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Abstract

In this paper, we consider a system of two degenerate wave equations coupled through the velocities, only one of them being controlled. We assume that the coupling parameter is sufficiently small and we focus on null controllability problem. To this aim, using multiplier techniques and careful energy estimates, we first establish an indirect observability estimate for the corresponding adjoint system. Then, by applying the Hilbert Uniqueness Method, we show that the indirect boundary controllability of the original system holds for a sufficiently large time.

Keywords: Controllability, degenerate hyperbolic systems, Hilbert uniqueness method, multiplier techniques, observability inequalities

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1 Introduction

Controllability problems for scalar nondegenerate hyperbolic equations have always been mainstream topics over the past several years, and numerous achievements have been made. We refer, for example, to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the articles citing them.

In the last decades, there has been a growing interest in the control of coupled hyperbolic systems. From the control theory point of view, it is important for practical applications and for cost reasons to control these systems by a reduced number of controls. This is called indirect controllability.

For such type of challenging indirect controllability issues, there is an extensive bibliography devoted to nondegenerate systems. See, in particular [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and the references therein.

Recently, several controllability results have also been obtained for scalar degenerate equations, see for example [24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. However, in the more complex situation of coupled degenerate hyperbolic equations, not many things are known. For this reason, we believe that it is natural to examine the exact controllability of coupled hyperbolic systems governed by linear one dimensional wave equations with degenerate variable coefficients.

More precisely, the question we deal with concerns the study of the indirect boundary controllability problem for the following degenerate hyperbolic system:

$$\begin{cases} z_{tt} - (a(x)z_x)_x + by_t = 0, & \text{in } (0,T)\times(0,1), \\ y_{tt} - (a(x)y_x)_x - bz_t = 0, & \text{in } (0,T)\times(0,1), \\ Bz(t,0) = By(t,0) = 0, & \text{on } (0,T), \\ z(t,1) = 0, & \text{on } (0,T), \\ y(t,1) = f(t), & \text{on } (0,T), \\ z(0,x) = z_0(x), \quad z_t(0,x) = z_1(x), & \text{in } (0,1), \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), & \text{in } (0,1), \end{cases} \tag{1.1}$$

where $a \in C([0,1]) \cap C^1((0,1])$ is positive on $]0,1]$ but vanishes at zero, (z, z_t, y, y_t) is the state variable, (z_0, z_1, y_0, y_1) is regarded as being the initial value, $T > 0$ stands for the length of the time-horizon, $b > 0$ is the coupling parameter and $f \in L^2(0,T)$ is the control function.

Further,

$$Bz(t,x) := \begin{cases} z(t,x), & \text{in the case,} \quad 0 < \mu_a < 1, \\ (a(x)z_x)(t,x), & \text{in the case,} \quad 1 \le \mu_a < 2. \end{cases}$$

The main novelty of this paper is that only one of the two components of the unknown is only controlled. The question is to determine if it is possible to control the full vector state solution of the coupled hyperbolic system (1.1) for nonzero values of the coupling parameter b by means of one boundary control. Because of the linearity and the time reversibility of the system (1.1), exact controllability is equivalent to null controllability, see [34, Proposition 3.1].

Hence, in the sequel, we shall focus on the following problem: For given $T > 0$ (sufficiently large) and initial data (z_0, z_1, y_0, y_1), does there exists a suitable control f that brings back the solution of the controlled system (1.1) to equilibrium at time T, that is such the corresponding solution of the system (1.1) satisfies

$$z(T,\cdot) = z_t(T,\cdot) = y(T,\cdot) = y_t(T,\cdot) = 0 \quad \text{on } (0,1).$$

Via some suitable estimates on the total energy of the associated homogeneous adjoint problem, we will give a positive answer to the above question under a smallness assumption on the coupling coefficient, for a time $T > 0$ sufficiently large.

To the best of our knowledge, the only paper in this framework, but for a certain class of systems of degenerate wave equations coupled in displacements is given in [35]. The authors established an interesting indirect exact controllability result with locally distributed control for a sufficiently large time T, provided that the coupling parameter is sufficiently small.

The point of view of this paper differs from that of [35]. Indeed, here we consider a system of two degenerate wave equations coupled through velocities with a boundary control acting on only one equation. Moreover, we will treat a generalization of the system considered in [35]: in place of the pure power x^α for some $\alpha \in (0,2)$, we consider a more general function

a which satisfies the following assumptions:

$$\begin{cases} \text{(i)} & a(x) > 0 \quad \forall x \in]0,1],\ a(0) = 0, \\ \text{(ii)} & \mu_a := \sup_{0<x\leq 1} \dfrac{x\,|a'(x)|}{a(x)} < 2, \ \text{and} \\ \text{(iii)} & a \in \mathcal{C}^{[\mu_a]}([0,1]), \end{cases} \tag{1.2}$$

where $[\cdot]$ stands for the integer part.

The paper is organized as follows. In Section 2, we briefly recall the appropriate functional spaces that are naturally associated with degenerate problems and prove the well-posedness of the adjoint problem. In Section 3, we establish two fundamental inequalities from below and from above on the total energy associated to the solution of the adjoint problem. In Section 4, thanks to these inequalities, we prove that the original control problem has a unique solution by transposition, which is null controllable by one boundary control force.

2 Functional setting and well-posedness of the homogeneous problem

Let $a \in \mathcal{C}([0,1]) \cap \mathcal{C}^1(]0,1])$ be a function satisfying assumptions (1.2). In order to study the well-posedness and controllability properties for (1.1), we shall need some basic properties for solutions of problems of the form

$$\begin{cases} v_{tt} - (a(x)v_x)_x + bu_t = 0, & \text{in } (0,T)\times(0,1), \\ u_{tt} - (a(x)u_x)_x - bv_t = 0, & \text{in } (0,T)\times(0,1), \\ Bv(t,0) = Bu(t,0) = 0, & \text{on } (0,T), \\ v(t,1) = u(t,1) = 0, & \text{on } (0,T), \\ v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), & \text{in } (0,1), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & \text{in } (0,1). \end{cases} \tag{2.1}$$

At first, as in [24], we introduce some weighted Sobolev spaces that are naturally associated with degenerate operators. We denote by $H^1_a(0,1)$ the space of all functions $u \in L^2(0,1)$ such that

$$\begin{cases} \text{(i)} & u \text{ is locally absolutely continuous in }]0,1], \text{ and} \\ \text{(ii)} & \sqrt{a}u_x \in L^2(0,1). \end{cases}$$

It is easy to see that $H^1_a(0,1)$ is a Hilbert space with the scalar product

$$\langle u, v\rangle_{H^1_a(0,1)} = \int_0^1 \left(a(x)u'(x)v'(x) + u(x)v(x)\right)\,dx \quad \forall u,v \in H^1_a(0,1)$$

and associated norm

$$\|u\|_{H^1_a(0,1)} = \left\{\int_0^1 \left(a(x)\,|u'(x)|^2 + |u(x)|^2\right)\,dx\right\}^{\frac{1}{2}} \quad \forall u \in H^1_a(0,1).$$

Next, we define

$$H^2_a(0,1) := \left\{u \in H^1_a(0,1) \mid au' \in H^1(0,1)\right\}.$$

Note that if $u \in H^2_a(0,1)$, then au' is continuous on $[0,1]$.

