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Performance evaluation and design criterion of a bistable nonlinear energy sink

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Abstract: In the evaluation of the vibration mitigation effect of a nonlinear energy sink (NES), the selection of performance measures is important. Based on a square root model, this paper aims to propose new reasonable performance measures and new design criteria of the NES in harmonic excitation. To broaden the application range of the slow invariant manifold, coordinate correction is carried out for the square root model of the bistable NES. Two original performance measures (the displacement amplitude of the main system and the energy dissipation rate of the NES) and two new performance measures (the mechanical energy and the input energy of the main system) are compared. The optimal geometric configuration and response form of the NES are determined in different excitation conditions. The results show that the energy dissipation rate of the NES is not competent to evaluate the performance of the NES in harmonic excitation, while the two new performance measures show competence. By adjusting the geometry parameter, the nonlinear vibration absorber with positive linear stiffness and the bistable NES show optimal vibration mitigation effect in small and large excitation amplitudes, respectively. The optimal response form is the periodic response rather than the strongly modulated response.

Keywords: Vibration control; nonlinear energy sink; slow invariant manifold; design criterion

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1 Introduction

Nonlinear vibration control devices have greater potential values in practical engineering because of broadband mitigation and high robustness in comparison with linear vibration control devices [1–4]. Nonlinear energy sink (NES), as a representative of nonlinear control devices, can trigger irreversible targeted energy transfer (TET) through nonlinear normal modes and passively absorb the energy of multiple modes with relatively small mass [5, 6]. To realize the vibration control in different working conditions, researchers have designed various types of non-smooth and non-polynomial models of the NES, such as magnetic models [7, 8], vibro-impact models [9, 10], piecewise models [11, 12], rotational models [13, 14], and track models [15, 16].

Compared with the above NES structures, a classical physical structure of the NES, namely a mass-damping and double-spring structure [17–19], is similar to the structure of linear vibration absorber and has a high application value. This physical structure can realize both a monostable NES with zero linear stiffness and a bistable NES with negative linear stiffness. Based on these two polynomial models, the dynamics and vibration mitigation effect of the NES are analyzed in the harmonic excitation. The complex dynamic phenomena in the monostable NES and the bistable NES include cyclic-fold and Hopf bifurcations, symmetry-breaking and period-doubling bifurcations, and chaos [20–24]. Apart from these phenomena, strongly modulated response (SMR) also plays an important role in the dynamics and vibration mitigation effect of the NES. Compared with periodic and weakly quasiperiodic responses, the SMR shows more phase-locked or chaotic characteristics [25]. Besides, better vibration mitigation effect of the SMR versus the periodic response was demonstrated in Ref. [26], so Qiu et al. [23] and Li et al. [24] tuned the parameters (especially the negative linear stiffness and nonlinear stiffness) of the bistable NES to activate the SMR as much as possible, which is the design criterion of the polynomial model of the bistable NES. Although researchers have studied the dynamics, vibration mitigation effect, and design criterion of the polynomial model of the NES, the awareness of the square root model of the NES which can better reflect the original physical structure and global dynamics is not deep enough. In the previous work, the authors have shown the differences between these two NES models in the potential energy surface, dynamics, and vibration mitigation effect of the impulsively excited autonomous system [27]. However, the differences in the harmonically excited non-autonomous system are not clear. On the other hand, the square root model may better reveal the response

characteristics of the NES with different geometric configurations and the response form with optimal vibration mitigation effect. These conclusions may differ from the polynomial model, and new design criteria of the NES may spring up.

When investigating the vibration mitigation effect of the NES, the selection of performance measures is very important. In different excitation conditions, researchers adopted different or even multiple indexes, e.g., energy dissipation rate and energy decaying duration in impulsive excitation [28–31], amplitude of the main system, energy dissipation rate of the NES, energy of the entire system, and transmissibility in harmonic excitation [23, 26, 32, 33], transition probability densities of transmissibility and input energy absorption percentage of the main system in combined harmonic and random excitation [34], and displacement, acceleration, and shear force of the main system in seismic excitation [9, 35]. From these studies, the vibration mitigation effect of the NES in harmonic excitation is mainly judged by the absolute number and rate of amplitude and energy. However, there is little discussion about the rationality of these indexes. It is necessary to assess the rationality of the indexes and propose some new reasonable performance measures for avoiding erroneous conclusions and suggestions.

According to the above discussion, this paper is structured as follows. Sect. 2 describes the square root model, potential geometrical configurations, and corresponding parameter values of the NES. Sect. 3 and Sect. 4 outline the slow invariant manifold (SIM) and generalized harmonic balance method (G-HBM) for analyzing the dynamics of the NES, and exhibit the time and frequency domain features of the modulated response in different geometrical configurations. Sect. 5 compares the excitation ranges of the modulated response in the square root and polynomial NES models and different geometrical configurations. In Sect. 6, two original indexes and two proposed indexes are compared for NES vibration mitigation evaluation and design criterion, and optimal geometrical configuration and response form are revealed in different excitation conditions. Finally, Sect. 7 addresses the conclusions.

2 Mechanical model

Fig. 1 presents the model of a linear oscillator (LO) coupled with an NES. Table 1 shows the meanings and values of notations. L_1 and L_2 are two characteristic lengths of NES springs. X_1 and X_2 are the displacement of the LO and the NES, respectively. F_0 and Ω are the amplitude and

frequency of harmonic force, respectively.

Fig. 1 Model of an LO coupled with an NES: **a** $L_1 \leq L$, $L_2 \geq 0$; **b** $L_1 > L$, $L_2 = 0$

Table 1 Physical parameters of the LO couple with the NES

Item	Notation	Value
Mass of the LO	M_1	2 kg
Mass of the NES	M_2	0.1 kg
Damping of the LO and NES	C_1 & C_2	0.3 N/m
Stiffness of the LO and NES spring	K_1 & K_s	2790 N/m
Free length of the NES spring	L	0.1 m

By utilizing Newton's law or the extended Hamilton principle, the dynamic equations of the LO-NES system can be derived as

$$\begin{aligned}
 M_1 \frac{d^2 X_1}{dT^2} + C_1 \frac{dX_1}{dT} + K_1 X_1 + C_2 \left(\frac{dX_1}{dT} - \frac{dX_2}{dT} \right) + 2K_s (X_1 - X_2 + L_2) \left[1 - \frac{\sqrt{L_1^2 + L_2^2}}{L_1^2 + (X_1 - X_2 + L_2)^2} \right] + F_0 \cos(\Omega T) &= 0, \\
 M_2 \frac{d^2 X_2}{dT^2} + C_2 \left(\frac{dX_2}{dT} - \frac{dX_1}{dT} \right) - 2K_s (X_1 - X_2 + L_2) \left[1 - \frac{\sqrt{L_1^2 + L_2^2}}{L_1^2 + (X_1 - X_2 + L_2)^2} \right] &= 0.
 \end{aligned} \tag{1}$$

