

Hilbert's First Problem and the New Progress of Infinity Theory

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Abstract: In the 19th century, Cantor created the infinite cardinal number theory based on the "1-1 correspondence" principle. The continuum hypothesis is proposed under this theoretical framework. In 1900, Hilbert made it the first problem in his famous speech on mathematical problems, which shows the importance of this question. We know that the infinitesimal problem triggered the second mathematical crisis in the 17-18th centuries. The Infinity problem is no less important than the infinitesimal problem. In the 21st century, Sergeyev introduced the Grossone method from the principle of "whole is greater than part", and created another ruler for measuring infinite sets. The development of the infinity theory provides new ideas for solving Hilbert's first problem, and provides a new mathematical foundation for Cosmic Continuum Theory.

Key words: Hilbert's first problem; Continuum hypothesis; Grossone method; Cosmic continuum; Infinity theory

1. Introduction

In 1874, Cantor introduced the concept of cardinal numbers based on the "1-1 correspondence" principle. Cantor proved that the cardinal number of the continuum, C , is equal to the cardinal number of the power set of the natural number set, 2^{\aleph_0} , where \aleph_0 is the cardinal number of the natural number set. Cantor arranges the cardinal number of infinities from small to large as $\aleph_0, \aleph_1, \dots, \aleph_a, \dots$. Among them, a is an arbitrary ordinal number, which means that the cardinal number of the natural number set, \aleph_0 , is the smallest infinity cardinal number. Cantor conjectured: $2^{\aleph_0} = \aleph_1$. This is the famous Continuum hypothesis (CH). For any ordinal a , $2^{\aleph_a} = \aleph_{a+1}$ holds, it is called the Generalized continuum hypothesis (GCH) [1].

In 1938 Gödel proved that the CH is not contradictory to the ZFC axiom system. In 1963, Cohen proved that the CH and the ZFC axiom system are independent of each other. Therefore, the CH cannot be proved in the ZFC axiom system [2]-[3].

However, people always have doubts about infinity theory. For example, in the study of Cosmic Continuum, the existing infinity theory shows great limitations[4]-[14].

In the 21st century, Sergeyev started from "the whole is greater than the part" and introduced a new method of counting infinity and infinitesimals, called the Grossone method. The introduced methodology (that is not related to non-standard analysis) gives the possibility to use the same numeral

38 system for measuring infinite sets, working with divergent series, probability, fractals, optimization
39 problems, numerical differentiation, ODEs, etc.[15]-[40]

40 The Grossone method introduced by Sergeyev takes the number of elements in the natural number
41 set as a total number, marked as $\textcircled{1}$, as the basic numeral symbol for expressing infinity and
42 infinitesimal, in order to more accurately describe infinity and infinitesimal.

43 The Grossone method was originally proposed as a Computational Mathematics, but its significance
44 has far exceeded the category of Computational Mathematics. In particular, the Grossone method
45 provides a new mathematical tool for the Cosmic Continuum Theory. A new infinity theory is about to
46 emerge.

47 **2. The traditional infinity paradox and the fourth mathematics crisis**

48 In the history of mathematics, there have been three mathematics crises, each of which involves the
49 foundation of mathematics. The first time was the discovery of irrational numbers, the second time was
50 the infinitesimal problem, and the third time was the set theory paradox[41]-[42]. However, no one
51 dare to say that the building of the mathematical theory system has been completed, and maybe the
52 fourth mathematical crisis will appear someday.

53 In fact, the fourth mathematics crisis is already on the way. This is the infinity problem. In 1900,
54 Hilbert put the Cantor continuum hypothesis as the first question in his famous lecture on 23
55 mathematics problems [43]. This will never be an impromptu work by an almighty mathematician.

56 The infinitesimal question unfold around whether the infinitesimal is zero or not. From the 1920s to
57 the 1970s, this problem has been initially solved through the efforts of generations of mathematicians.
58 However, there are still different opinions about the second mathematics crisis. I believe that the
59 infinitesimal problem has not been completely solved, otherwise there would be no infinity problem.
60 Because the infinity problem and the infinitesimal problem are actually two aspects of the same
61 problem.

62 Let us first look at what is problem with infinity.

63 The first is the expression of infinity. Now, there are two ways to express the infinity, one is to
64 express with infinity symbol \aleph , and the other is to express with infinity cardinal number. However,
65 neither the infinity symbol \aleph nor the infinity cardinal number can effectively express infinity and
66 infinitesimal.