In the following proposition, we collect useful properties of the above functional spaces which will play an important role in order to evaluate several boundary terms, see [24, Proposition 2.5].

Proposition 2.1. *Assume that a is a function satisfying (1.2). Then the following assertions hold true:*

1. *For every $u \in H^1_a(0,1)$*
$$\lim_{x\downarrow 0} x|u(x)|^2 = 0. \tag{2.2}$$
Moreover, if $\mu_a \in [0,1[$, then u is absolutely continuous in $[0,1]$.

2. *For every $u \in H^2_a(0,1)$*
$$\lim_{x\downarrow 0} xa(x)|u'(x)|^2 = 0. \tag{2.3}$$

3. *For all $u \in H^2_a(0,1)$ and for all $v \in H^1_a(0,1)$*
$$\lim_{x\downarrow 0} a(x)u'(x)v(x) = 0, \tag{2.4}$$
assuming, in addition,$v(0) = 0$ if $\mu_a \in [0,1[$.

4. *If $\mu_a \in [1,2[$, for every $u \in H^2_a(0,1)$*
$$\lim_{x\downarrow 0} a(x)u'(x) = 0. \tag{2.5}$$

In view of Proposition 2.1, we see that the boundary conditions imposed at $x = 0$ make sense for any classical solution of (2.1). Such conditions are of Dirichlet type if $\mu_a \in [0,1[$, whereas they are of Neumann/Dirichlet type at $x = 0$ and $x = 1$, respectively, if $\mu_a \in [1,2[$.

In order to express the boundary conditions in functional settings, we define the space $H^1_{a,0}(0,1)$ depending on the value of μ_a, as follows:

(i) For $0 \le \mu_a < 1$, we define
$$H^1_{a,0}(0,1) := \left\{u \in H^1_a(0,1) \mid u(0) = u(1) = 0\right\}.$$

(ii) For $1 \le \mu_a < 2$, we define
$$H^1_{a,0}(0,1) := \left\{u \in H^1_a(0,1) \mid u(1) = 0\right\}.$$

Let us recall the following version of Poincaré's inequality, which is proved in [24, Proposition 2.2].

Lemma 2.1. *Assume (1.2) holds. Then*
$$\int_0^1 |u(x)|^2\, dx \le C_a \int_0^1 a(x)\,|u'(x)|^2\, dx, \quad \forall u \in H^1_{a,0}(0,1), \tag{2.6}$$
where
$$C_a = \frac{1}{a(1)} \min\left\{4, \frac{1}{2-\mu_a}\right\}. \tag{2.7}$$

Let us set
$$\|u\|_{H^1_{a,0}(0,1)} := \left\{\int_0^1 a(x)\,|u'(x)|^2\, dx\right\}^{\frac{1}{2}} \quad \forall u \in H^1_{a,0}(0,1).$$
which, thanks to Lemmma 2.1, defines a norm on $H^1_{a,0}(0,1)$ that is equivalent to $\|\cdot\|_{H^1_a(0,1)}$.

Finally, we define
$$H^2_{a,0}(0,1) := H^2_a(0,1) \cap H^1_{a,0}(0,1).$$

Observe that all functions $u \in H^2_{a,0}(0,1)$ satisfy the above homogeneous boundary conditions at both $x = 0$ and $x = 1$.

Now, we will prove the well-posedness result for problem (2.1) using semigroup theory. For this purpose, we consider the following energy space associated to system (2.1):

$$\mathcal{H}_a = \left(H^1_{a,0}(0,1) \times L^2(0,1)\right)^2, \tag{2.8}$$

with the scalar product

$$\langle U, \widetilde{U} \rangle_{\mathcal{H}_a} = \int_0^1 (a(x) u_{1,x} \widetilde{u}_{1,x} + u_2 \widetilde{u}_2 + a(x) u_{3,x} \widetilde{u}_{3,x} + u_4 \widetilde{u}_4)\, dx,$$

for all $U = (u_1, u_2, u_3, u_4), \widetilde{U} = (\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}_4) \in \mathcal{H}_a$.

Next, we define the linear unbounded operator $\mathcal{A}_a : D(\mathcal{A}_a) \subset \mathcal{H}_a \to \mathcal{H}_a$ by

$$\begin{cases} D(\mathcal{A}_a) & = \left(H^2_{a,0}(0,1) \times H^1_{a,0}(0,1)\right)^2, \\ \mathcal{A}_a U & = (u_2, (a(x) u_{1,x})_x - b u_4, u_4, (a(x) u_{3,x})_x + b u_2), \quad \forall U = (u_1, u_2, u_3, u_4) \in D(\mathcal{A}_a). \end{cases} \tag{2.9}$$

Using this operator, we rewrite the system (2.1) as an abstract Cauchy problem. Indeed, setting $U(t) = (v(t), v_t(t), u(t), u_t(t))$, one has that (2.1) can be formulated as

$$\begin{cases} U'(t) & = \mathcal{A}_a U(t) \quad t \geq 0, \\ U(0) & = U_0, \end{cases} \tag{2.10}$$

where $U(0) = (v_0, v_1, u_0, u_1)$.

The existence and uniqueness result reads as follows.

Theorem 2.1. *For any $U_0 = (v_0, v_1, u_0, u_1) \in \mathcal{H}_a$, there exists a unique solution $U = (v, v_t, u, u_t) \in C^0([0, \infty); \mathcal{H}_a)$ of system (2.10). Moreover, if $U_0 = (v_0, v_1, u_0, u_1) \in D(\mathcal{A}_a)$, then*

$$U \in C^0([0, \infty); D(\mathcal{A}_a)) \cap C^1([0, \infty); \mathcal{H}_a).$$

Proof. We will prove that the operator $\mathcal{A}_a$ defined in (2.9) generates a contraction semigroup on the Hilbert space $\mathcal{H}_a$ given by (2.8). To this end, we show that the unbounded operator $\mathcal{A}_a$ is maximal dissipative on $\mathcal{H}_a$. According to [36, Proposition 2.2.6], it is sufficient to prove that $\mathcal{A}_a : D(\mathcal{A}_a) \to \mathcal{H}_a$ is dissipative and that $I - \mathcal{A}_a$ is surjective, where

$$I = \operatorname{diag}(Id, Id, Id, Id).$$

$\mathcal{A}_a$ is dissipative. Take $U = (v, v_t, u, u_t) \in D(\mathcal{A}_a)$. Then, by integrating by parts, we obtain

$$\begin{aligned} \langle \mathcal{A}_a U, U \rangle_{\mathcal{H}_a} &= \langle \left(v_t, (a(x) v_x)_x - b u_t, u_t, (a(x) u_x)_x + b v_t\right), \left(v, v_t, u, u_t\right) \rangle_{\mathcal{H}_a} \\ &= \int_0^1 \Big(a(x) v_{t,x} v_x + (a(x) v_x)_x v_t + a(x) u_{t,x} u_x + (a(x) u_x)_x u_t\Big) dx \\ &= [a v_x v_t]_0^1 + [a u_x u_t]_0^1. \end{aligned}$$

Using the fact that $U \in D(\mathcal{A}_a)$ and (2.4), we get that

$$\langle \mathcal{A}_a U, U \rangle_{\mathcal{H}_a} = 0.$$

Therefore the operator $\mathcal{A}_a$ is dissipative.

