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Dirac Equation *Redux* by Direct Quantization of the 4-Momentum Vector

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Abstract. The Dirac equation (DE) is a cornerstone of quantum physics. We prove that direct quantization of the 4-momentum *vector* p with modulus equal to the rest energy m ($c = 1$) yields a *coordinate-free* and *manifestly covariant* equation $\hat{\mathrm{p}}\psi = m\psi$ with $\hat{\mathrm{p}} = \hbar \mathrm{I}\nabla$. In coordinate representation, this is equivalent to DE with spacetime frame vectors x^{μ} replacing Dirac's γ^{μ}-matrices. Recall that standard DE is *not* manifestly covariant. Adding an independent Hermitian vector x^5 to the spacetime basis $\{\mathrm{x}^{\mu}\}$ allows accommodating the momentum operator in a *real* vector space with a complex structure arising alone from vectors and multivectors. The real vector space generated from the action of the Clifford or geometric product onto the quintet $\{\mathrm{x}^{\mu}, \mathrm{x}^5\}$ has dimension 32, the same as the equivalent real dimension for the space of Dirac matrices. x^5 proves defining for the combined CPT symmetry, distinction of axial vs. polar vectors, left and right handed rotors & spinors, etc. Therefore, we name it *reflector* and $\{\mathrm{x}^{\mu}, \mathrm{x}^5\}$ a basis for *spacetime-reflection* (STR). The pentavector $\mathrm{I} \equiv \mathrm{x}^{05123}$commutes with all elements of STR and depicts the *pseudoscalar* in STR. We develop the formalism by deriving all essential results from the novel STR DE: spin ½ magnetic angular momentum, conserved probability currents, symmetries and nonrelativistic approximation. In simple terms, we demonstrate how Dirac matrices are a redundant representation of spacetime-reflection frame vectors.

1. Introduction

The 93 years old Dirac equation (DE) [1,2] is one of the most far-reaching equations in physics: it is Lorentz covariant, electron spin springs from the equation and it predicted the first instance of antiparticle, the positron, discovered four years later by Anderson [3]. DE for a free electron is ($c = 1$) [4,5]:

$$(\gamma^{\mu} p_{\mu} - m)\psi = 0 \text{ (sum)};\ p_{\mu} = i\hbar\partial_{\mu} \equiv i\hbar\, \partial/\partial x^{\mu}\,; \{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu};\ \mu, \nu = 0,1,2,3 \quad (1)$$

m is the rest mass of the electron and $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the time-space signature. Dirac argued that γ^μ have to be 4×4 complex matrices "…describing some new degrees of freedom belonging to some internal motion in the electron" [1,2]. This remains the standard position today [4,5]. Dirac matrices and their generalizations are considered a fundamental representation to DE and to any domain of theoretical physics where spin 1/2 is relevant [4]. In the following we show that these extra degrees of freedom are redundant.

Based on the isomorphism of the Clifford algebras for Dirac γ^μ matrices and spacetime *vectors*, (see eq. (3) below) Hestenes [6] proposed a new version of DE in the formalism of his spacetime algebra STA [7]. There is no matrix and no imaginary number in STA DE. However, the spin has been put 'by hand' in the equation, thereby diminishing the predictive power and symmetry of STA DE compared to standard DE. The real vector space of STA has dimension 16, which is half of the equivalent real dimension for the space of 4×4 complex matrices. The Clifford algebra of STA is $\mathcal{C}\ell_{(1,3)}$, (1,3) essentially standing for timespace signature.

The Clifford algebras for spacetime frame vectors $\{\mathrm{x}^\mu\}$ and Dirac matrices $\{\gamma^\mu\}$ being the same, the *motivation* for the present work has been to show that direct quantization of the 4-momentum vector p of modulus equal to the rest energy m ($c = 1$), provides the simplest form of Dirac equation: $\hat{\mathrm{p}}\psi = m\psi$. This is a realization of the "square root" for the relativistic invariant $\mathrm{p}^2 = m^2$, with the spinor ψ 'intermediating' between a vectorial operator $\hat{\mathrm{p}}$ and *m*. As shown below, 'internal degrees of freedom' of the electron follow from the equation, extras, like matrices unneeded. This minimal approach together with the manifest covariance of the equation mark immediate improvements over the standard DE. After a swift introduction to the Clifford algebra $\mathcal{C}\ell_{(1,3)}$ over space-time, we expand to the $\mathcal{C}\ell_{(2,3)}$ over spacetime-reflection (STR) in order to accommodate the momentum operator in a *real* vector space. We develop the formalism in STR by demonstrating how it works. Detailed derivation of some additional results in STR appear in the Appendices.

This report comprises new material and some changes relative to deposited preprints [8] with the same title. Apart from rearrangements and enhanced section structure, an expanded analysis of time-reversal symmetry and its geometric meaning appears in Section 4.3 and in App. D. I have also added two tables, Table 1 giving an overview of the notation (this was suggested by one reviewer) and Table 2 summarizing the bilinears of STR DE. I have also slightly changed the form of the STR Dirac spinor, rendering it formally similar to the standard spinor. In addition, following the suggestion of another reviewer it appears much earlier in the report and with a more detailed explanation of its structure in eqs. (17 – 21) as well as in tab. 2.

2. Clifford product in spacetime.

The Clifford or geometric product [6, 7] of two vectors u, v combines the symmetric Hamilton's scalar product with the antisymmetric Grassmann's wedge product [8, 9]; in coordinate-free form it is:

$$\mathrm{uv} = \mathrm{u} \cdot \mathrm{v} + \mathrm{u} \wedge \mathrm{v}; \qquad (\mathrm{uv})\mathrm{w} = \mathrm{u}(\mathrm{vw}) = \mathrm{uvw}$$

$$2\mathrm{u} \cdot \mathrm{v} = (\mathrm{uv} + \mathrm{vu}) = \{\mathrm{u}, \mathrm{v}\}; \qquad 2\mathrm{u} \wedge \mathrm{v} = (\mathrm{uv} - \mathrm{vu}) = [\mathrm{u}, \mathrm{v}] \tag{2}$$

The two parts relate naturally to the anticommutator $\{\,,\}$ and commutator $[\,,]$. The bivector $\mathrm{u} \wedge \mathrm{v} = -\mathrm{v} \wedge \mathrm{u}$ represents the oriented area encompassed by the two vectors. The geometric product (2) is linear and if not zero, it is invertible. When normalized it renders rotors or boosts, the generators of Lorentz transformations. Examples of both will appear in the following. The geometric product of orthonormal frame vectors is:

$$\mathrm{x}^\mu \mathrm{x}^\nu \equiv \mathrm{x}^{\mu\nu} = \mathrm{x}^\mu \cdot \mathrm{x}^\nu + \mathrm{x}^\mu \wedge \mathrm{x}^\nu = \eta^{\mu\nu} + \mathrm{x}^\mu \wedge \mathrm{x}^\nu; \{\mathrm{x}^\mu, \mathrm{x}^\nu\} = 2\eta^{\mu\nu}; \mathrm{u} = u_\mu \mathrm{x}^\mu; \mu, \nu = 0,1,2,3 \tag{3}$$

Upright letters depict vectors; *italics* depict scalars. One could have used e^μ instead of x^μ [7]. From (3) $\mathrm{x}^\mu \cdot \mathrm{x}^\nu = \eta^{\mu\nu}$ defines the timespace signature. Signature can 'lift' indices of frame vectors up and down, thus connecting the reciprocal bases. The coincidence of Clifford algebras for x^μ and Dirac matrices γ^μ leads to STA - spacetime algebra [6, 7]. The real vector space arising from the action of Clifford product onto spacetime vectors (2, 3) has dimension 16 with basis elements comprising: the real scalar unit, four vectors x^μ, six bivectors $\mathrm{x}^{\mu\nu} = \mathrm{x}^\mu \wedge \mathrm{x}^\nu$ $(\mu \neq \nu)$, four trivectors $\mathrm{x}^{\lambda\mu\nu}$ $(\lambda \neq \mu \neq \nu \neq \lambda)$ and one tetravector x^{0123}:

$$\text{basis of STA vector space: } \{1, \mathrm{x}^\lambda, \mathrm{x}^{\lambda\mu}, \mathrm{x}^{\lambda\mu\nu}, \mathrm{x}^{0123}; \lambda, \mu, \nu = 0,1,2,3\}; \qquad \dim 16 \tag{4}$$

From (3) permutations of given orthogonal vectors can at most change the sign, not adding new basis elements. Similarly to the geometric interpretation of basis bivectors as oriented area elements, trivectors (tetra-vectors) in (4) are oriented 3-volume (resp. 4-volume) elements in spacetime, all unitless. Multivectors in STA correspond to tensors, represented by matrices in the standard formalism. Multivectors formed by wedge product alone, known as *blades*, are anti-symmetric relative to the exchange of any two indices. x^{0123} squares to -1 and defines the pseudo-scalar of STA. However, it commutes only with even grade elements of STA (scalars and bivectors) and *anti-commutes* with vectors and trivectors. We will define another real vector space in the following, therefore will not use here a specific symbol for x^{0123}. The STA basis (4) is defined in terms of upper indices, but one can *lift* indices up and down with the help of the appropriate form of spacetime signature and define a reciprocal basis relative to (4). Another operation is that of *reversing* (˜) the order of indices of a multivector, which corresponds to taking the transpose of a matrix. In STA the *Hermite conjugate* (†) of a basis vector is defined by the parity transformation (see eq. (26)) $\mathrm{x}^{\mu\dagger} = \mathrm{x}^0\mathrm{x}^\mu\mathrm{x}^0$; it works *without* shift to the reciprocal basis. The Hermite conjugate of any elements A, B of STA is:

$$\mathrm{A}^\dagger \equiv \mathrm{x}^0\widetilde{\mathrm{A}}\mathrm{x}^0, (\mathrm{A}+\mathrm{B})^\dagger = \mathrm{A}^\dagger + \mathrm{B}^\dagger;\ \text{e.g.} \left(a_{\mu\nu}\mathrm{x}^{\mu\nu}\right)^\dagger = a_{\mu\nu}\mathrm{x}^{0\nu\mu 0}\ (\text{sum over } \mu,\nu);\ a_{\mu\nu} \in \mathbb{R} \tag{5}$$

Alternatively, one could have defined Hermite conjugate by the combination of index lift and reversal, which changes a given basis vector to its reciprocal (co)vector. This is convenient when proving e.g. that the STA basis (4) is *orthonormal*, i.e. that each basis element has modulus 1 and the product of any pair of *different* elements has zero scalar part $\langle\ \ \rangle_0$. E.g. for fixed indices (no sum over repeated indices, so that $\mathrm{x}_\lambda\mathrm{x}^\lambda = 1$):

$$\langle\left(\mathrm{x}^{\lambda\mu}\right)^\dagger\mathrm{x}^{\lambda\nu}\rangle_0 = \langle\mathrm{x}_{\mu\lambda}\mathrm{x}^{\lambda\nu}\rangle_0 = \langle\mathrm{x}_\mu\mathrm{x}^\nu\rangle_0 = \delta_\mu^\nu,\ \text{or}\ \langle\left(\mathrm{x}^\lambda\right)^\dagger\mathrm{x}^{\mu\nu}\rangle_0 = \langle\mathrm{x}_\lambda\eta^{\mu\nu}\rangle_0 + \tfrac{1}{2}\langle\delta_\lambda^\mu\mathrm{x}^\nu - \delta_\lambda^\nu\mathrm{x}^\mu\rangle_0 = 0 \tag{6}$$

