

Two new Painlevé integrable KdV–Calogero-Bogoyavlenskii-Schiff (KdV-CBS) equation and new negative-order KdV-CBS equation

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Two new Painlevé integrable KdV–Calogero–Bogoyavlenskii–Schiff (KdV-CBS) equation and new negative-order KdV-CBS equation

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Abstract

In this work, we develop two new (3+1)-dimensional KdV–Calogero–Bogoyavlenskii–Schiff (KdV-CBS) equation and (3+1)-dimensional negative-order KdV-CBS (nKdV-nCBS) equation. The newly developed equations pass the Painlevé integrability test via examining the compatibility conditions for each developed model. We examine the dispersion relation and derive multiple soliton solutions for each new equation.

Keywords: KdV equation; Calogero–Bogoyavlenskii–Schiff equation; negative-order equations; multiple soliton solutions.

1 Introduction

The last few years have perceived an outburst in the study of higher-dimensional nonlinear systems, integrable in particular. These studies are motivated by the reality that solitary waves theory has practical relevance in scientific fields, such as communication technology, plasma physics, ocean waves, quantum mechanic, hydrodynamics, nonlinear optics and many others. Studies on developing higher-dimensional integrable equations have drawn more attention in recent decades, where several integrable models have been developed in the context of (2+1)- and (3+1)-dimensional equations. The field of integrable equations is an active area of research because it describes the real features and reveal the scientific nature of the nonlinearity in science areas. The concept of multiple soliton solutions is one important feature of completely integrable equations.

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The integrable KdV equation, that plays an important role in the solitary wave theory, reads

$$v_t + 6vv_x + v_{xxx} = 0. \quad (1)$$

The spatially one dimensional KdV equation is completely integrable [1–20], hence it admits multiple-soliton solutions and exhibits an infinite number of conservation laws of energy.

The integrable (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation [1–4] reads

$$v_t + 4vv_y + 2v_x \partial_x^{-1} v_y + v_{xxy} = 0, \quad (2)$$

The nonlinear CBS equation (2) describes the interaction between Riemann propagating wave along y -axis with long propagating wave along x -axis.

Recently, the great interest in the integrable KdV equation and the integrable CBS equation has led to the construction of a (2+1)-dimensional equation, by combining these two equations, that takes the form

$$4v_t + \alpha [4vv_y + 2v_x \partial_x^{-1} (v_y) + v_{xxy}] + \gamma [6vv_x + v_{xxx}] = 0, \quad (3)$$

called by many authors as the KdV–Calogero-Bogoyavlenskii-Schiff (KdV–CBS) equation, or the (2+1)-dimensional generalized KdV equation by others [18-28].

It is the aim of this work to extend the KdV–CBS equation (3) to a new (3+1)-dimensional KdV–CBS equation given as

$$4v_t + \alpha [4vv_y + 2v_x \partial_x^{-1} (v_y) + v_{xxy}] + \beta [4vv_z + 2v_x \partial_x^{-1} (v_z) + v_{xxz}] + \gamma [6vv_x + v_{xxx}] + av_x + bv_y + cv_z = 0, \quad (4)$$

where $\alpha, \beta, \gamma, a, b$, and c are arbitrary constants. This equation is obtained by adding the z -component of the CBS equation $[4vv_z + 2v_x \partial_x^{-1} (v_z) + v_{xxz}]$, in addition to three linear terms, namely v_x, v_y , and v_z . Equation (4) can be reduced to other integrable nonlinear equations with distinct significant physical features. For $\alpha \neq 0, \beta = \gamma = a = b = c = 0$, Eq. (4) is reduced to the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation. Moreover, for $\alpha \neq 0, \beta \neq 0, \gamma = a = b = c = 0$, Eq. (4) is reduced to the (3+1)-dimensional CBS equation. However, for $\gamma \neq 0, \alpha = \beta = a = b = c = 0$, Eq. (4) gives the standard KdV equation.

