariant module, then so is X.*

Proof. Let $i \in \mathrm{Aut}(X)$ and $\alpha \in \mathrm{End}(X)$. Then there exists a unique $j \in \mathrm{Aut}(H(X))$ such that $u = (i, j) \in \mathrm{Aut}((X, H(X)))$ and $\beta \in \mathrm{End}(H(X))$ such that $\Phi = (\alpha, \beta) \in \mathrm{End}((X, H(X)))$. Since $(X, H(X))$ is kernel-invariant, we obtain $u(\ker(\Phi)) \subseteq \ker(\Phi)$. Hence $(i(\ker(\alpha)), j(\ker(\beta))) \subseteq (\ker(\alpha), \ker(\beta))$ which implies $i(\ker(\alpha)) \subseteq \ker(\alpha)$. We deduce that X is a kernel-invariant module. □

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