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Research Article

Keywords: real vector space, Clifford algebra of 4-vectors, reflection / handedness as fifth dimension

Posted Date: September 3rd, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-313921/v4>

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Dirac Equation *Redux* by Direct Quantization of the 4-Momentum *Vector*

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Abstract. Dirac equation (DE) is a cornerstone of quantum physics. We prove that direct quantization of the 4-momentum *vector* $\mathbf{p}$ with modulus mc (m is rest mass) yields a *coordinate-free* and *manifestly covariant* equation. In coordinate representation, this is equivalent to DE with spacetime frame vectors $\mathbf{x}^\mu$ replacing Dirac's γ^μ -matrices. Remember that standard DE is *not* manifestly covariant. The two sets $\{\mathbf{x}^\mu\}$, $\{\gamma^\mu\}$ obey to the same Clifford algebra. Adding an independent Hermitian vector $\mathbf{x}^5$ to the spacetime basis $\{\mathbf{x}^\mu\}$ allows to accommodate the momentum operator in a *real* vector space with a complex structure generated alone by vectors and multivectors. The real vector space arising from the action of the Clifford product onto the quintet $\{\mathbf{x}^\mu, \mathbf{x}^5\}$ has dimension 32, the same as the equivalent real dimension for the space of Dirac matrices. $\mathbf{x}^5$ proves defining for the combined CPT symmetry, axial vs. polar vectors, left and right handed rotors & spinors, etc.; therefore, we name it *reflector* and $\{\mathbf{x}^\mu, \mathbf{x}^5\}$ – a basis for *spacetime-reflection* (STR). The pentavector $I \equiv \mathbf{x}^{05123}$ in STR substitutes the imaginary unit i . We develop the formalism by deriving all the essential results from the novel STR DE: conserved probability currents, symmetries, nonrelativistic approximation and spin 1/2 magnetic angular momentum. It will become clear that key symmetries follow more directly and with clearer geometric interpretation in STR than in the standard approach. In simple terms, we demonstrate how Dirac matrices are a redundant representation of spacetime-reflection directors.

1. Introduction

The 93 years old Dirac equation (DE)^{1,2} is one of the most far-reaching equations in quantum physics: it is compatible with special relativity, electron spin springs from the equation and it predicted the first instance of antiparticle, the positron, discovered four years later by Anderson.³ DE for a free electron is ($c = 1$):^{4,5}

$$(\gamma^\mu p_\mu - m)\psi = 0 \text{ (sum over } \mu), \text{ where: } p_\mu = i\hbar \partial_\mu = i\hbar \partial / \partial x^\mu; \gamma^\mu \cdot \gamma^\nu = \eta^{\mu\nu}; \mu, \nu = 0,1,2,3 \quad (1)$$

m is the rest mass of the electron and $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the time-space signature. Dirac argued that γ^μ have to be 4×4 complex matrices “...describing some new degrees of freedom belonging to some internal motion in the electron”.^{1,2} This remains the standard position today.^{4,5} Dirac matrices are considered an irreducible representation to DE, truly to any domain of theoretical physics where spin is relevant.⁴

Based on the isomorphism of the Clifford algebras for Dirac γ^μ matrices and spacetime *vectors*, (see eq. (3) below) Hestenes⁶ proposed a new version of DE in the formalism of his spacetime algebra STA.⁷ There is no matrix and no imaginary number in STA DE. However, the spin has been put ‘by hand’ in the equation with the spin vector defining a preferred direction, apparently breaking isotropy. The argument against this criticism⁷ is that the spin vector can take any direction and that one direction is chosen by convention, much in the same way as the matrix γ^0 (respectively σ_3) are choosen diagonal in the Dirac (Pauli) basis.^{4,5} Despite the proven equivalence with standard DE, apparently, by assuming spin the predictive power and symmetry of STA DE diminish compared to standard DE. The real vector space of STA has dimension 16, which is half of the equivalent real dimension for the space of 4×4 complex matrices under matrix multiplication.

The Clifford algebras for spacetime directors $\{x^\mu\}$ and Dirac matrices $\{\gamma^\mu\}$ being the same, the *motivation* for the present work has been to show that direct quantization of the 4-momentum *vector* p of modulus equal to the rest energy m ($c = 1$), provides the simplest form of Dirac equation: $p\psi = m\psi$. This is a realization of the vector p with modulus m being the natural “square root” for the relativistic invariant $p^2 = m^2$. As shown below, the internal degrees of freedom of the electron follow from this equation, extra entities, like matrices unneeded. This simplification together with the manifest covariance mark immediate improvements of the equation over the standard DE. After a swift introduction to Clifford algebra over spacetime, we expand to spacetime-reflection (STR) in order to accommodate the momentum operator in a *real* vector space. We develop the formalism in STR by first presenting STR-DE then demonstrating its workings. Some important results in the STR formalism appear in the Appendix.

2. Clifford product in spacetime

The Clifford or geometric product^{6,7} of two vectors u, v combines the symmetric Hamilton’s scalar product with the antisymmetric Grassmann’s wedge product^{8,9}; in coordinate-free form it is:

$$uv = u \cdot v + u \wedge v; \quad 2u \cdot v = (uv + vu) = \{u, v\}; \quad 2u \wedge v = (uv - vu) = [u, v] \quad (2)$$

The two parts relate naturally to the anticommutator $\{ , \}$ and commutator $[,]$. The bivector $u \wedge v = -v \wedge u$ represents the oriented area encompassed by the two vectors. The geometric product (2) is linear and if not zero, it is invertible. When normalized it renders rotors or boosts, the generators of Lorentz transformations. Examples of both will appear in the following. The geometric product of orthonormal frame vectors is:

$$x_\mu x_\nu \equiv x_{\mu\nu} = x_\mu \cdot x_\nu + x_\mu \wedge x_\nu = \eta_{\mu\nu} + x_\mu \wedge x_\nu; \eta_{\mu\nu} x^\nu = x_\mu; u = u^\mu x_\mu = u_\mu x^\mu; \mu, \nu = 0, 1, 2, 3 \quad (3)$$

Upright letters depict vectors; italics depict scalars. From (3) $x_\mu \cdot x_\nu = \eta_{\mu\nu}$ may be taken to define the time-space signature. As illustrated, signature can ‘lift’ up and down indices of frame vectors (and of scalar components). The coincidence of Clifford algebras for x_μ and Dirac matrices γ_μ leads to spacetime algebra, STA.^{6,7} The real vector space arising from the action of Clifford product onto spacetime vectors (2,3) has dimension 16 with basis elements comprising: the real scalar unit, four vectors x_μ , six bivectors $x_{\mu\nu} = x_\mu \wedge x_\nu$ ($\mu \neq \nu$), four trivectors $x_{\lambda\mu\nu}$ ($\lambda \neq \mu \neq \nu \neq \lambda$) and one tetravector x_{0123} :

$$\text{basis of STA vector space: } \{1, x_\lambda(4), x_{\lambda\mu}(6), x_{\lambda\mu\nu}(4), x_{0123}; \lambda, \mu, \nu = 0, 1, 2, 3\}; \quad \text{dim} 16 \quad (4)$$

From (3) permutations of given orthogonal vectors can at most change the sign, not adding new basis elements. Similarly to the geometric interpretation of bivectors as oriented area elements, trivectors (tetravectors) are oriented 3-volume (resp. 4-volume) elements in spacetime. Multivectors in STA correspond to tensors, represented by matrices in the standard formalism. Nonzero multivectors formed by the wedge product alone are anti-symmetric relative to the exchange of any two indices. x_{0123} squares to -1 and defines the pseudo-scalar of STA. However, it commutes only with even grade elements of STA, i.e. scalars and bivectors; it *anti-commutes* with the odd grade elements – vectors and trivectors. We will define another real vector space and pseudoscalar in the following, therefore will not use here a specific symbol for x_{0123} .

The STA basis (4) is defined in terms of lower indices, but as anticipated in (3) one can *lift* indices up and down with the help of the appropriate form of spacetime signature. Another operation is that of *reversing* the order of indices of a multivector, which corresponds to taking the transpose of a matrix. We identify the combined and commuting operations of index reverse ($\sim$) & lift ($\Uparrow$) with *Hermite conjugation*. Alternatively, in STA $\eta_{\mu\nu} x^\nu = x_\mu = x^0 x^\mu x^0$. For any multivector A in STA the Hermite conjugate $A^\dagger$ is:

$$A^\dagger \equiv (\Uparrow \widetilde{A}) = \Uparrow \widetilde{A} = x^0 \widetilde{A} x^0, \text{ e.g. : } x^{\mu\dagger} = \eta_{\mu\nu} x^\nu = x_\mu = x^0 x^\mu x^0; x_{\lambda\mu}^\dagger = x_\mu^\dagger x_\lambda^\dagger = \eta^{\mu\nu} x_\nu \eta^{\lambda\sigma} x_\sigma = x^{\mu\lambda} \quad (5)$$

The STA basis (4) is *orthonormal*, because each element has modulus 1 and the product of any pair of different elements has zero scalar part $\langle \cdot \rangle_0$, e.g. for fixed indices (no sum over dummy indices):

$$\langle x_{\lambda\mu}^\dagger x_{\lambda\nu} \rangle_0 = \langle x^{\mu\lambda} x_{\lambda\nu} \rangle_0 = \langle x^\mu x_\nu \rangle_0 = \delta_\nu^\mu \quad (6)$$

Now, the relativistic 4-momentum vector p of modulus m and different forms of its square, coordinate-free and coordinate-bound, are given in STA by ($c = 1$) (sum over repeated indices by default):

$$p = x^\mu p_\mu; \quad pp = p \cdot p + p \wedge p = x^{\mu\nu} p_\mu p_\nu = \eta^{\mu\nu} p_\mu p_\nu = p \cdot p = m^2 \quad (7)$$

Using same notation for corresponding quantum operators, (7) suggests a relativistic quantum *vector* equation of first order in spacetime derivatives for a particle of mass m :

$$p\psi = m\psi \text{ with } p = \hbar i \nabla = \hbar i x^\mu \partial_\mu; \quad \partial_\mu \equiv \partial / \partial x^\mu = \eta_{\mu\nu} \partial / \partial x_\nu; \text{ sum over repeated indices} \quad (8)$$

Due to the entrance of the imaginary i with the momentum operator, in passing from the classical (7) to the quantum equation (8) STA's field of scalars has to expand from real to complex numbers. Then, in analogy to the definition of Dirac's γ^5 matrix, we can define x^5 by:

$$x^5 = ix^{0123}; \quad x^5 x^5 = ix^{0123} ix^{0123} = 1; \quad (x^5)^\dagger = x_5 = -ix_{3210} = ix^{0123} = x^5 \quad (9)$$

i commutes with all the elements of STA in (4), therefore x^5 (like γ^5) is Hermitian, as shown. One can certainly build a formalism expanding the STA basis (4) on complex scalars, which as in (9) would *mix* the complex structures arising from on one side scalars and on the other side vectors & multivectors. However, we prefer taking an alternative path that as shown shortly, by 'inverting' (9) leads to a *real* vector space with the complex structure springing from vectors & multivectors alone, not from scalars.