We draw the attention that the KdV equation and the CBS equation can be obtained by using the KdV recursion operator

$$\Phi = \partial_x^2 + 4u + 2u_x \partial_x^{-1}. \quad (5)$$

In other words, we obtain the KdV and the CBS equations

$$u_t + \Phi(u_x) = \partial_x^2(u_x) + 4uu_x + 2u_x\partial_x^{-1}(u_x) = u_t + u_{xxx} + 6uu_x = 0. \quad (6)$$

and

$$u_t + \Phi(u_y) = \partial_x^2(u_y) + 4uu_y + 2u_x\partial_x^{-1}(u_y) = u_t + u_{xxy} + 4uu_y + 2u_x\partial_x^{-1}(u_y) = 0, \quad (7)$$

respectively.

Verosky [4] extended the Olver work in [3], and admitted the use of the negative direction to obtain a sequence of equations of increasingly negative orders. Verosky [4] elaborated that the hierarchy of evolution equations given as

$$v_t + \Phi(v_x) = 0, \quad (8)$$

and

$$v_t + \Phi(v_y) = 0, \quad (9)$$

for the construction of the standard KdV and CBS equations respectively, can be used in the negative order hierarchy in the form

$$v_t + \Phi^{-1}(v_x) = 0, \quad (10)$$

and

$$v_t + \Phi^{-1}(v_y) = 0, \quad (11)$$

or equivalently

$$\Phi(v_t) + v_x = 0, \quad (12)$$

and

$$\Phi(v_t) + v_y = 0, \quad (13)$$

where the power of Φ goes to the opposite direction for the KdV and the CBS equations respectively.

Using the negative order hierarchy (12) and (13) we obtain the integrable negative-order KdV (nKdV) equation and the integrable CBS (nCBS) equation given as

$$v_{xxt} + 4vv_t + 2v_x\partial_x^{-1}(v_t) + v_x = 0, \quad (14)$$

and

$$v_{xxt} + 4vv_t + 2v_x\partial_x^{-1}(v_t) + v_y = 0, \quad (15)$$

or equivalently

$$u_{xxxt} + 4u_xu_{xt} + 2u_{xx}u_t + u_{xx} = 0, \quad (16)$$

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$$u_{xxxxt} + 4u_x u_{xt} + 2u_{xx} u_t + u_{xy} = 0, \quad (17)$$

where we used $v = u_x$. In [1–3], we proved that these two equations (16) and (17) nicely pass the Painlevé integrability test.

In the present work, we follow the sense of combining the KdV and the CBS equations as explored earlier to establish a new combination of the negative-order KdV equation (16) and the negative-order CBS equation (17), hence we establish a new (3+1)-dimensional model

$$u_{xt} + u_{xxxxy} + 4u_x u_{xy} + 2u_{xx} u_y + \lambda u_{xx} + \mu u_{xy} + \nu u_{xz} = 0, \quad (18)$$

that will be called the negative-order KdV-CBS equation (nKdV-nCBS). It is obvious that for $\mu = 0$ and $\nu = 0$, Eq. (18) will be reduced to the negative-order KdV equation (16). However, $\lambda = 0$ and $\nu = 0$, Eq. (18) will be reduced to the negative-order CBS equation (17).

Our aim for this work first is to apply the Painlevé test to examine the integrability feature via determining the compatibility conditions for these two newly developed equations KdV-CBS equation (4) and the (nKdV-nCBS) equation (18). We next plan to derive N -soliton solutions for each developed equation. The simplified Hirota's method is a powerful technique due to its ease of use and does not require the use of the bilinear forms.

2 (3+1)-dimensional KdV–CBS equation

As stated earlier, we first employ the Painlevé analysis to confirm the integrability of the (3+1)-dimensional KdV–CBS equation (4). We next continue to derive the multiple soliton solutions for this equation.

