Study in two new patchy ecosystem models with commensalism

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Abstract

In this paper, two new patchy ecosystem models with commensalism are proposed. The environmental capacity of one species is enhanced due to commensalism. When the species has logical growth, commensalism can only increase the density. However, when the species has strong Allee effect, commensalism has great influence on the dynamic behavior of the system such as the emergence of the positive equilibrium and saddle-node bifurcation. Also, once the strong Allee effect is incorporated, the corresponding model has bistable attractors including a stable boundary equilibrium and a stable positive equilibrium. Commensalism is beneficial to the persistence of the species.

Keywords Commensalism; patch; Allee effect; stability; bifurcation

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1 Introduction

In biology, commensalism is an important and interesting relationship between two species. One species can get the food from the other species. However, the other species neither do harm to nor benefit the latter. A typical example is the relationship between cattle egret and cattle. When cattle is grazing, the insects are stirred up by cattle and then cattle egret can eat these insects. In other words, the cattle is not affected by cattle egret while cattle egret can gain food and benefit from the cattle. In 2003, Sun and Wei [1] firstly proposed the commensalism model as follows.

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= r_1 x\left(1 - \frac{x}{k_1}\right) + axy,\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= r_2 y\left(1 - \frac{y}{k_2}\right).
\end{aligned}
$$

The authors investigated the local stability property of all equilibria. During the last decades, lots of literatures have paid attention to investigate the commensalism models such as Michaelis-Menten harvesting [2], time delay case [3–5], the stochastic case [6, 7] and the discrete case [8]. We notice that in the above literatures, the influence of the second species on the first one usually obeys the functional response such as linear case, monotonic case and non-monotonic case. Actually, we think that to a certain extent, the second species can improve the environmental capacity of the first species due to commensalism. In this paper, we will investigate the following commensalism ecosystem:

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= rx\left(1 - \frac{x}{k + \lambda y}\right),\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= y(f - ey).
\end{aligned}
$$

Here the environmental capacity of the first species x is introduced by $k + \lambda y$. In other words, the species y is conductive to x species. And the species x has no impact on the population density of species y.

As is well known that the destruction of nature by human activities leads to the fragmentation of the habitat of many species into patches. To some extent, migration between different patches is conducive to the survival of the population, especially, for the conservation of endangered species. Recently, the two-patch ecosystem is widely investigated, one can see [9–12] for details. Motivated by the above, in this paper, we will discuss the following two patch commensalism model:

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= rx_1\left(1 - \frac{x_1}{k + \lambda y}\right) + e_2 x_2 - e_1 x_1,\\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= -dx_2 + e_1 x_1 - e_2 x_2, \qquad (1.1)\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= y(f - ey).
\end{aligned}
$$

Here the first species x can migrate between two patches. x_1 and x_2 are the population densities of x in two patches, respectively.

At the same time, in [13], Allee pointed out that mating between populations becomes difficult and then the birth rates will decrease when the density of the species is relatively sparse. In [14], Bazykin proposed the first single species model with strong Allee effect.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right)(x - m).$$

In last decades, there exist a large number of excellent literatures which investigate the ecological models with Allee effect [15–17]. Especially, the commensalism model with Allee effect is also proposed and investigated [8, 18–21]. Basis on the model (1.1), we will also investigate the dynamic behaviour of the following two patch commensalism model and strong Allee effect as follows.

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= rx_1\left(1 - \frac{x_1}{k + \lambda y}\right)\Big(x - m\Big) + e_2x_2 - e_1x_1, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -dx_2 + e_1x_1 - e_2x_2, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y(f - ey). \end{aligned} \tag{1.2}$$

In the above, m is the strong Allee constant and $0 < m < k$.

Different from the traditional model with commensalism, we have introduced the capacity of the species as $k + \lambda y$ and then several observations are presented in this paper. In detail, for the Allee effect case, when $\lambda = 0$, system (1.2) exists a boundary equilibrium which is globally asymptotically stable. However, when $\lambda > 0$, system (1.2) may have two positive equilibria. System (1.2) will exist a bistable phenomena and a saddle-node bifurcation. In other words, commensalism not only can keep two species from extinction but also give rise to more complex dynamical behaviours.

The rest of the paper is organized as follows. In Section 2 and Section 3, for the Logistic growth and strong Allee effect cases, we give the dynamical behavior of system (1.1) and (1.2), respectively. In Section 4, a brief discussion is presented.