Now, the relativistic 4-momentum vector p of modulus m and different forms of its square, coordinate-free and coordinate-bound, are given in STA by ($c = 1$) (sum over repeated indices by default):

$$\mathrm{p} = \mathrm{x}^\mu p_\mu;\ \ \mathrm{pp} = \mathrm{p}\cdot\mathrm{p} + \mathrm{p}\wedge\mathrm{p} = \mathrm{x}^{\mu\nu}p_\mu p_\nu = \eta^{\mu\nu}p_\mu p_\nu = \mathrm{p}\cdot\mathrm{p} = m^2 \tag{7}$$

A relativistic quantum *vector* equation of first order in spacetime derivatives for a particle of mass m is then:

$$\hat{\mathrm{p}}\psi = m\psi \text{ with } \hat{\mathrm{p}} = \hbar i\nabla = \hbar i\mathrm{x}^\mu\partial_\mu;\ \partial_\mu \equiv \partial/\partial x^\mu = \eta_{\mu\nu}\,\partial/\partial x_\nu;\ \text{sum over repeated indices} \tag{8}$$

In most cases we will drop the hat and depict the operators $\hat{\mathrm{p}}, \hat{p}_\mu$ by p, p_μ. Due to the imaginary i entering with the canonical momentum operator, STA's field of scalars has to expand from real to complex numbers. Then, in analogy to the definition of Dirac's γ^5 matrix, we can define x^5 by (note that $\mathrm{x}^{0123} = \mathrm{x}^{3210}$):

$$\mathrm{x}^5 = i\mathrm{x}^{0123};\ \mathrm{x}^5\mathrm{x}^5 = i\mathrm{x}^{0123}i\mathrm{x}^{0123} = 1 \Rightarrow (\mathrm{x}^5)^\dagger = -i(\mathrm{x}^{0123})^\dagger = -i\mathrm{x}^0\mathrm{x}^{3210}\mathrm{x}^0 = \mathrm{x}^5 \tag{9}$$

i commutes with all the elements of STA in (4), therefore x^5 (like γ^5) is Hermitian, as shown. One can certainly build a formalism expanding the STA basis (4) on complex scalars, which as in (9) would *mix* the complex structures arising from scalars and vectors & multivectors. However, we prefer taking an alternative path leading to a *real* vector space with the complex structure springing from vectors & multivectors alone.

3. The real vector space of spacetime-reflection, STR.

Using the same symbol as in (9) we assume a Hermitian vector x^5 to be a *basis frame* vector, on equal footing with $\{\mathrm{x}^\mu\}$, and name the quintet $\{\mathrm{x}^\mu, \mathrm{x}^5\}$ an orthonormal vector basis for *spacetime-reflection*, STR; the reason for the name will become clear in the following. Let X depict the *real* vector space generated by the action of the geometric product onto the basis $\{\mathrm{x}^\mu, \mathrm{x}^5\}$ of STR. An orthonormal basis for X (dim 32) is:

$$\text{X basis:} \left\{1, \mathrm{x}^\mu, \mathrm{x}^{\lambda\mu}, \mathrm{x}^{\lambda\mu\nu}, \mathrm{x}^{0123}, \mathrm{x}^5, \mathrm{x}^{\lambda 5}, \mathrm{x}^{\lambda\mu 5}, \mathrm{x}^{\lambda\mu\nu 5},\ \mathrm{x}^{01235}\right\};\quad \lambda, \mu, \nu = 0,1,2,3;\quad \lambda \neq \mu \neq \nu \neq \lambda \qquad (10)$$

$$\text{STR vector product:}\, \mathrm{x}^\tau \mathrm{x}^\upsilon \equiv \mathrm{x}^{\tau\upsilon} = \mathrm{x}^\tau \cdot \mathrm{x}^\upsilon + \mathrm{x}^\tau \wedge \mathrm{x}^\upsilon = \zeta^{\tau\upsilon} + \mathrm{x}^\tau \wedge \mathrm{x}^\upsilon;\ \tau, \upsilon = 0,1,2,3,5;\ \zeta^{\tau\upsilon} = \mathrm{x}^\tau \cdot \mathrm{x}^\upsilon = (+ - - - +) \qquad (11)$$

The first 16 elements in (10) are the basis elements of STA in (4) [7]. The other 16 elements are obtained by multiplying those with the frame vector x^5. Symbolically we could write $\mathrm{STR} = \mathrm{STA}(1 + \mathrm{x}^5)$. 16 elements in (10) square to –1, allowing for a rich complex structure in X. Of these only the element of highest grade, the pentavector is both *isotropic* (i.e. not privileging any spacetime director) and *commutes* with all elements of X; it constitutes the (geometric) pseudoscalar of STR, depicted by (compare with (9)):

$$\mathrm{I} \equiv \mathrm{x}^{05123} = -\mathrm{x}^{01235};\ \mathrm{I} = \tilde{\mathrm{I}};\ \mathrm{I}^2 = -1;\ \mathrm{I}^\dagger = -\mathrm{x}^{5\dagger}(\mathrm{x}^{0123})^\dagger = -\mathrm{I};\ \mathrm{I}\mathrm{x}^\tau = \mathrm{x}^\tau \mathrm{I};\ \tau = 0,1,2,3,5 \qquad (12)$$

A general element $\mathrm{y} \in \mathrm{X}$ is $\mathrm{y} = \mathrm{x}^\tau(a_\tau + \mathrm{I}b_\tau) + \mathrm{x}^{\tau\upsilon}(c_{\tau\upsilon} + \mathrm{I}d_{\tau\upsilon}); \tau, \upsilon = 0,1,2,3,5; a_\tau, b_\tau, c_{\tau\upsilon}, d_{\tau\upsilon} \in \mathbb{R}$ (there is redundancy; summation over dummy indices is on). The first two terms stand for vectors and tetravectors, respectively of grade 1 and 4. The last two terms stand for scalars & pseudoscalars ($\tau = \upsilon$) and bivectors & trivectors ($\tau \neq \upsilon$), respectively of grade 0, 5, 2, 3. Table 1 summarizes the STR notation; the position vector is $\mathrm{x} = x_\mu \mathrm{x}^\mu$ with the unit of length attached to the scalar components. We adapt (5) to define the Hermite conjugate of STR elements in (10) taking care for x^5 factors, as we did for $\mathrm{I}^\dagger$in (12); for a pure multivector:

$$\mathrm{A} = \mathrm{x}^{\tau\upsilon\omega\cdots}\!: \mathrm{A}^\dagger \equiv (-1)^s \mathrm{x}^0 \tilde{\mathrm{A}} \mathrm{x}^0; s = \text{nr. of}\ \mathrm{x}^5 \text{ in A};\ \text{e.g.:}\ (\mathrm{I}\mathrm{x}^{350})^\dagger = (-1)^2 \mathrm{x}^0 \mathrm{I}\mathrm{x}^{053} \mathrm{x}^0 = -\mathrm{I}\mathrm{x}^{350} \qquad (13)$$

3.1. Subspaces of 3D relative vectors.

Two subspaces of relative vectors in X as well as their union will be of special interest ($j, k = 1,2,3$):

$$\mathbf{X}\!: \left\{1(1), \mathbf{x}_j = \mathbf{x}^j \equiv \mathrm{x}^{j0}(3), \mathbf{x}_{jk} = \mathrm{x}^{kj}(3), \mathbf{x}_{123} = \mathrm{x}^{0123}(1)\right\};\ \text{generators:} \left\{\mathbf{x}_j\right\};\qquad \text{dim8}$$

$$\mathbf{\Sigma}\!: \left\{1(1), \boldsymbol{\sigma}_j = \boldsymbol{\sigma}^j \equiv \mathrm{x}^{j50}\,(3), \boldsymbol{\sigma}_{jk} = \mathbf{x}_{jk} = \epsilon_{jkl}\mathrm{I}\boldsymbol{\sigma}_l(3), \boldsymbol{\sigma}_{123} = \mathrm{I}(1)\right\};\ \text{generators:} \left\{\boldsymbol{\sigma}_j\right\};\ \text{dim8}^{¶} \qquad (14)$$

$$\mathbf{\Xi}\!: \left\{1, \mathrm{x}^5, \mathbf{x}_j, \boldsymbol{\sigma}_j, \mathrm{I}\mathbf{x}_j, \mathrm{I}\boldsymbol{\sigma}_j, \mathrm{I}\mathrm{x}^5, \mathrm{I}\right\};\quad \{\mathbf{X} \cup \mathbf{\Sigma}\} \subset \mathbf{\Xi};\quad \text{generators:} \left\{\mathrm{x}^5, \boldsymbol{\sigma}_j\right\} \text{or} \left\{\mathrm{x}^5, \mathbf{x}_j\right\};\qquad \text{dim16} \qquad (14')$$

¶ Similarly to the standard case [4], a vector space isomorphic to (10) occurs from the direct product $\mathrm{X} = \mathbf{\Sigma} \otimes \mathbf{\Sigma}$ with scalars and the pseudo-scalar $\mathrm{I} = \boldsymbol{\sigma}_{123}$ freely passing through $\otimes$. In addition to providing an alternative route leading to all the results presented here, this representation marks a further argument in favor of expanding ST to STR.

The subspaces $\mathbf{X}, \boldsymbol{\Sigma}$ share their even grade members, i.e. the scalar 1 (grade 0) and the three bivectors $\mathbf{x}_{jk} = \boldsymbol{\sigma}_{jk} = \mathrm{x}^{kj}$ ($j \neq k$ grade 2). We will present the parity transformation further down (eq. (26-27)), but remind that it changes sign to 3D polar vectors (parity-odd) and leaves unchanged axial vectors (parity-even). As shown in eq. (27) the vectors $\mathbf{x}_j$ behave as polar vectors (parity-odd), while $\boldsymbol{\sigma}_j$ behave as axial vectors (parity-even). The subspace of axial vectors, spin and rotor generators $\boldsymbol{\Sigma}$ is isomorphic to the space of Pauli matrices [7]. The subspace of polar vectors and boost generators **X** is the same as the even subspace of Hestenes' STA, in our notation having $\mathbf{x}_{123} = -\mathrm{I}\mathrm{x}^5$ as 'local' pseudoscalar [6,7,10]. The subspace $\boldsymbol{\Xi}$ in (14') comprises the Lorentz group and is home to Dirac spinors, as explained further down. Notice that the vectors $\mathbf{x}_j, \boldsymbol{\sigma}_j$ are Hermitian. In passing, the *bivectors* $\mathrm{x}^{j5} = \boldsymbol{\sigma}^j \mathrm{x}^0$ are also Hermitian and parity-even, see (27).

3.2. Momentum quantization in STR – the STR DE.

Now we can write the ***momentum quantization*** equation in STR (compare with (8)), obtaining the **STR DE**:

$$\mathrm{p}\psi = m\psi \text{ with } \mathrm{p} = \hbar \mathrm{I}\nabla = \hbar \mathrm{I}\mathrm{x}^{\mu}\partial_{\mu};\ \partial_{\mu} \equiv \partial/\partial x^{\mu} = \eta_{\mu\nu}\,\partial/\partial x_{\nu};\ \text{sum over repeated indices} \tag{15}$$

∇ is both coordinate-free and shorthand for $\mathrm{x}^{\mu}\partial_{\mu}$, therefore one can avoid the Feynman slash notation! ψ must have four components, i.e. it is a 4-spinor because of the four spacetime dimensions leading to four projectors (see eqs. (19-21) below). $\mathrm{p} = \mathrm{I}\hbar\nabla$ is a tetravector operator in X containing x^5. For the electron with charge e, in the presence of an electromagnetic 4-potential A, (15) generalizes to:

$$\mathrm{P}\psi = m\psi \quad \text{where } \mathrm{P} \equiv \mathrm{p} + e\mathrm{A} = \hbar \mathrm{I}\nabla + e\mathrm{A} = \mathrm{x}^{\mu}\left(\hbar \mathrm{I}\partial_{\mu} + eA_{\mu}\right) \tag{16}$$

This is the STR DE minimally coupled to electromagnetic (EM) field. The first equations in (15), (16) are frame-free, which underlines the manifest covariance of STR DE, as detailed further down by equation (37). The vector-operator ∇ obeys the chain rule $\nabla\mathrm{A} = \mathrm{x}^{\mu\nu}\partial_{\mu}A_{\nu} = \mathrm{x}^{\mu\nu}\left[\left(\partial_{\mu}A_{\nu}\right) + A_{\nu}\partial_{\mu}\right]$ and we need generalize the commutator in (2), as illustrated in Appendix A for $[\mathrm{x}, \mathrm{p}]$ (position-momentum operators). The generators of the Lorentz group appear naturally in $[\mathrm{x}, \mathrm{p}]$. Table 1 summarizes the notation in the STR formalism.