The dimensionless form of Eq. (1) is

$$\begin{aligned}
 \ddot{x}_1 + c_1 \dot{x}_1 + x_1 + c_2 (\dot{x}_1 - \dot{x}_2) + 2k_s (x_1 - x_2 + l_2) \left[1 - \frac{1}{\sqrt{l_1^2 + (x_1 - x_2 + l_2)^2}} \right] + f_0 \cos(\omega t) &= 0, \\
 \varepsilon \ddot{x}_2 + c_2 (\dot{x}_2 - \dot{x}_1) - 2k_s (x_1 - x_2 + l_2) \left[1 - \frac{1}{\sqrt{l_1^2 + (x_1 - x_2 + l_2)^2}} \right] &= 0,
 \end{aligned} \tag{2}$$

where the dot ($\dot{\bullet}$) is the derivative compared to t . The expressions of dimensionless variables and parameters are

$$\begin{aligned}
 \omega_0 &= \sqrt{\frac{K_1}{M_1}}, t = \omega_0 T, x_1 = \frac{X_1}{L}, x_2 = \frac{X_2}{L}, l_1 = \frac{L_1}{L}, l_2 = \frac{L_2}{L}, c_1 = \frac{C_1}{M_1 \omega_0}, \\
 c_2 &= \frac{C_2}{M_1 \omega_0}, k_s = \frac{K_s}{M_1 \omega_0^2}, \varepsilon = \frac{M_2}{M_1}, f_0 = \frac{F_0}{M_1 L \omega_0^2}, \omega = \frac{\Omega}{\omega_0}.
 \end{aligned} \tag{3}$$

According to the physical values in Table 1, some dimensionless parameter values are fixed, as shown in Table 2. These dimensionless parameter values are the same as those in the previous study [27]. The geometry parameter, namely l_1 , determines the property of the NES ($0 < l_1 < 1$ for the bistable NES, $l_1 = 1$ for the monostable NES, and $l_1 > 1$ for the nonlinear vibration absorber with positive linear stiffness). l_2 denotes the unstable static equilibrium position of the bistable NES, while 0 and $2l_2$ are two stable static equilibrium positions of the bistable NES. The depth of the potential well of the bistable NES is $k_s(1-l_1)^2$, which indicates that the closer l_1 gets to 0, the deeper the potential well is.

Table 2 Reduced parameters of the LO couple with the NES

Item	Notation	Value
Mass ratio of the NES to the LO	ε	5%
Dimensionless damping of the LO and the NES	c_1 & c_2	0.004
Spring stiffness ratio of the NES to the LO	k_s	1

When $|x_1 - x_2 + l_2| < l_1$, Eq. (2) can be converted into the dimensionless polynomial equations by using the Taylor series:

$$\begin{aligned}
\ddot{x}_1 + c_1 \dot{x}_1 + x_1 + c_2 (\dot{x}_1 - \dot{x}_2) + 2k_s \left(1 - \frac{1}{l_1}\right) (x_1 - x_2 + l_2) + \frac{k_s}{l_1^3} (x_1 - x_2 + l_2)^3 + f_0 \cos(\omega t) &= 0, \\
\varepsilon \ddot{x}_2 + c_2 (\dot{x}_2 - \dot{x}_1) - 2k_s \left(1 - \frac{1}{l_1}\right) (x_1 - x_2 + l_2) - \frac{k_s}{l_1^3} (x_1 - x_2 + l_2)^3 &= 0,
\end{aligned} \tag{4}$$

where Eq. (4) keeps the first two terms of the Taylor series. It can be seen that Eq. (4) can only describe local dynamic behaviors of the system, while Eq. (2) can describe global dynamic behaviors. In the following part, Eq. (2) and Eq. (4) will be handled in numerical and two analytical ways, namely, the SIM and the G-HBM. The authors will elaborate the two analytical ways in the next two sections.

3 Slow invariant manifold (SIM)

3.1 Derivation and stability analysis of the ε^0 -SIM

The SIM aims to reveal the modulated response in the vicinity of 1:1:1 resonance manifold, which denotes the vibrational frequencies of the LO and the NES are both close to the excitation frequency [36]. To better study the global and local dynamic characteristics and conduct the

complexification-averaging and multi-time scale processes of the SIM, the rescaling roles, coordinate correction, and complex variables are introduced:

$$\begin{aligned} c_1 &= \varepsilon c_3, c_2 = \varepsilon c_4, k_s = \varepsilon k_0, f_0 = \varepsilon f_1, x_3 = x_2 - l_2, y_1 = x_1 + \varepsilon x_3, y_2 = x_1 - x_3, \\ \phi_1 e^{i\omega t} &= \dot{y}_1 + i\omega y_1 = \dot{z}_1 + i\omega z_1, \phi_2 e^{i\omega t} = \dot{y}_2 + i\omega y_2 = \dot{z}_3 + i\omega z_3, \end{aligned} \quad (5)$$

where y_1 is the displacement of the mass center of the entire system, and y_2 is the relative displacement of the NES to the LO. The coordinate correction in Eq. (5) is further explained in Fig. 2. o_1 and o_2 are the origins of the coordinate of x_1 and x_2 , respectively. o_1 and o_2 are overlapped in the LO with the monostable NES, while they are different in the LO with the bistable NES. To determine y_1 and y_2 in the same coordinate system, o_2 is shifted to o_1 , and the position of the NES relative to o_1 is denoted as x_3 . Thus, y_1 and y_2 can be represented by x_1 and x_3 .