3. The real vector space of spacetime-reflection, STR

Using the same symbol as in (9) we assume a Hermitian vector x^5 to be a *basis* vector, on equal footing with $\{x^\mu\}$, and name the quintet $\{x^\mu, x^5\}$ an orthonormal vector basis for *spacetime-reflection*, STR. We call x^5 *reflector* because it turns out to be defining for axial vectors, CPT symmetry, left and right handed (chiral) rotors & spinors, etc. Let depict by X the *real* vector space generated by the action of the Clifford product onto the basis $\{x^\mu, x^5\}$ of STR. An orthonormal basis for X is (see (6) for orthonormality):

$$\begin{aligned} \mathbf{X}: \{1(1), \mathbf{x}^\mu(4), \mathbf{x}^\lambda \mathbf{x}^\mu \equiv \mathbf{x}^{\lambda\mu}(6), \mathbf{x}^{\lambda\mu\nu}(4), \mathbf{x}^{0123}(1), \mathbf{x}^5(1), \mathbf{x}^{\lambda 5}(4), \mathbf{x}^{\lambda\mu 5}(6), \mathbf{x}^{\lambda\mu\nu 5}(4), \mathbf{x}^{01235}(1); \lambda, \mu, \nu = \\ 0,1,2,3\}; \lambda \neq \mu \neq \nu \neq \lambda; \text{ STR vector product: } \mathbf{x}^\tau \mathbf{x}^\nu = \mathbf{x}^\tau \cdot \mathbf{x}^\nu + \mathbf{x}^\tau \wedge \mathbf{x}^\nu = \zeta^{\tau\nu} + \mathbf{x}^\tau \wedge \mathbf{x}^\nu; \tau, \nu = \\ 0,1,2,3,5; \zeta^{\tau\nu} = \mathbf{x}^\tau \cdot \mathbf{x}^\nu = (+ - - - +) \end{aligned} \quad \text{dim}32 \quad (10)$$

The first 16 elements in (10) are the basis elements of STA in (4).⁷ The other 16 elements are obtained by multiplying the STA elements with the reflector $\mathbf{x}_5$. 16 basis elements in (10) square to -1 , allowing for a rich complex structure in $\mathbf{X}$. Of these only the element of highest grade, the pentavector is both *isotropic* and *commutes* with all elements of $\mathbf{X}$; it constitutes the pseudoscalar of STR, depicted by (compare with (9)):

$$I \equiv -\mathbf{x}^{01235} = \mathbf{x}_{01235}; I^2 = -1; I^\dagger = -\mathbf{x}_{53210} = \mathbf{x}^{01235} = -I; I = \tilde{I}; I\mathbf{x}^\tau = \mathbf{x}^\tau I; \tau = 0,1,2,3,5 \quad (11)$$

In STR the geometric pseudoscalar I substitutes the algebraic i . In STR we adopt index lift-reverse (see (5)) to define the Hermite conjugate of elements of (10), as illustrated for the case of $I^\dagger$ in (11) and in general:

$$A^\dagger \equiv (\widetilde{\uparrow A}) = \uparrow \tilde{A}, \text{ e.g.: } \mathbf{x}^{\tau\dagger} = \mathbf{x}_\tau = \zeta_{\tau\nu} \mathbf{x}^\nu, (\mathbf{x}^{\tau\nu})^\dagger = (\mathbf{x}^\tau \cdot \mathbf{x}^\nu + \mathbf{x}^\tau \wedge \mathbf{x}^\nu)^\dagger = \zeta^{\tau\nu} + \mathbf{x}^{\nu\dagger} \wedge \mathbf{x}^{\tau\dagger} \quad (12)$$

The alternative definition of Hermite conjugate in STA, $A^\dagger = \mathbf{x}^0 \tilde{A} \mathbf{x}^0$ does *not* generalize to elements of STR containing $\mathbf{x}^5$. However, it allows taking the conjugate of quantities in the STA subspace of STR with no need to show indices. Two subspaces of $\mathbf{X}$ are of special interest in the following ($j, k = 1,2,3$):

$$\begin{aligned} \mathbf{X}: \{1(1), \mathbf{x}^j \equiv \mathbf{x}^{j0} = \mathbf{x}_{0j} = \mathbf{x}_j(3), \mathbf{x}^{jk} = \mathbf{x}_{jk}(3), \mathbf{x}^{123}(1)\}; \text{ dim}8 \\ \mathbf{\Sigma}: \{1(1), \boldsymbol{\sigma}^j \equiv \mathbf{x}^{j50} = \mathbf{x}_{j05} = \mathbf{x}^j \mathbf{x}^5 = \boldsymbol{\sigma}_j(3), \boldsymbol{\sigma}^{jk} = \epsilon_{jkl} I \boldsymbol{\sigma}^l(3), \boldsymbol{\sigma}^{123} = -\mathbf{x}_{123} \mathbf{x}_5 = I(1)\}; \text{ dim}8^\P \end{aligned} \quad (13)$$

^{\P} Similarly to the standard case,⁴ a vector space isomorphic to (10) occurs from the direct product $\mathbf{X} = \mathbf{\Sigma} \otimes \mathbf{\Sigma}$ with scalars and the pseudo-scalar $I = \boldsymbol{\sigma}_{123}$ freely passing through $\otimes$. In addition to providing an alternative route leading to all the results presented here, this representation marks a further argument in favor of expanding ST to STR (see also the last equation (42)).

The subspaces $\mathbf{X}, \mathbf{\Sigma}$ share their even grade members, i.e. the scalar 1 (grade 0) and the three bivectors $\mathbf{x}_{jk} = \boldsymbol{\sigma}_{jk} = \mathbf{x}_{kj}$ ($j \neq k$ grade 2). This intersection of subspaces $\mathbf{X}, \mathbf{\Sigma}$ can be succinctly expressed by the Clifford product: $\mathbf{X} \cap \mathbf{\Sigma} = \{\mathbf{x}_{jk} = \mathbf{x}_j \cdot \mathbf{x}_k + \mathbf{x}_j \wedge \mathbf{x}_k = \delta_{jk} + \mathbf{x}_j \wedge \mathbf{x}_k\} = \{\boldsymbol{\sigma}_{jk} = \delta_{jk} + \boldsymbol{\sigma}_j \wedge \boldsymbol{\sigma}_k = \delta_{jk} + \epsilon_{jkl} I \boldsymbol{\sigma}_l\}$, where $j = k$ yields the unit scalar and $j \neq k$ the three bivectors. We will present the parity transformation further down (eq. (20-21)), but remind that it changes sign to 3D polar vectors (parity-odd) and leaves unchanged axial vectors (parity-even). As shown in eq. (21) the vectors $\mathbf{x}_j$ behave as polar vectors (parity-

odd), while σ_j behave as axial vectors (parity-even). The space of ‘axial’ vectors, spin and rotor generators Σ is isomorphic to the space of Pauli matrices,⁷ therefore the notation. From (13) it is clear that each vector in one subspace multiplied by x_5 produces a corresponding vector in the other subspace. The subspace of ‘polar’ vectors and boost generators $\mathbf{X}$ is the same as the even subspace of Hestenes’ STA, in our notation having $\mathbf{x}_{123} = Ix_5$ as ‘local’ pseudoscalar.^{6,7,10} Notice that the vectors $\mathbf{x}_j, \sigma_j$ are Hermitian. In passing, we mention that the bivectors $x^{j5} = \sigma^j x^0$ are also Hermitian and parity-even or ‘axial’, see (21). *Index* operations refer to the form at the level of the full basis (10), e.g. $\tilde{\sigma}_j = \tilde{\sigma}^j = \widetilde{x_{j05}} = -\sigma_j; (\uparrow \sigma_j) = -\sigma_j$.

Now we can write the **momentum quantization** equation in STR (compare with (8)), obtaining the **STR-DE**:

$$p\psi = m\psi \text{ with } p = \hbar I \nabla = \hbar I x^\mu \partial_\mu; \partial_\mu \equiv \partial / \partial x^\mu = \eta_{\mu\nu} \partial / \partial x_\nu; \text{ sum over repeated indices} \quad (14)$$

∇ is both coordinate-free and shorthand for $x^\mu \partial_\mu$, therefore one can avoid the Feynman slash notation! ψ must have four components, i.e. it is a 4-spinor because of the four spacetime dimensions, as will become clear in the following. $p = \hbar I \nabla$ is a pseudovector (tetravector) operator in $\mathbf{X}$ containing x^5 . For the electron with charge e , in the presence of an electromagnetic 4-potential A , (14) generalizes to:

$$P\psi = m\psi \quad \text{where } P \equiv p + eA = \hbar I \nabla + eA = x^\mu (\hbar I \partial_\mu + eA_\mu) \quad (15)$$

This is the STR DE minimally coupled to electromagnetic (EM) field. The first equations in (14), (15) are frame-free, which underlines the manifest covariance of STR DE, as detailed further down by equation (28). The vector-operator ∇ obeys the chain rule $\nabla A = x^{\mu\nu} \partial_\mu A_\nu = x^{\mu\nu} [(\partial_\mu A_\nu) + A_\nu \partial_\mu]$ and we need to generalize the commutator in (2), as illustrated in the Appendix for $[x, p]$ (position-momentum operators). The generators of the Lorentz group appear naturally in $[x, p]$. Alike STA, STR promotes a geometric view of physics, where vectors and their combinations, not the scalar components set the complex structure.