2 Qualitative analysis for system (1.1)

In order to simplify system (1.1), we take the following transformations:

$$\bar{x}_1 = \frac{x}{k}, \ \bar{x}_2 = \frac{x_2}{k}, \ \bar{y} = \frac{ey}{f}, \ \tau = rt.$$

Let $\bar{\lambda} = \frac{f\lambda}{ek}$, $\bar{e}_1 = \frac{e_1}{r}$, $\bar{e}_2 = \frac{e_2}{r}$, $\bar{d} = \frac{d}{r}$, $\bar{f} = \frac{f}{r}$. Following we still reserve t, x_1, x_2, y to express τ, $\bar{x}_1$, $\bar{x}_2$, $\bar{y}$, respectively. Then we can obtain the system as follows.

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1\left(1 - \frac{x_1}{1 + \lambda y}\right) + e_2x_2 - e_1x_1, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -dx_2 + e_1x_1 - e_2x_2, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= fy(1 - y). \end{aligned} \tag{2.1}$$

System (2.1) has a trivial equilibrium $E_0(0,0,0)$ and a boundary equilibrium $E_1(0,0,1)$. Moreover, it may has another boundary equilibrium $E_2(x_{10}, x_{20}, 0)$ and a unique positive equilibrium $E_*(x_1^*, x_2^*, 1)$. We shall present the existence and stability of the equilibria as follows.

Theorem 2.1 For system (2.1),

(a) $E_0(0,0,0)$ is always unstable;

(b) $E_1(0,0,1)$ is stable when $e_1 > 1 + \frac{e_2}{d}$ and unstable when $e_1 < \frac{d+e_2}{d}$; Also, $E_1(0,0,1)$ is a saddle-node when $e_1 = \frac{d+e_2}{d}$.

(c) when $e_1 < 1+\frac{e_2}{d}$, there exists another boundary equilibrium $E_2(x_{10}, x_{20}, 0)$ which is always unstable;

(d) when $e_1 < 1+\frac{e_2}{d}$, there exists a unique positive equilibrium $E_*(x_1^*, x_2^*, 1)$ which is always stable.

Here $x_{10} = 1 - \frac{de_1}{d+e_2}$, $x_1^* = (1+\lambda)(1 - \frac{de_1}{d+e_2})$, $x_{20} = \frac{e_1}{d+e_2} x_{10}$, $x_2^* = \frac{e_1}{d+e_2} x_1^*$.

Proof The Jacobian matrix at $E(x,y)$ of system (2.1) is calculated as

$$J_E = \begin{bmatrix} 1 - \frac{2x_1}{k+\lambda y} - e_1 & e_2 & \frac{r\lambda x_1^2}{K+\lambda y} \\ e_1 & -(d+e_2) & 0 \\ 0 & 0 & f - 2fy \end{bmatrix}.$$

It is not difficult to obtain that the equilibria $E_0(0,0,0)$ and $E_2(x_{10}, x_{20}, 0)$ are always unstable since the corresponding Jacobian matrix has an eigenvalue which is $\lambda = f > 0$..

Notice that

$$J_{E_1} = \begin{bmatrix} 1 - e_1 & e_2 & 0 \\ e_1 & -(d+e_2) & 0 \\ 0 & 0 & -f \end{bmatrix}.$$

Obviously, one of the eigenvalues of J_{E_1} is $-f$. Let

$$J_{11} = \begin{bmatrix} 1 - e_1 & e_2 \\ e_1 & -(d+e_2) \end{bmatrix}.$$

We have $Det(J_{11}) = de_1 - (d+e_2)$ and $Tr(J_{11}) = 1 - (d+e_1+e_2)$. When $e_1 < 1 + \frac{e_2}{d}$, i.e., $Det(J_{11}) < 0$, the equilibrium $E_1(0,0,1)$ is unstable. However, when $e_1 > 1 + \frac{e_2}{d}$, i.e., $Det(J_{11}) > 0$ and $Tr(J_{11}) < 0$, the equilibrium $E_1(0,0,1)$ is stable. In the next, we will obtain that $E_1(0,0,1)$ is a saddle-node when $e_1 = 1+\frac{e_2}{d}$. We can obtain that the three eigenvalues of J_{E_1} are $\lambda_1 = 0$, $\lambda_2 = -f$ and $\lambda_3 = -(d+e_2) - \frac{e_1 e_2}{d+e_2}$ when $e_1 = 1 + \frac{e_2}{d}$. Make the following transformations:

$$\begin{bmatrix} X_1 \\ X_2 \\ Y \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ y-1 \end{bmatrix}$$

and

$$\begin{bmatrix} X_1 \\ X_2 \\ Y \end{bmatrix} = \begin{bmatrix} \frac{d+e_2}{e_1} & 0 & \frac{-e_2}{d+e_2} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ V \end{bmatrix}.$$