67 For example: when expressed in the infinity symbol \aleph , we cannot distinguish the size of the natural
68 number set and the real number set, nor can we distinguish the size of the natural number set and the
69 integer set, they are all \aleph . When expressed in infinity cardinal number, we can distinguish the size of
70 the natural number set and the real number set, because the cardinal number of the natural number set
71 is \aleph_0 , and the cardinal number of the real number set is $C = 2^{\aleph_0}$; but it is still impossible to
72 distinguish the size of the natural number set and the integer set, they are both \aleph_0 .

73 The second is the calculation of infinity. Whether it is the infinity symbol \aleph or the infinity cardinal
74 number, it cannot play a mathematically precise role in calculations. E.g:

75 $\aleph + 1 = \aleph$, $\aleph - 1 = \aleph$, $\aleph' \aleph = \aleph$, $\aleph^\aleph = \aleph$.

76 And $\frac{\aleph}{\aleph}$, $\aleph - \aleph$, etc. have no meaning at all.

77 Relative to infinity symbol \aleph , Cantor's infinite cardinal number is an improvement, but the
 78 cardinal number method of infinity can only be calculated qualitatively. The theory of infinity cardinal
 79 number is based on the principle of "1-1 correspondence". Although according to the principle of
 80 "power set is greater than the original set", infinite cardinal number can be compared in size, but it is
 81 only the size of classes of infinity, not the size of infinity individuals.

82 For example, according to the continuum hypothesis, the following equation holds:

$$83 \quad \aleph_0 + 1 = \aleph_0, \quad \aleph_0 + \aleph_0 = \aleph_0, \quad \aleph_0 + 2^{\aleph_0} = 2^{\aleph_0}, \quad 2^{\aleph_0} + 2^{\aleph_0} = 2^{\aleph_0}.$$

84 This obviously violates the calculation rules of finite numbers and does not meet the uniformity
 85 requirements of mathematical theory.

86 The reason for the infinity paradox in mathematical expressions and mathematical calculations is
 87 that the existing infinity theory does not need to follow the principle of "the whole is greater than the
 88 part", and this principle needs to be followed in the finite number theory. In this way, there is a problem
 89 of using different calculation rules in the same calculation formula.

90 Since there is an infinite problem, how can there be no infinitesimal problem?

91 For example: because the infinity and the infinitesimals are reciprocal of each other (when the
 92 infinitesimal is not zero), the following equation holds:

$$93 \quad \frac{1}{\aleph + 1} = \frac{1}{\aleph}, \quad \frac{1}{\aleph - 1} = \frac{1}{\aleph}, \quad \frac{1}{\aleph'} = \frac{1}{\aleph}, \quad \frac{1}{\aleph^{\aleph}} = \frac{1}{\aleph};$$

$$94 \quad \frac{1}{\aleph_0 + 1} = \frac{1}{\aleph_0}, \quad \frac{1}{\aleph_0 + \aleph_0} = \frac{1}{\aleph_0}, \quad \frac{1}{\aleph_0 + 2^{\aleph_0}} = \frac{1}{2^{\aleph_0}}, \quad \frac{1}{2^{\aleph_0} + 2^{\aleph_0}} = \frac{1}{2^{\aleph_0}}.$$

95 Obviously, in these equations, although mathematical calculations can also be performed, the
 96 mathematical accuracy is lost. At the same time, treating zero as a special infinitesimal is inconsistent
 97 with the concept of infinitesimals. Because in modern mathematics, the infinitesimal is not a number
 98 but a variable, and zero is a specific number, which is inconsistent with the definition of infinitesimal.

99 It can be seen that the problem of infinity involves many basic mathematics problems, and the
 100 mathematics crisis caused by it is no less than the previous three mathematics crises. No wonder
 101 Hilbert listed the continuum problem as the top of the 23 mathematical problems.

102 3. Grossone method and quantitative calculation of infinity

103 Sergeyev used Grossone $\textcircled{1}$ to represent the number of elements in set of natural numbers, which is
 104 similar to Kantor's cardinal number method. Kantor's cardinal number and Sergeyev's Grossone $\textcircled{1}$
 105 are superficially the same thing. Both represent the size of the set of natural numbers, but they are two
 106 completely different concepts.

107 The cardinal number represents the size of a type of set that satisfies the principle of "1-1
 108 correspondence". For a finite set, the cardinal number is the "number" of elements, but for an infinite
 109 set, the cardinal number is not the "number" of elements. Is the size of a class of infinite sets that are
 110 equivalent to each other. And Grossone $\textcircled{1}$ represents the "number" of elements in a natural number
 111 set, just like any finite set. Using this as a ruler, you can measure every infinity and infinitesimal.