Fig. 2 Coordinate correction of the dimensionless system

Substituting Eq. (5) into Eq. (2) and averaging over a period yield the following slowly modulated system:

$$\begin{aligned} \dot{z}_1 + \frac{\varepsilon c_3 z_1}{2(\varepsilon+1)} - \frac{[(\varepsilon+1)\omega^2 - 1]z_2}{2(\varepsilon+1)\omega} + \frac{\varepsilon^2 c_3 z_3}{2(\varepsilon+1)} + \frac{\varepsilon z_4}{2(\varepsilon+1)\omega} + \frac{\varepsilon f_1}{2} &= 0, \\ \dot{z}_2 + \frac{[(\varepsilon+1)\omega^2 - 1]z_1}{2(\varepsilon+1)\omega} + \frac{\varepsilon c_3 z_2}{2(\varepsilon+1)} - \frac{\varepsilon z_3}{2(\varepsilon+1)\omega} + \frac{\varepsilon^2 c_3 z_4}{2(\varepsilon+1)} &= 0, \\ \dot{z}_3 + \frac{\varepsilon c_3 z_1}{2(\varepsilon+1)} + \frac{z_2}{2(\varepsilon+1)\omega} + \frac{[\varepsilon^2 c_3 + (\varepsilon+1)^2 c_4]z_3}{2(\varepsilon+1)} - \left[\frac{\omega}{2} - \frac{2(\varepsilon+1)^2 k_0 + \varepsilon}{2(\varepsilon+1)\omega} \right] z_4 - \frac{4(\varepsilon+1)k_0 I_1 z_4}{\pi \sqrt{z_3^2 + z_4^2}} + \frac{\varepsilon f_1}{2} &= 0, \\ \dot{z}_4 - \frac{z_1}{2(\varepsilon+1)\omega} + \frac{\varepsilon c_3 z_2}{2(\varepsilon+1)} + \left[\frac{\omega}{2} - \frac{2(\varepsilon+1)^2 k_0 + \varepsilon}{2(\varepsilon+1)\omega} \right] z_3 + \frac{[\varepsilon^2 c_3 + (\varepsilon+1)^2 c_4]z_4}{2(\varepsilon+1)} + \frac{4(\varepsilon+1)k_0 I_1 z_3}{\pi \sqrt{z_3^2 + z_4^2}} &= 0, \end{aligned} \quad (6)$$

where

$$I_1 = \int_0^{\pi/2} \frac{z_5 \cos^2(\tau)}{\sqrt{l_1^2 + z_5^2 \cos^2(\tau)}} d\tau, z_5 = \frac{\sqrt{z_3^2 + z_4^2}}{\omega}. \quad (7)$$

Here, the authors introduce multi-scale variables and a detuning parameter σ which represents the distance of the excitation frequency ω to the reduced natural frequency of the LO:

$$\begin{aligned}
\omega &= 1 + \varepsilon\sigma, T_0 = t, T_1 = \varepsilon t, \frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \dots, \\
\phi_i(t) &= \phi_{i,0}(T_0, T_1, \dots) + \varepsilon \phi_{i,1}(T_0, T_1, \dots) + \dots, i = 1, 2, \\
z_j(t) &= z_{j,0}(T_0, T_1, \dots) + \varepsilon z_{j,1}(T_0, T_1, \dots) + \dots, j = 1, 2, \dots, 5, \\
z_{5,0} &= \sqrt{z_{3,0}^2 + z_{4,0}^2}, z_{5,1} = \frac{z_{3,0}z_{3,1} + z_{4,0}z_{4,1}}{z_{5,0}} - \sigma z_{5,0}, \\
I_1 &= I_{1,0} + \varepsilon I_{1,1} + \dots, I_{1,0} = z_{5,0} I_2, I_{1,1} = z_{5,1} (I_2 - z_{5,0}^2 I_3), \\
I_2 &= \int_0^{\pi/2} \frac{\cos^2(\tau)}{\sqrt{I_1^2 + z_{5,0}^2 \cos^2(\tau)}} d\tau, I_3 = \int_0^{\pi/2} \frac{\cos^4(\tau)}{[I_1^2 + z_{5,0}^2 \cos^2(\tau)]^{3/2}} d\tau,
\end{aligned} \tag{8}$$

Then Eq. (6) can be divided into different time scales. The approximate equations of ε^0 order are

$$\begin{aligned}
\frac{\partial z_{1,0}}{\partial T_0} &= 0, \frac{\partial z_{3,0}}{\partial T_0} + \frac{z_{2,0}}{2} + \frac{c_4 z_{3,0}}{2} + \left(k_0 - \frac{1}{2} - \frac{4k_0 I_2}{\pi}\right) z_{4,0} = 0, \\
\frac{\partial z_{2,0}}{\partial T_0} &= 0, \frac{\partial z_{4,0}}{\partial T_0} - \frac{z_{1,0}}{2} - \left(k_0 - \frac{1}{2} - \frac{4k_0 I_2}{\pi}\right) z_{3,0} + \frac{c_4 z_{4,0}}{2} = 0.
\end{aligned} \tag{9}$$

The approximate equations of ε^1 order are

$$\begin{aligned}
\frac{\partial z_{1,0}}{\partial T_1} + \frac{\partial z_{1,1}}{\partial T_0} + \frac{c_3 z_{1,0}}{2} - \left(\frac{1}{2} + \sigma\right) z_{2,0} + \frac{z_{4,0}}{2} + \frac{f_1}{2} &= 0, \frac{\partial z_{2,0}}{\partial T_1} + \frac{\partial z_{2,1}}{\partial T_0} + \left(\frac{1}{2} + \sigma\right) z_{1,0} + \frac{c_3 z_{2,0}}{2} - \frac{z_{3,0}}{2} = 0, \\
\frac{\partial z_{3,0}}{\partial T_1} + \frac{\partial z_{3,1}}{\partial T_0} + \frac{c_3 z_{1,0}}{2} - \frac{(\sigma+1)z_{2,0}}{2} + \frac{c_4 z_{3,0}}{2} - (\sigma-1) \left(k_0 + \frac{1}{2}\right) z_{4,0} + \frac{z_{2,1}}{2} + \frac{c_4 z_{3,1}}{2} + \left(k_0 - \frac{1}{2}\right) z_{4,1} \\
+ \frac{4k_0 [z_{4,0}^2 z_{4,1} + z_{4,0} (z_{3,0} z_{3,1} - z_{5,0} - z_{5,1}) - z_{4,1} z_{5,0}]}{\pi z_{5,0}^2} + \frac{4k_0 z_{4,0} z_{5,1} I_3}{\pi} + \frac{f_1}{2} &= 0, \\
\frac{\partial z_{4,0}}{\partial T_1} + \frac{\partial z_{4,1}}{\partial T_0} + \frac{(\sigma+1)z_{1,0}}{2} + \frac{c_3 z_{2,0}}{2} + (\sigma-1) \left(k_0 + \frac{1}{2}\right) z_{3,0} + \frac{c_4 z_{4,0}}{2} - \frac{z_{1,1}}{2} - \left(k_0 - \frac{1}{2}\right) z_{3,1} + \frac{c_4 z_{4,1}}{2} \\
- \frac{4k_0 [z_{3,0} (z_{4,0} z_{4,1} - z_{5,0} - z_{5,1}) - z_{4,0}^2 z_{3,1}]}{\pi z_{5,0}^2} - \frac{4k_0 z_{3,0} z_{5,1} I_3}{\pi} &= 0.
\end{aligned} \tag{10}$$