4. STR at work

With the above preliminaries in place, it is instructive to calculate at first the **square of the STR DE** (15), which will also illustrate how in the presence of external EM field the spin magnetic moment of the electron arises from spacetime-reflection (see also the derivation of the STR Pauli equation (38)):

$$\begin{aligned}
P\psi &= m\psi \Rightarrow PP\psi = m^2\psi \Rightarrow [(\hbar I\nabla + eA)(\hbar I\nabla + eA) - m^2]\psi = \left(\eta^{\mu\nu}(\hbar I\partial_\mu + eA_\mu)(\hbar I\partial_\nu + eA_\nu) - m^2 + \right. \\
&e\hbar I(\nabla \wedge A + A \wedge \nabla)\left.\right)\psi = \left(KG + e\hbar I(-(\nabla A_0) - (\partial_0 \mathbf{A}) + Ix_5(\nabla \times \mathbf{A}))\right)\psi = (KG + e\hbar I(\mathbf{E} + Ix_5\mathbf{B}))\psi \equiv \\
&\equiv (KG + e\hbar I(\mathbf{E} + I(\boldsymbol{\sigma}, \mathbf{B})))\psi \equiv (KG + e\hbar IF)\psi = 0; \text{ Brackets in e. g. } (\nabla A_0) \text{ limit the action of } \nabla \text{ to } A_0 \\
\mathbf{A} &= A_j \mathbf{x}_j \text{ (vector potential); } \mathbf{E} = E_j \mathbf{x}_j \text{ (electric field, polar 3D vector); } (\boldsymbol{\sigma}, \mathbf{B}) \equiv \boldsymbol{\sigma}_j B_j = \\
&x_5 \mathbf{B} \text{ (magnetic field, 'axial' 3D vector); } F \equiv \mathbf{E} + I(\boldsymbol{\sigma}, \mathbf{B}) = \nabla \wedge A + A \wedge \nabla \text{ (Faraday)} \quad (16)
\end{aligned}$$

KG stands for the Klein-Gordon term; it comprises only scalar and pseudoscalar components, including operators. The term $\mathbf{E} + Ix_5\mathbf{B} = \mathbf{E} + I(\boldsymbol{\sigma}, \mathbf{B})$ is independent of x_5 and is known as the *Faraday* F. It depicts the relativistic invariant *EM field strength* experienced by the electron, as signaled by the prefactor $e\hbar$. F is a tensor in the standard formalism;^{4,11} in STR and STA it is a 4D bivector.^{7,10} The term $e\hbar IF$ distinguishes the squared DE from the KG equation. It represents the ‘*internal degrees of freedom*’ of the electron – the spin, interacting with the EM field. Indeed, in the nonrelativistic regime, equation (16) (or the STR Pauli equation (38)) yields equation (17) below, as shown in Appendix D (orbital and spin angular momentum vectors: $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$; $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$; φ is a Pauli spinor approximating Dirac’s spinor ψ for a slow electron):

$$(P \cdot P - m^2 + e\hbar IF)\psi = 0 \xrightarrow{\text{nonrelativistic approx.}} \left[I\hbar\partial_t - \frac{\mathbf{p}^2}{2m} + eA_0 - \frac{e}{2m}\mathbf{B} \cdot (\mathbf{L} + 2\mathbf{S}) \right] \varphi = 0 \quad (17)$$

This is the famous prediction from DE that the unit of spin angular momentum interacts twice as strongly with the magnetic field as the unit of orbital angular momentum. The derivation of (16-17) proves that spin springs from 4-momentum in spacetime-reflection, without any preconceived internal degrees of freedom. The spin magnetic moment of the electron from (17) is $-\hbar e\boldsymbol{\sigma}/2mc$ with modulus a factor of ~ 1.00118 smaller than the experimental figure, the gap arising from QED corrections, beyond the scope of DE.^{4,5}

Conserved currents^{2,4} of STR DE. From (12,13) $\mathbf{x}_j^\dagger = \mathbf{x}_j$, which simplifies the derivation of the Hermite conjugate for DE and the Dirac conjugate $\bar{\psi}$ for the spinor ψ :

$$\begin{aligned}
(P - m)\psi &= (x^\mu(\hbar I\partial_\mu + eA_\mu) - m)\psi = \left((\hbar I\partial_0 + eA_0) + \mathbf{x}_j(\hbar I\partial_j + eA_j) - mx_0\right)x^0\psi = 0 \\
\rightarrow \psi^\dagger x_0 &\left((-\hbar I\partial_0 + eA_0) + \mathbf{x}_j(-\hbar I\partial_j + eA_j) - mx_0\right) \equiv \bar{\psi} (x^\mu(-\hbar I\partial_\mu + eA_\mu) - m)x_0 = 0 \quad (18)
\end{aligned}$$

After Hermite conjugation $\dagger$, ∂_μ act to the left. Left-multiply the DE by $\bar{\psi} = \psi^\dagger x_0$ and right-multiply the last

equation in (18) by $\bar{\psi}^\dagger = x^0\psi$; then subtract it from the first, obtaining the conservation of probability current (the angled brackets below are shorthand for $\langle \cdot \rangle_0$ in (6), e.g. for α scalar $\langle \alpha(1 + x_\mu) \rangle = \alpha$):

$$(\partial_\mu \bar{\psi})x^\mu\psi + \bar{\psi}x^\mu(\partial_\mu\psi) = \partial_\mu(\bar{\psi}x^\mu\psi) = 0 \text{ with probability density } \rho = \langle \bar{\psi}x^0\psi \rangle = \langle \psi^\dagger\psi \rangle \geq 0 \quad (19)$$

We will look more closely to the use of angled brackets^{6,7}, the values of the Lorentz scalar $\langle \bar{\psi}\psi \rangle$ and the probability density ρ after having specified a form for the spinor ψ in STR, see (36, 40).

Symmetries of STR DE. Let us look now at the *symmetries* parity $\mathcal{P}$, time reversal $\mathcal{T}$ and charge conjugation $\mathcal{C}$ for the STR DE and for the STR Dirac spinor ψ . In the following we will present a direct approach to $\mathcal{T}$ and $\mathcal{C}$ in STR that does not require further preparation. A second approach resembling the standard treatment of these two symmetries can be found in Appendix B. As both approaches yield the same operator part of STR DE we also discuss the relation between the two forms of spinors resulting from the two approaches.

Parity $\mathcal{P}$: $x^\mu \rightarrow x^0x^\mu x^0 = x_\mu^\mu$. As in the standard treatment left-multiply STR DE by x^0 :

$$(x^\mu P_\mu - m)\psi \xrightarrow{\mathcal{P}} \{x^0(x^\mu P_\mu - m)\psi = (x^0x^\mu x^0 P_\mu - m)x^0\psi \equiv (x^\mu P_{\mu\mathcal{P}} - m)\psi_{\mathcal{P}}\} = [x^0(I\partial_0 + eA_0) + x^j(-I\partial_j - eA_j) - m]\psi_{\mathcal{P}} \Rightarrow P_{\mu\mathcal{P}} = \eta_{\mu\nu}P_\nu = \eta_{\mu\nu}(I\partial_\nu + eA_\nu); \psi_{\mathcal{P}} = x^0\psi \quad (20)$$

Applying the parity transformation (20) to the 3D vectors from (13) we clearly see that:

$$\mathbf{x}_j \xrightarrow{\mathcal{P}} x^0\mathbf{x}_j x^0 = -\mathbf{x}_j \text{ are } odd; \boldsymbol{\sigma}_j \xrightarrow{\mathcal{P}} x^0\boldsymbol{\sigma}_j x^0 = \boldsymbol{\sigma}_j \text{ and } x^{5j} \xrightarrow{\mathcal{P}} x^0x^{5j}x^0 = x^0\boldsymbol{\sigma}_j x^0 x^0 = x^{5j} \text{ are } even \quad (21)$$

Time reversal $\mathcal{T}$: $x_0 \rightarrow -x_0 \Leftrightarrow \partial_0 \rightarrow \partial'_0 = -\partial_0$. Starting with STR DE in (15), first Hermite conjugate ($\dagger$) as in (18) then reverse ($\sim$) it. From (12) this combination results in the single transformation of index lift $\Uparrow$:

$$\mathcal{T}: \text{DE} \rightarrow \Uparrow [(x^\mu(I\partial_\mu + eA_\mu) - m)\psi] = [x_\mu(-I\partial_\mu + eA_\mu) - m](\Uparrow\psi) = [x^0(I\partial'_0 + eA_0) + x'^j(-I\partial_j + eA_j) - m]x^0\widetilde{\psi} = 0 \Rightarrow \psi_T \equiv (\Uparrow\psi) = \widetilde{\psi}^\dagger = x^0\widetilde{\psi}; x'^j = -x^j \quad (22)$$

From (22): $TIT^{-1} = \Uparrow I \Uparrow = -I$; $T^2 = \Uparrow\Uparrow = 1$. T keeps the position *components* x_j unchanged and reverses the components of momentum operator $I\partial_j$. From (22) T reverses the spatial directors. In the absence of external fields, it is straightforward to show that the above transformation conserves the Dirac Hamiltonian:

$$\mathcal{T}: I\partial_0\psi = H\psi = (x^{j0}I\partial_j + mx^0)\psi \rightarrow I\partial'_0\psi_T = H\psi_T = (x^{j0}I\partial_j + mx^0)\psi_T \quad (23)$$

What about spin? We will see further down that the Dirac spinor ψ can be expressed as a linear combination of two Pauli spinors φ and χ . In the Pauli basis each of these, e.g. φ , can be arranged in STR into the form $\varphi = (\sigma_3 \varphi_u - I \sigma_2 \varphi_d)(1 + \sigma_3)$, where $\sigma_j \in \Sigma$ from (13) and φ_u, φ_d are probability amplitudes for spin up, resp. spin down *before* the action of T . T transforms each of the Pauli spinors in the same way, as below:

$$T: \varphi \rightarrow T\varphi = \uparrow \varphi = [(-\sigma_3)\varphi_u - I\sigma_2\varphi_d][1 + (-\sigma_3)], \text{ with } \uparrow \sigma_3 = -\sigma_3, \uparrow (I\sigma_2) = I\sigma_2 \quad (24)$$

In a separate publication I have shown that in a Pauli spinor representation in orientation space, as is the case here, the two terms within the first square bracket in (24) represent *opposite* spin parts of the spinor. This means that if the first term stands for the spin up part (σ_3), as in φ , then the second term ($I\sigma_2$) stands for spin down; if, on the other side the first term stands for spin down ($-\sigma_3$), as in $T\varphi$, then the second term stands for spin up. This means that T effectively *flips* spin.