Multivector	Space X [(a)]	Spacetime [(b)]	Subspace $\mathbf{X}$ [(c)]	Subspace $\boldsymbol{\Sigma}$ [(c)]
	$(\tau, \upsilon = 0,1,2,3,5)$	$(\mu, \nu = 0,1,2,3)$	$(j, k = 1,2,3)$	$(j, k = 1,2,3)$

Scalars [d]	$\mathrm{x}^\tau \cdot \mathrm{x}^\upsilon = \zeta^{\tau\upsilon}$	$\mathrm{x}^\mu \cdot \mathrm{x}^\nu = \eta^{\mu\nu}$	$\mathbf{x}_j \cdot \mathbf{x}_k = \delta_{jk}$	$\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k = \delta_{jk}$
Basis vectors	x^τ (one could use e^τ)	x^μ (or e^μ, etc. not used here)	$\mathbf{x}_j = \mathbf{x}^j = \mathrm{x}^{j0}$ (...$\mathbf{e}_j$)	$\boldsymbol{\sigma}_j = \boldsymbol{\sigma}^j = \mathrm{x}^{j50} = -\mathrm{x}^5\mathbf{x}^j$
Vectors [e]	$\mathrm{x}' = x_\tau \mathrm{x}^\tau$	$\mathrm{x} = x_\mu \mathrm{x}^\mu; \mathrm{p} = p_\mu \mathrm{x}^\mu; \mathrm{A} = A_\mu \mathrm{x}^\mu$	$\mathbf{x} = x_j \mathbf{x}_j;\ \mathbf{E} = E_j \mathbf{x}_j$	$(\boldsymbol{\sigma}, \mathbf{B}) \equiv B_j \boldsymbol{\sigma}_j;\ \mathbf{s} = \frac{\hbar}{2}\boldsymbol{\sigma}_3$
Basis bivectors	$\mathrm{x}^\tau \wedge \mathrm{x}^\upsilon$	$\mathrm{x}^\mu \wedge \mathrm{x}^\nu$	$\mathbf{x}_j \wedge \mathbf{x}_k = \mathrm{x}^k \wedge \mathrm{x}^j = \boldsymbol{\sigma}_j \wedge \boldsymbol{\sigma}_k = \mathrm{I}\varepsilon_{jkl}\boldsymbol{\sigma}_l$	
Bivectors [f]	x^{50} in $\mathrm{R}_\omega = e^{-\mathrm{x}^{50}\frac{\omega}{2}}$	$\mathrm{F} = (\nabla \wedge \mathrm{A})$	$\mathrm{F} = \mathbf{E} + \mathrm{I}(\boldsymbol{\sigma}, \mathbf{B})$	
Trivectors	$\mathrm{I}\mathrm{x}^\tau \wedge \mathrm{x}^\upsilon$	$\mathrm{I}\mathrm{x}^{\mu 5}$	$\mathbf{x}_{123} = -\mathrm{I}\mathrm{x}^5$	$\boldsymbol{\sigma}_{123} = \mathrm{I}$
Tetravectors	$\mathrm{I}\mathrm{x}^\tau, \hat{\mathrm{p}} = \mathrm{I}\hbar\nabla = \mathrm{I}\hbar\mathrm{x}^\mu\partial_\mu$	$\mathrm{x}^{0123} = -\mathrm{I}\mathrm{x}^5$	-	-
Pseudoscalar	$\mathrm{I} = \mathrm{x}^{05123}$	$\mathrm{I}\mathrm{x}^5 = -\mathrm{x}^{0123}$	$\mathrm{I}\mathrm{x}^5 = -\mathbf{x}_{123}$	$\mathrm{I} = \boldsymbol{\sigma}_{123}$

Table 1. Summary of notation in STR. Upright letters depict vectorial quantities, letters in *italics* depict real scalars.

[a] From the definition of the orthonormal basis for the full STR vector space X in Equation (10);

[b] From Equations (3, 4);

[c] From Equation (14);

[d] The signature scalars for the different spaces in STR: $\zeta^{\tau\upsilon}(+---+)$, $\eta^{\mu\nu}(+---)$, $\delta_{jk}(+++)$;

[e] Examples of shown vectors: the 4-position vector $\mathrm{x} = x_\mu \mathrm{x}^\mu$, 4-momentum p, EM 4-potential A, relative position $\mathbf{x}$, electrical field $\mathbf{E}$, magnetic field as an axial vector $(\boldsymbol{\sigma}, \mathbf{B})$ and the spin vector $\mathbf{s}$ along $\boldsymbol{\sigma}_3$;

[f] The generalized rotor R_ω is relevant e.g. in passing from the Dirac to the Weyl basis (see eq. (7A) in Appendix B). The invariant bivector Faraday F is shown in two forms, see eq. (22, 22'). In $(\nabla \wedge \mathrm{A})$ the operator ∇ is confined within the brackets.

The form of the STR Dirac spinor ψ as expressed by two Pauli spinors φ, χ resembles the standard case in the Dirac basis, but of course, without any matrices here (as we will see further down, Lorentz covariance eq. (37) and charge conjugation symmetry eq. (34) impose $\psi \in \boldsymbol{\Xi}$ with $\boldsymbol{\Xi}$ already presented in (14')):

$$\psi = \tfrac{1}{2}[(1 + \mathrm{x}^0)\psi + (1 - \mathrm{x}^0)\psi] \equiv 2(\varphi + \chi);\ \text{with}\ \varphi \equiv \tfrac{1}{4}(1 + \mathrm{x}^0)\psi;\ \chi \equiv \tfrac{1}{4}(1 - \mathrm{x}^0)\psi;\ \ \varphi, \mathrm{x}^5\chi \in \boldsymbol{\Sigma} \qquad (17)$$

It is useful to explicate projectors in the form:

$$\psi = (1 + \mathrm{x}^0)\varphi + (1 - \mathrm{x}^0)\chi \qquad (17')$$

As standard, depending on the physical problem at hand one may choose to explicate a scalar + pseudoscalar (grade 0&5) factor in ψ. E.g., in the case of free field, one expands ψ in plane waves of positive and negative energy and a constant spinor (with p the 4-momentum vector and s the spin degrees of freedom):

$$\psi_+ = e^{-\mathrm{I}\mathrm{p}\cdot\mathrm{x}/\hbar}\mathrm{u}(\mathrm{p}, s)\ \text{ and }\ \psi_- = e^{\mathrm{I}\mathrm{p}\cdot\mathrm{x}/\hbar}\mathrm{v}(\mathrm{p}, s) \qquad \mathrm{u}(\mathrm{p}, s), \mathrm{v}(\mathrm{p}, s)\ \text{satisfy DE} \qquad (18)$$

Or, when deriving the Pauli equation as non-relativistic approximation to DE, following Feynman we split the Dirac spinor $\psi = \rho\psi_P$, isolating in ρ the fast oscillating part and in ψ_P the Pauli spinors proper φ_P, χ_P:

$$\psi = \tfrac{1}{2}\rho[(1+\mathrm{x}^0)\psi_P + (1-\mathrm{x}^0)\psi_P] = e^{-\mathrm{I}mt/\hbar}(\varphi_P + \chi_P) \tag{19}$$

We will adopt in the following a form of the two Pauli spinors φ and χ that is analogous to (17), now in the Pauli basis. φ, χ, not being limited to the slow particle regime are more general than φ_P, χ_P above. The Pauli spinor representation for spin up and spin down, as in STA [7], are 1 and $-\mathrm{I}\boldsymbol{\sigma}_2$, respectively (see also [8]). The probability amplitudes for spin up and spin down are proportional to $\varphi_u, \varphi_d;\ \chi_u, \chi_d$:

$$\varphi = (1+\boldsymbol{\sigma}_3)\varphi_u + (1-\boldsymbol{\sigma}_3)(-\mathrm{I}\boldsymbol{\sigma}_2)\varphi_d = \rho_\varphi \mathrm{R}_\varphi(1+\boldsymbol{\sigma}_3)$$

$$\text{with } \mathrm{R}_\varphi = e^{-\mathrm{I}\boldsymbol{\sigma}_2\varphi/2} \quad \text{and} \quad \rho_\varphi \cos\tfrac{\varphi}{2} \sim \varphi_u;\ \ \rho_\varphi \sin\tfrac{\varphi}{2} \sim \varphi_d \tag{20}$$

$$\text{Similarly: } \mathrm{x}^5\chi = \rho_\chi \mathrm{R}_\chi(1+\boldsymbol{\sigma}_3);\quad \mathrm{R}_\chi = e^{-\mathrm{I}\boldsymbol{\sigma}_2\chi/2} \quad \text{and} \quad \rho_\chi \cos\tfrac{\chi}{2} \sim \chi_u;\quad \rho_\chi \sin\tfrac{\chi}{2} \sim \chi_d \tag{20'}$$

Note that φ, χ (upright) are spinors, while φ, χ (*italics*) are angles; ρ_φ, ρ_χ have grade 0&5 and may comprise normalization factors. The overall result of (17', 20, 20') is a Dirac spinor with four projectors (note, χ anti-commutes with x^0, i.e. $-\mathrm{x}^0\chi = \chi\mathrm{x}^0$):

$$\psi \equiv (1+\mathrm{x}^0)\varphi + (1-\mathrm{x}^0)\chi = (\varphi+\chi)(1+\mathrm{x}^0);\quad \varphi = \rho_\varphi \mathrm{R}_\varphi(1+\boldsymbol{\sigma}_3);\quad \mathrm{x}^5\chi = \rho_\chi \mathrm{R}_\chi(1+\boldsymbol{\sigma}_3) \tag{21}$$

4. STR at work

With the above preliminaries in place, it is instructive to start by showing how in the presence of external EM field the spin magnetic moment of the electron arises from spacetime-reflection.