112 In Grossone theory, infinity and infinitesimal are not variables, but definite quantities. Infinity and
 113 infinitesimal are the reciprocal of each other. For example, the number of elements $\textcircled{1}$ of the natural

114 number set is an infinity, and its reciprocal $\frac{1}{\textcircled{1}}$ is an infinitesimal. Obviously, zero is not an

115 infinitesimal.

116 Let us see how numbers are expressed. The decimal numeral we generally use now are:
117 1,2,3,4,5,6,7,8,9,0. Among these 10 numeral, the largest numeral is 9, but we can use them to express
118 all finite numbers, whether it is ten thousand digits, billion digits, or larger numbers.

119 As the number of elements in the natural number set, Grossone, together with 1,2,3,4,5,6,7,8,9,0, can
120 express any finite number and infinity.

121 For example, according to the principle of "whole is greater than part", we can get:

122 $\textcircled{1} + 1 > \textcircled{1}$, $\textcircled{1} + \textcircled{1} = 2\textcircled{1}$, $\textcircled{1} + 2^{\textcircled{1}} > 2^{\textcircled{1}}$, $2^{\textcircled{1}} + 2^{\textcircled{1}} = 2' 2^{\textcircled{1}}$

123 The Grossone method can not only accurately express infinity, but also accurately express
124 infinitesimal. E.g:

125 $\frac{1}{2\textcircled{1}}$, $\frac{2}{3\textcircled{1}^2}$, $\frac{3}{2^{\textcircled{1}}}$

126 For example, infinity can be operated like a finite number:

127 $0 \times \textcircled{1} = \textcircled{1} \times 0 = 0$, $\textcircled{1} - \textcircled{1} = 0$, $\frac{\textcircled{1}}{\textcircled{1}} = 1$, $\textcircled{1}^0 = 1$, $1^{\textcircled{1}} = 1$, $0^{\textcircled{1}} = 0$

128 $\lim_{x \textcircled{1}} \frac{1}{x} = \frac{1}{\textcircled{1}}$, $\lim_{x \textcircled{2}^{\textcircled{1}}} \frac{1}{x} = \frac{1}{2^{\textcircled{1}}}$, $\lim_{x \textcircled{\frac{1}{\textcircled{1}}}} x^3 = \frac{1}{\textcircled{1}^3}$

129 $\overset{\textcircled{1}}{\underset{\textcircled{0}}{\int}} x^2 dx = \frac{1}{3} \textcircled{1}^3$, $\overset{\textcircled{1}^2}{\underset{\textcircled{0}}{\int}} x^2 dx = \frac{1}{3} (\textcircled{1}^6 - \textcircled{1}^3)$, $\overset{2^{\textcircled{1}}}{\underset{\textcircled{0}}{\int}} x^2 dx = \frac{1}{3} \times 2^{3\textcircled{1}}$

130 More importantly, the Grossone method solves the calculation problems of $\frac{\textcircled{\neq}}{\textcircled{\neq}}$, $\textcircled{\neq} - \textcircled{\neq}$, etc. that
131 cannot be performed in the infinity theory.

132 For example, the following calculations are possible:

133 $\frac{\textcircled{1}}{2\textcircled{1}} = \frac{1}{2}$, $\frac{2\textcircled{1}}{3\textcircled{1}^3} = \frac{2}{3\textcircled{1}^2}$, $3\textcircled{1} - \textcircled{1} = 2\textcircled{1}$

134 It can be seen that the Grossone method meets the requirements of the unity of mathematical theory.
135 From the above discussion, we can see that the cardinal method uses the "1-1 correspondence"
136 principle but violates the "whole is greater than the part" principle, while the Grossone method uses the
137 "whole is greater than the part" principle, but does not violate the "1-1 correspondence" principle.