Then system (2.1) can be rewritten as

$$\begin{aligned}
\frac{\mathrm{d}U_1}{\mathrm{d}t} &= -\frac{d^2(d+e_2)}{(1+\lambda)(d^2+de_2+e_2)}U_1^2 + \frac{2de_2}{(1+\lambda)(d^2+de_2+e_2)}U_1V \\
&\quad +O(|U_1,U_2,V|^3), \\
\frac{\mathrm{d}U_2}{\mathrm{d}t} &= -fU_2 - eU_2^2, \\
\frac{\mathrm{d}V}{\mathrm{d}t} &= -\frac{(d+e_2)^2+e_1e_2}{d+e_2}V - \frac{2de_2}{(1+\lambda)(d^2+de_2+e_2)}U_1V \\
&\quad +O(|U_1,U_2,V|^3),
\end{aligned} \tag{2.2}$$

System (2.2) reduced on the one-dimensional center manifold $U_2 = O(|U_1|^3)$, $V = O(|U_1|^3)$ is given by $\frac{\mathrm{d}U_1}{\mathrm{d}t} = -\frac{d^2(d+e_2)}{(1+\lambda)(d^2+de_2+e_2)}U_1^2 + O(|U_1|^3)$. The coefficient of term U_1^2 is $-\frac{d^2(d+e_2)}{(1+\lambda)(d^2+de_2+e_2)} \neq 0$. Thus $E_1(0,0,1)$ is a saddle-node when $e_1 = 1 + \frac{e_2}{d}$.

Also,

$$J_{E_*} = \begin{bmatrix} -x_1^* - \frac{e_1e_2}{d+e_2} & e_2 & cx_1^* \\ e_1 & -(d+e_2) & 0 \\ 0 & 0 & -f \end{bmatrix}.$$

Define

$$J_{11}^* = \begin{bmatrix} -x_1^* - \frac{e_1e_2}{d+e_2} & e_2 \\ e_1 & -(d+e_2) \end{bmatrix}.$$

one can have $Det(J_{11}^*) > 0$ and $Tr(J_{11}^*) < 0$. Thus, the positive equilibrium $E_*(x_1^*, x_2^*, 1)$ is always stable.

Following we present Theorem 7.1 in [22] on asymptotically autonomous system which will be applied when the global asymptotically stability os model (2.1) is obtained.

Consider the differential systems

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t) \tag{2.3}$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x). \tag{2.4}$$

with t being the independent variable, $t \in R$, and $x \in R^n$. We say that (2.3) is asymptotically autonomous with limiting system (2.4) if $f(x,t) \to g(x)$ as $t \to \infty$, locally uniformly with respect to $x \in R^n$.

Lemma 2.1 Let e be a locally asymptotically stable equilibrium of (2.4) and W be its basin of attraction. Then every pre-compact orbit of (2.3) whose ω-limit set intersects W converges to e.

Theorem 2.2 When $e_1 > 1 + \frac{e_2}{d}$, the equilibrium $E_1(0,0,1)$ is globally asymptotically stable.

Proof According to the third equation of system (2.1), i.e., $\frac{dy}{dt} = fy(1-y)$, one can deduce that $\lim\limits_{t\to+\infty} y(t) = 1$. Thus the first two equations of system (2.1) become an asymptotical autonomous differential equation as follows.

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1\left(1 - \frac{x_1}{1+\lambda y}\right) + e_2x_2 - e_1x_1 \triangleq f_1(t, x_1, x_2), \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -dx_2 + e_1x_1 - e_2x_2 \triangleq f_2(t, x_1, x_2) \end{aligned} \tag{2.5}$$

whose limiting system is

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1\left(1 - \frac{x_1}{1+\lambda}\right) + e_2x_2 - e_1x_1 \triangleq g_1(x_1, x_2), \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -dx_2 + e_1x_1 - e_2x_2 \triangleq g_2(x_1, x_2). \end{aligned} \tag{2.6}$$

Denote

$$f(t, x_1, x_2) = \begin{bmatrix} f_1(t, x_1, x_2) \\ f_2(t, x_1, x_2) \end{bmatrix}$$

and

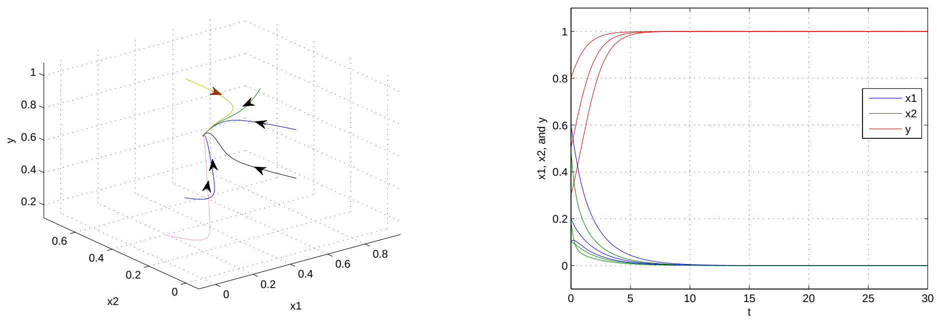
$$g(x_1, x_2) = \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{bmatrix}.$$