Then the authors introduce new variables (Z_1, Z_2, N_1, N_2) representing the amplitude of $\phi_{1,0}$ and $\phi_{2,0}$:

$$|\phi_{1,0}|^2 = Z_1 = N_1^2 = z_{1,0}^2 + z_{2,0}^2, |\phi_{2,0}|^2 = Z_2 = N_2^2 = z_{3,0}^2 + z_{4,0}^2 = z_{5,0}^2. \tag{11}$$

By utilizing Eq. (11) and considering the steady state of Eq. (9), the expression of the ε^0 -SIM as well as the relation among $z_{1,0}$, $z_{2,0}$, $z_{3,0}$, and $z_{4,0}$ can be obtained:

$$Z_1 = Z_2 \left[\left(\frac{8k_0 I_2}{\pi} - 2k_0 + 1 \right)^2 + c_4^2 \right], \tag{12}$$

$$z_{2,0} = -c_4 z_{3,0} + \left(\frac{8k_0 I_2}{\pi} - 2k_0 + 1 \right) z_{4,0}, z_{1,0} = \left(\frac{8k_0 I_2}{\pi} - 2k_0 + 1 \right) z_{3,0} + c_4 z_{4,0}. \tag{13}$$

It can be seen that the expression of the ε^0 -SIM (12) only relates to the parameters (k_0 and c_4) of the NES and does not depend on the parameters of the LO and excitation.

To judge the stability of the ε^0 -SIM, the perturbed solutions (δ_1 and δ_2) are assumed:

$$z_{3,0}(T_0, T_1) = z_{3,0}(T_1) + \delta_1(T_0), z_{4,0}(T_0, T_1) = z_{4,0}(T_1) + \delta_2(T_0). \quad (14)$$

By substituting Eq. (14) into Eq. (9) and subtracting the new equation from Eq. (9), the equations of perturbed system can be derived:

$$\begin{aligned} \frac{\partial \delta_1}{\partial T_0} &= \frac{4k_0(z_{4,0} + \delta_2)}{\pi} \int_0^{\pi/2} \frac{\cos^2(\tau)}{\sqrt{l_1^2 + [(z_{3,0} + \delta_1)^2 + (z_{4,0} + \delta_2)^2] \cos^2(\tau)}} d\tau - \frac{4k_0 z_{4,0} I_2}{\pi} - \frac{c_4 \delta_1}{2} - \left(k_0 - \frac{1}{2}\right) \delta_2, \\ \frac{\partial \delta_2}{\partial T_0} &= -\frac{4k_0(z_{3,0} + \delta_1)}{\pi} \int_0^{\pi/2} \frac{\cos^2(\tau)}{\sqrt{l_1^2 + [(z_{3,0} + \delta_1)^2 + (z_{4,0} + \delta_2)^2] \cos^2(\tau)}} d\tau + \frac{4k_0 z_{3,0} I_2}{\pi} + \left(k_0 - \frac{1}{2}\right) \delta_1 - \frac{c_4 \delta_2}{2}. \end{aligned} \quad (15)$$

It is a remarkable coincidence that the determinant of the Jacobi matrix of Eq. (15) equals $dZ_1/(4dZ_2)$. In all the trials, if $dZ_1/dZ_2 > 0$, there will be two negative eigenvalues, which correspond to the stable part of the ε^0 -SIM, while $dZ_1/dZ_2 < 0$ denotes the unstable part of the ε^0 -SIM. Therefore, the solution satisfying $dZ_1/dZ_2 = 0$ denotes the bifurcation of the ε^0 -SIM, which may deeply affect the motion of the system.

3.2 Effects of the geometry parameter on the response of the NES

Previous studies show that by combining the ε^0 -SIM and the results computed by the Runge-Kutta method (RKM), qualitative distinctions can be made between the modulated response and other types of response [23, 37]. Here, the results of the ε^0 -SIM, the RKM, and the fast Fourier transformation (FFT) are compared to reveal the characteristics of responses in the square root model of the NES with fixed excitation frequency ($\omega=1$) and different geometry (l_1) and excitation (f_0) conditions. The calculation duration of the RKM is 1000 periods corresponding to the excitation frequency, and the time step is 1/200 of one period. Both the initial displacement and velocity of the LO and the NES are set as 0. The last 20-period results of the RKM will be presented on the ε^0 -SIM plane. R_ω denotes the ratio of the FFT frequency to the excitation frequency.

Firstly, the authors concentrate on the modulated response of the monostable and bistable NESs. In Fig. 3a, Fig. 4a, and Fig. 5a, the projections of the modulated response on the ε^0 -SIM plane shuttle between the two stable manifolds due to the coexistence of large and small amplitude motion. Compared with the FFT result of the monostable NES in Fig. 3c, the FFT results of the

bistable NES in Fig. 4c and Fig. 5c contain more zero-drift components, which are related to the differences of their potential energy surfaces (numbers of static equilibrium positions). In Fig. 4b, the small amplitude motion of the bistable shallow-well NES is the cross-well motion around the unstable static equilibrium position and the time-domain features of the modulated response are similar to those in Fig. 3b. However, in Fig. 5b, the small amplitude motion of the bistable deep-well NES is the in-well motion around either one of the stable static equilibrium positions, which will cause much disorder in the modulated response of the bistable deep-well NES. In previous studies [23, 37], the results in Fig. 3b and Fig. 4b are named as the strongly modulated responses (SMR). To distinguish the SMR, the result in Fig. 5b is named as the chaotically modulated response (CMR).

Fig. 3 SMR of the monostable unstretched NES: **a** ε^0 -SIM projections; **b** displacement time histories; **c** FFT

Fig. 4 SMR of the bistable shallow-well NES: **a** ε^0 -SIM projections; **b** displacement time histories; **c** FFT

Fig. 5 CMR of the bistable deep-well NES: **a** ε^0 -SIM projections; **b** displacement time histories; **c** FFT

Apart from the modulated response, there are also periodic responses and cross-well chaos in the square root model of the NES. Their characteristics may differ. Since the bistable NES contains

more abundant dynamical behaviors (e.g., the CMR and cross-well chaos) than the monostable NES, the authors will stress the characteristics of the response of the bistable NES. Since the local and global periodic motions (LPM and GPM) are the motions with only small and large amplitudes, respectively, and the large amplitude of cross-well chaos cannot reach the large amplitude of modulated responses, the projections of periodic response and cross-well chaos on the ε^0 -SIM plane just remain near one of the stable manifolds in Fig. 6a, Fig. 7a, and Fig. 8a. From the results of Fig. 4c, Fig. 5c, and Fig. 6c, it can be seen that the zero-drift components take the lead in the cross-well chaos, which differs from the modulated response. The reason is the cross-well chaos contains more local information than the modulated response. The results of Fig. 7c and Fig. 8c show the dominance of the excitation frequency component in both the LPM and the GPM. However, suffering from multiple static equilibrium positions, the GPM possesses more zero-drift and high-frequency components than the LPM.