4.1. The square of the STR DE.

$$\mathrm{P}\psi = m\psi \Rightarrow \mathrm{PP}\psi = m^2\psi \Rightarrow [(\hbar\mathrm{I}\nabla + e\mathrm{A})(\hbar\mathrm{I}\nabla + e\mathrm{A}) - m^2]\psi = \Big(\eta^{\mu\nu}(\hbar\mathrm{I}\partial_\mu + eA_\mu)(\hbar\mathrm{I}\partial_\nu + eA_\nu) - m^2 + e\hbar\mathrm{I}(\nabla\wedge\mathrm{A} + \mathrm{A}\wedge\nabla)\Big)\psi = [\mathrm{KG} + e\hbar\mathrm{I}[(\nabla\wedge\mathrm{A})]]\psi = \Big[\mathrm{KG} + e\hbar\mathrm{I}[-(\boldsymbol{\nabla}A_0) - (\partial_0\mathbf{A}) - \mathrm{I}\mathrm{x}^5(\boldsymbol{\nabla}\times\mathbf{A})]\Big]\psi = [\mathrm{KG} + e\hbar\mathrm{I}(\mathbf{E} - \mathrm{I}\mathrm{x}^5\mathbf{B})]\psi \equiv [\mathrm{KG} + e\hbar\mathrm{I}[\mathbf{E} + \mathrm{I}(\boldsymbol{\sigma},\mathbf{B})]]\psi \equiv (\mathrm{KG} + e\hbar\mathrm{I}\mathrm{F})\psi = 0 \tag{22}$$

Brackets in e.g. $(\nabla\wedge\mathrm{A})$ or $(\boldsymbol{\nabla}A_0)$ confine the action of the operator;

$$\mathbf{A} = A_j\mathbf{x}_j \text{ (vector potential)};\ \mathbf{E} = E_j\mathbf{x}_j \text{ (electric field, polar 3D vector)};\ (\boldsymbol{\sigma},\mathbf{B}) \equiv \boldsymbol{\sigma}_j B_j = -\mathrm{x}^5\mathbf{B} \text{ (magnetic field, axial 3D vector)};\ \mathrm{F} \equiv \mathbf{E} + \mathrm{I}(\boldsymbol{\sigma},\mathbf{B}) = \nabla\wedge\mathrm{A} + \mathrm{A}\wedge\nabla = (\nabla\wedge\mathrm{A}) \text{ (Faraday)} \tag{22'}$$

KG stands for the Klein-Gordon term; it comprises grade 0, 5, 0&5 components, including operators. The

x^5-independent term $(\nabla \wedge \mathrm{A}) = \mathbf{E} + \mathrm{I}(\boldsymbol{\sigma}, \mathbf{B})$ is the *Faraday* F, depicting the relativistic invariant *EM field strength* experienced by the electron, as marked by the prefactor $e\hbar$. F is a tensor in the standard formalism [4, 5]; in STR and STA it is a 4D bivector as clearly seen in (22') [7, 11]. The term $e\hbar$IF distinguishes the squared DE from the KG equation. It represents the '*internal degrees of freedom*' of the electron – the spin, interacting with the EM field. Indeed, in the nonrelativistic regime, equation (22) (or the STR Pauli equation (42)) yields equation (23) below, as shown in Appendix C (orbital and spin angular momentum vectors: $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$; $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$. $\mathbf{S}$ is the *symbolic* Pauli's spin operator in STR – meaningful only as part of a product, e.g. $\mathbf{B} \cdot 2\mathbf{S} \equiv \hbar(\boldsymbol{\sigma}, \mathbf{B}) = \hbar B_j \boldsymbol{\sigma}_j$; φ is a Pauli spinor; for a slow electron $\psi \approx \varphi$, see (41-42), (11A-12A)):

$$(\mathrm{P} \cdot \mathrm{P} - m^2 + e\hbar\mathrm{IF})\psi = 0 \xRightarrow{\text{nonrelativistic approx.}} \left[\mathrm{I}\hbar\partial_t - \frac{\mathbf{p}^2}{2m} + eA_0 - \frac{e}{2m}\mathbf{B} \cdot (\mathbf{L} + 2\mathbf{S})\right]\varphi = 0 \tag{23}$$

This is the famous prediction from DE that the unit of spin angular momentum interacts twice as strongly with the magnetic field as the unit of orbital angular momentum. The derivation of (22-23) proves that spin springs from 4-momentum in spacetime-reflection, without any preconceived internal degrees of freedom. The spin magnetic moment of the electron from (23) has modulus $\hbar|e|/2m_e$, a factor ~1.00116 smaller than the experimental figure, the gap arising from QED corrections, beyond the scope of DE [4, 5].

4.2. Conserved currents [2, 4] **of STR DE.**

From (13, 14) $\mathbf{x}_j^\dagger = \mathbf{x}_j$, which simplifies the derivation of the Hermite conjugate for DE and the Dirac conjugate $\bar{\psi}$ for the spinor ψ:

$$(\mathrm{P} - m)\psi = \left[\mathrm{x}^\mu\left(\hbar\mathrm{I}\partial_\mu + eA_\mu\right) - m\right]\psi = \left[(\hbar\mathrm{I}\partial_0 + eA_0) + \mathbf{x}_j\left(\hbar\mathrm{I}\partial_j + eA_j\right) - m\mathrm{x}^0\right]\mathrm{x}^0\psi = 0$$

$$\xrightarrow{\dagger} \psi^\dagger\mathrm{x}^0\left[(-\hbar\mathrm{I}\partial_0 + eA_0) + \mathbf{x}_j\left(-\hbar\mathrm{I}\partial_j + eA_j\right) - m\mathrm{x}^0\right] \equiv \bar{\psi}\left[\mathrm{x}^\mu\left(-\hbar\mathrm{I}\partial_\mu + eA_\mu\right) - m\right]\mathrm{x}^0 = 0 \tag{24}$$

After Hermite conjugation †, ∂_μ act to the left. Left-multiply the DE by $\bar{\psi} = \psi^\dagger\mathrm{x}^0$ and right-multiply the last equation in (24) by $\bar{\psi}^\dagger = \mathrm{x}^0\psi$; then subtract it from the first, obtaining the conservation of probability current (the angled brackets below remove the projectors, e.g. for $\alpha \in \mathrm{X}$, $\langle\alpha(1 + \mathrm{x}^0)\rangle = \alpha$):

$$\left(\partial_\mu\bar{\psi}\right)\mathrm{x}^\mu\psi + \bar{\psi}\mathrm{x}^\mu\left(\partial_\mu\psi\right) = \partial_\mu(\bar{\psi}\mathrm{x}^\mu\psi) = 0 \text{ with probability density } \langle\bar{\psi}\mathrm{x}^0\psi\rangle = \langle\psi^\dagger\psi\rangle \geq 0 \tag{25}$$

Adapting the form of φ, χ from (20, 20') one can go further with the calculation in the Pauli basis, eventually deriving expressions for the components $\langle\bar{\psi}\mathrm{x}^j\psi\rangle$ consisting of only the probability amplitudes $\varphi_u, \varphi_d\ \chi_u, \chi_d$

(not shown here). Table 2 summarizes the Dirac bilinears of STR DE.

Bilinear	Standard form	STR form	Expanded form in STR (with ψ from eq. (17))
Scalar	$\langle\bar{\psi}\vert\psi\rangle$	$\langle\bar{\psi}\psi\rangle$	$\langle(1+\mathrm{x}^0)(\varphi^\dagger\varphi-\chi^\dagger\chi)\rangle=\rho_\varphi^2-\rho_\chi^2\equiv\rho^2\cos\beta$ (a)
Conserved 4-current	$\langle\bar{\psi}\vert\gamma^\mu\vert\psi\rangle$	$\langle\bar{\psi}\mathrm{x}^\mu\psi\rangle$	$\delta^{\mu0}\rho^2-\langle\delta^{\mu j}(\varphi^\dagger\mathbf{x}_j\chi+\chi^\dagger\mathbf{x}_j\varphi)\rangle$
Tensor / Bivector	$\langle\bar{\psi}\vert\sigma^{\mu\nu}\vert\psi\rangle$ (b)	$\langle\bar{\psi}\mathrm{x}^\mu\wedge\mathrm{x}^\nu\psi\rangle$	$\langle-\varepsilon(\varphi^\dagger\mathbf{x}_j\chi+\chi^\dagger\mathbf{x}_j\varphi)-\mathrm{I}\delta\varepsilon_{jkl}(\varphi^\dagger\boldsymbol{\sigma}_l\varphi-\chi^\dagger\boldsymbol{\sigma}_l\chi)\rangle$ (c)
Pseudo (axial) vector	$\langle\bar{\psi}\vert\gamma^\mu\gamma^5\vert\psi\rangle$	$\langle\bar{\psi}\mathrm{x}^{\mu5}\psi\rangle$	$\langle\delta^{\mu0}\mathrm{x}^5(\varphi^\dagger\chi+\chi^\dagger\varphi)+\delta^{\mu j}(\varphi^\dagger\boldsymbol{\sigma}_j\varphi+\chi^\dagger\boldsymbol{\sigma}_j\chi)\rangle$ (d)
Pseudoscalar	$\langle\bar{\psi}\vert\gamma^5\vert\psi\rangle$	$\langle\bar{\psi}\mathrm{x}^5\psi\cancel{\mathrm{x}^5}\rangle$	$\langle\varphi^\dagger\mathrm{x}^5\chi-\chi^\dagger\mathrm{x}^5\varphi\rangle=\langle R_\varphi^\dagger R_\chi\rangle_0(\rho_\varphi^\dagger\rho_\chi-\rho_\chi^\dagger\rho_\varphi)$ (e)

Table 2. Dirac bilinears in standard and STR formalisms. Expanded forms of the STR Dirac bilinears appear in the last column, in terms of the Pauli spinors (20, 20'). The angled brackets ⟨ ⟩ in STR remove common projectors after completed projection as illustrated in (a). The remaining ⟨ ⟩ in the last column have still 'to handle' projector $(1+\boldsymbol{\sigma}_3)$.

(a) The angle β is standard in the STA literature. As shown, it is defined in STR by the relative moduli of the Pauli spinors, with $\rho^2=\rho^\dagger\rho=\langle\bar{\psi}\mathrm{x}^0\psi\rangle=\langle\psi^\dagger\psi\rangle=\rho_\varphi^\dagger\rho_\varphi+\rho_\chi^\dagger\rho_\chi=\rho_\varphi^2+\rho_\chi^2$. All ρ-s are sums of a scalar and a pseudoscalar! See also footnote (e) below.

(b) The standard antisymmetric traceless tensor is defined by the commutator of Dirac matrices $\sigma^{\mu\nu}\equiv\frac{i}{2}[\gamma^\mu,\gamma^\nu]$;

(c) $\varepsilon\equiv(\delta^{\mu0}-\delta^{0\nu})$ and $\delta\equiv\delta^{\mu j}\delta^{\nu k}$;

(d) As anticipated in eq. (14), $\boldsymbol{\sigma}_j$ are axial, therefore they appear naturally here. For $m=0,$ the axial currents $\bar{\psi}\mathrm{x}^{\mu5}\psi$ are conserved.

(e) In STR $\langle\bar{\psi}\mathrm{x}^5\psi\rangle=\langle\varphi^\dagger\chi-\chi^\dagger\varphi\rangle=\langle R_\varphi^\dagger R_\chi\rangle_0(\rho_\varphi^\dagger\rho_\chi-\rho_\chi^\dagger\rho_\varphi)\sim\mathrm{I}\sin\beta$, which changes sign under Hermite conjugation, as a pseudoscalar should. The corresponding expression in STA is in our notation $\rho^2\sin\beta$, which is a scalar and hints to a simplistic definition of ψ in STA. $\langle R_\varphi^\dagger R_\chi\rangle_0$ extracts the scalar part of $R_\varphi^\dagger R_\chi$.

4.3. Symmetries of STR DE.

Let us look now at the *symmetries* parity $\mathcal{P}$, time reversal $\mathcal{T}$and charge conjugation $\mathcal{C}$ for the STR DE and for the STR Dirac spinor ψ. In the following we will present three forms $\mathcal{T},\mathcal{T}',\mathcal{T}''$ for time-reversal symmetry in STR and discuss the relation between the three (see also Appendix D).