Notice that $|f(t, x_1, x_2) - g(x_1, x_2)| = \frac{x_1^2}{\lambda^2}|y-1|$. $x_1(t)$ is bounded and $\lim\limits_{t\to+\infty} y(t) = 1$. Thus $f(t, x_1, x_2)$ converges locally uniformly to $g(x_1, x_2)$. According to Theorem 2.2 in [12], it follows that when $e_1 > 1 + \frac{e_2}{d}$, for the limiting ODE (2.6), $O(0,0)$ is a globally stable equilibrium. Thus from Lemma 2.1, one can conclude that the equilibrium $E_1(0,0,1)$ is globally asymptotically stable. □

Similarly, we can have the following result:

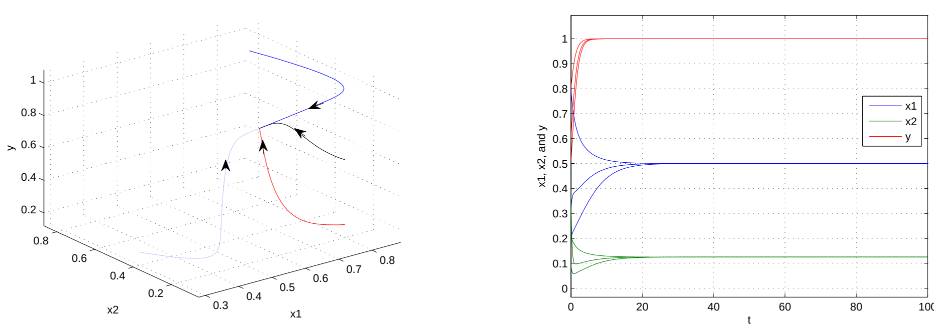
Theorem 2.3 When $e_1 < 1 + \frac{e_2}{d}$, the equilibrium $E_*(x_1^*, x_2^*, 1)$ is globally asymptotically stable.

From Theorem 2.1-2.3, we obtain that commensalism makes no difference to the existence and stability of equilibria $E_1(0,0,1)$ and $E_*(x_1^*, x_2^*, 1)$. The commensalism coefficient λ only improves the prey's survival density. We fix the coefficients $e_2 = 1,\ d = 3,\ f = 1,\ \lambda = 1$. In Figure 2.1 and 2.2, we choose $e_1 = 2$ and $e_1 = 1$, respectively. It follows that the equilibrium $E_1(0,0,1)$ and $E_*(x_1^*, x_2^*, 1)$ are globally asymptotically stable, respectively.



(a) (b)

Figure 2.1: The globally asymptotical stability of $E_0(0, 0, 1)$ when $e_1 = 2$.



(a) (b)

Figure 2.2: The globally asymptotical stability of $E^*(x_1^*, x_2^*, 1)$ when $e_1 = 1$

3 Qualitative analysis for system (1.2)

Similar to that in Section 2, we can simplify system (1.2) and obtain the system as follows.

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1\left(1-\frac{x_1}{1+\lambda y}\right)\left(x-m\right)+e_2x_2-e_1x_1,\\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= -dx_2+e_1x_1-e_2x_2,\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= fy(1-y),
\end{aligned} \tag{3.1}$$

It is not difficult to conclude that system (3.1) has a trivial equilibrium $\bar{E}_0(0,0,0)$ and a boundary equilibrium $\bar{E}_1(0,0,1)$. For other possible boundary equilibrium $\bar{E}(x_{10},x_{20},0)$, it is satisfied with the following equation:

$$\begin{cases} x_{10}(1-x_{10})(x_{10}-m)+e_2x_{20}-e_1x_{10}=0,\\ -dx_{20}+e_1x_{10}-e_2x_{10}=0. \end{cases}$$

The above can deduce the following equation:

$$x_{10}^2-(m+1)x_{10}+m+\tfrac{de_1}{d+e_2}=0. \tag{3.2}$$

Let the discriminant of equation (3.2) denoted by Δ_1 as follows.

$$\Delta_1=(1-m)^2-4\tfrac{de_1}{d+e_2}.$$

When $\Delta_1>0$, i.e., $e_1<\frac{(1-m)^2}{4}(1+\frac{e_2}{d})$, it has two boundary equilibria $\bar{E}_{10}(x_{11},x_{12},0)$, $\bar{E}_{20}(x_{21},x_{22},0)$. When $\Delta_1=0$, i.e., $e_1=\frac{(1-m)^2}{4}(1+\frac{e_2}{d})$, it only has another boundary equilibrium $\bar{E}_{30}(x_{31},x_{32},0)$. When $\Delta_1<0$, there exists no other boundary equilibrium. Here $x_{11}=\frac{m+1+\sqrt{\Delta_1}}{2}$, $\bar{x}_{21}=\frac{m+1-\sqrt{\Delta_1}}{2}$ and $x_{31}=\frac{m+1}{2}$.