<u>$I-\mathcal{A}_a$ surjective.</u> Given $F=(f_1,f_2,f_3,f_4)\in\mathcal{H}_a$, we seek $U=(u_1,u_2,u_3,u_4)\in D(\mathcal{A}_a)$ such that

$$U-\mathcal{A}_aU=F\Leftrightarrow\begin{cases}u_1-u_2 & =f_1,\\ u_2-(a(x)u_{1,x})_x+bu_4 & =f_2,\\ u_3-u_4 & =f_3,\\ u_4-(a(x)u_{3,x})_x-bu_2 & =f_4.\end{cases}\tag{2.11}$$

Eliminating u_2 and u_4, we get the following system

$$\begin{cases}u_1-(a(x)u_{1,x})_x+bu_3 & =f_1+f_2+bf_3,\\ u_3-(a(x)u_{3,x})_x-bu_1 & =-bf_1+f_3+f_4.\end{cases}\tag{2.12}$$

Consider the bilinear form $\Gamma:\left(H^1_{a,0}(0,1)\times H^1_{a,0}(0,1)\right)^2\to\mathbb{R}$ given by

$$\begin{aligned}\Gamma\left((u_1,u_3),(\phi_1,\phi_2)\right)=&\int_0^1(u_1\phi_1+a(x)u_{1,x}\phi_{1,x}+u_3\phi_2+a(x)u_{3,x}\phi_{2,x})dx\\ &+b\int_0^1\left(u_3\phi_1-u_1\phi_2\right)dx\end{aligned}$$

and the linear form $L:H^1_{a,0}(0,1)\times H^1_{a,0}(0,1)\to\mathbb{R}$ given by

$$L\left(\phi_1,\phi_2\right)=\int_0^1\left(f_1+f_2+bf_3\right)\phi_1\,dx+\int_0^1\left(-bf_1+f_3+f_4\right)\phi_2\,dx.$$

In view of Poincaré inequality (2.6), Γ is a continuous bilinear form on $H^1_{a,0}(0,1)\times H^1_{a,0}(0,1)$ and L is a continuous linear functional on $H^1_{a,0}(0,1)\times H^1_{a,0}(0,1)$. Moreover, it is easy to check that Γ is also coercive on $H^1_{a,0}(0,1)\times H^1_{a,0}(0,1)$. So applying the Lax–Milgram theorem, we deduce that there exists a unique solution $(u_1,u_3)\in H^1_{a,0}(0,1)\times H^1_{a,0}(0,1)$ of

$$\Gamma\left((u_1,u_3),(\phi_1,\phi_2)\right)=L\left(\phi_1,\phi_2\right)\text{ for all }(\phi_1,\phi_2)\in H^1_{a,0}(0,1)\times H^1_{a,0}(0,1).\tag{2.13}$$

Now, take $(u_2,u_4):=(u_1-f_1,u_3-f_3)$; then $(u_2,u_4)\in\left(H^1_{a,0}(0,1)\right)^2$. It remains to prove that $(u_1,u_3)\in\left(H^2_{a,0}(0,1)\right)^2$ and solves (2.12). Since $C_c^\infty(0,1)\subset H^1_{a,0}(0,1)$, the equation (2.13) holds for all $(\phi_1,\phi_2)\in C_c^\infty(0,1)\times C_c^\infty(0,1)$, namely

$$\begin{aligned}&\int_0^1\left(u_1+bu_3\right)\phi_1\,dx+\int_0^1a(x)u_{1,x}\phi_{1,x}\,dx+\int_0^1\left(u_3-bu_1\right)\phi_2\,dx+\int_0^1a(x)u_{3,x}\phi_{2,x}\,dx\\ &=\int_0^1\left(f_1+f_2+bf_3\right)\phi_1\,dx+\int_0^1\left(-bf_1+f_3+f_4\right)\phi_2\,dx,\\ &\forall(\phi_1,\phi_2)\in C_c^\infty(0,1)\times C_c^\infty(0,1).\end{aligned}\tag{2.14}$$

Using the fact that, for all $(\phi_1,\phi_2)\in C_c^\infty(0,1)\times C_c^\infty(0,1)$, we have

$$\begin{aligned}\int_0^1\left(u_1+bu_3\right)\phi_1\,dx+\int_0^1a(x)u_{1,x}\phi_{1,x}\,dx&=\int_0^1\left(u_1-(a(x)u_{1,x})_x+bu_3\right)\phi_1\,dx\\ \int_0^1\left(u_3-bu_1\right)\phi_2\,dx+\int_0^1a(x)u_{3,x}\phi_{2,x}\,dx&=\int_0^1\left(u_3-(a(x)u_{3,x})_x-bu_1\right)\phi_2\,dx,\end{aligned}\tag{2.15}$$

it follows that the integral representation (2.14) is equivalent to

$$\begin{aligned}\int_0^1\left(u_1-(a(x)u_{1,x})_x+bu_3\right)\phi_1\,dx&=\int_0^1\left(f_1+f_2+bf_3\right)\phi_1\,dx,\quad\forall\phi_1\in C_c^\infty(0,1)\\ \int_0^1\left(u_3-(a(x)u_{3,x})_x-bu_1\right)\phi_2\,dx&=\int_0^1\left(-bf_1+f_3+f_4\right)\phi_2\,dx,\quad\forall\phi_2\in C_c^\infty(0,1).\end{aligned}\tag{2.16}$$

Thus $(u_1, u_3) \in H^2_{a,0}(0,1) \times H^2_{a,0}(0,1)$ and

$$\begin{aligned} u_1 - (a(x)u_{1,x})_x + bu_3 &= f_1 + f_2 + bf_3 \quad \text{ a.e. in } (0,1) \\ u_3 - (a(x)u_{3,x})_x - bu_1 &= -bf_1 + f_3 + f_4 \quad \text{ a.e. in } (0,1). \end{aligned}$$

So, we have found $U = (u_1, u_2, u_3, u_4) \in D(\mathcal{A}_a)$, which satisfies (2.11). Hence, the well-posedness result follows from the Hille-Yosida theorem (see [37, Theorem 4.5.1] or [38, Theorem A.11]). □

For any regular solution of problem (2.1) we define the associated energy by:

$$E_{v,u}(t) = \frac{1}{2}\int_0^1 \left\{v_t^2(t,x) + a(x)v_x^2(t,x) + u_t^2(t,x) + a(x)u_x^2(t,x)\right\} dx, \quad \forall t \geq 0. \tag{2.17}$$

Then, by a direct derivation, it is easy to show that the energy is conserved:

$$E_{v,u}(t) = E_{v,u}(0) = \frac{1}{2}\|U\|^2_{\mathcal{H}_a} \quad \forall t \geq 0. \tag{2.18}$$

3 Two key inequalities

In this section, by means of the so-called multiplier method [4, 39, 40], we prove some estimates from above and below for the energy associated to the solution of the adjoint system (2.1), which are crucial to prove that the control problem (1.1) has a unique solution and this solution is null controllable.

3.1 Direct inequality

This subsection is devoted to prove the so-called direct inequality of the solutions of (2.1). More precisely, we prove the following.