Fig. 6 Chaos of the bistable shallow-well NES: **a** ε^0 -SIM projections; **b** displacement time histories; **c** FFT

Fig. 7 LPM of the bistable deep-well NES: **a** ε^0 -SIM projections; **b** displacement time histories; **c** FFT

Fig. 8 GPM of the bistable deep-well NES: **a** ε^0 -SIM projections; **b** displacement time histories; **c** FFT

Fig. 3–Fig. 8 present the ε^0 -SIM curves of the bistable and monostable NESs. So what about the ε^0 -SIM curve of the nonlinear vibration absorber (NVA) with positive linear stiffness? Fig. 9 will explain that. In Fig. 9, with the increase of the geometry parameter l_1 , the distance between the two bifurcation points of the stable and unstable manifolds becomes smaller on the Z_2 scale, and the system has a unique stable manifold when $l_1 \geq 1.03$. Since the existence of the unstable manifold represents the possibility of the modulated response, the results in Fig. 9 indicate that the system may not generate the modulated response and only have periodic response when the springs of the NVA is extended to a certain extent. Besides, the differences on the distance between the two bifurcation points on the Z_2 scale may affect the excitation conditions of the modulated response in the NVA with different geometry parameter, which needs further investigation.

Fig. 9 ε^0 -SIM curves of the NVA with positive linear stiffness: **a** $l_1=1.01$; **b** $l_1=1.02$; **c** $l_1=1.03$

3.3 Derivation of the ε^1 -SIM

Back to Eq. (12), the expression of the ε^0 -SIM does not include the excitation parameters (σ, f_1) . Without the help of the RKM, further treatment on the SIM is needed for quantifying the excitation condition of the modulated response. By considering the steady-state of T_0 time scale of the first two ε^1 -SIM equations (10) and utilizing Eq. (13) and the polar forms of $z_{3,0}$ and $z_{4,0}$ ($z_{3,0}=N_2\cos(\theta_2)$, $z_{4,0}=N_2\sin(\theta_2)$), the ε^1 -SIM equation will be rewritten:

$$\frac{\partial N_2}{\partial T_1} = \frac{g_1(N_2, \theta_2)}{h(N_2)}, \quad \frac{\partial \theta_2}{\partial T_1} = \frac{g_2(N_2, \theta_2)}{h(N_2)}, \quad (16)$$

where

$$\begin{aligned} g_1 = & \left[-\pi^2 (4k_0^2 c_3 + c_3 c_4^2 - 4k_0 c_3 + c_3 + c_4) + \pi c_3 (32k_0^2 - 16k_0) I_2 - 64k_0^2 c_3 I_2^2 \right] N_2 \\ & + \pi f_1 \left[\pi (2k_0 - 1) - 8k_0 I_2 \right] \cos(\theta_2) - \pi^2 c_4 f_1 \sin(\theta_2), \\ g_2 = & \left[16k_0 (2\sigma + 1) (4I_2 - \pi) + 8\pi (2\sigma - c_3 c_4) \right] k_0 I_3 N_2^3 - \pi^2 c_4 f_1 \cos(\theta_2) - \pi f_1 \left[\pi (2k_0 - 1) - 8k_0 (I_2 - I_3 N_2^2) \right] \sin(\theta_2) \\ & - \left\{ 64(2\sigma + 1) k_0^2 I_2^2 + 8\pi k_0 (8\sigma k_0 - 4\sigma + 4k_0 - 1) I_2 + \pi^2 \left[(2\sigma + 1) c_4^2 + 2\sigma (2k_0 - 1)^2 + 2k_0 (2k_0 - 1) \right] \right\} N_2, \\ h = & 2 \left[\pi (2k_0 - 1) - 8k_0 I_2 \right]^2 + 16 \left[\pi (2k_0 - 1) - 8k_0 I_2 \right] k_0 I_3 N_2^2 + 2\pi^2 c_4^2 = 2\pi^2 \frac{dZ_1}{dZ_2}. \end{aligned} \quad (17)$$

It can be seen from Eqs. (16) and (17) that singular points emerge when $dZ_1/dZ_2=0$, which corresponds to the bifurcation points of the ε^0 -SIM. Then by considering the steady-state of T_1 time scale ($g_1=g_2=0$) and eliminating the phase angle θ_2 , the ε^1 -SIM equation can be derived:

$$\left(2\pi\sigma c_4 - 2\pi c_3 k_0 + 8c_3 k_0 I_2 + \pi c_3 + \pi c_4\right)^2 + \left(4\pi\sigma k_0 + \pi c_3 c_4 - 16\sigma k_0 I_2 - 2\pi\sigma + 2\pi k_0 - 8k_0 I_2\right)^2 = \frac{\pi^2 f_1^2}{Z_2}. \quad (18)$$

Compared with the ε^0 -SIM equation (12), the ε^1 -SIM equation (18) includes the LO damping (c_3) and excitation parameters (σ, f_1). Therefore, the excitation conditions of the modulated response may be quantified by Eq. (18) and the bifurcation points of the ε^0 -SIM.

4 Generalized harmonic balance method (G-HBM)

Compared with the SIM, the G-HBM proposed by Luo and Huang [38] places more emphases on solving accurate analytical periodic solutions of nonlinear dynamical systems and can address the issue of non-polynomial nonlinear vibration. In the G-HBM, the approximate periodic solution of Eq. (2) is assumed as

$$x_i(t) = a_{i,0} + \sum_{j=1}^n \left[a_{i,2j-1} \cos(j\omega t) + a_{i,2j} \sin(j\omega t) \right], i=1,2, \quad (19)$$

where $a_{i,0}, a_{i,1}, \dots, a_{i,2n}$ represent undetermined amplitudes of each harmonic component. Here, the G-HBM is simplified by utilizing the Galerkin procedure. The orthogonal trial functions are introduced as $[1, \cos(j\omega t), \sin(j\omega t)]$, where $j=1, 2, \dots, n$. n is set as 3 for high accuracy. Substituting Eq. (19) into Eq. (2) and averaging over a period with the trial functions yield the Galerkin conditions:

$$\int_0^{2\pi/\omega} R_i(t) dt = 0, \int_0^{2\pi/\omega} R_i(t) \cos(j\omega t) dt = 0, \int_0^{2\pi/\omega} R_i(t) \sin(j\omega t) dt = 0, i=1,2, j=1,2,\dots,n, \quad (20)$$

where $R_1(t)$ and $R_2(t)$ are the left side of Eq. (2) and denote the residual force. Then the arc-length continuation method can be adopted to solve Eq. (20). To judge the stability of periodic solutions, the perturbed amplitudes $[\Delta a_{i,0}(t), \Delta a_{i,1}(t), \dots, \Delta a_{i,2n}(t)]$ are added into Eq. (19):

$$x_i(t) = a_{i,0} + \Delta a_{i,0}(t) + \sum_{j=1}^n \left\{ \left[a_{i,2j-1} + \Delta a_{i,2j-1}(t) \right] \cos(j\omega t) + \left[a_{i,2j} + \Delta a_{i,2j}(t) \right] \sin(j\omega t) \right\}, i=1,2. \quad (21)$$