Charge conjugation $C: e \rightarrow -e$. First, as in (22) index-lift STR DE (15), then left-multiply by x^{123} :

$$C: (x^\mu P_\mu - m)\psi = 0 \rightarrow C[(x^\mu P_\mu - m)\psi] = x^{123} \uparrow [(x^\mu (I\partial_\mu + eA_\mu) - m)\psi] = x^{123} [-x^0 (I\partial_0 - eA_0) + x^j (I\partial_j - eA_j) - m]x^0 \tilde{\psi} = [x^\mu (I\partial_\mu - eA_\mu) - m]Ix^5 \tilde{\psi} = 0 \Rightarrow \psi_C \equiv Ix^5 \tilde{\psi} = -x^{0123} \tilde{\psi} \quad (25)$$

Notice that compared to T , $CIC^{-1} = x^{123} \uparrow I \uparrow x^{123} = -I$; $CC = x^{123} \uparrow x^{123} \uparrow = -1$. With (15) describing the electron, (25) describes its antiparticle, the *positron*. A study of T, C in (22, 25) will appear elsewhere.

Now we can apply all the three symmetries (20-25) in series onto the STR DE (15).

$$\begin{aligned} \mathcal{CPT}: \text{Applying the three operations in series yields } \mathcal{CPT}(\text{DE}): (x^\mu (I\partial_\mu + eA_\mu) - m)\psi &= 0 \xrightarrow{T} \uparrow \\ (x^\mu (I\partial_\mu + eA_\mu) - m)\psi &= 0 \xrightarrow{P} x^0 \uparrow (x^\mu (I\partial_\mu + eA_\mu) - m)\psi = 0 \xrightarrow{C} x^{1230} \uparrow \uparrow (x^\mu (I\partial_\mu + eA_\mu) - m)\psi = \\ Ix^5 (x^\mu (I\partial_\mu + eA_\mu) - m)\psi &= (x^\mu (-I\partial_\mu - eA_\mu) - m)Ix^5 \psi = 0; \quad \psi_{\mathcal{CPT}} = Ix^5 \psi. \end{aligned} \quad (26)$$

(26) is identical to eq. (8A) from the Appendix. Within an overall phase: $\psi_{\mathcal{CPT}} = x^5 \psi$, which has the upper and lower Pauli spinors in ψ swapped (see (33)). Transformations (20, 22, 25, 26) above treat the spacetime directors x^μ on equal footing, signature taken into account. They do not rely on K -conjugation from (4A) in the symmetry transformations $\mathcal{T}', \mathcal{C}'$ in the Appendix, eqs. (6A, 7A), which treat x^2 differently from the other directors, just like the standard complex conjugation ultimately relies on γ^2, σ_2 being the only imaginary matrices in the Dirac, resp. Pauli basis.^{4,5} An advantage of the $\mathcal{T}'$ symmetry in (6A) is that it renders explicit

Kramer's degeneracy.¹¹ The parity transformation is the same for the two approaches, therefore:

$$\begin{aligned} \mathcal{CPT}(DE) = \mathcal{C}'\mathcal{P}\mathcal{T}'(DE) \Rightarrow \mathcal{CT}(DE) = \mathcal{C}'\mathcal{T}'(DE) \text{ and } \psi_{\mathcal{CT}} = \psi_{\mathcal{C}'\mathcal{T}'}, \text{ i.e. the duet } \mathcal{CT} \text{ is the same for both} \\ \text{approaches. } \mathcal{CT}(DE) = \mathcal{C}'\mathcal{T}'(DE): (x^\mu(I\partial_\mu + eA_\mu) - m)\psi = 0 \rightarrow x^{123} \Updownarrow (x^\mu(I\partial_\mu + eA_\mu) - m)\psi = \\ (x^0(I\partial'_0 - eA_0) + (-x^j)(-I\partial_j - eA_j) - m)Ix^{50}\psi = 0; \psi_{\mathcal{CT}} = \psi_{\mathcal{C}'\mathcal{T}'} = Ix^{50}\psi = x^{123}\psi \end{aligned} \quad (27)$$

The independence of $\mathcal{CT}$ from the approach hints to a deeper connection between time reversal and charge conjugation symmetries.

Now we will touch shortly on the **relativistic covariance** of the STR DE (15). We have already seen that the geometric product of two vectors comprises a scalar and a bivector, see (3), (13). The generators of Lorentz boosts and rotations are contained in the bivector (a factor 2 ensures consistency with the definition in (1A)):

$$x^\mu x^\nu = \eta^{\mu\nu} + x^\mu \wedge x^\nu = \eta^{\mu\nu} + \frac{1}{2}[x^\mu, x^\nu]; \frac{1}{2}[x^\mu, x^\nu] = \begin{cases} 2\mathbf{K}_j = \mathbf{x}_j \text{ for } \mu = j, \nu = 0 \text{ Boosts} \\ 2\mathbf{J}_l = -\varepsilon_{jkl}I\boldsymbol{\sigma}_l \text{ for } \mu = j, \nu = k \text{ Rotors} \end{cases} \quad (28)$$

Note that I is both parity and Lorentz invariant. The *operator* $(P - m)$ is co-ordinate-free and relativistic-invariant, i.e. it remains unchanged as a whole under a Lorentz transformation $\mathcal{L}$. On the other end, the spinor ψ transformation is a one-sided generalized rotor S , i.e. a rotor and/or a boost comprising the exponentiated generators $\mathbf{J}_j, \mathbf{K}_j$ in (28) (see (29) below and (1A), (2A) in the Appendix) with the respective rotation angle and/or rapidity from the transformation $\mathcal{L}$ at hand. Therefore, we can write:

$$\begin{aligned} \psi \xrightarrow{\mathcal{L}} S\psi \Rightarrow (P - m)\psi = 0 \xrightarrow{\mathcal{L}} (P - m)S\psi = 0 \Leftrightarrow S(P - m)S^{-1}S\psi = S[(P - m)\psi] = 0, \text{ where} \\ S = e^{S_j\omega_j} \text{ with } \begin{cases} \mathbf{S}_j = \mathbf{K}_j = \mathbf{x}_j/2; \omega_j = \alpha_j \text{ (rapidity, hyp. angle) boosts} \\ \mathbf{S}_j = \mathbf{J}_j = -I\boldsymbol{\sigma}_j/2; \omega_j = \vartheta_j \text{ (Euclidian angle) rotors} \end{cases}; S^\dagger x^0 = e^{S_j^\dagger\omega_j}x^0 = x^0 S^{-1} \end{aligned} \quad (29)$$

STR DE transforms as a 4-spinor, as it should. Compare the simplicity of the STR derivation (29) with the linearized infinitesimal Lorentz transformation in the standard approach!^{2,4,5} The two-sided rotor around the operator $S(P - m)S^{-1}S\psi = 0$ is useful to calculate the transformed frame vectors and components of 4-momentum and 4-potential. Under Lorentz transformation the Dirac spinor and bilinears transform as:

$$\psi \xrightarrow{\mathcal{L}} \psi' = S\psi; \bar{\psi} \xrightarrow{\mathcal{L}} \bar{\psi}' = \bar{\psi}S^{-1}; \bar{\psi}\psi \xrightarrow{\mathcal{L}} \bar{\psi}'\psi' = \bar{\psi}\psi; \bar{\psi}x^\mu\psi \xrightarrow{\mathcal{L}} \bar{\psi}'Sx^\mu S^{-1}\psi' = \bar{\psi}'x'^\mu\psi' \quad (30)$$

(29, 30) illustrate how boosts and rotations actually take place in the physical spacetime. STR DE stands at an advantage point when infinitesimal Lorentz transformations become essential, like in curved spacetime.⁴

How do we change basis in STR? (29) hints to the answer: by two sided general rotor transformations, i.e. $x^\mu \rightarrow x'^\mu = Sx^\mu S^{-1}$. One can realize the general transformations S in X by exponentiation. For example, in the deep relativistic regime ($|p| \gg m$) one can neglect the mass term in DE and adapt a more convenient basis and corresponding spinor than the STR Dirac basis adapted here, known as the Weyl basis and Weyl spinors. The generalized rotor $R_\omega = e^{x_5 \omega/2}$ acting one-sided to spinors and two-sided to vectors, can swap x_0 with x_5 leaving x_j unchanged: $R_\omega x^j R_\omega^{-1} = x^j$, as recounted in Appendix C. The symmetries above help us specify a form for the spinor ψ in STR expressed by a couple of Pauli spinors as now shown.