Lemma 3.1. *Let $T > 0$ and assume (1.2) holds. Then there is a constant $c_1 = c_1(a,b,T) > 0$ such that for every $U_0 = (v_0, v_1, u_0, u_1) \in \mathcal{H}_a$ the solution of the homogeneous system (2.1) satisfies the following direct inequality*

$$\int_0^T u_x^2(t,1)\, dt \leq c_1 E_{v,u}(0), \tag{3.1}$$

where

$$c_1 := \left[\left(6 + \frac{2b}{\min\{1, a(1)\}}\right)T + \frac{4}{\min\{1, a(1)\}}\right]. \tag{3.2}$$

Proof. Using multiplier technique, we establish the direct inequality (3.1) for a regular solution $U \in D(\mathcal{A}_a)$, the case $U \in \mathcal{H}_a$ then follows by a density argument. Let $U \in D(\mathcal{A}_a)$ be a function satisfying (2.1). We start by proving that the following identity holds true:

$$\begin{aligned} a(1)\int_0^T u_x^2(t,1)\, dt = &\int_0^T\!\!\int_0^1 \left\{u_t^2(t,x) + (a(x) - xa'(x))\, u_x^2(t,x)\right\} dx\, dt \\ &+ 2\left[\int_0^1 xu_x(t,x)u_t(t,x)\, dx\right]_{t=0}^{t=T} - 2b\int_0^T\!\!\int_0^1 xu_x(t,x)v_t(t,x)\, dx\, dt. \end{aligned} \tag{3.3}$$

We multiply the second equation of (2.1) by xu_x and we integrate by parts over $(0,T)\times(0,1)$ as follows:

$$\begin{aligned}
0 &= \int_0^T\int_0^1 xu_x(t,x)\Big(u_{tt}(t,x) - (a(x)u_x(t,x))_x - bv_t(t,x)\Big)\,dx\,dt \\
&= \left[\int_0^1 xu_x(t,x)u_t(t,x)\,dx\right]_{t=0}^{t=T} - \int_0^T\int_0^1 xu_{tx}(t,x)u_t(t,x)\,dx\,dt \\
&\quad - \int_0^T\int_0^1 \left(xa'(x)u_x^2(t,x) + xa(x)u_x(t,x)u_{xx}(t,x)\right)\,dx\,dt - b\int_0^T\int_0^1 xu_x(t,x)v_t(t,x)\,dx\,dt \\
&= \left[\int_0^1 xu_x(t,x)u_t(t,x)\,dx\right]_{t=0}^{t=T} - \int_0^T\int_0^1 xa'(x)u_x^2(t,x)\,dx\,dt \\
&\quad - \int_0^T\int_0^1 \left(x\left(\frac{u_t^2(t,x)}{2}\right)_x + xa(x)\left(\frac{u_x^2(t,x)}{2}\right)_x\right)\,dx\,dt - b\int_0^T\int_0^1 xu_x(t,x)v_t(t,x)\,dx\,dt.
\end{aligned} \tag{3.4}$$

On the other hand, by integrating by parts and owing to (2.2)-(2.3), we have

$$\begin{aligned}
&\int_0^T\int_0^1 x\left(\frac{u_t^2(t,x)}{2}\right)_x\,dx\,dt = -\frac{1}{2}\int_0^T\int_0^1 u_t^2(t,x)\,dx\,dt, \\
&\int_0^T\int_0^1 xa(x)\left(\frac{u_x^2(t,x)}{2}\right)_x\,dx\,dt = \frac{1}{2}\int_0^T a(1)u_x^2(t,1)\,dt - \frac{1}{2}\int_0^T\int_0^1 (xa(x))'u_x^2(t,x)\,dx\,dt.
\end{aligned} \tag{3.5}$$

Inserting (3.5) into (3.4), we get the desired identity (3.3).

Now, observe that condition (1.2) (ii) yields:

$$a(x) \geq a(1)x^{\mu_a} \quad \forall x \in [0,1]. \tag{3.6}$$

From Young ineqality and (3.6) together with the constancy of the energy, it follows that:

$$\begin{aligned}
\left|\int_0^1 xu_x(t,x)u_t(t,x)\,dx\right| &\leq \frac{1}{2}\int_0^1 \left\{u_t^2(t,x) + x^2u_x^2(t,x)\right\}\,dx \\
&\leq \frac{E_{v,u}(0)}{\min\{1,a(1)\}} \qquad \forall t \geq 0, \\
\left|b\int_0^T\int_0^1 xu_x(t,x)v_t(t,x)\,dx\,dt\right| &\leq \frac{b}{2}\int_0^T\int_0^1 \left\{v_t^2(t,x) + x^2u_x^2(t,x)\right\}\,dx\,dt \\
&\leq \frac{b}{\min\{1,a(1)\}}\int_0^T E_{v,u}(0)\,dt = \frac{b}{\min\{1,a(1)\}}TE_{v,u}(0).
\end{aligned} \tag{3.7}$$

Finally, combining (3.3), (3.7) and the inequality $x|a'(x)| < 2a(x)$, one obtains (3.1). □

3.2 Inverse inequality

The aim of this subsection is to establish, under some further hypotheses, the inverse inequality of the direct inequality obtained in Lemma 3.1. Let us first introduce the following notations

$$M_{a,b} = \min\left\{1 - \frac{b}{2}C_a, \frac{1}{2}\left(1 - \frac{6-\mu_a}{2-\mu_a}b\right), 2\left(1 - \frac{4}{(2-\mu_a)a(1)}b\right)\right\}, \tag{3.8}$$

$$b_a = \min\left\{\frac{2}{C_a}, \frac{2-\mu_a}{6-\mu_a}, \frac{(2-\mu_a)a(1)}{4}\right\},$$

and

$$T_{a,b} = \frac{\frac{3(2-\mu_a)}{2b} + 2\left(1 + \frac{4}{\min\{1,a(1)\}}\right) + (2-\mu_a)\sqrt{C_a}}{(2-\mu_a)M_{a,b}}. \tag{3.9}$$

Remark 1. From condition (1.2) (ii) and assuming that $0 < b < b_a$, one can see that $M_{a,b}, T_{a,b} > 0$.

Now, using the multiplier method, we are going to prove the following inverse inequality result:

Theorem 3.1. *Let $T > T_{a,b}$ and assume (1.2) holds. Suppose that $0 < b < b_a$. Then there is a constant $c_2 = c_2(a, b, T) > 0$ such that for every $U_0 = (v_0, v_1, u_0, u_1) \in \mathcal{H}_a$ the solution of the homogeneous system (2.1) satisfies the following inverse inequality*

$$E_{v,u}(0) \leq c_2 \int_0^T u_x^2(t,1)\, dt, \tag{3.10}$$

where

$$c_2 := \frac{2a(1)}{(2-\mu_a)M_{a,b}T - \left(\frac{3(2-\mu_a)}{2b} + 2\left(1 + \frac{4}{\min\{1,a(1)\}}\right) + (2-\mu_a)\sqrt{C_a}\right)}. \tag{3.11}$$

Proof. Once again we suppose U is a regular solution of (2.1). For simplicity, we divide the proof into several steps.
Step 1. As a first step, let us prove that the following estimation holds true:

$$\begin{aligned}\frac{2-\mu_a}{2} \int_0^T\!\!\int_0^1 u_t^2(t,x)\, dx\, dt + \left(\frac{2-\mu_a}{2} - \frac{2b}{a(1)}\right) \int_0^T\!\!\int_0^1 a(x)u_x^2(t,x)\, dx\, dt \\ \leq a(1) \int_0^T u_x^2(t,1)\, dt + \frac{4}{\min\{1,a(1)\}} E_{v,u}(0) + \frac{b}{2} \int_0^T\!\!\int_0^1 v_t^2(t,x)\, dx\, dt, \quad \forall T > 0.\end{aligned} \tag{3.12}$$