Similarly, a group of nonlinear differential integral equations in a perturbed dynamic system is obtained with the Galerkin procedure. The eigenvalues of Jacobi matrix of the perturbed dynamic system will be used in the stability judgement of the periodic solutions.

5 Result comparison among different configurations, methods, and models

Based on the above introduction of the SIM, the G-HBM, and the RKM, the frequency responses of the square root model of the NES can be obtained with varied geometry parameter l_1 , as shown in Fig. 10. A_{NES} is the steady-state displacement amplitude of the NES. PM is the abbreviation of periodic motion. The upward and backward forms of sweeping frequency will be both considered in the RKM. The calculation duration and time step of the RKM are the same as those in Subsection 3.2. The last 200-period results of the RKM are left for computing the displacement amplitude of the NES. The frequency responses of the RKM show the maximum and average amplitudes of the NES. In the results of the ε^1 -SIM, the frequency ranges of the modulated response are determined by the bifurcation points of the ε^0 -SIM, and the shapes of response curves are similar in different geometry configurations. In the results of the G-HBM and the RKM, the frequency ranges of the modulated response are determined by the stability of periodic solutions and the features of time-domain responses. Since the bistable shallow-well NES contains cross-well chaos, its frequency response shows more complexity than the monostable NES and the bistable deep-well NES.

Fig. 10 Frequency responses of the square root model of the NES obtained by three methods with varied geometry parameter l_1 and fixed excitation amplitude ($f_0=0.005$): **a** ε^1 -SIM, $l_1=1$; **b** G-HBM and RKM, $l_1=1$; **c** ε^1 -SIM, $l_1=0.99$; **d** G-HBM and RKM, $l_1=0.99$; **e** ε^1 -SIM, $l_1=0.9$; **f** G-HBM and RKM, $l_1=0.9$

Table 3 summarizes the results of excitation frequency ranges of the modulated response in Fig. 10. The results show that when the excitation amplitude is 0.005, the excitation frequency range of the modulated response becomes narrower with the decrease of the geometry parameter l_1 from 1. This can be explained by the results of ε^0 -SIM in Subsection 3.2. When the monostable NES changes to the bistable NES, the distance between the two stable manifolds of ε^0 -SIM on the Z_2 scale will increase, and the time domain response will be more likely to stay near a certain stable manifold and more difficult to shuttle between the two stable manifolds.

Table 3 also compares the results obtained by the three methods in Fig. 10. For the monostable NES ($l_1=1$), the results of the three methods are almost the same, while for the bistable NES ($l_1<1$), the results of the ε^1 -SIM are significantly different from those of the other two methods. This can be traced back to the hypothesis of the SIM, namely the excitation frequency dominating the vibration of the LO and the NES. As the results in Subsection 3.2 show multiple considerable frequency components in the modulated response and the GPM of the bistable NES, this hypothesis will cause a huge deviation for the SIM to predict the modulated response of the bistable NES. Therefore, the SIM may only be applicable to quantitative calculations of the response of the monostable NES and not to those of the bistable NES. However, compared with the G-HBM, the SIM can save a lot of computation time and also reflect the intrinsic properties of the NES (the variation trend of the excitation range of the modulated response with the geometry parameter), so the SIM still plays an important role in predicting the modulated response of the NES.

The results in Fig. 10 and Table 3 mainly focus on the frequency response of the square root

model of the NES with different geometry configurations and methods. Here, the authors further compare the differences of the response between the square root and polynomial models of the NES, as shown in Fig. 11. In the monostable situation of Fig. 11a, b, there is no difference between the two NES models. Yet there are some certain differences in the bistable situation of Fig. 11c–f. In the bistable shallow-well situation of Fig. 11c, d, the square root model has a larger excitation amplitude range of the modulated response than the polynomial model. In the bistable deep-well situation of Fig. 11e, f, the square root model requires larger excitation amplitude to achieve the modulated response. This also indicates that the deeper the potential well is, the more differences on the excitation conditions of the modulated response can be discovered in these two NES models.

Table 3 Frequency range of modulated response obtained by three methods with varied geometry parameter l_1 and fixed excitation amplitude ($f_0=0.005$)

Geometry parameter l_1	Method	Frequency range of modulated response
1	SIM	0.9886–1.0365
	G-HBM	0.9861–1.0314
	RKM	0.985–1.04
0.99	SIM	0.9874–1.0165
	G-HBM	0.9652–1.0546
	RKM	0.975–1.05
0.9	SIM	0.9811–0.9829
	G-HBM	0.9690–0.9753
	RKM	0.97–0.975

Besides, there are loads of jump phenomena in Fig. 11, like the coexistence of the modulated and periodic responses in the same excitation amplitude, which represents the great influence of initial values on the NES response. Then by referring to the results in Fig. 11b, d, f, the features of the amplitude of the modulated response, cross-well chaos, and periodic motion can be extracted. Specifically, there is an obvious gap between the amplitude region of the LPM and the GPM. Cross-well chaos fills the gap in the bistable shallow-well NES. Most amplitudes of the modulated response exceed the lower limit of the amplitude of the GPM. These response characteristics in Fig. 11 can well match the results in Subsection 3.2 and Fig. 10.

Fig. 11 Displacement amplitude of two NES models with varied geometry parameter l_1 , varied excitation amplitude f_0 , and fixed excitation frequency ($\omega=1$): **a** square root model, $l_1=1$; **b** polynomial model, $l_1=1$; **c** square root model, $l_1=0.99$; **d** polynomial model, $l_1=0.99$; **e** square root model, $l_1=0.9$; **f** polynomial model, $l_1=0.9$

6 Performance measures and design criterion of vibration mitigation

The above analysis shows the complex dynamic behaviors of the NES in different models and geometry and excitation conditions, and the LO would also have modulated response and chaotic motion. Are these behaviors good or bad for the vibration mitigation effect of the NES? In this section, the effects of the geometry parameter l_1 and response form on the vibration mitigation of the square root model of the NES are discussed by the RKM results of the LO. The initial displacement and velocity of the system are constantly set as 0 in different excitation conditions.