Parity $\mathcal{P}$: $\mathrm{x}^\mu\rightarrow\mathrm{x}^0\mathrm{x}^\mu\mathrm{x}^0=\mathrm{x}_{\mathcal{P}}^\mu$. As in the standard treatment left-multiply STR DE by x^0 (below $\hbar=1$):

$$\mathcal{P}\colon(\mathrm{P}-m)\psi=0\rightarrow\mathrm{x}^0(\mathrm{P}-m)\psi=(\mathrm{x}^0\mathrm{P}\mathrm{x}^0-m)\mathrm{x}^0\psi\equiv(\mathrm{P}_{\mathcal{P}}-m)\psi_{\mathcal{P}}=\big[\mathrm{x}^0(\mathrm{I}\partial_0+eA_0)-\mathrm{x}^j(\mathrm{I}\partial_j+eA_j)-m\big]\psi_{\mathcal{P}}=\big[\mathrm{x}_{\mathcal{P}}^\mu(\mathrm{I}\partial_\mu+eA_\mu)-m\big]\psi_{\mathcal{P}}\Rightarrow\mathrm{P}_{\mathcal{P}}=\eta_{\mu\nu}\mathrm{x}^\nu(\mathrm{I}\partial_\nu+eA_\nu);\ \psi_{\mathcal{P}}=\mathrm{x}^0\psi \qquad (26)$$

It is clear from (26) that both 3-momentum and 3-position, therefore also the vector potential, change sign in STR DE under parity. Applying the parity transformation (26) to the 3D vectors from (14) we see that:

$$\mathbf{x}_j \overset{\mathcal{P}}{\rightarrow} \mathrm{x}^0\mathbf{x}_j\mathrm{x}^0 = -\mathbf{x}_j \text{ are } odd;\ \boldsymbol{\sigma}_j \overset{\mathcal{P}}{\rightarrow} \mathrm{x}^0\boldsymbol{\sigma}_j\mathrm{x}^0 = \boldsymbol{\sigma}_j \text{ and } \mathrm{x}^{j5} \overset{\mathcal{P}}{\rightarrow} \mathrm{x}^0\mathrm{x}^{j5}\mathrm{x}^0 = \mathrm{x}^0\boldsymbol{\sigma}_j\mathrm{x}^0\mathrm{x}^0 = \mathrm{x}^{j5} \text{ are } even \tag{27}$$

Time reversal $\mathcal{T}$: $x_0 \rightarrow -x_0 \Leftrightarrow \partial_0 \rightarrow \partial_0' = -\partial_0$. We write first DE in terms of the time-reversed quantities:

$$\text{DE: } \left(\mathrm{x}^\mu\left(p_\mu + eA_\mu\right) - m\right)\psi = \left[\mathrm{x}^0(-p_0' + eA_0) + \mathrm{x}^j\left(-p_j' - eA_j'\right) - m\right]\psi = 0 \tag{28}$$

In order to render the energy term positive in the time-reversed system we need to invert the sign of $-p_0'$ in (28) without changing eA_0. This can be achieved in different ways, e.g. by Hermite conjugation & reversal ($\tilde{\dagger}$) ($\mathcal{T}$), by K_3- or K_5- conjugations (leading respectively to $\mathcal{T}', \mathcal{T}''$) of DE as now shown. We first present the three transformations then discuss and compare them. The Hermite conjugate of DE was shown in (24).

$$\mathcal{T}: \mathrm{DE} \xrightarrow{T=\tilde{\dagger}} \mathrm{x}^0\left[\mathrm{x}^\mu\left(-\hbar\mathrm{I}\partial_\mu + eA_\mu\right) - m\right]\tilde{\bar{\psi}} = \left[\mathrm{x}^0(\mathrm{I}\partial_0' + eA_0) + \mathrm{x}^j\left(\mathrm{I}\partial_j - eA_j\right) - m\right]\mathrm{x}^0\tilde{\bar{\psi}} = 0 \Rightarrow$$

$$\Rightarrow \psi_T = \mathrm{x}^0\tilde{\bar{\psi}} = \widetilde{\psi^\dagger};\ T\mathrm{I}T^{-1} = \tilde{\mathrm{I}}^\dagger = -\mathrm{I};\ T^2 = 1 \tag{29}$$

Alternatively, we introduce K_ω-conjugation that inverts only the x^ω (below we look at $\omega = 3$ or 5):

$$K_\omega: \{\mathrm{x}^\tau \rightarrow K_\omega\mathrm{x}^\tau K_\omega = (1 - 2\delta_{\omega\tau})\mathrm{x}^\tau,\ \text{or}\ \mathrm{x}^\omega \rightarrow -\mathrm{x}^\omega;\ \mathrm{x}^{\tau\neq\omega} \rightarrow \mathrm{x}^\tau;\ \tau = 0,1,2,3,5\} \tag{30}$$

First the case $\omega = 3$. We define $\mathcal{T}'$ by the product of a unitary transformation and K_3, $T' = UK_3 = \mathrm{x}^{21}K_3$:

$$\mathcal{T}': \mathrm{DE} \rightarrow \mathrm{x}^{21}K_3\left[\mathrm{x}^\mu\left(\mathrm{I}\partial_\mu + eA_\mu\right) - m\right]\psi = \left[\mathrm{x}^0(\mathrm{I}\partial_0' + eA_0) + \mathrm{x}^j\left(\mathrm{I}\partial_j - eA_j\right) - m\right]\mathrm{x}^{21}K_3\psi = 0 \ \Rightarrow$$

$$\Rightarrow \psi_{T'} \equiv T'\psi = \mathrm{x}^{21}K_3\psi = \mathrm{I}\boldsymbol{\sigma}_3K_3\psi \equiv \mathrm{I}\boldsymbol{\sigma}_3\psi^{K_3};\quad T'\mathrm{I}T'^{-1} = \mathrm{x}^{21}K_3\mathrm{I}K_3\mathrm{x}^{12} = -\mathrm{I};\quad T'^2 = -1 \tag{31}$$

$$\text{For } \omega = 5,\ \mathcal{T}'': \mathrm{DE} \xrightarrow{T''=K_5} K_5\left[\mathrm{x}^\mu\left(\mathrm{I}\partial_\mu + eA_\mu\right) - m\right]\psi = \left[\mathrm{x}^0(\mathrm{I}\partial_0' + eA_0) - \mathrm{x}^j\left(\mathrm{I}\partial_j - eA_j\right) - m\right]K_5\psi = 0 \ \Rightarrow$$

$$\psi_{T''} \equiv T''\psi = K_5\psi \equiv \psi^{K_5};\qquad T''\mathrm{I}T''^{-1} = K_5\mathrm{I}K_5 = -\mathrm{I};\qquad T''^2 = 1 \tag{32}$$

Note that the final *operators* in (29) and (31) are identical, while that in (32) has opposite sign at the spatial part (see below). Let list in bullet form few salient properties of the transformations (29), (31), (32).

- All the three transformations invert the pseudoscalar I and therefore are *antiunitary* [12] in STR.
- The vector *operators* for 3-momentum and position transform opposite (according) to the expectation for $\mathcal{T}, \mathcal{T}'$ (resp. $\mathcal{T}''$), which is clear by inspection of the transformed operators. Similarly to the cases of $\mathcal{T}, \mathcal{T}'$ the standard *t*-reversed DE has the wrong sign in front of the 3-momentum.
- In the absence of magnetic field ($A_j = 0$) the Dirac Hamiltonian remains unchanged under $\mathcal{T}, \mathcal{T}'$

$$\mathrm{I}\partial_0\psi = H\psi = \left(\mathrm{x}^{j0}\mathrm{I}\partial_j - eA_0 + m\mathrm{x}^0\right)\psi \xrightarrow{T,T'} \mathrm{I}\partial_0'\psi_{T,T'} = H\psi_{T,T'} \tag{33}$$

The 3-momentum term in the same Hamiltonian changes sign under $\mathcal{T}''$ (as it should under *t*-reversal!):

$$\mathrm{I}\partial_0\psi = H\psi \xrightarrow{T''=K_5} \mathrm{I}\partial_0'\psi^{K_5} = K_5 H K_5 \psi^{K_5} = \left(-\mathrm{x}^{j0}\mathrm{I}\partial_j - eA_0 + m\mathrm{x}^0\right)\psi^{K_5};\ \ \mathrm{x}^0 K_5 H K_5 \mathrm{x}^0 = H \qquad (33')$$

- The final DE operators in (29), (31) differ by a parity transformation compared to the *proper* time-reversal operator (32). This leads to vectors for 3-momentum and position bearing wrong sign in (29), (31) – the price to pay for preserving the Hamiltonian (33) above; see also the last equation in (33').
- The vector potential $\mathbf{A} = A_j\mathbf{x}_j$ does *not* invert under the proper *t*-reversal (32). An additional parity transformation is required, as in (29), (31) in order to invert **A**. In the following, keeping with tradition we will identify *t*-reversal with (29), (31), which both invert **A** and preserve the Hamiltonian (33).
- The transformed operator parts being the same in (29) and (31) one expects ψ_T and $\psi_{T'}$ to represent the same state. Indeed, the $T = \tilde{\dagger}$ transformation is equivalent to a reversal of the spatial directors alone $\mathrm{x}^j \to -\mathrm{x}^j$, leaving $\mathrm{x}^0, \mathrm{x}^5$ unchanged; we call it K_j-conjugation ($K_j = K_1K_2K_3$). The property $K_j\mathrm{x}^5K_j = \mathrm{x}^5$ distinguishes it from the parity transformation (26). K_3, K_j preserve the Clifford algebra $\{\mathrm{x}'^\mu, \mathrm{x}'^\nu\} = 2\eta^{\mu\nu}$. Both reverse handedness of spatial coordinates, therefore the resulting bases from the two conjugations differ by a proper rotation, as illustrated in Figure 1. In particular, $\mathcal{T}, \mathcal{T}'$ both flip spin.
- Kramers' degeneracy [13] for a system consisting of an odd number of spin ½ particles in an electric field is more directly proved for the transformation $\mathcal{T}'$ in (31) with $T'^2 = -1$, just as in the standard case [4]. However, it is of course valid for $\mathcal{T}$ in (29), which as argued above, leads to the same state.

The discussion above illustrates the power of STR to elucidate the geometric meaning of *t*-reversal, a subject that is quite involved in the standard formalism [4]. Two remarks before closing the paragraph. First, the standard *t*-reversal transformation corresponds to $\omega = 2$ in (30) with $T_{st} = \mathrm{x}^{13}K_2$. Second, conjugations K_j, K_1, K_2, K_3 from (30) (see also (16-17A)) relate in 3D to inversion and three frame-plane reflections, the same *improper* rotations characterizing the four maximally entangled states of pairs of spin ½ particles [14].