Multiplying the second equation of (2.1) by u, integrating by parts over $(0,T) \times (0,1)$, we get

$$\begin{aligned}0 &= \int_0^T\!\!\int_0^1 u(t,x)\Big(u_{tt}(t,x) - (a(x)u_x(t,x))_x - bv_t(t,x)\Big)\, dx\, dt \\ &= \left[\int_0^1 u_t(t,x)u(t,x)\, dx\right]_{t=0}^{t=T} - \int_0^T\!\!\int_0^1 u_t^2(t,x)\, dx\, dt - \int_0^T [a(x)u_x(t,x)u(t,x)]_{x=0}^{x=1}\, dt \\ &\quad + \int_0^T\!\!\int_0^1 a(x)u_x^2(t,x)\, dx\, dt - b\int_0^T\!\!\int_0^1 v_t(t,x)u(t,x)\, dx\, dt.\end{aligned}$$

Using the boundary conditions together with (2.4), this gives

$$\begin{aligned}\int_0^T\!\!\int_0^1 \left\{a(x)u_x^2(t,x) - u_t^2(t,x)\right\}\, dx\, dt + \left[\int_0^1 u_t(t,x)u(t,x)\, dx\right]_{t=0}^{t=T} \\ - b\int_0^T\!\!\int_0^1 v_t(t,x)u(t,x)\, dx\, dt = 0.\end{aligned} \tag{3.13}$$

By adding to the right-hand side of (3.3) the left side of (3.13) multiplied by $\frac{\mu_a}{2}$, we obtain

$$
\begin{aligned}
a(1)\int_0^T u_x^2(t,1)dt &= \int_0^T\!\!\int_0^1 \left\{(1-\frac{\mu_a}{2})u_t^2(t,x) + \left[(1+\frac{\mu_a}{2})a(x) - xa'(x)\right] u_x^2(t,x)\right\} dx\,dt \\
&\quad + \left[\int_0^1 u_t(t,x)\left(2xu_x(t,x) + \frac{\mu_a}{2}u(t,x)\right) dx\right]_0^T \\
&\quad - b\int_0^T \int_0^1 v_t(t,x)\left(2xu_x(t,x) + \frac{\mu_a}{2}u(t,x)\right) dx\,dt \\
&\geq \frac{2-\mu_a}{2}\int_0^T\!\!\int_0^1 \left\{u_t^2(t,x) + a(x)u_x^2(t,x)\right\} dx\,dt \\
&\quad + \left[\int_0^1 u_t(t,x)\left(2xu_x(t,x) + \frac{\mu_a}{2}u(t,x)\right) dx\right]_0^T \\
&\quad - b\int_0^T \int_0^1 v_t(t,x)\left(2xu_x(t,x) + \frac{\mu_a}{2}u(t,x)\right) dx\,dt,
\end{aligned}
\tag{3.14}
$$

where we have used the inequality $x|a'(x)| \leq \mu_a a(x)$.

We proceed to estimate the last two terms above. Using the Young inequality we have

$$
\left|\int_0^1 u_t(t,x)\left(xu_x(t,x) + \frac{\mu_a}{4}u(t,x)\right) dx\right| \leq \frac{1}{2}\|u_t\|^2_{L^2(0,1)} + \frac{1}{2}\|xu_x + \frac{\mu_a}{4}u\|^2_{L^2(0,1)}. \tag{3.15}
$$

Next, we compute:

$$
\begin{aligned}
\|xu_x + \frac{\mu_a}{4}u\|^2_{L^2(0,1)} &= \int_0^1 x^2u_x^2\,dx + \frac{\mu_a}{4}\int_0^1 x(u^2)_x\,dx + \frac{\mu_a^2}{16}\int_0^1 u^2\,dx \\
&= \int_0^1 x^2u_x^2\,dx + \frac{\mu_a}{4}\left[xu^2\right]_{x=0}^{x=1} + \frac{\mu_a}{4}\left(\frac{\mu_a}{4} - 1\right)\int_0^1 u^2\,dx \\
&\leq \int_0^1 x^2u_x^2\,dx + \frac{\mu_a}{4}\left[xu^2\right]_{x=0}^{x=1}.
\end{aligned}
$$

By using the boundary conditions, (2.2) and (3.6), we obtain that

$$
\|xu_x + \frac{\mu_a}{4}u\|^2_{L^2(0,1)} \leq \int_0^1 x^2u_x^2\,dx \leq \frac{1}{a(1)}\int_0^1 a(x)u_x^2\,dx. \tag{3.16}
$$

We combine (3.15) and (3.16) to get

$$
\left|\int_0^1 u_t(t,x)\left(xu_x(t,x) + \frac{\mu_a}{4}u(t,x)\right) dx\right| \leq \frac{1}{\min\{1,a(1)\}}E_{v,u}(0).
$$

Thus

$$
\left|\left[\int_0^1 u_t(t,x)\left(2xu_x(t,x) + \frac{\mu_a}{2}u(t,x)\right) dx\right]_0^T\right| \leq \frac{4}{\min\{1,a(1)\}}E_{v,u}(0). \tag{3.17}
$$

By the same technique, we obtain

$$
\begin{aligned}
&\left|b\int_0^T\!\!\int_0^1 v_t(t,x)\left(2xu_x(t,x) + \frac{\mu_a}{2}u(t,x)\right) dx\,dt\right| \\
&\leq \frac{b}{2}\int_0^T\!\!\int_0^1 v_t^2(t,x)\,dx\,dt + \frac{2b}{a(1)}\int_0^T\!\!\int_0^1 au_x^2(t,x)\,dx\,dt.
\end{aligned}
\tag{3.18}
$$

Finally, combining (3.14) together with (3.17) and (3.18) one obtains the desired estimate (3.12).

Step 2. In a second step, we prove that the following estimation holds true:

$$\begin{aligned}\left(\frac{2-\mu_a}{2}-\frac{b}{2}\right)\int_0^T\!\!\int_0^1 u_t^2(t,x)\,dx\,dt+\left(\frac{2-\mu_a}{2}-\frac{2b}{a(1)}\right)\int_0^T\!\!\int_0^1 a(x)u_x^2(t,x)\,dx\,dt\\ \leq a(1)\int_0^T u_x^2(t,1)\,dt+\left(1+\frac{4}{\min\{1,a(1)\}}\right)E_{v,u}(0),\quad \forall T>0.\end{aligned} \tag{3.19}$$

To this purpose, we will first establish the following inequality:

$$b\int_0^T\!\!\int_0^1 v_t^2(t,x)\,dx\,dt\leq b\int_0^T\!\!\int_0^1 u_t^2(t,x)\,dx\,dt+2E_{v,u}(0),\quad \forall T>0. \tag{3.20}$$

Multiplying the second equation of (2.1) by v_t, integrating by parts over $(0,T)\times(0,1)$, we get

$$\begin{aligned}0&=\int_0^T\!\!\int_0^1 v_t(t,x)\Big(u_{tt}(t,x)-(a(x)u_x(t,x))_x-bv_t(t,x)\Big)\,dx\,dt\\ &=\int_0^T\!\!\int_0^1 v_t(t,x)u_{tt}(t,x)\,dx\,dt-\int_0^1[a(x)u_x(t,x)v_t(t,x)]_{x=0}^{x=1}\,dt\\ &\quad+\int_0^T\!\!\int_0^1 a(x)u_x(t,x)v_{tx}(t,x)\,dx\,dt-b\int_0^T\!\!\int_0^1 v_t^2(t,x)\,dx\,dt\\ &=\int_0^T\!\!\int_0^1 v_t(t,x)u_{tt}(t,x)\,dx\,dt+\int_0^T\!\!\int_0^1 a(x)u_x(t,x)v_{tx}(t,x)\,dx\,dt-b\int_0^T\!\!\int_0^1 v_t^2(t,x)\,dx\,dt,\end{aligned}$$

because $a(x)u_x(t,x)v_t(t,x)$ vanishes at $x=1$ and, owing to (2.4), also at $x=0$.