The strategy consists in finding a form of DE that is **entirely expressed in the basis of subspace Σ** in (13), i.e. make disappear any explicit dependence from x_0, x_5 by, as in the standard procedure, splitting the Dirac spinor ψ into two Pauli spinors φ, χ .^{2,4-6} By inspection of STR DE operator in the form (Σ) below, it becomes clear that in addition to φ, χ the form of ψ we are looking for has to depend explicitly only on x_0 and x_5 that will eventually eliminate the x_{50} factor (below $P_j x^j = P_j x_j x^0 = \mathbf{P}x_0 = \mathbf{P}x_5 x_5 x_0 = P_j \sigma_j x_{50} \equiv (\sigma, \mathbf{P})x_{50}$):

$$\text{DE: } (x^\mu P_\mu - m)\psi \stackrel{(\mathbf{X})}{=} (P_0 x^0 + \mathbf{P}x^0 - m)\psi \stackrel{(\Sigma)}{=} (P_0 x_0 + x_{50}(\sigma, \mathbf{P}) - m)\psi = 0; \psi = \psi(x_0, x_5, \varphi, \chi) \quad (31)$$

An alternative form of DE in (31) is to write it together with the parity transformed $\mathcal{P}(\text{DE})$ as follows:

$$\text{DE} = \frac{1}{2}[\text{DE} + \mathcal{P}(\text{DE}) + \text{DE} - \mathcal{P}(\text{DE})] = \frac{1}{2}[(1 + x_0)\text{DE} + (1 - x_0)\text{DE}] \stackrel{(\mathbf{X})}{\Rightarrow} \frac{1}{2}[(P_0 - m + \mathbf{P})(1 + x_0) - (P_0 + m + \mathbf{P})(1 - x_0)]\psi \stackrel{(\Sigma)}{=} \frac{1}{2}[(P_0 - m + x_5(\sigma, \mathbf{P}))(1 + x_0) - (P_0 + m + x_5(\sigma, \mathbf{P}))(1 - x_0)]\psi = 0 \quad (32)$$

The two projectors $(1 \pm x_0)$ are mutually orthogonal. From the form ($\mathbf{X}$) in (32) we choose provisionally $(1 + x_0)\psi \sim \varphi$; $(1 - x_0)\psi \sim \chi$. From the last form (Σ), in order to be able at the end to remove the x_5 factor from the $x_5(\sigma, \mathbf{P})$ term we need to specify the provisional form to, e.g. $(1 + x_0)\psi \sim \varphi$; $(1 - x_0)\psi \sim x_5 \chi$. Projectors being idempotent within constant factors (therefore we use $\sim$ instead of $=$), can spread from one side onto both sides of an equation, e.g. $(1 + x_0)\psi \sim \varphi \Rightarrow (1 + x_0)\psi \sim (1 + x_0)\varphi$, so that we can write:

$$\psi \sim (1 + x_0)\psi + (1 - x_0)\psi \sim \rho((1 + x_0)\varphi + (1 - x_0)x_5 \chi) = \rho(\varphi + x_5 \chi)(1 + x_0) \quad (33)$$

In (33) ρ is a (pseudo)scalar and the compact form at the right follows from x_0 commuting with the Pauli spinors and anti-commuting with x_5 . Note that the 4-vector x_0 in the projectors corresponds to the diagonal matrix γ_0 in the Dirac basis. This is a general trend in STR, e.g. x_5 appears in the projector of the spinor in

the Weyl basis (12A), while σ_3 appears in the projector of the spinor in the Pauli basis (39, 39'). Applying one of the last two forms in (33) to the last equation in (32), or directly to the DE in (31) yields:

$$\text{DE: } (P_0 x_0 + x_5 (\sigma, \mathbf{P}) - m) \psi = 0 \xRightarrow{(33)} \{[(P_0 - m) \varphi - (\sigma, \mathbf{P}) \chi] + x_5 [(-P_0 - m) \chi + (\sigma, \mathbf{P}) \varphi]\} \rho (1 + x_0) = \rho (1 + x_0) [(P_0 - m) \varphi - (\sigma, \mathbf{P}) \chi] + \rho (1 - x_0) x_5 [(-P_0 - m) \chi + (\sigma, \mathbf{P}) \varphi] = 0 \quad (34)$$

The quantity inside curly brackets cannot be proportional to $(1 - x_0)$, because apart from x_5 all terms in it belong to the subspace Σ . Therefore, it has to be zero. Due to independence of the two terms (1 and x_5 are orthonormal basis elements in (10)), (34) is then equivalent to the standard-looking pair of coupled equations below, entirely written in the subspace Σ :

$$\begin{cases} (P_0 - m) \rho \varphi - (\sigma, \mathbf{P}) \rho \chi = 0 \\ (P_0 + m) \rho \chi - (\sigma, \mathbf{P}) \rho \varphi = 0 \end{cases} \quad (35)$$

Now, from (34) we calculate the Lorentz scalar $\langle \bar{\psi} \psi \rangle$ and the probability density $\langle \psi^\dagger \psi \rangle$ from (19), showing how to extract the scalar part (the final scalar value will appear further down, see (40)) by use of the angled brackets, and at the same time casting new light onto the role of projectors in STR spinors.

$$\begin{aligned} \{\bar{\psi} = \psi^\dagger x_0 = \rho^\dagger (1 + x_0) (\varphi^\dagger + x_5 \chi^\dagger) x_0 = \rho^\dagger (1 + x_0) (\varphi^\dagger - x_5 \chi^\dagger)\} &\Rightarrow \begin{cases} \langle \psi^\dagger \psi \rangle \\ \langle \bar{\psi} \psi \rangle \end{cases} = \rho^\dagger \rho \langle (1 + x_0) (\varphi^\dagger \pm x_5 \chi^\dagger) (\varphi + x_5 \chi) (1 + x_0) \rangle \\ &= 2 \rho^\dagger \rho \langle \varphi^\dagger \varphi \pm \chi^\dagger \chi \rangle \sim \rho^2 \langle \varphi^\dagger \varphi \pm \chi^\dagger \chi \rangle \end{aligned} \quad (36)$$

Apart from a normalization factor, the last expression in (36) correctly yields real numbers, non-negative in the case of the probability density (+ sign). The projectors in (36) are essential to projecting out the cross-product terms comprising x_5 . However, when transformations of the whole DE are at stake, like Lorentz transformations (29), the projectors of the spinor (33) either absorbed or annihilate into the operator in (32). In these cases the spinor behaves as if it did not comprise projectors.

We now derive the **STR Pauli equation** as **non-relativistic** approximation to STR DE in (35) (see also Appendix D). We adopt a procedure from Feynman consisting in first isolating the fast oscillating part of ψ in the factor $\rho = \rho(t)$, then cancelling it after the action of $p_0 = I \hbar \partial_t$:

$$\begin{cases} (P_0 - m) \rho \varphi - (\sigma, \mathbf{P}) \rho \chi = 0 \\ (P_0 + m) \rho \chi - (\sigma, \mathbf{P}) \rho \varphi = 0 \end{cases} \xrightarrow{\rho = e^{-Imt/\hbar}} \begin{cases} (I \hbar \partial_t + e A_0) \varphi - (\sigma, \mathbf{P}) \chi = 0 \\ (I \hbar \partial_t + e A_0 + 2m) \chi - (\sigma, \mathbf{P}) \varphi = 0 \end{cases} \quad (37)$$

For $|(I\hbar\partial_t + eA_0)\chi| \ll 2m|\chi|$ (nonrelativistic regime) the lower of the two coupled equations approximates in lowest order to: $\chi \approx (\boldsymbol{\sigma}, \mathbf{P})\varphi/2m$. I.e. for slow electrons $|\chi| \ll |\varphi|$. Substituting into the upper equation one obtains the Pauli Hamiltonian H_P (below: $\mathbf{P}^2 \equiv (\mathbf{p} + e\mathbf{A}) \cdot (\mathbf{p} + e\mathbf{A}) = (-\hbar(I)\nabla + e\mathbf{A}) \cdot$

$(-\hbar(I)\nabla + e\mathbf{A})$ is a (pseudo) scalar operator):

$$I\hbar\partial_t\varphi = H_P\varphi = \left(\frac{\mathbf{p}^2}{2m} - eA_0 + \frac{\hbar e}{2m}(\boldsymbol{\sigma}, \mathbf{B}) \right) \varphi \quad (38)$$

This is the STR Pauli equation (PE), which looks formally like the standard PE,¹² but here without matrices and with a complex structure surging from the real vector space (10)! The term $\frac{\hbar e}{2mc}(\boldsymbol{\sigma}, \mathbf{B})$, which we met earlier in eq. (17) marks the additional potential energy due to the spin magnetic moment of a slow electron. Of course, eq. (17) can also be derived from (38). The Stern-Gerlach term distinguishes STR PE from the STR Schrödinger equation¹³, which is obtained from (38) by removing it (no spin) and by freezing the spinor φ , let say to spin up. The Pauli spinor φ below expressed as a linear combination of the spin wavefunctions φ_u, φ_d resolves the STR PE (38); it is analogue to the STR Dirac spinor (33), now in the subspace Σ :

$$\varphi = \rho_\varphi(\varphi_u + \boldsymbol{\sigma}_1\varphi_d)(1 + \boldsymbol{\sigma}_3) = \rho_\varphi(\varphi_u - I\boldsymbol{\sigma}_2\varphi_d)(1 + \boldsymbol{\sigma}_3) = \rho_\varphi R_\varphi(1 + \boldsymbol{\sigma}_3); R_\varphi = e^{-I\boldsymbol{\sigma}_2\vartheta_\varphi/2} \quad (39)$$

Again here, the vector in the projector corresponds to the diagonal Pauli matrix $\boldsymbol{\sigma}_3$. The form of spinor with a rotor R_φ comprising the probability amplitudes for spin up and spin down $\cos(\vartheta_\varphi/2) = \varphi_u$; $\sin(\vartheta_\varphi/2) = \varphi_d$, allows us to treat a number of systems more directly⁷ than by first substituting (39) into (38) then solving the resulting pair of coupled equations. The other Pauli spinor χ takes the same form as φ in (39):

$$\chi = \rho_\chi(\chi_u - I\boldsymbol{\sigma}_2\chi_d)(1 + \boldsymbol{\sigma}_3) = \rho_\chi R_\chi(1 + \boldsymbol{\sigma}_3); R_\chi = e^{-I\boldsymbol{\sigma}_2\vartheta_\chi/2} \quad (39')$$