For possible positive equilibrium $\bar{E}^*(x_1,x_2,1)$, it must satisfy

$$x_1^2-(m+1+\lambda)x_1+(1+\lambda)(m+\tfrac{de_1}{d+e_2})=0. \tag{3.3}$$

For equation (3.3), notice that the discriminant Δ_2 is as follows.

$$\Delta_2=(\lambda+1-m)^2-4(\lambda+1)\tfrac{de_1}{d+e_2}.$$

When $e_1<\frac{(\lambda+1-m)^2}{4(\lambda+1)}(1+\frac{e_2}{d})$, system (3.1) has two positive equilibria $\bar{E}_1^*(x_{11}^*,x_{12}^*,1)$ and $\bar{E}_2^*(x_{21}^*,x_{22}^*,1)$. When $e_1=\frac{(\lambda+1-m)^2}{4(\lambda+1)}(1+\frac{e_2}{d})$, system (3.1) has a unique positive equilibrium $\bar{E}_3^*(x_{31}^*,x_{32}^*,1)$. When $e_1>\frac{(\lambda+1-m)^2}{4(\lambda+1)}(1+\frac{e_2}{d})$, system (3.1) has no positive equilibrium. Here $x_{11}^*=\frac{m+1+\lambda+\sqrt{\Delta_2}}{2}$, $x_{21}^*=\frac{m+1+\lambda-\sqrt{\Delta_2}}{2}$ and $x_{31}^*=\frac{m+1+\lambda}{2}$. .

The existence and stability of the equilibria can be summarized as follows.

Theorem 3.1 For system (3.1),

(a) the trivial equilibrium $\bar{E}_0(0,0,0)$ is always unstable;

(b) the boundary equilibrium $\bar{E}_1(0,0,1)$ is always stable;

(c) when $e_1 < \frac{(1-m)^2}{4}(1+\frac{e_2}{d})$, there exist two boundary equilibria $\bar{E}_{10}(x_{11}, x_{12}, 0)$ and $\bar{E}_{20}(x_{21}, x_{22}, 0)$. When $e_1 = \frac{(1-m)^2}{4}(1+\frac{e_2}{d})$, there exists another boundary equilibrium $\bar{E}_{30}(x_{31}, x_{32}, 0)$. $\bar{E}_{i0}(x_{i1}, x_{i2}, 0)(i = 1,\ 2,\ 3)$ are always unstable;

(d) when $e_1 < \frac{(\lambda+1-m)^2}{4(\lambda+1)}(1+\frac{e_2}{d})$, there exist two positive equilibria $\bar{E}_1^*(x_{11}^*, x_{12}^*, 1)$ and $\bar{E}_2^*(x_{21}^*, x_{22}^*, 1)$, Here $\bar{E}_1^*(x_{11}^*, x_2^*, 1)$ is stable and $\bar{E}_2^*(x_1^*, x_2^*, 1)$ is unstable. When $e_1 = \frac{(\lambda+1-m)^2}{4(\lambda+1)}(1+\frac{e_2}{d})$, there exist a unique positive equilibrium $\bar{E}_3^*(x_{31}^*, x_{32}^*, 1)$ which is a saddle-node.

Proof The Jacobian matrix at $\bar{E}(x_1, x_2, y)$ of system (3.1) is calculated as

$$J_{\bar{E}} = \begin{bmatrix} *1 - e_1 & e_2 & \frac{\lambda x_1^2(x_1-m)}{(1+\lambda y)^2} \\ e_1 & -(d+e_2) & 0 \\ 0 & 0 & f(1-2y) \end{bmatrix}.$$

Here $*1 = (1-\frac{x_1}{1+\lambda y})(x_1-m) - \frac{x_1}{1+\lambda y}(x_1-m) + x_1(1-\frac{x_1}{1+\lambda y})$. It is easy to obtain that for both $J_{\bar{E}_0}$ and $J_{\bar{E}_{i0}}(i = 1,\ 2,\ 3)$ one of their eigenvalues is $\lambda = f > 0$. Thus the equilibria $\bar{E}_0(0,0,0)$ and $\bar{E}_{i0}(x_{i1}, x_{i2}, 0)(i = 1,\ 2,\ 3)$ are always unstable.