After integrating by parts on time, this gives

$$\begin{aligned}0&=\left[\int_0^1 v_t(t,x)u_t(t,x)\,dx\right]_0^T-\int_0^T\!\!\int_0^1 v_{tt}(t,x)u_t(t,x)dx\,dt+\left[\int_0^1 a(x)u_x(t,x)v_x(t,x)\,dx\right]_0^T\\ &\quad-\int_0^T\!\!\int_0^1 a(x)v_x(t,x)u_{tx}(t,x)\,dx\,dt-b\int_0^T\!\!\int_0^1 v_t^2(t,x)\,dx\,dt.\end{aligned} \tag{3.21}$$

Next, we multiply the first equation of (2.1) by u_t and integrate the resulting equation over $(0,T)\times(0,1)$. This gives, after a suitable integration by parts,

$$0=\int_0^T\!\!\int_0^1 u_t(t,x)v_{tt}(t,x)dx\,dt+\int_0^T\!\!\int_0^1 a(x)v_x(t,x)u_{tx}(t,x)\,dx\,dt+b\int_0^T\!\!\int_0^1 u_t^2(t,x)\,dx\,dt. \tag{3.22}$$

Combining (3.21) and (3.22), we obtain

$$\begin{aligned}b\int_0^T\!\!\int_0^1 v_t^2(t,x)\,dx\,dt-b\int_0^T\!\!\int_0^1 u_t^2(t,x)\,dx\,dt\\ =\left[\int_0^1 v_t(t,x)u_t(t,x)\,dx\right]_0^T+\left[\int_0^1 a(x)u_x(t,x)v_x(t,x)\,dx\right]_0^T.\end{aligned} \tag{3.23}$$

Moreover, taking into account the fact that

$$\left|\int_0^1 v_t(t,x)u_t(t,x)\,dx\right|+\left|\int_0^1 a(x)u_x(t,x)v_x(t,x)\,dx\right|\leq E_{v,u}(t),$$

and the constancy of the energy, it follows that

$$\left[\int_0^1 v_t(t,x)u_t(t,x)\,dx\right]_0^T + \left[\int_0^1 a(x)u_x(t,x)v_x(t,x)\,dx\right]_0^T \le 2E_{v,u}(0). \tag{3.24}$$

Then, inserting (3.24) into (3.23), we get the required estimation (3.20). Finally, (3.19) follows by inserting (3.20) into (3.12).

Step 3. In the third step, we are going to prove that the following estimation holds true:

$$\begin{aligned}&-\int_0^T\int_0^1 v_t^2(t,x)\,dx\,dt + \left(1-\frac{b}{2}C_a\right)\int_0^T\int_0^1 a(x)v_x^2(t,x)\,dx\,dt - \frac{b}{2}\int_0^T\int_0^1 u_t^2(t,x)\,dx\,dt\\ &\le 2\sqrt{C_a}E_{v,u}(0) \quad \forall T>0,\end{aligned} \tag{3.25}$$

where C_a is defined in (2.6).

We multiply the first equation of (2.1) by v and integrate the resulting equation over $(0,T)\times(0,1)$. After suitable integrations by parts, this gives

$$\begin{aligned}0 = &\left[\int_0^1 v(t,x)v_t(t,x)\,dx\right]_0^T - \int_0^T\int_0^1 v_t^2(t,x)\,dx\,dt - \int_0^T [a(x)v_x(t,x)v(t,x)]_{x=0}^{x=1}\,dt\\ &+ \int_0^T\int_0^1 a(x)v_x^2(t,x)\,dx\,dt + b\int_0^T\int_0^1 v(t,x)u_t(t,x)\,dx\,dt.\end{aligned}$$

Using the fact that $a(x)v_x(t,x)v(t,x)$ vanishes at $x=1$ and, owing to (2.4) also at $x=0$, we get

$$\begin{aligned}&-\int_0^T\int_0^1 v_t^2(t,x)\,dx\,dt + \int_0^T\int_0^1 a(x)v_x^2(t,x)\,dx\,dt\\ &\qquad = -\left[\int_0^1 v(t,x)v_t(t,x)\,dx\right]_0^T - b\int_0^T\int_0^1 v(t,x)u_t(t,x)\,dx\,dt.\end{aligned} \tag{3.26}$$

On the other hand, using Young's inequality and the Poincaré inequality (2.6), we have

$$\begin{aligned}\left|\int_0^1 v(t,x)v_t(t,x)\,dx\right| &\le \frac{1}{2}\int_0^1 \left(\frac{1}{\sqrt{C_a}}v^2(t,x) + \sqrt{C_a}v_t^2(t,x)\right)dx\\ &\le \frac{\sqrt{C_a}}{2}\int_0^1 \left(a(x)v_x^2(t,x) + v_t^2(t,x)\right)dx\\ &\le \sqrt{C_a}E_{v,u}(0).\end{aligned}$$

Thus

$$\left|\left[\int_0^1 v(t,x)v_t(t,x)\,dx\right]_0^T\right| \le 2\sqrt{C_a}E_{v,u}(0), \tag{3.27}$$

One can show similarly that

$$\left|b\int_0^T\int_0^1 v(t,x)u_t(t,x)\,dx\,dt\right| \le \frac{b}{2}\int_0^T\int_0^1 u_t^2(t,x)\,dx\,dt + \frac{b}{2}C_a\int_0^T\int_0^1 a(x)v_x^2(t,x)\,dx\,dt. \tag{3.28}$$

Then, inserting (3.27) and (3.28) into (3.26), we arrive at the desired inequality (3.25).