Charge conjugation $\mathcal{C}: e \to -e$. Each form of time-reversal symmetry has its counterpart here, e.g. the form $\mathcal{C}$ corresponding to (29) is obtained by first K_j-conjugating STR DE (16), then left-multiplying by x^{123}:

$$\mathcal{C}: \left(\mathrm{x}^\mu P_\mu - m\right)\psi = 0 \to C\left(\mathrm{x}^\mu P_\mu - m\right)\psi = \mathrm{x}^{123}\mathrm{x}^0\left[-\mathrm{x}^\mu\left(\hbar\mathrm{I}\partial_\mu - eA_\mu\right) - m\right]\tilde{\bar{\psi}} = \left[\mathrm{x}^\mu\left(\mathrm{I}\partial_\mu - eA_\mu\right) - m\right]\psi_{\mathcal{C}} = 0 \ \ \Rightarrow \ \ \psi_{\mathcal{C}} \equiv \mathrm{x}^{123}\widetilde{\psi^\dagger} = \mathrm{x}^{123}\psi^{K_j};\ \ \ CIC^{-1} = \mathrm{x}^{123}K_j\mathrm{I}K_j\mathrm{x}^{123} = -\mathrm{I};\ \ \ CC = \mathrm{x}^{123}K_j\mathrm{x}^{123}K_j = -1 \qquad (34)$$

With (16) describing the electron, (34) describes the *positron*. Notice the role of x^5 in $\mathrm{x}^{123} = I\mathrm{x}^{50}$

$\boldsymbol{\mathcal{CPT}}$**.** We can apply the three symmetries (26), (29), (34) in series onto the STR DE (16) (notice x^5again!)

$$\mathcal{CPT}: \left(\mathrm{x}^\mu\left(\mathrm{I}\partial_\mu + eA_\mu\right) - m\right)\psi = 0 \to \mathrm{x}^{123}K_j\mathrm{x}^0K_j\left(\mathrm{x}^\mu\left(\mathrm{I}\partial_\mu + eA_\mu\right) - m\right)\psi = \mathrm{Ix}^5\left(\mathrm{x}^\mu\left(\mathrm{I}\partial_\mu + eA_\mu\right) - m\right)\psi =$$

$$\left(\mathrm{x}^\mu\left(-\mathrm{I}\partial_\mu - eA_\mu\right) - m\right)\mathrm{Ix}^5\psi = 0 \quad \Rightarrow \quad \psi_{\mathcal{CPT}} = \mathrm{Ix}^5\psi = -\mathrm{x}^{0123}\psi \tag{35}$$

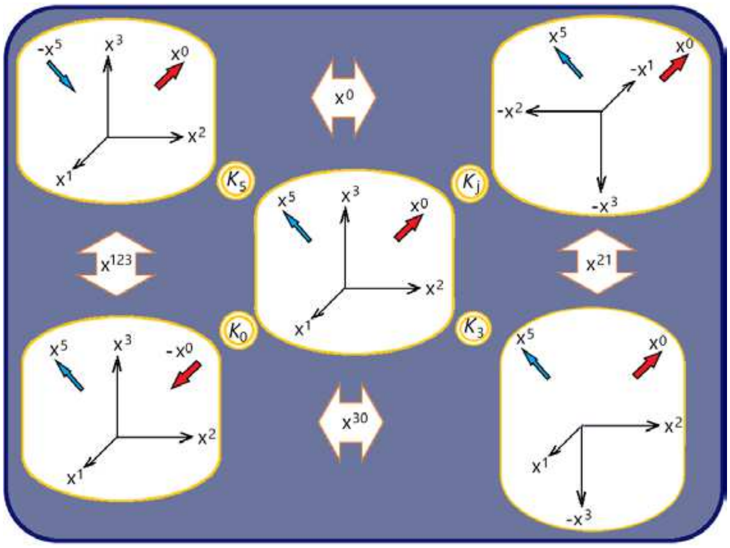

Figure 1. Effect of conjugations K_0, K_3, K_j, K_5 onto the STR directors. The transformed frames and operators are related as shown by the two-sided arrows. From (29-32), $T = K_j; T' = \mathrm{I}\sigma_3K_3; T'' = K_5$; we add here $T''' = \mathrm{x}^{123}K_0$. All T, T', T'', T''' invert I and thereby time; of these only T'', T''' keep 3-position unchanged and invert 3-momentum. However, as shown above, the Hamiltonian (33) preserves form under *t*-reversal when DE transforms by $T = K_j$, or $T' = \mathrm{I}\sigma_3K_3$; we accept these by convention to stand for *t*-reversal.

$\mathcal{CPT}$ reverses time, 3-momentum and charge, leaving invariant the Dirac Lagrangian (see Appendix D). The discussion above makes it clear that $\mathcal{CPT} = \mathcal{CT}$", with $\mathcal{T}$" the *proper* time-reversal (see (32) and fig. 1). A study of the richness of STA conjugations (30) and their effect on DE symmetries will appear elsewhere.

Relativistic covariance of the STR DE (16). We have already seen that the geometric product of two vectors comprises a scalar and a bivector, see (3), (11). The generators of Lorentz boosts and rotations are contained in the bivector (a factor of 2 ensures consistency with the definition in (1A)):

$$\mathrm{x}^\mu\mathrm{x}^\nu = \eta^{\mu\nu} + \mathrm{x}^\mu \wedge \mathrm{x}^\nu = \eta^{\mu\nu} + \tfrac{1}{2}[\mathrm{x}^\mu, \mathrm{x}^\nu];\ \mathrm{x}^\mu \wedge \mathrm{x}^\nu = \begin{cases} 2\mathbf{K}_j = \mathbf{x}_j \text{ for } \mu = j, \nu = 0 \ \ Boosts \\ 2\mathbf{J}_l = -\varepsilon_{jkl}\mathbf{I}\boldsymbol{\sigma}_l \text{ for } \mu = j, \nu = k \ Rotors \end{cases} \tag{36}$$

Note that I is both parity and Lorentz invariant. The *operator* $(\mathrm{P} - m)$ is coordinate-free and relativistic-invariant, i.e. it remains unchanged as a whole under a Lorentz transformation $\mathcal{L}$. On the other end, the spinor ψ transformation is a one-sided generalized rotor S, i.e. a rotor and/or a boost comprising the exponentiated generators $\mathbf{J}_j, \mathbf{K}_j$ in (36) (see (37) below and (1A), (2A) in Appendix A) with the respective rotation angle and/or rapidity from the transformation $\mathcal{L}$ at hand. Therefore, we can write (DE transforms as a spinor!):

$\mathcal{L}: (\mathrm{P} - m)\psi = 0 \to \mathrm{S}(\mathrm{P} - m)\mathrm{S}^{-1}\mathrm{S}\psi = \mathrm{S}[(\mathrm{P} - m)\psi] = (\mathrm{P} - m)\mathrm{S}\psi = 0 \Rightarrow \psi_{\mathcal{L}} = \mathrm{S}\psi,$ where

$$\mathrm{S} = e^{\mathbf{S}_j\omega_j} \text{ with} \begin{cases} \mathbf{S}_j = \mathbf{K}_j = \mathbf{x}_j/2\,;\omega_j = \alpha_j \text{ (rapidity, hyp. angle) boosts} \\ \mathbf{S}_j = \mathbf{J}_j = -\,\mathrm{I}\boldsymbol{\sigma}_j/2\,;\omega_j = \vartheta_j \text{ (Euclidian angle) rotors} \end{cases} ;\mathrm{S}^\dagger \mathrm{x}^0 = e^{\mathbf{S}_j^\dagger \omega_j}\mathrm{x}^0 = \mathrm{x}^0\mathrm{S}^{-1} \quad (37)$$

Compare the simplicity of the STR in the first line of (37) with the infinitesimal Lorentz transformation in the standard approach [2, 4, 5], the standard DE *not* being manifestly covariant! The two-sided rotor around the operator $\mathrm{S}(\mathrm{P}-m)\mathrm{S}^{-1} = \mathrm{P}-m$ is useful to calculate the transformed frame vectors and components of 4-momentum and 4-potential. Lorentz transformation of the Dirac spinor and two bilinears take the form:

$$\psi \xrightarrow{\mathcal{L}} \psi' = \mathrm{S}\psi;\ \bar{\psi} \xrightarrow{\mathcal{L}} \bar{\psi}' = \bar{\psi}\mathrm{S}^{-1}; \bar{\psi}\psi \xrightarrow{\mathcal{L}} \bar{\psi}'\psi' = \bar{\psi}\psi; \bar{\psi}\mathrm{x}^\mu\psi \xrightarrow{\mathcal{L}} \bar{\psi}'\mathrm{x}^\mu\psi' \text{ (from } \mathcal{L}\text{-invariance of } \nabla \text{ in (25))} \quad (38)$$

(37, 38) illustrate how boosts and rotations actually take place in physical spacetime. STR DE stands at an advantage point when infinitesimal Lorentz transformations become essential, like in curved spacetime [4].

How do we **change basis** in STR? (37) hints to the answer: by two sided general rotor transformations, i.e. $\mathrm{x}^\mu \to \mathrm{x}'^\mu = \mathrm{S}\mathrm{x}^\mu\mathrm{S}^{-1}$. One can realize the general transformations S in X by exponentiation. For example, in the deep relativistic regime ($|\mathrm{p}| \gg m$) one can neglect the mass term in DE and adapt a more convenient basis and corresponding spinor than the STR Dirac basis adapted here, known as the Weyl basis and Weyl spinors. The generalized rotor $R_\omega = e^{\mathrm{x}^{50}\omega/2}$ acting one-sided to spinors and two-sided to vectors, can swap x^0 with x^5 leaving x^j unchanged: $R_\omega \mathrm{x}^j R_\omega^{-1} = \mathrm{x}^j$, as recounted in Appendix B.

4.4. STR DE in terms of Pauli spinors and its nonrelativistic limit – the STR Pauli equation.

Now we turn to the form (17) of the STR Dirac spinor and rewrite the Dirac equation in terms of φ and χ:

$$\left(P_0\mathrm{x}^0 + P_j\mathrm{x}^j - m\right)\psi = 0 \Rightarrow (1+\mathrm{x}^0)\left[(P_0-m)\varphi + P_j\mathrm{x}^j\chi\right] + (1-\mathrm{x}^0)\left[(-P_0-m)\chi + P_j\mathrm{x}^j\varphi\right] =$$

$$(1+\mathrm{x}^0)[(P_0-m)\varphi - \mathbf{P}\chi] + (1-\mathrm{x}^0)[(-P_0-m)\chi + \mathbf{P}\varphi] = 0 \quad (39)$$

The quantities inside square brackets cannot be proportional to $(1 \pm \mathrm{x}_0)$, because all terms in them belong to the subspace Ξ from (14'). Therefore, they have to be zero. Due to the independence of the two terms, (39) is then equivalent to the standard-looking pair of coupled equations below, written in the subspace Ξ:

$$\begin{cases} (P_0 - m)\varphi - \mathbf{P}\chi = 0 \\ (P_0 + m)\chi - \mathbf{P}\varphi = 0 \end{cases} \quad (40)$$

We now derive the **STR Pauli equation** as **non-relativistic** approximation to STR DE in (40) (see also Appendix C). As anticipated in (19), following Feynman, we isolate the fast oscillating part of ψ as a common factor $\rho = \rho(t)$ to φ and χ in (40), leaving behind the nonrelativistic Pauli spinors proper φ_P, χ_P:

$$\begin{cases}(P_0 - m)\rho\varphi_P - \mathbf{P}\rho\chi_P = 0 \\ (P_0 + m)\rho\chi_P - \mathbf{P}\rho\varphi_P = 0\end{cases} \xRightarrow{\rho = e^{-\mathrm{I}mt/\hbar}} \begin{cases}(\mathrm{I}\hbar\partial_t + eA_0)\varphi_P - \mathbf{P}\chi_P = 0 \\ (\mathrm{I}\hbar\partial_t + eA_0 + 2m)\chi_P - \mathbf{P}\varphi_P = 0\end{cases} \tag{41}$$

For $|(\mathrm{I}\hbar\partial_t + eA_0)\chi| \ll 2m|\chi|$ (nonrelativistic regime) the lower of the two coupled equations approximates in lowest order to: $\chi_P \approx \mathbf{P}\varphi_P/2m$. I.e. for slow electrons $|\chi_P| \ll |\varphi_P|$. Substituting into the upper equation one obtains the Pauli Hamiltonian H_P (below: $\mathbf{PP} = \mathbf{P}\cdot\mathbf{P} + \mathbf{P}\wedge\mathbf{P} = \mathbf{P}\cdot\mathbf{P} + \hbar e(\boldsymbol{\sigma}, \mathbf{B})$, where $\mathbf{P}\cdot\mathbf{P} = (\mathbf{p} + e\mathbf{A})\cdot(\mathbf{p} + e\mathbf{A}) = (-\hbar(\mathrm{I})\boldsymbol{\nabla} + e\mathbf{A})\cdot(-\hbar(\mathrm{I})\boldsymbol{\nabla} + e\mathbf{A}) \equiv \mathbf{P}^2$ is a grade 0, 5, 0&5 operator):

$$I\hbar\partial_t\varphi_P = H_P\varphi_P = \left(\frac{\mathbf{P}^2}{2m} - eA_0 + \frac{\hbar e}{2m}(\boldsymbol{\sigma}, \mathbf{B})\right)\varphi_P \tag{42}$$

This is the STR Pauli equation (PE), identical in form to the standard PE [15], but here without matrices and with a complex structure surging from the real vector space (10)! The term $(\hbar e/2m)(\boldsymbol{\sigma}, \mathbf{B})$, which we met earlier in eq. (23) marks the additional potential energy due to the spin magnetic moment of a slow electron. It distinguishes STR PE from the STR Schrödinger equation [16], which is obtained from (42) by removing it (no spin) and by freezing the spinor φ, let say to spin up. Of course, eq. (23) can also be derived from (42).