Notice that

$$J_{\bar{E}_1} = \begin{bmatrix} -(m+e_1) & e_2 & 0 \\ e_1 & -(d+e_2) & 0 \\ 0 & 0 & -f \end{bmatrix}.$$

Notice that for $J_{\bar{E}_1}$, $-f$ is one of its eigenvalues. Let

$$\bar{J}_{11} = \begin{bmatrix} -(m+e_1) & e_2 \\ e_1 & -(d+e_2) \end{bmatrix}.$$

We obtain that $Det(\bar{J}_{11}) = (m+e_1)d + me_2 > 0$ and $Tr(\bar{J}_{11}) = -(m+d+e_1+e_2) < 0$. Then the equilibrium $\bar{E}_1(0,0,1)$ is always stable.

Also,

$$J_{\bar{E}_i^*} = \begin{bmatrix} \bar{*}_i - e_1 & e_2 & \frac{\lambda x_{i1}^2(x_{11}-m)}{(1+\lambda)^2} \\ e_1 & -(d+e_2) & 0 \\ 0 & 0 & -f \end{bmatrix}.$$

Here $\bar{*}_i = \frac{3de_1}{d+e_2} + 2m - (1+\frac{m}{1+\lambda})x_{i1}^*$ due to $(1-\frac{x_{i1}^*}{1+\lambda})(x_{i1}^* - m) = \frac{de_1}{d+e_2}$. We can obtain that $-f$ is one of the eigenvalue of $J_{\bar{E}_i^*}$. Define

$$\bar{J}_i^* = \begin{bmatrix} \bar{*}_i - e_1 & e_2 \\ e_1 & -(d+e_2) \end{bmatrix}.$$

Moreover, one can have $Det(\bar{J}_2^*) = \frac{(d+e_2)\sqrt{\Delta_2}}{2(1+\lambda)}(\sqrt{\Delta_2} - (m+1+\lambda)) < 0$. Thus, the equilibrium $\bar{E}_2^*(x_{11}^*, x_{i2}^*, 1)$ is always unstable. Also, $Det(\bar{J}_1^*) = \frac{(d+e_2)\sqrt{\Delta_2}}{2(1+\lambda)}(\sqrt{\Delta_2} + (m+1+\lambda)) > 0$ and $Tr(\bar{J}_1^*) = \frac{1}{d+e_2}\big[3de_1 + (2m-d-e_1-e_2)(d+e_2) - \frac{m+1+\lambda}{1+\lambda}(d+e_2)x_{11}^*\big] < -(d+e_2) - \frac{e_1e_2}{d+e_2} < 0$ due to $\frac{m+1+\lambda}{1+\lambda}(d+e_2)x_{11}^* > 2(de_1 + m(d+e_2))$. As a result, the equilibrium $\bar{E}_1^*(x_{11}^*, x_{12}^*, 1)$ is always stable.

Also, when $e_1 = \frac{(\lambda+1-m)^2}{4(\lambda+1)}(1+\frac{e_2}{d})$, system (3.1) has a unique positive equilibrium $\bar{E}_1^*(x_{31}^*, x_{32}^*, 1)$. Following we will obtain that $\bar{E}_3^*(x_{31}^*, x_{32}^*, 1)$ is a saddle-node. It is easy to obtain that the three eigenvalues of $J_{\bar{E}_3}^*$ are $\lambda_1 = 0$, $\lambda_2 = -f$ and $\lambda_3 = -(d+e_2) - \frac{e_1e_2}{d+e_2}$ when $e_1 = \frac{(\lambda+1-m)^2}{4(\lambda+1)}(1+\frac{e_2}{d})$. Introduce two transformations

$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{Y} \end{bmatrix} = \begin{bmatrix} x_1 - x_{31}^* \\ x_2 - x_{32}^* \\ y - 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{Y} \end{bmatrix} = \begin{bmatrix} \frac{d+e_2}{e_1} & -\frac{(d+e_2)(-f+d+e_2)}{f(d^2+2de_2-df+e_1e_2+e_2^2-e_2f)} & \frac{-e_2}{d+e_2} \\ 1 & -\frac{e_1(d+e_2)}{f(d^2+2de_2-df+e_1e_2+e_2^2-e_2f)} & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \\ \bar{V} \end{bmatrix}.$$