138 Therefore, the new infinity theory can integrate the infinity cardinal number method with the
139 Grossone method. But when using the infinity cardinal number theory to calculate, we should not use
140 the "=" symbol, but can use "⁰" to indicate that it is equivalent under the "1-1
141 correspondence" principle. E.g:

142 $\overset{\textcircled{1}}{\underset{\textcircled{0}}{\int}} + 1^{\textcircled{0}} \overset{\textcircled{1}}{\underset{\textcircled{0}}{\int}}$, $\overset{\textcircled{1}}{\underset{\textcircled{0}}{\int}} + \overset{\textcircled{1}}{\underset{\textcircled{0}}{\int}} \overset{\textcircled{0}}{\underset{\textcircled{0}}{\int}} \overset{\textcircled{1}}{\underset{\textcircled{0}}{\int}}$, $\overset{\textcircled{1}}{\underset{\textcircled{0}}{\int}} + 2^{\overset{\textcircled{1}}{\underset{\textcircled{0}}{\int}}} \overset{\textcircled{0}}{\underset{\textcircled{0}}{\int}} 2^{\overset{\textcircled{1}}{\underset{\textcircled{0}}{\int}}}$, $2^{\overset{\textcircled{1}}{\underset{\textcircled{0}}{\int}}} + 2^{\overset{\textcircled{1}}{\underset{\textcircled{0}}{\int}}} \overset{\textcircled{0}}{\underset{\textcircled{0}}{\int}} 2^{\overset{\textcircled{1}}{\underset{\textcircled{0}}{\int}}}$;

143 $\frac{1}{\mathbb{A}_0+1} \circ \frac{1}{\mathbb{A}_0}, \frac{1}{\mathbb{A}_0+\mathbb{A}_0} \circ \frac{1}{\mathbb{A}_0}, \frac{1}{\mathbb{A}_0+2^{\mathbb{A}_0}} \circ \frac{1}{2^{\mathbb{A}_0}}, \frac{1}{2^{\mathbb{A}_0}+2^{\mathbb{A}_0}} \circ \frac{1}{2^{\mathbb{A}_0}};$

144 $\mathbb{1}+1 \circ \mathbb{A}_0, \mathbb{1}+\mathbb{1} \circ \mathbb{A}_0, \mathbb{1}+2^{\mathbb{1}} \circ 2^{\mathbb{A}_0}, 2^{\mathbb{1}}+2^{\mathbb{1}} \circ 2^{\mathbb{A}_0};$

145 $\frac{1}{\mathbb{1}+1} \circ \frac{1}{\mathbb{A}_0}, \frac{1}{\mathbb{1}+\mathbb{1}} \circ \frac{1}{\mathbb{A}_0}, \frac{1}{\mathbb{1}+2^{\mathbb{1}}} \circ \frac{1}{2^{\mathbb{A}_0}}, \frac{1}{2^{\mathbb{1}}+2^{\mathbb{1}}} \circ \frac{1}{2^{\mathbb{A}_0}}.$

146 However, things are not so simple. Sergeyev also encountered a mathematical problem, which is the
 147 "maximal number paradox." Just imagine, if $\mathbb{1}$ represents the number of elements in a set of natural
 148 numbers, is $\mathbb{1}+1$ a natural number? If $\mathbb{1}+1$ is a natural number, because of $\mathbb{1}+1 > \mathbb{1}$, then
 149 the number of elements in the natural number set is not $\mathbb{1}$.

150 Sergeyev thought $\mathbb{1}+1 \notin N$, and the number greater than $\mathbb{1}$ is called an extended number [40].

151 But this is hard to make sense, because $\mathbb{1}+1$ fully conforms to the definition of natural numbers,
 152 and the extended natural numbers are still natural numbers. We will discuss this issue later.

153 **4. Grossone is a number-like symbol used for calculations**

154 In Cantor's infinite cardinal theory, the cardinal number of the natural number set, \mathbb{A}_0 , is the
 155 smallest infinite cardinal number. Using Grossone method, the set of natural numbers can also be
 156 decomposed into smaller sets of infinity. For example: the natural numbers set N can be divided into
 157 two infinite sets, the odd set and the even set. Let O be the odd set and E be the even set. Then
 158 there are:

159 $O = \{1,3,5, \frac{1}{4}, \mathbb{1}-3, \mathbb{1}-1\}, E = \{2,4,6, \frac{1}{4}, \mathbb{1}-2, \mathbb{1}\}$

160 $N = O \cup E = \{1,2,3, \frac{1}{4}, \mathbb{1}-3, \mathbb{1}-2, \mathbb{1}-1, \mathbb{1}\}$

161 Obviously, the number of elements in the odd number set and the even number set is $\frac{\mathbb{1}}{2}$, which is

162 less than the number of elements $\mathbb{1}$ in the natural number set.

163 Sergeyev also created a method of constructing an infinite subset of the natural number set [40]. He

164 uses $N_{k,n}$ ($1 \leq k \leq n, n \in N, n$ is a finite number) to indicate a set that the first number is k ,

165 and equal difference is n , and the size of the set is $\frac{\mathbb{1}}{n}$.