The maximum displacement amplitude of the LO will be shown. The calculation duration, time step, and truncated period number are consistent with those in Sect. 5.

Since the vibration on the resonance frequency of the LO is the most dangerous situation, the authors will exhibit the results of the LO displacement amplitude with resonance frequency ($\omega=1$) and varied excitation amplitude f_0 in the first place. These results are shown in Fig. 12 and Table 4. It can be concluded from the LO amplitude (A_{LO}) results that the NVA with positive linear stiffness ($l_1>1$) and the bistable NES ($l_1<1$) have optimal vibration mitigation effect in small and large excitation amplitude (the left and right side of the vertical dashed line in Fig. 12), respectively. Specifically, the optimal geometry parameter l_1 with different excitation amplitude is 1.03 with $f_0=0.002-0.004$, 1.02 with $f_0=0.006-0.016$, 0.99 with $f_0=0.018$, and 0.97 with $f_0=0.02-0.04$. Combined with Table 4, the LO responses in these optimal geometry conditions are all periodic responses. In addition, the results in Fig. 12 and Table 4 show that the monostable NES ($l_1=1$) causes the large-amplitude modulated response of the LO within a large excitation amplitude range, which is not conducive to vibration mitigation.

Fig. 12 Displacement amplitude of the LO with fixed excitation frequency ($\omega=1$) and varied excitation amplitude f_0 and geometry parameter l_1 : **a** $l_1=0.97\sim 1$; **b** $l_1=1\sim 1.03$

Table 4 Excitation amplitude range of aperiodic response of the LO with fixed excitation frequency ($\omega=1$) and varied excitation amplitude f_0 and geometry parameter l_1

l_1	Excitation amplitude range of aperiodic response
0.97	0.008–0.018
0.98	0.004–0.018
0.99	0.002–0.016
1	0.002–0.026
1.01	0.004–0.02
1.02	0.004
1.03	–

Fig. 12 and Table 4 present the vibration mitigation on the resonance frequency of the LO. So what about the results of other excitation frequencies? Fig. 13 and Table 5 show the frequency response of the LO in two kinds of excitation amplitude. Two vertical dashed lines in Fig. 13 divide the whole frequency band into the low-frequency region ($\omega < 0.98$), the resonance region ($\omega = 0.98 \sim 1.02$), and the high-frequency region ($\omega > 1.02$). Similarly, the NVA with positive linear stiffness ($l_1 > 1$) and the bistable NES ($l_1 < 1$) have the advantage of the vibration mitigation of the resonance region in small and large excitation amplitudes, respectively. And almost all the optimal responses of the LO in the resonance region are periodic responses.

Fig. 13 Frequency response of the LO with varied excitation amplitude f_0 and geometry parameter l_1 : **a** $f_0=0.005$, $l_1=0.97 \sim 1$; **b** $f_0=0.005$, $l_1=1 \sim 1.03$; **c** $f_0=0.02$, $l_1=0.97 \sim 1$; **d** $f_0=0.02$, $l_1=1 \sim 1.03$

Besides, Fig. 13 and Table 5 provide additional useful information. When $f_0=0.005$, the bistable shallow-well NES with $l_1=0.98$ or 0.99 would cause the aperiodic response of the LO in the three frequency regions, while the bistable NES with $l_1=0.97$ would cause the modulated response of the LO in a narrower low-frequency region. The modulated response of the LO with the monostable NES ($l_1=1$) would generate at the junction of the resonance and high-frequency regions. As the geometry parameter l_1 increases more, the modulated response of the LO would firstly expand its territory in the high-frequency region ($l_1=1.01$ or 1.02) and then disappear ($l_1=1.03$).

These results can well match the analysis of the SIM in Fig. 9. Because of the decreasing distance of the bifurcation points of the ε^0 -SIM on the Z_2 scale with increasing l_1 , the NVA with positive linear stiffness can be easier to produce the modulated response compared with the monostable NES and the bistable NES. However, only one stable invariant manifold is witnessed when $l_1=1.03$, so the modulated response is hard to be produced in this situation. As for the bistable shallow-well NES, its cross-well chaos prefers low-frequency components and does not require very large amplitude. So the system with the bistable shallow-well NES contains the chaotic motion in a wide low-frequency region when $f_0=0.005$. When $f_0=0.02$, the NES will make the quasi-periodic responses of the LO concentrate in the high-frequency region, and the low-frequency and resonance regions are mainly composed of periodic responses. In this case, the amplitude of the system in the low-frequency region will be quite large, making it difficult for the bistable NES to generate chaotic motion.

Table 5 Excitation frequency range of aperiodic response of the LO with varied excitation amplitude f_0 and geometry parameter l_1

f_0	l_1	Excitation frequency range of aperiodic response
0.005	0.97	0.94–0.98
	0.98	0.9–1.08
	0.99	0.8–0.86 & 0.94–1.12
	1	1–1.04
	1.01	1–1.08
	1.02	1–1.14
	1.03	–
0.02	0.97	0.94–0.96 & 1–1.2
	0.98	0.94–0.96 & 1.02–1.2
	0.99	0.96 & 1.04~1.2
	1	0.96 & 1~1.14
	1.01	1~1.2
	1.02	1.04~1.2
	1.03	1.1 & 1.14~1.2

Summarizing the results in Fig. 13 and Table 5, the vibration mitigation of the LO in the resonance region can be realized in the aperiodic or periodic response form, where the vibration

mitigation effect of periodic response is better than that of aperiodic response. Large amplitude can be witnessed in the aperiodic response caused by the bistable NES and the periodic response caused by the NVA with positive linear stiffness in the low-frequency region and the aperiodic response in the high-frequency region. These responses are not conducive to vibration mitigation. While reducing the amplitude of the LO in the resonance region as much as possible, these detrimental responses should be avoided as much as possible.