Then we can change system (3.1) as

$$\begin{aligned} \frac{\mathrm{d}\bar{U}_1}{\mathrm{d}t} &= a_{200}\bar{U}_1^2 + a_{110}\bar{U}_1\bar{V} + a_{011}\bar{U}_2\bar{V} + O(|\bar{U}_1, \bar{U}_2, \bar{V}|^3), \\ \frac{\mathrm{d}\bar{U}_2}{\mathrm{d}t} &= -f\bar{U}_2 - f\bar{U}_2^2, \\ \frac{\mathrm{d}\bar{V}}{\mathrm{d}t} &= -\frac{(d+e_2)^2 + e_1e_2}{d+e_2}\bar{V} + c_{200}\bar{U}_1^2 + c_{110}\bar{U}_1\bar{V} + c_{011}\bar{U}_2\bar{V} + O(|\bar{U}_1, \bar{U}_2, \bar{V}|^3). \end{aligned} \tag{3.4}$$

Here $a_{200} = -\frac{8d^2(d+e_2)(1+\lambda)(1+\lambda+m)}{(1+\lambda-m)^2(4d(d+e_2)(\lambda+1)+e_2(\lambda+1-m)^2)} \neq 0$. Similar to that in Theorem 2.1, it follows that $\bar{E}_3^*(x_{31}^*, x_{32}^*, 1)$ is a saddle-node.

□

Theorem 3.2 When $e_1 > \frac{(\lambda+1-m)^2}{4(\lambda+1)}(1+\frac{e_2}{d})$, the equilibrium $\bar{E}_1(0,0,1)$ is globally asymptotically stable.

Proof For the first two equations of system (3.1), we can obtain the following asymptotical autonomous differential equation:

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1\Big(1 - \frac{x_1}{1+\lambda y}\Big)\Big(x_1 - m\Big) + e_2x_2 - e_1x_1 \triangleq h_1(t, x_1, x_2), \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -dx_2 + e_1x_1 - e_2x_2 \triangleq h_2(t, x_1, x_2) \end{aligned} \tag{3.5}$$

whose limiting system is

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1\Big(1-\frac{x_1}{1+\lambda}\Big)\Big(x_1-m\Big)+e_2x_2-e_1x_1 \triangleq I_1(x_1,x_2),\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= -dx_2+e_1x_1-e_2x_2 \triangleq I_2(x_1,x_2).\end{aligned} \tag{3.6}$$

Denote

$$h(t,x_1,x_2)=\begin{bmatrix} h_1(t,x_1,x_2) \\ h_2(t,x_1,x_2)\end{bmatrix}$$

and

$$I(x_1,x_2)=\begin{bmatrix} I_1(x_1,x_2) \\ I_2(x_1,x_2)\end{bmatrix}.$$

It follows that $|h(t,x_1,x_2)-I(x_1,x_2)| \le \frac{x_1^2|x_1-m|}{(\lambda+1)^2}|y-1|$. As a result, $h(t,x_1,x_2)$ converges locally uniformly to $I(x_1,x_2)$ due to the boundedness of $x_1(t)$ and $\lim\limits_{t\to+\infty} y(t)=1$. For the limiting ODE (3.6), it is not difficult to obtain the local asymptotical stability of the unique equilibrium $E_0(0,0)$. Moreover, since system (3.6) has no other equilibrium, there exists no limit cycle. The above can deduce that $E_0(0,0)$ is globally asymptotically stable when $e_1 > \frac{(\lambda+1-m)^2}{4(\lambda+1)}(1+\frac{e_2}{d})$. From Lemma 1, the proof is complete. □

From Theorem 3.1, we know that if $e_1 < \frac{(d+e_2)(\lambda+1-m)^2}{4d(\lambda+1)}$, system (3.1) has two positive equilibria, i.e., $\bar{E}_1^*(x_{11}^*,x_{21}^*,1)$ and $\bar{E}_2^*(x_{12}^*,x_{22}^*,1)$; if $e_1 = \frac{(d+e_2)(\lambda+1-m)^2}{4d(\lambda+1)}$, system (3.1) has a unique positive equilibrium $\bar{E}_3^*(x_{13}^*,x_{23}^*,1)$; if $e_1 > \frac{(d+e_2)(\lambda+1-m)^2}{4d(\lambda+1)}$, system (3.1) has no positive equilibria. Following we will prove that system (3.1) undergoes the saddle-node bifurcation when $e_1 = e_1^{SN} \triangleq \frac{(d+e_2)(\lambda+1-m)^2}{4d(\lambda+1)}$.

Theorem 3.3 When $e_1 = \frac{(d+e_2)(\lambda+1-m)^2}{4d(\lambda+1)}$, system (3.1) undergoes saddle-node bifurcation.