166 $N_{k,n} = \{k, k+n, k+2n, k+3n, \frac{1}{4}\}$

167
$$N = \bigcup_{k=1}^n N_{k,n}$$

168 For example:

169
$$N_{1,2} = \{1,3,5,\frac{1}{4}\} = O, \quad N_{2,2} = \{2,4,6,\frac{1}{4}\} = E$$

170
$$N = N_{1,2} \cup N_{2,2} = O \cup E$$

171 Or:

172
$$N_{1,3} = \{1,4,7,\frac{1}{4}\}, \quad N_{2,3} = \{2,5,8,\frac{1}{4}\}, \quad N_{3,3} = \{3,6,9,\frac{1}{4}\}$$

173
$$N = N_{1,3} \cup N_{2,3} \cup N_{3,3}$$

174 Grossone $\textcircled{1}$ is a numeral symbol that represents the number of elements in natural numbers set.

175 However, the set of integers and real numbers are larger than the set of natural numbers. According to
 176 the principle of "the whole is greater than the part", does it mean that there are integers and real
 177 numbers greater than $\textcircled{1}$?

178 Below we use Grossone method to examine the integer set Z and real number set R .

179
$$Z = \{-\textcircled{1}, -\textcircled{1}+1, \frac{1}{4}, 2, 1, 0, 1, 2, \frac{1}{4}, \textcircled{1}-1, \textcircled{1}\}$$

180
$$R = [-\textcircled{1}, -\textcircled{1}+1) \cup \frac{1}{4} \cup [1,0) \cup \{0\} \cup (0,1] \cup \frac{1}{4} \cup (\textcircled{1}-1, \textcircled{1}]$$

181 It is easy to see that there are no integers and real numbers exceeding $\textcircled{1}$ in both the integer set
 182 and the real number set.

183 The number of elements in the integer set is $2\textcircled{1}+1$; because the number of elements in $(0,1]$ is
 184 $10^{\textcircled{1}}$, the number of elements in the real number set is $C = 2\textcircled{1} \times 10^{\textcircled{1}} + 1$. It can be seen that the set
 185 of real numbers is not the power set of the set of natural numbers. Obviously, Integer set and real
 186 number set the number of elements in are all greater than $\textcircled{1}$.

187 The integer set and real number set are larger than the natural number set, which refers to the number
 188 of elements, rather than the existence of numbers exceeding $\textcircled{1}$ in the integer set and real number set.

189 In fact, $\textcircled{1}$ is not a number, but infinity. No number can exceed infinity, and $\textcircled{1}$ is a symbol for
 190 infinity.

191 Looking back at the problem of the "maximum number paradox" now, it is not difficult to solve it.

192 The problem lies in the qualitative aspect of A. In fact, A is just a number-like symbol used for
 193 infinity calculations, and is a ruler used to measure all infinity sets.

194 Take $\textcircled{1}+1$ as an example. First, $\textcircled{1}+1$, like $\textcircled{1}$, is infinity, not a numeral. Second,

195 $\textcircled{1} + 1 > \textcircled{1}$, indicating that this infinite set exceeds a single Grossone $\textcircled{1}$. Exceeding does not mean
 196 that it cannot be expressed. It is like measuring an object with a ruler. It does not matter if the object
 197 exceeds the ruler. You can measure a few more times. $\textcircled{1}$ is the ruler for measuring the infinite set.
 198 An infinite set is 1 more than this ruler. You can measure it more. After the measurement is accurate,
 199 mark it as $\textcircled{1} + 1$.

200 Let A be an infinite set of $\textcircled{1} + 1$ elements, then A can be written as:

201
$$A = N \cup \{1\} = \{1, 2, \frac{1}{4}, \textcircled{1} - 1, \textcircled{1}, \mathbf{1}\}$$

202 Or:

203
$$A = N \cup \{\textcircled{1} + 1\} = \{1, 2, \frac{1}{4}, \textcircled{1} - 1, \textcircled{1}, \textcircled{1} + 1\}$$

204 Or:

205
$$A = \{a_1, a_2, \frac{1}{4}, a_{\textcircled{1}-1}, a_{\textcircled{1}}, a_{\textcircled{1}+1}\}$$

206 It can be seen that the so-called "maximum number paradox" does not exist for Grossone method.

207 **5. The size of the point is not zero and the continuity of the set**

208 The continuum originally refers to the real numbers set. Since the real number corresponds to the
 209 point 1-1 on the straight line, the straight line is intuitively composed of continuous and unbroken
 210 points, so the real number set is called the continuum. In the number sequence, the set that satisfies the
 211 "1-1 correspondence" relationship with the interval (0, 1) is called the continuum.