5. Final remarks and conclusions.

As the fast oscillating part of the Dirac spinor can be localized at the scalar prefactor ρ , the *form* of the two Pauli spinors in (39-39') is independent of the nonrelativistic approximation adopted to derive the PE (38). By substituting (39-39') into the couple of equations to the left (prior to the nonrelativistic approximation) in (37) one can write down four coupled first order equations in terms of $\varphi_u, \varphi_d, \chi_u, \chi_d$ and with the x_μ absent; these are equivalent to STR DE. Let us now finish the calculation of the scalar quantities in (36):

$$\rho^2\langle\varphi^\dagger\varphi \pm \chi^\dagger\chi\rangle = \rho^2(\rho_\varphi^2 \pm \rho_\chi^2) \quad (40)$$

The scalar quantities in brackets in (40) are the square moduli of the two Pauli spinors. Finally, the similarity between the first equalities in (39-39') and the Dirac spinor (33) is not a coincidence. For the first, (33) has the form of a 'rotor' $\varphi + x_5\chi$ with $x_5^2 = -1$; however it cannot be a proper rotor, because φ, χ are not scalars. (33) can be seen as the $\omega = 0$ limit of the generalized rotor below, the same rotor that for $\omega = \pi/2$ transforms Dirac's to Weyl's basis (See equations (12A-13A) in the Appendix):

$$\psi = \rho(\varphi + x_5\chi)(1 + x_0) \rightarrow R_\omega\psi = \rho R_\omega[(1 + x_0)\varphi + (1 - x_0)x_5\chi] \quad (41)$$

For the second, there is a symmetry underlying the transition from Dirac's equation (14,15), written in the full basis X in (10) to its equivalent form as a pair of coupled equations in (35), written in Pauli's subspace Σ from (13). The symmetry becomes apparent by the isomorphism between X and the direct product $\Sigma \otimes \Sigma$, $X \leftrightarrow \Sigma \otimes \Sigma$, mentioned in the note to definition (13). From the correspondence:

$$x_0 \leftrightarrow 1 \otimes \sigma_3; x_5 \leftrightarrow 1 \otimes \sigma_1; x_{50} \leftrightarrow 1 \otimes (-I\sigma_2) \quad (42)$$

it becomes clear that what we have achieved by going from (15) to (35) and from (33) to (39-39') is just to reverse the direct products (42)! A kind of reflection, which encapsulates the necessity of $x_5 \leftrightarrow 1 \otimes \sigma_1$.

In conclusion, the definition of STR DE shows its clear physical and geometrical meaning as the most direct quantization of the classical 4-momentum vector with modulus m . Its demonstrated working hints to the expectation that all the formal machinery developed in nine decades to handle DE and its generalizations, adapt easily to the STAR formalism. In many instances, results are obtained more directly than in standard DE, as shown for the $\mathcal{T}, \mathcal{C}, \mathcal{CPT}$ symmetries or the Lorentz covariance in (22-29). Spacetime-reflection stands on clear physical grounds in contrast to the abstract 'internal degrees of freedom' represented by γ -matrices; therefore, it becomes relevant to question the role of the same matrices in areas of modern physics treating them as a fundamental representation. The STR formalism with its inborn distinction between 3D boost and rotation/spin vectors, can potentially affect the formalism of many areas of physics. By developing Dirac's ideas we come to recognize handedness/reflection as fundamental, on equal footing with spacetime.

Appendix

A) Commutation relations of boost and rotor generators and left and right rotors in STR;

B) Symmetries $\mathcal{T}$ and $\mathcal{C}$ in STR revisited;

C) STR Weyl spinors;

D) Derivation of eq. (17) and the Lagrangian of STR DE.

A) More about STAR rotor and boost generators. We first calculate the commutator $[x, p]$ (below $x =$

$$x_\mu x^\mu = x^\mu x_\mu; p = I\hbar\nabla = I\hbar x^\mu \partial_\mu; \nabla = x^{j0} \partial_{x_j} = \mathbf{x}_j \partial / \partial x_j; \mathbf{L} = \mathbf{x} \times \mathbf{p}; (\boldsymbol{\sigma}, \mathbf{L}) = \boldsymbol{\sigma} = \boldsymbol{\sigma}_j \mathbf{x}_j L_j \mathbf{x}_j = L_j \boldsymbol{\sigma}_j):$$

$$\begin{aligned} [x, p] &= I\hbar(x\nabla - \nabla x) = I\hbar(x \cdot \nabla - \nabla \cdot x + x \wedge \nabla - \nabla \wedge x) = I\hbar(-(\nabla \cdot x) + 2x \wedge \nabla) = I\hbar[-\partial_\mu(x^\mu) + \\ &2x^{j0}(x_j \partial_0 - x_0 \partial_j) + 2x^{jk}(x_j \partial_k - x_k \partial_j)] = I\hbar[-4 + 2\mathbf{x}_j(x_0 \partial_{x_j} + x_j \partial_0) + 2\epsilon_{jkl} I \boldsymbol{\sigma}_l(x_j \partial_{x_k} - x_k \partial_{x_j})] = \\ &I\hbar[-4 + 2(x_0 \nabla + \mathbf{x} \partial_0) + 2I(\boldsymbol{\sigma}, (\mathbf{x} \times \nabla))] = -4I\hbar + 2(\mathbf{x} p_0 - x_0 \mathbf{p}) - 2I(\boldsymbol{\sigma}, \mathbf{L}) = 4I\hbar[-1 + \\ &\mathbf{K}_j(x_j \partial_t + t \partial_{x_j}) - \epsilon_{jkl} \mathbf{J}_l(x_j \partial_{x_k} - x_k \partial_{x_j})] = 4I\hbar[-1 + \mathbf{K} + \mathbf{J}] \end{aligned} \quad (1A)$$

The commutator (1A) unites (minus) dimensionality (cumulative ‘uncertainty relation’), boost $\mathbf{K}$ and rotation $\mathbf{J}$ generators with $\mathbf{K}_j \equiv x^{j0}/2 \equiv \mathbf{x}_j/2$, $\mathbf{J}_l \equiv \epsilon_{jkl} x^{jk}/2 = -\epsilon_{jkl} \mathbf{x}_{jk}/2 = -\epsilon_{jkl} \boldsymbol{\sigma}_{jk}/2 = -I \boldsymbol{\sigma}_l/2$, which generate proper Lorentz transformations. $\mathbf{K}_j, \mathbf{J}_j$ in (1A) can be considered as directors for the Killing vectors (the components $x_j \partial_t + t \partial_{x_j}$ and $x_j \partial_{x_k} - x_k \partial_{x_j}$ are components of the Killing vector in spacetime). Notice the similarity with the electric and magnetic fields from the potentials in (16).

The commutators of boost and rotor generators $\mathbf{K}_j, \mathbf{J}_j$ from (1A) are:

$$\begin{aligned} [\mathbf{J}_j, \mathbf{J}_k] &= -\frac{1}{2} \boldsymbol{\sigma}_{jk} = -\frac{1}{2} \epsilon_{jkl} I \boldsymbol{\sigma}_l = \epsilon_{jkl} \mathbf{J}_l; [\mathbf{K}_j, \mathbf{K}_k] = \frac{1}{2} \mathbf{x}_{jk} = -\epsilon_{jkl} \mathbf{J}_l; [\mathbf{J}_j, \mathbf{K}_k] = -\frac{1}{2} I x_5 \boldsymbol{\sigma}_{jk} = \frac{1}{2} \epsilon_{jkl} \mathbf{x}_l = \\ &\epsilon_{jkl} \mathbf{K}_l \end{aligned} \quad (2A)$$

By the definition of $\mathbf{J}_j, \mathbf{K}_j$ it is clear that they do not comprise x_5 . The rotor-boost space can split into *right* and *left* handed *disjoint* subspaces, S_{+j} and S_{-j} , illustrating the double coverage of the group $SO(1,3)$ by the spinor group $SU(2)$, $SO(1,3) = SU(2) \otimes SU(2)$ (where $=$ stands for *isomorphic to*):

$$\begin{aligned} \{S_{\pm j} \equiv \frac{1}{2}(\mathbf{J}_j \pm I \mathbf{K}_j) = \frac{1}{2} \mathbf{J}_j(1 \mp x_5)\} &\Rightarrow \{[S_{\pm j}, S_{\pm k}] = \frac{1}{4} \epsilon_{jkl} \mathbf{J}_l(1 \mp x_5)^2 = \frac{1}{2} \epsilon_{jkl} \mathbf{J}_l(1 \mp x_5) = \\ &\epsilon_{jkl} S_{\pm l}; [S_{+j}, S_{-k}] = 0\} \end{aligned} \quad (3A)$$

Notice the entrance with I of x_5 into $S_{\pm j}$.

B) Symmetries $\mathcal{T}$ and $\mathcal{C}$ in STR revisited.

As the alternative forms of $\mathcal{T}$ and $\mathcal{C}$ symmetries in STR resemble the corresponding symmetries in the standard formalism, we need to refresh few things from the last. In contrast to the NonRelativistic QM (NRQM) the operations of Hermite ($\dagger$) and complex ($*$) conjugate are not uniquely related in RQM. E.g. in the Dirac basis, *spacetime signature* defines $\dagger: \gamma^{\mu\dagger} = \gamma^0 \gamma^{\mu} \gamma^0$; had we used the NR prescript, transpose & complex conjugate, we would have found the *incorrect* result for $(\gamma^{\mu=1,3})^{\dagger} = \overline{\gamma^{\mu\dagger}} = \gamma^{\mu}$ and the correct result for $\mu = 0, 2$ (T stands for Transpose). On the other side, when it comes to $\gamma^5 = i\gamma^{0123}$ the prescript $\gamma^{5\dagger} = \overline{\gamma^5} \gamma^5 \gamma^0 = -\gamma^5$ gives the *wrong* result; here we had better used the NR prescript! This opportunistic approach relates to the non-covariance of $\dagger$ as well as to the definition of γ matrices in the Dirac basis: $\{\gamma^0 = 1 \otimes \sigma^3; \gamma^j = \sigma^j \otimes i\sigma^2\} \Rightarrow \gamma^5 = 1 \otimes \sigma^1$, with σ^2 being the only imaginary matrix. Keeping *complex conjugation* in RQM is useful as it can provide antiunitary operations that as shown by Wigner¹⁴ characterize $\mathcal{T}$ in both NRQM & RQM. In the presence of spin 1/2, $\mathcal{T}$ flips spin.