2.1 Painlevé Analysis

To emphasize the integrability of the developed (3+1)-dimensional KdV-CBS equations (4), where it is assumed to have a solution as a Laurent expansion about a singular manifold $\psi = \psi(x, y, z, t)$ as

$$u(x, y, z, t) = \sum_{k=0}^{\infty} u_k(x, y, z, t) \psi^{k-\gamma_1}, \quad (19)$$

where $u_k(x, y, z, t)$, $k = 0, 1, 2, \dots$, are functions of x, y, z , and t . We follow the analysis we presented in our work in [2] to get a characteristic equation for resonances with one branch with four resonances at $k = -1, 1, 4$, and 6 . Proceeding as in [2], we observed

explicit expressions for u_2, u_3 , and u_5 , where we found that u_1, u_4, u_6 turn out to be arbitrary functions and also compatibility conditions, for $k = 1, 4, 6$, are satisfied identically which implies that equations (4) passes the Painlevé test for complete integrability. Note that integrability of equation (4) is confirmed without any restriction on the parameters, $\alpha, \beta, \gamma, a, b$, and c , and therefore does not depend on these parameters.

2.2 Multiple soliton solutions for the KdV-CBS equation

In this section, we employ the simplified Hirota's method to determine the multiple soliton solutions for the new (3+1)-dimensional KdV-CBS equation that reads

$$4v_t + \alpha [4vv_y + 2v_x \partial_x^{-1}(v_y) + v_{xxy}] + \beta [4vv_z + 2v_x \partial_x^{-1}(v_z) + v_{xxz}] + \gamma [6vv_x + v_{xxx}] + av_x + bv_y + cv_z = 0, \quad (20)$$

or equivalently

$$4u_{xt} + \alpha [4u_x u_{xy} + 2u_{xx} u_y + u_{xxx}] + \beta [4u_x u_{xz} + 2u_{xx} u_z + u_{xxx}] + \gamma [6u_x u_{xx} + u_{xxxx}] + au_{xx} + bu_{xy} + cu_{xz} = 0, \quad (21)$$

obtained upon using the potential

$$v(x, y, z, t) = u_x(x, y, z, t). \quad (22)$$

Substituting

$$u(x, y, z, t) = e^{\theta_i}, \theta_i = k_i x + r_i y + s_i z - c_i t, \quad (23)$$

into the linear terms of (21), and solving the resulting equation for dispersion relation c_i we obtain

$$c_i = \frac{k_i^2(\alpha r_i + \beta s_i + \gamma k_i) + (ak_i + br_i + cs_i)}{4}, i = 1, 2, \dots, N, \quad (24)$$

and hence the wave variable θ_i becomes

$$\theta_i = \frac{k_i^2(\alpha r_i + \beta s_i + \gamma k_i) + (ak_i + br_i + cs_i)}{4} t. \quad (25)$$

We next substitute

$$u(x, y, z, t) = 2(\ln(f(x, y, z, t)))_x, \quad (26)$$

where the auxiliary function is given by

$$f(x, y, z, t) = 1 + e^{k_1 x + r_1 y + s_1 z - \frac{k_1^2(\alpha r_1 + \beta s_1 + \gamma k_1) + (ak_1 + br_1 + cs_1)}{4} t}. \quad (27)$$

Consequently, the single soliton solution for the (3+1)-dimensional nonlinear evolution equation (21) is given by

$$u(x, y, z, t) = \frac{2k_1 e^{k_1 x + r_1 y + s_1 z - \frac{k_1^2(\alpha r_1 + \beta s_1 + \gamma k_1) + (ak_1 + br_1 + cs_1)}{4} t}}{1 e^{k_1 x + r_1 y + s_1 z - \frac{k_1^2(\alpha r_1 + \beta s_1 + \gamma k_1) + (ak_1 + br_1 + cs_1)}{4} t}}. \quad (28)$$

Recall from (22) that $v(x, t) = u_x(x, t)$.

To derive the two-soliton solutions, we substitute

$$f(x, y, z, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \quad (29)$$

into Eq. (21), where θ_1 and θ_2 are given in (25), we find that the phase shift a_{12} is given by

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad (30)$$

and hence

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, 1 \leq i < j \leq N. \quad (31)$$

To determine the two-soliton solutions explicitly, we substitute (29) into (26).