212 Traditional mathematics has an axiom: a point has no size. This can be proved by continuum theory.

213 Suppose that the interval (0,1] on the number axis is the continuum, and the points on the number
 214 axis are "dense and no holes", so the distance between points is zero. Since the cardinal number of the
 215 continuum is c , the interval (0,1] point size is:

216
$$s = \lim_{c \textcircled{\neq} c} \frac{1}{c} = 0$$

217 However, according to the Grossone method, because the number of elements in (0,1] is $10^{\textcircled{1}}$, the
 218 size of the point in the interval (0,1] is:

219
$$s = \frac{1}{10^{\textcircled{1}}} > 0$$

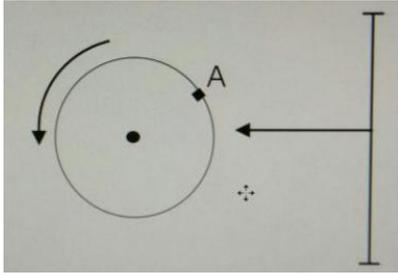
220 Not only does the dot have a size, but the size of the dot is related to the decimal or binary system of
 221 the number on the number axis. For example, when using binary system, the number of elements in
 222 (0,1] is $2^{\textcircled{1}}$, and the size of the point in the interval (0,1) is:

223
$$s = \frac{1}{2^{\textcircled{1}}}$$

224 Imagine that one-dimensional straight lines, two-dimensional planes, three-dimensional and
 225 multi-dimensional spaces, etc. are all composed of points. If the size of a point is zero, how to form a

226 straight line, plane and space with size? The Grossone method solves this infinitesimal puzzle.

227 We use a probability problem exemplified by Sergeev to illustrate [40].



228

229 As shown in the figure above, suppose the radius of the disc in the figure is r , and the disc is
 230 rotating. We want to ask a probabilistic event E : What is the probability that point A on the disk stops
 231 just in front of the fixed arrow on the right? According to the traditional calculation method, point A
 232 has no size, so the probability of occurrence of E is:

233
$$P(E) = \lim_{h \rightarrow 0} \frac{h}{2\pi r} = 0$$

234 This is obviously contrary to experience and common sense. And if the size of the point is solved,

235 such as $s = \frac{1}{10} \text{①}$, then you can get:

236
$$P(E) = \frac{1}{2\pi r > 10 \text{①}}$$

237 This is the logical result.

238 Although point has a size, point is isolated, so its dimension is zero. We know that straight lines and
 239 planes are one-dimensional and two-dimensional continuums, corresponding to real numbers and
 240 complex numbers, respectively. There are also three-dimensional and multi-dimensional space
 241 continuums, such as Euclidean space, Minkowski space, Riemann space, etc. In this sense, a point can
 242 also be regarded as a special continuum, that is, a zero-dimensional continuum.

243 In Sergeev's view, the "dense and no-hole" continuum is an absolute continuity. Correspondingly,
 244 he proposed a relative continuity theory [40]. In order to distinguish between the two continuums, I call
 245 the traditional continuum collectively the absolute continuum, and the set of relative continuity as the
 246 relative continuum.

247 Sergeev established the relative continuity on the function $f(x)$. The point that stipulates the

248 range of the independent variable $[a, b]_S$ of $f(x)$ can be a finite number or an infinity, but the set

249 $[a, b]_S$ is always discrete, where S represents a certain numeral system. In this way, for any point

250 $x \hat{\in} [a, b]_S$, its nearest left and right neighbors can always be determined:

251
$$x^+ = \min\{z : z \hat{\in} [a, b]_S, z > x\}, \quad x^- = \max\{z : z \hat{\in} [a, b]_S, z < x\}$$

252 Suppose a set $X = [a, b]_S = \{x_0, x_1, \frac{1}{4}, x_{n-1}, x_n\}_S$, where $a = x_0$, $b = x_n$, and the numeral
 253 system S allow a certain unit of measure m to be used to calculate the coordinates of the elements in

254 the set. If for any $x \hat{\in} (a, b)_s$, $x^+ - x$ and $x - x^-$ are infinitesimal, then the set X is said to be
 255 continuous in the unit of measure \mathfrak{m} . Otherwise, set X is said to be discrete in the unit of measure
 256 \mathfrak{m} .