In contrast to the standard formalism, the definition of Hermite conjugate (12) in STR applies to *all* elements of X . We can realize in STR a transformation similar to ($*$) by allowing for *improper* rotations; we define below one improper rotation (reflection onto a space axis) – a *K-conjugation* by:

$$K: \{x^{\tau} \rightarrow Kx^{\tau}K \equiv x^{\tau K} \text{ with } x^{2K} = -x^2; x^{(\tau \neq 2)K} = x^{\tau}; \tau = 0, 1, 2, 3, 5\} \quad (4A)$$

Few examples of the action of K onto different elements of the bases in (10, 13) are:

$$K1K = 1; KIK = -I; Kx^{13}K = x^{13} = I\sigma^2; K\sigma^jK = (-1)^{j-1}\sigma^j; j = 1, 2, 3 \quad (5A)$$

Therefore, K will not change (respectively, will reverse) the components of position x_{μ} (momentum $I\partial_{\mu}$).

With this preparation in place, we can now discuss the alternative $\mathcal{T}'$ and $\mathcal{C}'$ symmetries of STR DE (15).

Time reversal $\mathcal{T}': x_0 \rightarrow -x_0 \Leftrightarrow \partial_0 \rightarrow \partial'_0 = -\partial_0$. As in the standard approach, we apply an operator $T' = UK$ comprising the K -conjugation from (4A) and a unitary operator, here $U = x^{13}, U^{-1} = x^{31}$ (below $\hbar = 1$):

$$\begin{aligned} \mathcal{T}': (P - m)\psi = 0 \rightarrow T'[(P - m)\psi] &= UK[x^{\mu}(I\partial_{\mu} + eA_{\mu}) - m]\psi = x^{13}[x^{\mu K}(-I\partial_{\mu} + eA_{\mu}) - m]K\psi = \\ &= x^{13}[x^0(I\partial'_0 + eA_0) - x^1(I\partial_1 - eA_1) + x^2(I\partial_2 - eA_2) - x^3(I\partial_3 - eA_3) - m]K\psi = [x^0(I\partial'_0 + eA_0) + \\ &+ x^j(I\partial_j - eA_j) - m]x^{13}K\psi = 0 \Rightarrow \psi_{T'} \equiv T'\psi = x^{13}K\psi = I\sigma_2 K\psi \equiv I\sigma_2 \psi^K \end{aligned} \quad (6A)$$

Note that the T' -transformed *operator* part of DE in (6A) is identical to the T -transformed operator part in (22); this is not the case for the corresponding spinors. It is evident that T' is antiunitary: $T'IT'^{-1} =$

$UKIKU^{-1} = -I; T'^2 = UKUK = UU^K = x^{13}x^{13} = -1 = I\sigma_2 I\sigma_2$. As shown in (33, 39, 39') the Dirac

spinor ψ (and its K -conjugate ψ^K) can be expressed as linear combinations of two Pauli spinors, each of the form $\rho_\varphi(\varphi_u - I\sigma_2\varphi_d)(1 + \sigma_3)$, with φ_u, φ_d proportional to the probability amplitudes for spin up and spin down, respectively. Within a total phase factor, the time reversal $\psi \rightarrow \psi_T = I\sigma_2\psi^K = \rho_\varphi(I\sigma_2\varphi_u + \varphi_d)(1 + \sigma_3)$ in (6A) swaps the two spins $\varphi_u \leftrightarrow \varphi_d$ for each of the Pauli spinors, as expected.

Charge conjugation C' : $e \rightarrow -e$. First K -conjugate STR DE (15) as in (6A) then left-multiply by x^2 :

$$\begin{aligned} (x^\mu P_\mu - m)\psi &= 0 \xrightarrow{C'} [(x^\mu P_\mu - m)\psi] = x^2 K [(x^\mu P_\mu - m)\psi] = x^2 (-x^{\mu\neq 2} (I\partial_\mu - eA_\mu) + \\ x^2 (I\partial_2 - eA_2) - m)\psi^K &= (x^\mu (I\partial_\mu - eA_\mu) - m)x^2\psi^K = 0 \Rightarrow \psi_{C'} \equiv x^2\psi^K \end{aligned} \quad (7A)$$

Note that contrary to $T'T'$, $C'IC'^{-1} = I$; $C'C' = x^2 K x^2 K = 1$. Again the operator parts but not the spinors for the two alternative charge conjugation transformations of STR DE in (25) and (7A) are the same. DE (15) describing the electron, (7A) describes its antiparticle, the *positron*. It is clear that the specific form of the transformed STR DE and ψ in (6A, 7A) depend on the choice of K -conjugation in (4A), in the same way as the corresponding symmetry transformations in the Dirac or Weyl bases of the standard formalism ultimately rely on σ_2 being the only imaginary matrix in the Pauli basis.^{4,5}

Now we can apply all the three symmetries onto STR DE (15), the parity being the same in the two cases.

$$\begin{aligned} \mathcal{C}'\mathcal{P}\mathcal{T}': \text{Applying the three operations in series yields } (\mathcal{C}'\mathcal{P}\mathcal{T}')\text{DE: } (x^\mu(I\partial_\mu + eA_\mu) - m)\psi &= \\ 0 \xrightarrow{T'} x^{13} (x^{\mu K} (-I\partial_\mu + eA_\mu) - m)\psi^K &= 0 \xrightarrow{\mathcal{P}} x^{013} (x^{\mu K} (-I\partial_\mu + eA_\mu) - m)\psi^K = 0 \xrightarrow{C'} x^{2013} (x^\mu (I\partial_\mu + \\ eA_\mu) - m)\psi &= Ix^5 (x^\mu (I\partial_\mu + eA_\mu) - m)\psi = (x^\mu (-I\partial_\mu - eA_\mu) - m)Ix^5\psi = 0; \psi_{C'\mathcal{P}\mathcal{T}'} = Ix^5\psi. \end{aligned} \quad (8A)$$

Both the operator and spinor parts resulting from $\mathcal{C}'\mathcal{P}\mathcal{T}'$ in (9A) and from $\mathcal{C}\mathcal{P}\mathcal{T}$ in (26) are identical.

C) Weyl spinors in STR. STR renders explicit the similarity between the chiral representation of rotors in (3A) and Right and Left handed Weyl spinors that are orthonormal $(1 \pm x_5)/2$ projections of Dirac spinors:

$$\psi = \psi_L + \psi_R = \frac{1}{2}(1 - x_5)\psi + \frac{1}{2}(1 + x_5)\psi \quad (9A)$$

Let see where this form is useful. The Dirac spinor and basis are preferred to describe slow electrons. For fast electrons, $p \gg m$, we can neglect the mass term; then DE in momentum representation takes the form:

$$(p - m)\psi(p) = 0 \xrightarrow{p \gg m} p\psi(p) \approx 0 \quad (10A)$$

If $\psi(p)$ is a solution to the STR DE at the r.h.s. then $x_5\psi(p)$ will also be a solution, x_5 anticommuting with p . Now, proceeding in the same way as we did to recover the form of the Dirac spinor in (33), write $\psi(p)$ as:

$$\psi(p) = \frac{1}{2}[(1 + x_5)\psi(p) + (1 - x_5)\psi(p)] \quad (11A)$$

By precisely the same reasoning as applied to obtain (33), obviously with the exchange $x_0 \rightleftharpoons x_5$, we can write down the following form, comprising the Weyl spinors in (9A):

$$\psi \stackrel{(4A)}{=} \psi_R + \psi_L = \rho[(1 + x_5)(\varphi + x_0\chi) + (1 - x_5)(\varphi + x_0\chi)] = \rho(\varphi + x_0\chi)(1 + x_5) \quad (12A)$$

This form of the spinor is convenient to describe fast electrons or massless fermions. The compact form at the right in (12A) is possible because x_5 commutes with the Pauli spinors and anticommutes with x_0 .

Alternatively, we can complete the spinor transformation (41) related to the change of basis in (13A):

$$\begin{aligned} \psi \rightarrow R_\omega \psi &= \rho R_\omega [(1 + x_0)\varphi + (1 - x_0)x_5\chi] = \rho[(1 + R_\omega x_0 R_\omega^{-1})R_\omega \varphi + (1 - \\ R_\omega x_0 R_\omega^{-1})R_\omega x_5 R_\omega^{-1} R_\omega \chi] &\stackrel{\omega=\pi/2}{=} \rho[(1 + x_5)\varphi - (1 - x_5)x_0\chi] R_\omega = \rho(\varphi - x_0\chi)(1 + x_5) R_\omega \end{aligned} \quad (12'A)$$

In (12'A) R_ω commutes with φ, χ and when the transformed spinor engages to DE it can be removed by a right multiplication with R_ω^{-1} . The spinor (12'A) converts to the form in (12A) by left multiplication with x_5 .

The swap $x_0 \rightleftharpoons x_5$, already shown in (12'A) constitutes the passage from the Dirac (x_0 in the projectors) to Weyl basis (x_5 in the projectors). As mentioned in the main text in the discussion following eq. (30) the change of basis in STAR is achieved by two-sided generalized rotor transformations. Let illustrate it here in relation to the passage from Dirac to Weyl basis. It takes the form:

$$\{x_\tau\} \rightarrow \{x'_\tau\}: x'_\tau = R_\omega x_\tau R_\omega^{-1} = e^{x_{50}\omega/2} x_\tau e^{-x_{50}\omega/2} \stackrel{\omega=\pi/2}{=} \begin{cases} x'_0 = x_5 \\ x'_j = x_j \quad ; \quad \tau = 0,1,2,3,5; j = 1,2,3 \\ x'_5 = -x_0 \end{cases} \quad (13A)$$

Within a sign (not relevant for a basis element) we have realized by (13A) the mentioned transformation of basis. Notice that $x_{50}^2 = -1$, which makes exponentiation in (13A) meaningful.