Step 4. The sum of $\frac{2-\mu_a}{2}\times$(3.25) and $2\times$(3.19) gives that

$$\begin{aligned}
&-\frac{2-\mu_a}{2}\int_0^T\!\!\int_0^1 v_t^2(t,x)\,dx\,dt+\frac{2-\mu_a}{2}\left(1-\frac{b}{2}C_a\right)\int_0^T\!\!\int_0^1 a(x)v_x^2(t,x)\,dx\,dt\\
&-\frac{2-\mu_a}{2}\frac{b}{2}\int_0^T\!\!\int_0^1 u_t^2(t,x)\,dx\,dt+2\left(\frac{2-\mu_a}{2}-\frac{b}{2}\right)\int_0^T\!\!\int_0^1 u_t^2(t,x)\,dx\,dt\\
&+2\left(\frac{2-\mu_a}{2}-\frac{2b}{a(1)}\right)\int_0^T\!\!\int_0^1 a(x)u_x^2(t,x)\,dx\,dt\\
&\le(2-\mu_a)\sqrt{C_a}E_{v,u}(0)+2a(1)\int_0^T u_x^2(t,1)\,dt+2\left(1+\frac{4}{\min\{1,a(1)\}}\right)E_{v,u}(0),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\frac{2-\mu_a}{2}\left(1-\frac{b}{2}C_a\right)\int_0^T\!\!\int_0^1 a(x)v_x^2\,dx\,dt+2\left(\frac{2-\mu_a}{2}-\frac{2b}{a(1)}\right)\int_0^T\!\!\int_0^1 a(x)u_x^2\,dt\,dx\\
&+\frac{2-\mu_a}{4}\int_0^T\!\!\int_0^1 v_t^2\,dx\,dt+\frac{2-\mu_a}{4}\left(1-\frac{6-\mu_a}{2-\mu_a}b\right)\int_0^T\!\!\int_0^1 u_t^2\,dx\,dt\\
&+\frac{3(2-\mu_a)}{4}\int_0^T\!\!\int_0^1\left(u_t^2-v_t^2\right)\,dx\,dt\\
&\le 2a(1)\int_0^T u_x^2(t,1)\,dt+(2-\mu_a)\sqrt{C_a}E_{v,u}(0)+2\left(1+\frac{4}{\min\{1,a(1)\}}\right)E_{v,u}(0).
\end{aligned}\tag{3.29}$$

On the other hand, using (3.20), we can see that

$$\frac{3(2-\mu_a)}{4}\int_0^T\!\!\int_0^1\left(u_t^2-v_t^2\right)\,dx\,dt\ge-\frac{3(2-\mu_a)}{2b}E_{v,u}(0).\tag{3.30}$$

Inserting (3.30) into (3.29) and using the definition of $M_{a,b}$, then we have

$$\begin{aligned}
&(2-\mu_a)M_{a,b}\int_0^T E_{v,u}(t)\,dt-\frac{3(2-\mu_a)}{2b}E_{v,u}(0)\\
&\le 2a(1)\int_0^T u_x^2(t,1)\,dt+(2-\mu_a)\sqrt{C_a}E_{v,u}(0)+2\left(1+\frac{4}{\min\{1,a(1)\}}\right)E_{v,u}(0).
\end{aligned}$$

Finally, using the constancy of the energy, we get

$$\begin{aligned}
&\left((2-\mu_a)M_{a,b}T-\left(\frac{3(2-\mu_a)}{2b}+2\left(1+\frac{4}{\min\{1,a(1)\}}\right)+(2-\mu_a)\sqrt{C_a}\right)\right)E_{v,u}(0)\\
&\le 2a(1)\int_0^T u_x^2(t,1)\,dt.
\end{aligned}$$

This leads to the inverse inequality (3.10). The proof is thus complete. □

4 Boundary controllability

In this section, by using the Hilbert Uniqueness Method introduced by J.-L. Lions [40, 41], we prove the null controllability of the degenerate hyperbolic system (1.1). At first, let us study well-posedness and regularity results for a such boundary control system.

4.1 Solutions by transposition

This subsection is devoted to preliminary results concerning the well-posedness of (1.1). Thus, as usual when dealing with boundary controls, we need to define the solution of (1.1) by applying the so called transposition method in the spirit of [40] (see also [42], Chapter 3, Section 9). To this aim, we define $H_a^{-1}(0,1)$ as the dual space of $H^1_{a,0}(0,1)$ and we denote the dual of $\mathcal{H}_a$ by $\mathcal{H}'_a$.

Thanks to the direct inequality (3.1), the following definition in the sense of transposition of solution of (1.1) makes sense:

Definition 4.1. Let $f \in L^2(0,T)$ and $(z_0, z_1, y_0, y_1) \in \mathcal{H}'_a$ be given. We say that $Z = (z, z_t, y, y_t) \in C^0([0,T], \mathcal{H}'_a)$. is a solution of problem (1.1), in the sense of transposition, if it satisfies the identity

$$\begin{aligned} &\int_0^1 \left(z_t(T)\psi_T^0 + y_t(T)\varphi_T^0 - z(T)\psi_T^1 - y(T)\varphi_T^1 - bz(T)\varphi_T^0 + by(T)\psi_T^0\right) dx \\ &= \int_0^1 \left(z_t(0)\psi(0) + y_t(0)\varphi(0) - z(0)\psi_t(0) - y(0)\varphi_t(0) - bz(0)\varphi(0) + by(0)\psi(0)\right) dx \\ &\quad - a(1)\int_0^T f(t)\varphi_x(t,1)\, dt, \end{aligned} \tag{4.1}$$

for all $T > 0$ and all $\left(\psi_T^0, \psi_T^1, \varphi_T^0, \varphi_T^1\right) \in \mathcal{H}_a$, where (ψ, φ) is the solution of the backward problem

$$\begin{cases} \psi_{tt} - (a(x)\psi_x)_x + b\varphi_t = 0, & \text{in } (0,T)\times(0,1), \\ \varphi_{tt} - (a(x)\varphi_x)_x - b\psi_t = 0, & \text{in } (0,T)\times(0,1), \\ B\psi(0) = B\varphi(0) = 0, & \text{on } (0,T), \\ \psi(t,1) = \varphi(t,1) = 0, & \text{on } (0,T), \\ \psi(T,x) = \psi_T^0(x), \psi_t(T,x) = \psi_T^1(x), & \text{on } (0,1) \\ \varphi(T,x) = \varphi_T^0(x), \varphi_t(T,x) = \varphi_T^1(x), & \text{on } (0,1). \end{cases} \tag{4.2}$$

Note that according to Theorem 2.1, the backward system (4.2) admits a unique solution $\Psi = (\psi, \psi_t, \varphi, \varphi_t) \in C^0([0,T]; \mathcal{H}_a)$. Moreover, this solution depends continuously on $\left(\psi_T^0, \psi_T^1, \varphi_T^0, \varphi_T^1\right) \in \mathcal{H}_a$ and the energy $E_{\psi,\varphi}$ of (ψ, φ) is conserved through time.

Let us consider the linear form $\mathcal{L}$ defined by

$$\begin{aligned} \mathcal{L}\left(\psi_T^0, \psi_T^1, \varphi_T^0, \varphi_T^1\right) &= \left\langle (z_1, -z_0, y_1, -y_0), (\psi(0), \psi_t(0), \varphi(0), \varphi_t(0)) \right\rangle_{\mathcal{H}'_a, \mathcal{H}_a} \\ &\quad - a(1)\int_0^T f(t)\varphi_x(t,1)\, dt, \end{aligned}$$

for all $T > 0$, where

$$\begin{aligned} &\left\langle (z_1, -z_0, y_1, -y_0), (\psi(0), \psi_t(0), \varphi(0), \varphi_t(0)) \right\rangle_{\mathcal{H}'_a, \mathcal{H}_a} \\ &= \int_0^1 \left(z_t(0)\psi(0) + y_t(0)\varphi(0) - z(0)\psi_t(0) - y(0)\varphi_t(0) - bz(0)\varphi(0) + by(0)\psi(0)\right) dx \end{aligned} \tag{4.3}$$

In view of the direct inequality (3.1), for all $T > 0$, there exists a constant $D > 0$ such that

$$\int_0^T f(t)\varphi_x(t,1)\, dt \le DE_{\psi,\varphi}(0) = DE_{\psi,\varphi}(T),$$

where

$$E_{\psi,\varphi}(T) = \frac{1}{2}\left[\|\psi_T^1\|^2_{L^2} + \|\psi_T^0\|^2_{H^1_{a,0}(0,1)} + \|\varphi_T^1\|^2_{L^2} + \|\varphi_T^0\|^2_{H^1_{a,0}(0,1)}\right].$$