To determine the three-soliton solutions we set

$$u(x, y, z, t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1 + \theta_2} + a_{23} e^{\theta_2 + \theta_3} + a_{13} e^{\theta_1 + \theta_3} + a_{12} a_{23} a_{13} e^{\theta_1 + \theta_2 + \theta_3}. \quad (32)$$

We proceed as before to formally derive the three soliton solutions.

3 (3+1)-dimensional negative-order nKdV–nCBS equation

We first follow the analysis presented earlier to confirm the Painlevé integrability of the (3+1)-dimensional nKdV–nCBS equation (18). Proceeding as before, we obtained the same resonant points $k = 1, 4, 6$, and the equation passes the Painlevé test for complete integrability without any restriction on the parameters, λ, μ , and ν .

3.1 Multiple soliton solutions for the nKdV–nCBS equation

In this section, we employ the simplified Hirota's method to determine the multiple soliton solutions for the new (3+1)-dimensional nKdV–nCBS equation that reads

$$u_{xt} + u_{xxxxy} + 4u_x u_{xy} + 2u_{xx} u_y + \lambda u_{xx} + \mu u_{xy} + \nu u_{xz} = 0, \quad (33)$$

Substituting

$$u(x, y, z, t) = e^{\theta_i}, \theta_i = k_i x + r_i y + s_i z - c_i t, \quad (34)$$

into the linear terms of (33), and solving the resulting equation for dispersion relation c_i we obtain

$$c_i = k_i^2 r_i + \lambda k_i + \mu r_i + \nu s_i, i = 1, 2, \dots, N, \quad (35)$$

and hence the wave variable θ_i becomes

$$\theta_i = (k_i^2 r_i + \lambda k_i + \mu r_i + \nu s_i) t. \quad (36)$$

We next substitute

$$u(x, y, z, t) = 2(\ln(f(x, y, z, t)))_x, \quad (37)$$

where the auxiliary function is given by

$$f(x, y, z, t) = 1 + e^{k_1 x + r_1 y + s_1 z - (k_1^2 r_1 + \lambda k_1 + \mu r_1 + \nu s_1) t}. \quad (38)$$

Consequently, the single soliton solution for the (3+1)-dimensional nonlinear evolution equation (21) is given by

$$u(x, y, z, t) = \frac{2k_1 e^{k_1 x + r_1 y + s_1 z - (k_1^2 r_1 + \lambda k_1 + \mu r_1 + \nu s_1) t}}{1 + e^{k_1 x + r_1 y + s_1 z - (k_1^2 r_1 + \lambda k_1 + \mu r_1 + \nu s_1) t}}. \quad (39)$$

Recall that $v(x, y, z, t) = u_x(x, y, z, t)$.

To derive the two-soliton solutions, we substitute

$$f(x, y, z, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \quad (40)$$

into Eq. (37), and then into (33), where θ_1 and θ_2 are given in (36), we find that the phase shift a_{12} is given by

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad (41)$$

and hence

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, 1 \leq i < j \leq N. \quad (42)$$

To determine the two-soliton solutions explicitly, we substitute (40) into (37).

To determine the three-soliton solutions we set

$$u(x, y, z, t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1 + \theta_2} + a_{23} e^{\theta_2 + \theta_3} + a_{13} e^{\theta_1 + \theta_3} + a_{12} a_{23} a_{13} e^{\theta_1 + \theta_2 + \theta_3}. \quad (43)$$

We proceed as before to formally derive the three soliton solutions.

4 Discussion

In this work, we examined two new (3+1)-dimensional KdV-CBS equation and the negative-order of the KdV-CBS equation. We showed that both equations pass nicely the Painlevé test without any restriction on the compatibility conditions or the parameters involved in each equation. We noticed that the dispersion relations of the two models are distinct, whereas the phase shifts remain the same for the two developed models. The phase shift retained the KdV-type phase shift. Both equations were handled by using a simplified form of the Hirota method. Multiple soliton solutions for each equation were formally derived.

Compliance with ethical standards

Conflict of interest We declare we have no conflict of interests.

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