D) Derivation of Eq. (17). For convenience, let us rewrite the squared DE (16) here:

$$\left(KG + e\hbar I(\mathbf{E} + I(\boldsymbol{\sigma}, \mathbf{B})) \right) \psi = 0 \quad (14A)$$

If $\psi = \rho(\varphi + x_5\chi)$ is a solution to DE then it is also a solution to its square, eq. (16A). We have removed the projectors $(1 \pm x_0)$ from the spinor, because there is no free x_0 in (14A). Now we follow the same strategy as earlier and try to write (14A) completely in the Σ subspace with the help of the two Pauli spinors:

$$\begin{aligned} \psi &= \rho(\varphi + x_5\chi) \stackrel{(14A)}{\implies} \left(KG + e\hbar I(x_5(\boldsymbol{\sigma}, \mathbf{E}) + I(\boldsymbol{\sigma}, \mathbf{B})) \right) \rho(\varphi + x_5\chi) = \\ \{ [KG - e\hbar(\boldsymbol{\sigma}, \mathbf{B})]\rho\varphi + e\hbar I(\boldsymbol{\sigma}, \mathbf{E})\rho\chi &= 0 \\ [KG - e\hbar(\boldsymbol{\sigma}, \mathbf{B})]\rho\chi + e\hbar I(\boldsymbol{\sigma}, \mathbf{E})\rho\varphi &= 0 \end{aligned} \quad (15A)$$

The coupled equations (15A) are symmetric relative to the exchange $\varphi \leftrightarrow \chi$; they make a good starting point for investigating the interaction of both $\mathbf{E}$ and $\mathbf{B}$ fields with the electron, with the caution that the solutions found be solutions for DE. As pointed out by Hestenes, Dirac just ignored the $\mathbf{E}$ term in (14A) under the unjustified excuse that it is imaginary. This will be the subject of future work. We take the nonrelativistic limit of (15A). Under the derivation of the Pauli equation we saw that in this limit $|\chi| \ll |\varphi|$. This is expected, as by writing DE in the rest frame of the electron in the momentum representation one finds $\chi = 0$. Therefore, in this regime we can take $\psi \approx \rho\varphi$ (this can be done also after removing the fast oscillations, which we do after (16A)) and the two equations in (15A) decouple:

$$\begin{cases} [\text{KG} - e\hbar(\boldsymbol{\sigma}, \mathbf{B})]\rho\varphi = 0 \\ e\hbar I(\boldsymbol{\sigma}, \mathbf{E})\rho\varphi = 0 \end{cases} \quad (16A)$$

In the following we will look only at the top equation. As done earlier we isolate the fast oscillations in $\rho = e^{-Imt/\hbar}$, so that second derivative $\partial_t^2 \varphi$ of φ from the KG term can be ignored after differentiating ρ and removing the exponential. We also assume a weak constant EM field and in this way can also ignore the A^2 terms. These simplifications will make treatment easier allowing to focus on few essential properties of DE:

$$\begin{aligned} (\mathbf{p} \cdot \mathbf{p} - m^2 - e\hbar(\boldsymbol{\sigma}, \mathbf{B}))\rho\varphi &= \left(-\hbar^2 \partial_t^2 - \mathbf{p}^2 + e^2(A_0^2 - \mathbf{A}^2) + e\hbar I(2A_0\partial_0 + (\partial_0 A_0) - 2A_j\partial_j - \right. \\ & \left. (\partial_j A_j)) - m^2 - e\hbar(\boldsymbol{\sigma}, \mathbf{B}) \right) \exp(-Imt/\hbar) \varphi \approx \exp(-Imt/\hbar) \left(-\hbar^2 \left(-2\frac{Im}{\hbar} \frac{\partial \varphi}{\partial t} - \frac{m^2}{\hbar^2} + \frac{\partial^2 \varphi}{\partial t^2} \right) - \mathbf{p}^2 + \right. \\ & \left. e\hbar I \left(-2A_0 \frac{Im}{\hbar} + (\partial_t A_0) - 2A_j\partial_j - (\partial_j A_j) \right) - m^2 - e\hbar(\boldsymbol{\sigma}, \mathbf{B}) \right) \varphi \Rightarrow \left\{ I\hbar\partial_t - \frac{\mathbf{p}^2}{2m} + eA_0 - \right. \\ & \left. \frac{e\hbar I}{2m} [2A_j\partial_j + (\partial_j A_j)] - \frac{e\hbar}{2m}(\boldsymbol{\sigma}, \mathbf{B}) \right\} \varphi \approx 0 \end{aligned} \quad (17A)$$

The last equation can be derived from the Pauli equation (38) in the main text, as well. Now, we know that the Schrödinger equation¹² can handle magnetic orbital momentum, which must be the term within square brackets in (17A). Let us render it explicit by trying to express A_j as a function of the components of the magnetic field B_k . Start with $\mathbf{B} = \nabla \times \mathbf{A} \Rightarrow B_j = (\partial_{j+1}A_{j+2}) - (\partial_{j+2}A_{j+1})$ with indices varying cyclically *mod*3, e.g. $B_2 = \partial_3A_1 - \partial_1A_3$. The brackets in e.g. $(\partial_{j+2}A_{j+1})$ mean that the derivative operates only to that term. Taking $A_j = (B_{j+1}x_{j+2} - B_{j+2}x_{j+1})/2$, or $2\mathbf{A} = \mathbf{B} \times \mathbf{x}$ satisfies the equation for $\mathbf{B}$, remembering that the B_j are constant. With this choice the term $(\partial_j A_j) = 0$ in (17A) and $2\hbar I A_j \partial_j = 2\mathbf{A} \cdot \mathbf{p} = \mathbf{B} \times \mathbf{x} \cdot \mathbf{p} = \mathbf{B} \cdot \mathbf{x} \times \mathbf{p} \equiv \mathbf{B} \cdot \mathbf{L}$, where $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ is the orbital angular momentum. The spin vector being $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$, we put it all together into the last equation in (17A) and obtain eq. (17) from the main text:

$$\left[I\hbar\partial_t - \frac{\mathbf{p}^2}{2m} + eA_0 - \frac{e}{2m} \mathbf{B} \cdot (\mathbf{L} + 2\mathbf{S}) \right] \varphi = 0 \quad (18A)$$

The STR DE Lagrangian is:

$$\mathcal{L} = \bar{\Psi}(\mathbf{p} - m)\Psi = \bar{\Psi}(I\hbar\nabla - m)\Psi = \bar{\Psi}(I\hbar x^\mu \partial_\mu - m)\Psi \quad (19A)$$

As in the standard case $\bar{\Psi}$ and Ψ can be considered as independent (see (32,36), where we can make $\langle \bar{\Psi}\Psi \rangle = 0$ by choosing $\rho_\varphi^2 = \rho_\chi^2$ in (36)) and one can recover STR DE from the field equations most directly by differentiating $\mathcal{L}$ in (19A) with respect to $\partial_\mu \bar{\Psi}$ and $\bar{\Psi}$ obtaining (∇ in (19A) acts to the right):

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \bar{\Psi}} - \frac{\delta \mathcal{L}}{\delta \bar{\Psi}} = 0 \Rightarrow \frac{\delta \mathcal{L}}{\delta \bar{\Psi}} = (I\hbar\nabla - m)\Psi = 0 \quad (20A)$$

The conservation of probability currents follows from the gauge symmetry of the Lagrangian and Noether's

$$\text{theorem. Gauge symmetry: } \{\Psi \rightarrow e^{i\theta}\Psi; \mathcal{L} \rightarrow \mathcal{L}\} \Rightarrow \partial_\mu \bar{\Psi} x^\mu \Psi = 0 \quad (21A)$$

References

- ¹ P. A. M. Dirac, "The quantum theory of the electron", Proc. Roy. Soc. Lon. A, 117, 610 (1928).
- ² P. A. M. Dirac, Principles of Quantum Mechanics: International Series of Monographs on Physics 4th ed., Ch. XI (Oxford University Press, Oxford, 1958).
- ³ C. D. Anderson, "The Positive Electron", Phys. Rev. 43, 491 (1933).
- ⁴ A. Zee, "Quantum Field Theory in a Nutshell", Part II, III, VIII, (Princeton University Press, 2nd ed. 2010).
- ⁵ F. Schwabl, "Advanced Quantum Mechanics", Ch. 15 (Springer-Verlag, Berlin, Heidelberg, New York, 2nd ed. 2004).
- ⁶ D. Hestenes, "Space-Time Algebra" (Gordon and Breach, New York, 1966).
- ⁷ C. Doran, A. Lasenby, "Geometric Algebra for Physicists", Ch. 8, (Cambridge University Press, Cambridge 2007).
- ⁸ Grassmann, H. *Die Ausdehnungslehre*. (Enslin, Berlin, 1862).
- ⁹ Clifford, W. K. Applications of Grassmann's extensive algebra. *Am. J. Math.*, **1**, 350 (1878).
- ¹⁰ Dressel, J., Bliokh, K., Nori, F. Spacetime algebra as a powerful tool for electromagnetism, *Physics Reports* 589, 1–71 (2015).
- ¹¹ Kramers, H. A. Théorie générale de la rotation paramagnétique dans les cristaux. *Proceedings of the Royal Netherlands Academy of Arts and Sciences*. **33** (6–10): 959-972 (1930).

¹² Pauli, W. Zur Quantenmechanik des magnetischen Elektrons. *Zeitschrift für Physik*, **43**, 601 (1927).

¹³ Schrödinger, E. An Undulatory Theory of the Mechanics of Atoms and Molecules. *Phys. Rev.*, **28**, 1049–1070 (1926)

¹⁴ Wigner, E. P. “Gruppentheorie und ihre Anwendung auf die Quanten mechanik der Atomspektren”, (Braunschweig, Germany: Friedrich Vieweg und Sohn. 251–254, 1931)