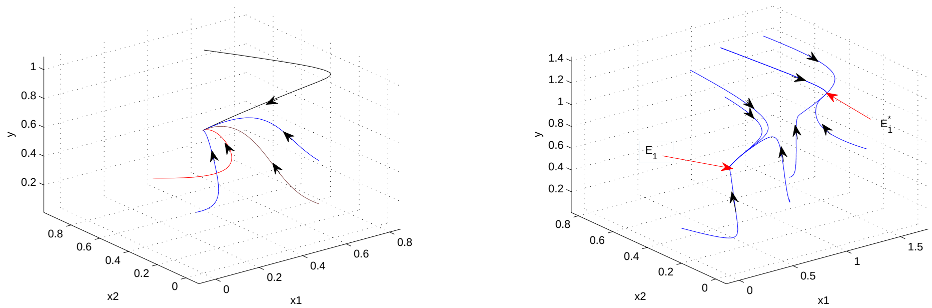
Proof Here we will use Sotomayor's theorem in [23] to obtain the occurrence of saddle-node bifurcation at $e_1 = e_1^{SN}$. When $e_1 = \frac{(d+e_2)(\lambda+1-m)^2}{4d(\lambda+1)}$, the corresponding Jacobian matrix $J_{\bar{E}_3^*}$ has a unique zero eigenvalue, denoted by λ_1.

For the eigenvalue λ_1 of the matrices $J_{\bar{E}_3^*}$ and $J_{\bar{E}_3^*}^T$, we can get the corresponding eigenvectors V and W, respectively. In fact,

$$V=\begin{pmatrix} V_1\\ V_2\\ V_3\end{pmatrix}=\begin{pmatrix}\frac{d+e_2}{e_1}\\ 1\\ 0\end{pmatrix},\quad W=\begin{pmatrix} W_1\\ W_2\\ W_3\end{pmatrix}=\begin{pmatrix}\frac{8f(\lambda+1)^2}{\lambda(\lambda+1+m)^2(\lambda+1-m)}\\ \frac{fe_2}{d+e_2}\\ 1\end{pmatrix}.$$

Moreover, we have

$$F_m(\bar{E}_3^*;e_1^{SN})=\begin{pmatrix}-\frac{m+1+\lambda}{2}\\ -\frac{m+1+\lambda}{2}\\ 0\end{pmatrix}, \tag{3.7}$$



(a) $\lambda = 0$ (b) $\lambda = 1$

Figure 3.1: The dynamical behavior of system (1.2) when $e_1 = 0.5,\ e_2 = 1,\ d = 1,\ f = 1,\ m = 0.5$.

$$D^2F(\bar{E}_3^*; e_1^{SN})(V, V) = \begin{pmatrix} -\frac{m+1+\lambda}{1+\lambda} \\ 0 \\ 0 \end{pmatrix}. \tag{3.8}$$

Here $F_1 = x_1(1 - \frac{x_1}{1+\lambda y})(x_1 - m) + e_2x_2 - e_1x_1,\ F_2 = -dx_2 + e_1x_1 - e_2x_2$ and $F_3 = fy(1-y)$. According to (3.7) and (3.8), it follows that

$$W^T F_{e_1}(\bar{E}_3^*; e_1^{SN}) = -\frac{df(m+1+\lambda)}{2*(d+e_2)} \neq 0,$$

$$W^T[D^2F(\bar{E}_3^*; e_1^{SN})(V, V)] = -\frac{(m+1+\lambda)(d+e_2)^2}{e_1^2(1+\lambda)} \neq 0.$$

The above shows that the saddle-node bifurcation occurs at $\bar{E}_3^*$ when $e_1 = e_1^{SN}$. Thus we complete the proof. □

When $\lambda = 0$, system (1.2) has no positive equilibrium and the boundary equilibrium $\bar{E}_1(0, 0, 1)$ is globally asymptotically stable(see Figure 3.1(a)). However, when $\lambda = 1$, there exist two positive equilibria $\bar{E}_i^*(x_{i1}^*, x_{i2}^*, 1)(i = 1, 2)$ and $\bar{E}_1^*(x_{11}^*, x_{12}^*, 1)$ is locally asymptotically stable. In other words, commensalism leads to the bistability of system (1.2) which shows that both the species may be permanent(see Figure 3.1(b)). In all, commensalism is conductive to the species' survival.

4 Conclusion

In this paper, we have investigated the dynamic behaviors of two new commensalism models. For the Logistic growth case, commensalism has little influence on the dynamic behavior of the above system such as the existence and stability of equilibria. However, for the strong Allee case, when commensalism is introduced, the corresponding model may admit at most two positive equilibria and it may undergo saddle-node bifurcation. Commensalism will lead to bistable phenomenon, that is, a stable boundary equilibrium and a stable positive equilibrium state. Commensalism is beneficial to the survival of the species.

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