257 For example, if the unit of measure \mathfrak{m} is used to calculate that the position difference between
 258 adjacent elements of set X is equal to $\textcircled{1}^{-1}$, then set X is continuous in the unit of measure \mathfrak{m} ;
 259 but if the unit of measure $\mathfrak{u} = \mathfrak{m} \times \textcircled{1}^{-3}$ is used instead, calculate that the position difference between
 260 adjacent elements of the set X is equal to $\textcircled{1}^2$, then the set X is discrete in the unit of measure
 261 \mathfrak{u} . Therefore, whether the set X is continuous or discrete depends on the size of the unit of measure
 262 \mathfrak{m} .

263 Function $f(x)$ is continuous in the unit of measure at some point $x \hat{\in} (a, b)_s$ in $[a, b]_s$, if
 264 $f(x^+) - f(x)$ and $f(x) - f(x^-)$ are both infinitesimal. If only one is infinitesimal, it can be
 265 called left continuous or right continuous. If function $f(x)$ is continuous in the unit of measure \mathfrak{m}
 266 at each point of $[a, b]_s$, then $f(x)$ is said to be continuous in the unit of measure \mathfrak{m} on set
 267 $X = [a, b]_s$.

268 Relative continuum theory provides a new way to solve the problem of continuity of sets other than
 269 absolute continuum.

270 **6. The mathematical continuum is the foundation of the cosmic continuum**

271 The cosmic continuum is a scientific foundation theory based on the mathematical continuum
 272 [4]-[14]. The cosmic continuum theory believes that there are three basic entities in the universe: mass
 273 body, energy body and dark mass body. Their smallest units are elementary particle m_{\min} , elementary
 274 quantum q_{\min} , and elementary dark particle d_{\min} .

275 In this way, we get three countably-infinite sets in the universe: elementary particle set M ,
 276 elementary quantum set Q , and elementary dark particle set D . Suppose m_i is a elementary particle,
 277 q_i is a elementary quantum, d_i is a elementary dark particle, i is a natural number, then:

$$278 \quad M = \{m_1, m_2, \frac{1}{4}, m_i, \frac{1}{4}\}, Q = \{q_1, q_2, \frac{1}{4}, q_i, \frac{1}{4}\}, D = \{d_1, d_2, \frac{1}{4}, d_i, \frac{1}{4}\}$$

279 Thus, we obtain the following basic existence set E :

$$280 \quad E = M \mathbf{U} Q \mathbf{U} D = \{m_1, m_2, \frac{1}{4}, m_i, \frac{1}{4}\} \mathbf{U} \{q_1, q_2, \frac{1}{4}, q_i, \frac{1}{4}\} \mathbf{U} \{d_1, d_2, \frac{1}{4}, d_i, \frac{1}{4}\}$$

281 According to cosmic continuum theory, the coupling energy quantum connects all entities to form

282 the universe as a whole, and any changes in the universe affect the whole. Due to the effect of the
 283 coupling energy quantum, the basic existence set E will form the following power set, namely the
 284 existence continuum $P(E)$:

$$285 \quad P(E) = 2^E = \{e|e \dot{\mid} E\}$$

286 Since space is the existence dimension of the mass body, time is the existence dimension of the
 287 energy body, and dark space is the existence dimension of the dark mass body, correspondingly, we
 288 obtain the other three countable infinite sets in the universe: basic space set S , basic time set T and
 289 basic dark space set G . Suppose s_i is the elementary space, t_i is the elementary time, g_i is the
 290 elementary dark space, i is a natural number, then:

$$291 \quad S = \{s_1, s_2, \frac{1}{4}, s_i, \frac{1}{4}\}, T = \{t_1, t_2, \frac{1}{4}, t_i, \frac{1}{4}\}, G = \{g_1, g_2, \frac{1}{4}, g_i, \frac{1}{4}\}$$

292 Thus, we obtain the following basic dimension set W :

$$293 \quad W = S \mathbf{U} T \mathbf{U} G = \{s_1, s_2, \frac{1}{4}, s_i, \frac{1}{4}\} \mathbf{U} \{t_1, t_2, \frac{1}{4}, t_i, \frac{1}{4}\} \mathbf{U} \{g_1, g_2, \frac{1}{4}, g_i, \frac{1}{4}\}$$

294 The basic dimension set W will correspondingly form the following power set, namely the
 295 dimension continuum $P(W)$:

$$296 \quad P(W) = 2^W = \{w|w \dot{\mid} W\}$$

297 According to Cosmic continuum hypothesis: the universe is a continuum consisting of an existence
 298 continuum and an existing dimension continuum. The existence continuum is composed of mass bodies,
 299 energy bodies and dark massbodies. The existing dimension continuum is composed of space, time and
 300 dark space.