Final remarks and conclusions.

The couple of equations (40) is independent of the nonrelativistic approximation adopted to derive the PE (42). By substituting φ, χ from (20-20') into (40) one can write down four coupled first order partial differential equations in terms of $\varphi_u, \varphi_d, \chi_u, \chi_d$ and with the x^μ absent; these are equivalent to STR DE.

In **conclusion**, alike STA, STR promotes a geometric view of physics, where vectors and their Clifford combinations, *not* the scalar components, set the complex structure. The definition of STR DE $\mathrm{p} = \hbar\mathrm{I}\nabla = \hbar\mathrm{I}\mathrm{x}^\mu\partial_\mu$ shows its clear physical and geometric meaning as minimal path to quantization of the classical 4-momentum vector with modulus *m*. Its demonstrated working hints to the expectation that all the formal machinery developed in nine decades to handle DE and its generalizations, adapt easily to the STR formalism. Spacetime-reflection stands on clear physical grounds in contrast to the fictional 'internal degrees of freedom' represented by γ-matrices; therefore, it becomes relevant to question the role of the same matrices in areas of modern physics taking them as a fundamental representation. With its inborn distinction between (polar) boost and (axial) rotation/spin vectors, the STR formalism might be relevant to other areas of physics. By developing Dirac's ideas, we recognize reflection / handedness as fundamental, side by side with space & time.

Appendices

Appendix A. Commutation relations of boost and rotor generators and left and right rotors in STR

We first calculate the commutator [x, p] (below $\mathrm{x} = x_\mu \mathrm{x}^\mu = x^\mu \mathrm{x}_\mu$; $\mathrm{p} = I\hbar\nabla = I\hbar \mathrm{x}^\mu \partial_\mu$; $\boldsymbol{\nabla} = \mathrm{x}^{j0}\partial_{x_j} = \mathbf{x}_j\, \partial/\partial x_j$; $\mathbf{L} = \mathbf{x} \times \mathbf{p}$; $(\boldsymbol{\sigma}, \mathbf{L}) = \boldsymbol{\sigma}_j \mathbf{x}_j L_j \mathbf{x}_j = L_j \boldsymbol{\sigma}_j$):

$[\mathrm{x}, \mathrm{p}] = I\hbar(\mathrm{x}\nabla - \nabla\mathrm{x}) = I\hbar(\mathrm{x}\cdot\nabla - \nabla\cdot\mathrm{x} + \mathrm{x}\wedge\nabla - \nabla\wedge\mathrm{x}) = I\hbar(-(\nabla\cdot\mathrm{x}) + 2\mathrm{x}\wedge\nabla) = I\hbar\big[-\partial_\mu(x^\mu) + 2\mathrm{x}^{j0}\big(x_j\partial_0 - x_0\partial_j\big) + 2\mathrm{x}^{jk}\big(x_j\partial_k - x_k\partial_j\big)\big] = I\hbar\left[-4 + 2\mathbf{x}_j\left(x_0\partial_{x_j} + x_j\partial_0\right) + 2\epsilon_{jkl} I\boldsymbol{\sigma}_l\left(x_j\partial_{x_k} - x_k\partial_{x_j}\right)\right] = I\hbar\big[-4 + 2(x_0\boldsymbol{\nabla} + \mathbf{x}\partial_0) + 2I\big(\boldsymbol{\sigma}, (\mathbf{x}\times\boldsymbol{\nabla})\big)\big] = -4I\hbar + 2(\mathbf{x}p_0 - x_0\mathbf{p}) - 2I(\boldsymbol{\sigma}, \mathbf{L}) = 4I\hbar\left[-1 + \mathbf{K}_j\left(x_j\partial_t + t\partial_{x_j}\right) - \epsilon_{jkl}\mathbf{J}_l\left(x_j\partial_{x_k} - x_k\partial_{x_j}\right)\right] = 4I\hbar[-1 + \mathbf{K} + \mathbf{J}]$ (1A)

The commutator (1A) unites dimensionality (cumulative 'uncertainty relation'), boost **K** and rotation **J** – the generators of proper Lorentz transformations, with $\mathbf{K}_j \equiv \mathrm{x}^{j0}/2 \equiv \mathbf{x}_j/2$, $\mathbf{J}_l \equiv \epsilon_{jkl}\,\mathrm{x}^{jk}/2 = -\epsilon_{jkl}\,\boldsymbol{\sigma}_{jk}/2 = -I\boldsymbol{\sigma}_l/2$. $\mathbf{K}_j, \mathbf{J}_l$ appear as directors for the Killing vectors, $x_j\partial_t + t\partial_{x_j}$ and $x_j\partial_{x_k} - x_k\partial_{x_j}$being components of the Killing vector in spacetime. Translation and rotation symmetries are the basic symmetries of space-time. Notice the well-known similarity with the electric and magnetic fields from the potentials in (22).

The commutators of boost and rotor generators $\mathbf{K}_j, \mathbf{J}_j$ from (1A) are:

$\left[\mathbf{J}_j, \mathbf{J}_k\right] = -\frac{1}{2}\boldsymbol{\sigma}_{jk} = -\frac{1}{2}\epsilon_{jkl} I\boldsymbol{\sigma}_l = \epsilon_{jkl}\mathbf{J}_l$; $\left[\mathbf{K}_j, \mathbf{K}_k\right] = \frac{1}{2}\mathbf{x}_{jk} = -\epsilon_{jkl}\mathbf{J}_l$; $\left[\mathbf{J}_j, \mathbf{K}_k\right] = -\frac{1}{2}I\mathrm{x}^5\boldsymbol{\sigma}_{jk} = \frac{1}{2}\epsilon_{jkl}\mathbf{x}_l = \epsilon_{jkl}\mathbf{K}_l$ (2A)

By the definition of $\mathbf{J}_j, \mathbf{K}_j$ it is clear that they do not comprise x^5. The rotor-boost space can split into *right* and *left* handed *disjoint* subspaces, S_{+j} and S_{-j}, illustrating the double coverage of the group $SO(1,3)$ by the spinor group $SU(2)$, i.e. $SO(1,3) = SU(2)\otimes SU(2)$ (where = stands for *isomorphic to*):

$\left\{\mathrm{S}_{\pm j} \equiv \frac{1}{2}\big(\mathbf{J}_j \pm I\mathbf{K}_j\big) = \frac{1}{2}\mathbf{J}_j(1 \pm \mathrm{x}^5)\right\} \Rightarrow \left\{\left[\mathrm{S}_{\pm j}, \mathrm{S}_{\pm k}\right] = \frac{1}{4}\epsilon_{jkl}\mathbf{J}_l(1 \pm \mathrm{x}^5)^2 = \frac{1}{2}\epsilon_{jkl}\mathbf{J}_l(1 \pm \mathrm{x}^5) = \epsilon_{jkl}\mathrm{S}_{\pm l}; \left[\mathrm{S}_{+j}, \mathrm{S}_{-k}\right] = 0\right\}$; the two projectors $(1 \pm \mathrm{x}^5)$ are orthogonal. Note the entrance of x^5 into $\mathrm{S}_{\pm j}$ (3A)

Appendix B. Weyl spinors in STR.

STR renders explicit the similarity between the chiral representation of rotors in (3A) and Right and Left handed Weyl spinors that are orthonormal $(1 \pm \mathrm{x}^5)/2$ projections of Dirac spinors:

$$\psi = \psi_L + \psi_R = \frac{1}{2}(1 - \mathrm{x}^5)\psi + \frac{1}{2}(1 + \mathrm{x}^5)\psi \tag{4A}$$

Let see where this form is useful. The Dirac spinor and basis are preferred to describe slow electrons. For fast electrons, $\mathrm{p} \gg m$, we can neglect the mass term; then DE in momentum representation takes the form:

$$(\mathrm{p} - m)\psi(\mathrm{p}) = 0 \xrightarrow{\mathrm{p} \gg m} \mathrm{p}\psi(\mathrm{p}) \approx 0 \tag{5A}$$

If $\psi(\mathrm{p})$ is a solution to the STR DE at the r.h.s. then $\mathrm{x}^5\psi(\mathrm{p})$ will also be a solution, x^5 anticommuting with p. Now, in analogy to the Dirac spinor in (17) we write $\psi(\mathrm{p})$ as:

$$\psi(\mathrm{p}) = \frac{1}{2}[(1 + \mathrm{x}^5)\psi(\mathrm{p}) + (1 - \mathrm{x}^5)\psi(\mathrm{p})] \equiv (1 + \mathrm{x}^5)\psi_R(\mathrm{p}) + (1 - \mathrm{x}^5)\psi_L(\mathrm{p}) \tag{6A}$$

This form of the spinor is convenient to describe fast electrons or massless fermions.

The swap $\mathrm{x}^0 \rightleftarrows \mathrm{x}^5$ constitutes the passage from the Dirac (x^0 in the projectors) to Weyl basis (x^5 in the projectors). As mentioned in the main text in the discussion following eq. (38) the change of basis in STR is achieved by two-sided generalized rotor transformations. Let illustrate it here in relation to the passage from Dirac to Weyl basis. It takes the form:

$$\{\mathrm{x}^5\} \to \{\mathrm{x}'^5\}\colon\ \mathrm{x}'^\tau = R_\omega \mathrm{x}^\tau R_\omega^{-1} = e^{\mathrm{x}^{50}\omega/2}\mathrm{x}^\tau e^{-\mathrm{x}^{50}\omega/2} \overset{\omega=\pi/2}{=} \begin{cases} \mathrm{x}'^0 = \mathrm{x}^5 \\ \mathrm{x}'^j = \mathrm{x}^j \\ \mathrm{x}'^5 = -\mathrm{x}^0 \end{cases};\ \tau = 0,1,2,3,5;\ j = 1,2,3 \tag{7A}$$

Within a sign (not relevant for a basis element) we have realized by (8A) the mentioned transformation of basis. Notice that $(\mathrm{x}^{50})^2 = -1$, which makes exponentiation in (7A) meaningful.

Appendix C. Derivation of Eq. (23).