Thus, the mapping $\mathcal{L}$ is a continuous linear form with respect to $\left(\psi_T^0, \psi_T^1, \varphi_T^0, \varphi_T^1\right) \in \mathcal{H}_a$. Therefore, we can use the Riesz Theorem obtaining that for any $T > 0$ there exists a unique 4-uplet $\left(z_0^T, z_1^T, y_0^T, y_1^T\right) \in \mathcal{H}'_a$ such that

$$\begin{aligned}
&\mathcal{L}\left(\psi_T^0, \psi_T^1, \varphi_T^0, \varphi_T^1\right) \\
&= \left\langle \left(z_1^T, -z_0^T, y_1^T, -y_0^T\right), \left(\psi_T^0, \psi_T^1, \varphi_T^0, \varphi_T^1\right)\right\rangle_{\mathcal{H}'_a, \mathcal{H}_a}, \quad \forall \left(\psi_T^0, \psi_T^1, \varphi_T^0, \varphi_T^1\right).
\end{aligned}$$

Hence, one can prove the following existence result (see [40, Theorem 4.2, page 46–54] for a complete proof).

Theorem 4.1. *Assume the hypotheses of Lemma 3.1. For each $Z_0 = (z_0, z_1, y_0, y_1) \in \mathcal{H}'_a$ and $f \in L^2(0,T)$, system (1.1) has a unique solution*

$$Z \in C^0\left([0,T], \mathcal{H}'_a\right).$$

4.2 Controllability result

The main result of this paper is the following.

Theorem 4.2. *Assume (1.2) holds and suppose that $0 < b < b_a$. For all $T > T_{a,b}$ and all initial condition $U_0 \in \mathcal{H}'_a$, there exists a control $f \in L^2(0,T)$, such that the associated solution $Z = (z, z_t, y, y_t)$ of (1.1) (defined in the sense of transposition) satisfies*

$$z(T,x) = z_t(T,x) = y(T,x) = y_t(T,x) = 0 \quad \forall x \in (0,1). \tag{4.4}$$

Proof. Our approach consists in applying the Hilbert Uniqueness Method, introduced by J.-L. Lions in [40]. Let $(z_0, z_1, y_0, y_1) \in \mathcal{H}'_a$ be given. Let us consider the bilinear form Λ on $\mathcal{H}_a$ defined by

$$\Lambda\left(\Psi^T, \widetilde{\Psi}^T\right) := a(1)\int_0^T \varphi_x(t,1)\widetilde{\varphi}_x(t,1)\,dt \quad \forall \Psi^T, \widetilde{\Psi}^T \in \mathcal{H}_a,$$

where $\Psi^T = (\psi_T^0, \psi_T^1, \varphi_T^0, \varphi_T^1)$, $\widetilde{\Psi}^T = (\widetilde{\psi}_T^0, \widetilde{\psi}_T^1, \widetilde{\varphi}_T^0, \widetilde{\varphi}_T^1)$, and $\Psi = (\psi, \psi_t, \varphi, \varphi_t)$ is the solution of (4.2), and $\widetilde{\Psi} = (\widetilde{\psi}, \widetilde{\psi}_t, \widetilde{\varphi}, \widetilde{\varphi}_t)$ is defined similarly by replacing the final condition $\Psi^T = (\psi_T^0, \psi_T^1, \varphi_T^0, \varphi_T^1)$ by $\widetilde{\Psi}^T = (\widetilde{\psi}_T^0, \widetilde{\psi}_T^1, \widetilde{\varphi}_T^0, \widetilde{\varphi}_T^1)$.

From the direct inequality (3.1), it is clear that Λ is continuous on $\mathcal{H}_a$. Moreover, thanks to the observability inequality (3.10), Λ is coercive on $\mathcal{H}_a$ for $T > T_{a,b}$.

Next, we also define the continuous linear map

$$\ell\left(\widetilde{\Psi}^T\right) := \left\langle (z_1, -z_0, y_1, -y_0), \left(\widetilde{\psi}(0), \widetilde{\psi}_t(0), \widetilde{\varphi}(0), \widetilde{\varphi}_t(0)\right)\right\rangle_{\mathcal{H}'_a, \mathcal{H}_a} \quad \forall \widetilde{\Psi}^T \in \mathcal{H}_a,$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}'_a, \mathcal{H}_a}$ is defined by (4.3).

Since Λ is continuous and coercive on $\mathcal{H}_a$, and ℓ is continuous on the Hilbert space $\mathcal{H}_a$, by the Lax-Milgram Theorem, there exists a unique $\widehat{\Psi}^T \in \mathcal{H}_a$ such that

$$\Lambda\left(\widehat{\Psi}^T, \widetilde{\Psi}^T\right) = \ell\left(\widetilde{\Psi}^T\right) \quad \forall \widetilde{\Psi}^T \in \mathcal{H}_a.$$

Set $f := \widehat{\varphi}_x(t,1)$, where $\widehat{\Psi} = (\widehat{\psi}, \widehat{\psi}_t, \widehat{\varphi}, \widehat{\varphi}_t)$ is the unique solution of (4.2) associated to $\widehat{\Psi}^T$. Then we have

$$\begin{aligned}
&a(1)\int_0^T f(t)\widetilde{\varphi}_x(t,1)\,dt \\
&= a(1)\int_0^T \widehat{\varphi}_x(t,1)\widetilde{\varphi}_x(t,1)\,dt \\
&= \Lambda\left(\widehat{\Psi}^T, \widetilde{\Psi}^T\right) \\
&= \left\langle (z_1, -z_0, y_1, -y_0), \left(\widetilde{\psi}(0), \widetilde{\psi}_t(0), \widetilde{\varphi}(0), \widetilde{\varphi}_t(0)\right)\right\rangle_{\mathcal{H}'_a, \mathcal{H}_a}, \quad \forall \widetilde{\Psi}^T \in \mathcal{H}_a.
\end{aligned} \tag{4.5}$$

Finally, denote by $Z = (z, z_t, y, y_t)$ the solution by transposition of (1.1) associated to the function f introduced above. We have that

$$\begin{aligned} -a(1)\int_0^T f(t)\widetilde{\varphi}_x(t,1)\,dt = &\left\langle (z_t(T), -z(T), y_t(T), -y(T))\,, \widetilde{\Psi}^T \right\rangle_{\mathcal{H}_a', \mathcal{H}_a} \\ &- \left\langle (z_1, -z_0, y_1, -y_0)\,, \left(\widetilde{\psi}(0), \widetilde{\psi}_t(0), \widetilde{\varphi}(0), \widetilde{\varphi}_t(0)\right) \right\rangle_{\mathcal{H}_a', \mathcal{H}_a}. \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), it follows that

$$\left\langle (z_t(T), -z(T), y_t(T), -y(T))\,, \widetilde{\Psi}^T \right\rangle_{\mathcal{H}_a', \mathcal{H}_a} = 0, \quad \forall \widetilde{\Psi}^T \in \mathcal{H}_a.$$

Thus, we have (4.4). □

Conflicts of Interest

The authors declare that they have no conflict of interest.

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