301 Let the cosmic continuum be U , then:

$$302 \quad U = P(E) \mathbf{U} P(W) = 2^E \mathbf{U} 2^W = \{e|e \dot{\mid} E\} \mathbf{U} \{w|w \dot{\mid} W\}$$

303 Therefore, the cosmic continuum is actually a multi-dimensional continuum composed of the power
 304 set of the basic existence set and the power set of the basic dimensional set. It is an absolute continuum
 305 that conforms to the Cantor cardinal number method.

306 In Cosmic Continuum Theory, the dimensional continuum $P(W)$ is the mirror image of the
 307 existence continuum $P(E)$.

308 For example, in the history of physics, three fundamental theories of physics, classical mechanics,
 309 relativity, and quantum mechanics have appeared successively. The corresponding Euclidean space,
 310 Minkowski space, Riemann space, matrix space, and probability space are equivalent to Space-time
 311 mirroring of physics events.

312 In this sense, the cosmic continuum is actually a continuum composed of all physical events in the
 313 universe. In other words, every element of the cosmic continuum is a physical event.

314 Next we examine how to describe physical events in the cosmic continuum.

315 According to Variation axiom of the cosmic continuum: the change of energy is the cause of the state

316 change of all cosmic systems.

317 Suppose the energy change in a universe system is ΔE , and the corresponding change in the state
318 of the universe system is Δx , then $\Delta E \propto \Delta x$. In the universe continuum, energy is the independent
319 variable, and the state of the universe system is the dependent variable.

320 Relative continuum provides us with a mathematical tool for describing physical events.

321 Variation axiom tells us that the essence of all physical events is that energy changes cause the state
322 of the universe to change. Suppose the dependent variable $f(x)$ is the state of the universe system,
323 the independent variable x is the energy, and $[a, b]_s$ is the range of the independent variable that
324 can be described by a certain digital system S, then any physical event, including all interactions and
325 quantum phenomena, can be described by the above-mentioned relative continuum.

326 Example 1, the physical event of classical mechanics: car action. The dependent variable $f(x)$ is
327 the car coordinates, the independent variable x is gasoline, and $x \in (a, b)_s$ represents the range of
328 gasoline amount.

329 Example 2: Relativistic physical event: "deflection of light" under the action of gravity. The
330 dependent variable $f(x)$ is the ray coordinate, the independent variable x is the gravitational field,
331 and $x \in (a, b)_s$ represents the range of the gravitational field.

332 Example 3: Quantum mechanics physical event: photoelectric effect. The dependent variable A is the
333 electronic state, the independent variable B is the photon frequency, and C represents the photon
334 frequency range.

335 **7. Conclusion**

336 This article discusses several aspects of the development of infinity theory.

337 (1) Qualitative calculation and quantitative calculation. Cantor used the cardinal number method to
338 solve the problem of comparing infinity. Sergeyev used Grossone method to solve the problem of
339 unifying the calculation rules of infinity and finite numbers.

340 (2) Absolute continuum and relative continuum. The continuum in traditional mathematics refers to
341 a collection of "dense and no holes", the relative continuum is a continuum that changes with the
342 change of measurement units.

343 (3) Cosmic continuum and continuum theory. The universe continuum is a mathematical model of
344 existence and its dimensions in the universe. The development of continuum theory provides a new
345 mathematical foundation for the Cosmic Continuum Theory.

346 The discussion in this article shows that:

347 (1) The theory of infinity will usher in a new era of development. Grossone method is a scientific
348 infinity theory like the cardinal number method; in the new infinity theory, infinity and infinity can be
349 mathematically calculated like finite numbers.

350 (2) New developments in the theory of infinity will promote the development of fundamental
351 scientific theories. The basic theories of mathematics and physics have always been intertwined and
352 developed, such as classical mechanics and calculus, relativity and non-Euclidean geometry, etc.,
353 which are all good stories in the history of science. The relative continuum theory provides a new path

354 for the study of the cosmic continuum.

355 (3) The essence of Hilbert's first problem "continuum hypothesis" is that the infinity theory is not yet
356 mature. The development of Grossone theory makes this problem self-explanatory. According to the
357 principles of "power set is greater than original set" and "whole is greater than part", there is neither the
358 largest infinity and infinitesimal, nor the smallest infinity and infinitesimal.

359

360 Competing interests statement: I declare that there is no competing Interest.

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