For convenience, let us rewrite the squared DE (22) here:

$$\Big(\mathrm{KG} + e\hbar I\big(\mathbf{E} + I(\boldsymbol{\sigma}, \mathbf{B})\big)\Big)\psi = 0 \tag{8A}$$

If ψ is a solution to DE then it is also a solution to its square, eq. (8A). We have removed the projectors $(1 \pm \mathrm{x}^0)$ from the spinor, because there is no free x^0 in (8A). Now we follow the same strategy as earlier and try to write (8A) completely in the $\boldsymbol{\Sigma}$ subspace with the help of the two Pauli spinors:

$$\psi = (1 + \mathrm{x}^0)\varphi + (1 - \mathrm{x}^0)\chi \overset{(9\mathrm{A})}{\Longrightarrow} \Big(\mathrm{KG} + e\hbar I\big(\mathbf{E} + I(\boldsymbol{\sigma}, \mathbf{B})\big)\Big)\big((1 + \mathrm{x}^0)\varphi + (1 - \mathrm{x}^0)\chi\big) =$$

$$\begin{cases} [\mathrm{KG} - e\hbar(\boldsymbol{\sigma}, \mathbf{B})]\varphi + e\hbar I\mathbf{E}\chi = 0 \\ [\mathrm{KG} - e\hbar(\boldsymbol{\sigma}, \mathbf{B})]\chi + e\hbar I\mathbf{E}\varphi = 0 \end{cases} \tag{9A}$$

The coupled equations (9A) are symmetric relative to the exchange $\varphi \leftrightarrow \chi$. We take the nonrelativistic limit of (9A). Under the derivation of the Pauli equation we saw that in this limit $|\chi| \ll |\varphi|$. This is expected, as by writing DE in the rest frame of the electron in the momentum representation one finds $\chi = 0$. Therefore, in this regime we can take $\psi \approx \rho\varphi$ (this can be done also after removing the fast oscillations, which we do after (10A)) and the two equations in (9A) decouple:

$$\begin{cases}[\mathrm{KG} - e\hbar(\boldsymbol{\sigma}, \mathbf{B})]\rho\varphi = 0 \\ \quad e\hbar I(\boldsymbol{\sigma}, \mathbf{E})\rho\varphi = 0\end{cases} \tag{10A}$$

In the following we will look only at the top equation. As done earlier we isolate the fast oscillations in $\rho = e^{-Imt/\hbar}$, so that second derivative $\partial_t^2\varphi$ of φ from the KG term can be ignored after differentiating ρ and removing the exponential. We also assume a weak constant EM field and in this way can also ignore the A^2 terms. These simplifications will make treatment easier allowing to focus on few essential properties of DE:

$$\left(\mathrm{P}\cdot\mathrm{P} - m^2 - e\hbar(\boldsymbol{\sigma}, \mathbf{B})\right)\rho\varphi = \left(-\hbar^2\,\partial_t^2 - \mathbf{p}^2 + e^2(A_0^2 - \mathbf{A}^2) + e\hbar I\left(2A_0\partial_0 + (\partial_0 A_0) - 2A_j\partial_j - \left(\partial_j A_j\right)\right) - m^2 - e\hbar(\boldsymbol{\sigma}, \mathbf{B})\right)\exp(-Imt/\hbar)\,\varphi \approx \exp(-Imt/\hbar)\left(-\hbar^2\left(-2\frac{Im}{\hbar}\frac{\partial\varphi}{\partial t} - \frac{m^2}{\hbar^2} + \frac{\partial^2\varphi}{\partial t^2}\right) - \mathbf{p}^2 + e\hbar I\left(-2A_0\frac{Im}{\hbar} + (\partial_t A_0) - 2A_j\partial_j - \left(\partial_j A_j\right)\right) - m^2 - e\hbar(\boldsymbol{\sigma}, \mathbf{B})\right)\varphi \Rightarrow \left\{I\hbar\partial_t - \frac{\mathbf{p}^2}{2m} + eA_0 - \frac{e\hbar I}{2m}\left[2A_j\partial_j + \left(\partial_j A_j\right)\right] - \frac{e\hbar}{2m}(\boldsymbol{\sigma}, \mathbf{B})\right\}\varphi \approx 0 \tag{11A}$$

The last equation can be derived from the Pauli equation (42) in the main text, as well. Now, we know that the Schrödinger equation [16] can handle magnetic orbital momentum, which must be the term within square brackets in (11A). Let us render it explicit by trying to express A_j as a function of the components of the magnetic field B_k. Start with $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \Rightarrow B_j = \left(\partial_{j+1}A_{j+2}\right) - \left(\partial_{j+2}A_{j+1}\right)$ with indices varying cyclically *mod*3, e.g. $B_2 = \partial_3 A_1 - \partial_1 A_3$. The brackets in e.g. $\left(\partial_{j+2}A_{j+1}\right)$ mean that the derivative operates only to that term. Taking $A_j = \left(B_{j+1}x_{j+2} - B_{j+2}x_{j+1}\right)/2$, or $2\mathbf{A} = \mathbf{B} \times \mathbf{x}$ satisfies the equation for $\mathbf{B}$, remembering that the B_j are constant. With this choice the term $\left(\partial_j A_j\right) = 0$ in (11A) and $2\hbar I A_j\partial_j = 2\mathbf{A}\cdot\mathbf{p} = \mathbf{B}\times\mathbf{x}\cdot\mathbf{p} = \mathbf{B}\cdot\mathbf{x}\times\mathbf{p} \equiv \mathbf{B}\cdot\mathbf{L}$, where $\mathbf{L} = \mathbf{x}\times\mathbf{p}$ is the orbital angular momentum. The Pauli spin vector being $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$, we put it all together into the last equation in (11A) and obtain eq. (23) from the main text:

$$\left[I\hbar\partial_t - \frac{\mathbf{p}^2}{2m} + eA_0 - \frac{e}{2m}\mathbf{B}\cdot(\mathbf{L} + 2\mathbf{S})\right]\varphi = 0 \tag{12A}$$

Appendix D. Lagrangian of STR DE and orthogonality of direct vs. time-reversed STR Dirac spinors.

The Lagrangian in STR is (the last equality shows its invariance under CPT transformation):

$$\mathcal{L} = \overline{\psi}(\mathrm{p} - m)\psi = \overline{\psi}(I\hbar\nabla - m)\psi = \overline{\psi}\left(I\hbar \mathrm{x}^{\mu}\,\partial_{\mu} - m\right)\psi = \overline{\psi}\mathrm{x}^{5}\left(-I\hbar \mathrm{x}^{\mu}\,\partial_{\mu} - m\right)\mathrm{x}^{5}\psi \tag{13A}$$

As in the standard case $\overline{\psi}$ and ψ are independent and one can recover STR DE from the field equations most directly by differentiating $\mathcal{L}$ in (13A) with respect to $\partial_{\mu}\overline{\psi}$ and $\overline{\psi}$ obtaining (∇ in (14A) acts to the right):

$$\partial_{\mu}\frac{\delta\mathcal{L}}{\delta\,\partial_{\mu}\overline{\psi}} - \frac{\delta\mathcal{L}}{\delta\overline{\psi}} = 0 \Rightarrow \frac{\delta\mathcal{L}}{\delta\overline{\psi}} = (I\hbar\nabla - m)\psi = 0 \tag{14A}$$

The conservation of probability currents follows from the gauge symmetry of the Lagrangian and Noether's theorem. Gauge symmetry: $\left\{\psi \to e^{I\theta}\psi \colon \mathcal{L} \to \mathcal{L}\right\} \Rightarrow \partial_{\mu}\overline{\psi}\mathrm{x}^{\mu}\psi = 0$ (15A)

Orthogonality of Dirac spinor $\psi = (1 + \mathrm{x}^0)\varphi + (1 - \mathrm{x}^0)\chi \equiv (1 \pm \mathrm{x}^0)\psi_{\pm};\ \ \psi_{\pm} = \rho_{\psi_{\pm}}\mathrm{R}_{\psi_{\pm}}(1 + \boldsymbol{\sigma}_3)$ vs. time-reversed spinors.

$$\mathcal{T}^{(\omega)}\colon \psi \to T^{(\omega)}\psi = \begin{cases} K_j\psi = \mathrm{x}^0 K_5\psi = \mathrm{x}^0(1\pm\mathrm{x}^0)\psi_{\pm}^{K_5}; & \psi_{\pm}^{K_5} = \rho_{\psi_{\pm}}^{\dagger}\mathrm{R}_{\psi_{\pm}}(1-\boldsymbol{\sigma}_3)\ \text{ for } \omega=5 \\ I\boldsymbol{\sigma}_1 K_1\psi = \mathrm{I}\boldsymbol{\sigma}_1(1\pm\mathrm{x}^0)\psi_{\pm}^{K_1}; & \psi_{\pm}^{K_1} = \rho_{\psi_{\pm}}^{\dagger}\mathrm{R}_{\psi_{\pm}}^{\dagger}(1+\boldsymbol{\sigma}_3)\ \text{ for } \omega=1 \\ I\boldsymbol{\sigma}_2 K_2\psi = \mathrm{I}\boldsymbol{\sigma}_2(1\pm\mathrm{x}^0)\psi_{\pm}^{K_2}; & \psi_{\pm}^{K_2} = \rho_{\psi_{\pm}}^{\dagger}\mathrm{R}_{\psi_{\pm}}(1+\boldsymbol{\sigma}_3)\ \text{ for } \omega=2 \\ I\boldsymbol{\sigma}_3 K_3\psi = \mathrm{I}\boldsymbol{\sigma}_3(1\pm\mathrm{x}^0)\psi_{\pm}^{K_3}; & \psi_{\pm}^{K_3} = \rho_{\psi_{\pm}}^{\dagger}\mathrm{R}_{\psi_{\pm}}^{\dagger}(1-\boldsymbol{\sigma}_3)\ \text{ for } \omega=3 \end{cases} \tag{16A}$$

Now we can prove the orthogonality relations (below $\mathrm{I}\boldsymbol{\sigma}_{\omega} = \mathrm{x}^0, \mathrm{I}\boldsymbol{\sigma}_1, \mathrm{I}\boldsymbol{\sigma}_2, \mathrm{I}\boldsymbol{\sigma}_3$ for $\omega = 5,1,2,3,$ respectively): $\langle\overline{\psi}T^{(\omega)}\psi\rangle = \psi_{\pm}^{\dagger}(1 \pm \mathrm{x}^0)\mathrm{x}^0\mathrm{I}\boldsymbol{\sigma}_{\omega}(1 \pm \mathrm{x}^0)\psi_{\pm}^{K_{\omega}} = \langle\varphi^{\dagger}\mathrm{I}\boldsymbol{\sigma}_{\omega}\varphi^{K_{\omega}} \pm \chi^{\dagger}\mathrm{I}\boldsymbol{\sigma}_{\omega}\chi^{K_{\omega}}\rangle$; $\varphi^{\dagger}\mathrm{I}\boldsymbol{\sigma}_{\omega}\varphi^{K_{\omega}} =$

$$(1 + \boldsymbol{\sigma}_3)\rho_{\varphi}^{\dagger}\mathrm{R}_{\varphi}^{\dagger}\begin{cases} \rho_{\varphi}^{\dagger}\mathrm{R}_{\varphi}(1-\boldsymbol{\sigma}_3) = \rho_{\varphi}^{\dagger 2}(1+\boldsymbol{\sigma}_3)(1-\boldsymbol{\sigma}_3) = 0 & \text{for } \omega=5 \\ \rho_{\varphi}^{\dagger}I\boldsymbol{\sigma}_1 R_{\varphi}^{\dagger}(1+\boldsymbol{\sigma}_3) = \rho_{\varphi}^{\dagger 2}(1+\boldsymbol{\sigma}_3)(1-\boldsymbol{\sigma}_3)I\boldsymbol{\sigma}_1 = 0 & \text{for } \omega=1 \\ \rho_{\varphi}^{\dagger}I\boldsymbol{\sigma}_2 R_{\varphi}(1+\boldsymbol{\sigma}_3) = \rho_{\varphi}^{\dagger 2}(1+\boldsymbol{\sigma}_3)(1-\boldsymbol{\sigma}_3)I\boldsymbol{\sigma}_2 = 0 & \text{for } \omega=2 \\ \rho_{\varphi}^{\dagger}I\boldsymbol{\sigma}_3 R_{\varphi}^{\dagger}(1-\boldsymbol{\sigma}_3) = \rho_{\varphi}^{\dagger 2}(1+\boldsymbol{\sigma}_3)(1-\boldsymbol{\sigma}_3)I\boldsymbol{\sigma}_3 = 0 & \text{for } \omega=3 \end{cases}$$

; similarly, $\chi^{\dagger}I\boldsymbol{\sigma}_{\omega}\chi^{K_{\omega}} = \rho_{\chi}^{\dagger 2}(1 + \boldsymbol{\sigma}_3)(1 - \boldsymbol{\sigma}_3)I\boldsymbol{\sigma}_{\omega} = 0$, which completes the proof. (17A)

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