In the previous study [23], not only the displacement amplitude of the LO but the energy dissipation rate of the NES was used to evaluate the vibration mitigation effect of the NES in harmonic excitation. Here, the expressions of the kinetic energy ($E_{ki,LO}$ & $E_{ki,NES}$), potential energy ($E_{po,LO}$ & $E_{po,NES}$), dissipated energy ($E_{di,LO}$ & $E_{di,NES}$), and total energy ($E_{al,LO}$ & $E_{al,NES}$) of the LO and the NES are presented:

$$\begin{aligned} E_{ki,LO} &= \frac{\dot{x}_1^2}{2}, E_{ki,NES} = \frac{\varepsilon \dot{x}_2^2}{2}, E_{po,LO} = \frac{x_1^2}{2}, E_{po,NES} = k_s \left[\sqrt{l_1^2 + (x_1 - x_2 + l_2)^2} - 1 \right]^2, \\ E_{di,LO} &= c_1 \int_0^t \dot{x}_1^2 dt, E_{di,NES} = c_2 \int_0^t (\dot{x}_1 - \dot{x}_2)^2 dt, \\ E_{al,LO} &= E_{ki,LO} + E_{po,LO} + E_{di,LO}, E_{al,NES} = E_{ki,NES} + E_{po,NES} + E_{di,NES}. \end{aligned} \quad (22)$$

The expression of the energy dissipation rate of the NES η_{NES} is

$$\eta_{NES} = \frac{E_{di,NES}(t_{end})}{E_{al,LO}(t_{end}) + E_{al,NES}(t_{end})}, \quad (23)$$

where $t_{end}=2000\pi/\omega$.

Fig. 14 shows the results of the energy dissipation rate of the NES in the excitation frequency domain. Surprisingly, in Fig. 14, both the bistable NES ($l_1=0.99$) and the NVA with positive linear stiffness ($l_1=1.02$) have higher energy dissipation rates than the monostable NES ($l_1=1$) at each excitation frequency. In other words, the vibration mitigation effect of the NES in these two geometric configurations is better than that of the monostable NES in the whole frequency domain. This result is quite different from the result of the LO frequency response in Fig. 13. The reason is that the system energy will be different in different excitation and geometry conditions; even if the energy dissipation efficiency of the NES is very high, the amplitude of the LO may still be large if the total energy of the system is large. In this case, the energy dissipation rate of the NES overestimates the vibration mitigation effect. Therefore, it is necessary to find indexes that can reflect the vibrational energy of the LO.

Fig. 14 Energy dissipation rate of the NES with fixed excitation amplitude ($f_0=0.005$) and varied excitation frequency ω and geometry parameter l_1 : **a** $l_1=0.97\sim 1$; **b** $l_1=1\sim 1.03$

The average mechanical energy of the LO ($E_{\text{me,LO}}$) and the input energy of the LO ($E_{\text{in,LO}}$) can reflect the vibrational energy of the LO. Their expressions are

$$E_{\text{me,LO}} = \frac{1}{t_{\text{end}}} \int_0^{t_{\text{end}}} (E_{\text{ki,LO}} + E_{\text{po,LO}}) dt, E_{\text{in,LO}} = \int_{t_{\text{start}}}^{t_{\text{end}}} \max[E_{\text{al,LO}}(t) - E_{\text{al,LO}}(t - \tau), 0] dt, \quad (24)$$

where $t_{\text{start}}=1600\pi/\omega$ and $\tau=\pi/(100\omega)$. The input energy of the LO represents the input amount of mechanical and dissipated energy in a duration. The smaller the input energy of the LO is, the smaller its contribution to vibration will be. At this time, more energy will be transferred to the NES, which is more conducive to vibration mitigation.

Fig. 15 shows the results of mechanical energy and input energy of the LO in the excitation frequency domain. Compared with the results of the LO displacement amplitude in Fig. 13, both the results of mechanical energy and input energy of the LO can reflect the vibration mitigation effect of the NES in the resonance region, the aggravation of the vibration of the LO in the low- and high-frequency regions, and the resonance peak drifting to the low-frequency region. However, the results of the LO displacement amplitude, mechanical energy, and input energy have a few differences. For example, in Fig. 13 and Fig. 15c, d, the NVA with positive linear stiffness ($l_1=1.01$) has the best vibration mitigation effect on the whole frequency band, while in Fig. 15a, b, the bistable NES ($l_1=0.99$) is the best. In addition, as the resonance region of the LO input energy in Fig. 15c, d is narrower than that of the other two indexes, higher aggravation of the vibration of the LO in the low- and high-frequency regions can be demonstrated in the results of the LO input energy. Although there are slight differences in the results of the three indexes, all of them can reflect the LO response characteristics with different NES geometry parameters and the optimal vibration mitigation effect near the LO resonance frequency.

Fig. 15 Two energy indexes of the LO with fixed excitation amplitude ($f_0=0.005$) and varied excitation frequency ω and geometry parameter l_1 : **a** and **b** average mechanical energy of the LO; **c** and **d** input energy of the LO

The above analysis elaborates the optimal vibration mitigation effect and NES geometric configuration in different excitation conditions by using the three LO indexes. Here, the decent TET effect of the NES is further revealed by the energy transfer time history of the system. The instantaneous transfer energy of the LO and the NES ($E_{tr,LO}$ and $E_{tr,NES}$) is defined as

$$E_{tr,LO}(t) = E_{al,LO}(t) - E_{al,LO}(t - \tau), E_{tr,NES}(t) = E_{al,NES}(t) - E_{al,NES}(t - \tau). \quad (25)$$

According to the results in Fig. 13 and Fig. 15, the excitation condition ($f_0=0.005$ and $\omega=0.98$) and three geometry conditions ($l_1=0.99$, 1, or 1.02) are selected to calculate the energy transfer time history of the system. The results are shown in Fig. 16, where positive and negative E_{tr} indicate input and output energy, respectively. It can be seen that, for the modulated response of the bistable NES ($l_1=0.99$), the input and output energy of the LO is equivalent, while the input energy takes the lead in the NES. For the periodic response of the monostable NES ($l_1=1$), poor performance of the NES can be seen in both the input and output energy. For the periodic response of the NVA with positive linear stiffness ($l_1=1.02$), the energy of the excitation is mainly transferred to the NVA, but almost no leakage from the NVA, while the LO hardly takes part in the energy transfer process, which is the desired TET effect. The results in Fig. 16 also indicate that the modulated response

may not have the optimal TET effect, while the periodic response can achieve the optimal TET effect.

Fig. 16 Energy transfer time history of the system with fixed excitation parameters ($f_0=0.005$ and $\omega=0.98$) and varied geometry parameter l_1 : **a** $l_1=0.99$; **b** $l_1=1$; **c** $l_1=1.02$

7 Conclusions

The main purpose of this paper is to propose new reasonable performance measures and new design criteria of the nonlinear energy sink (NES) in a square root model and reveal the response form and geometric configuration of the optimal targeted energy transfer effect in harmonic excitation. Two original performance measures (the displacement amplitude of the main system and the energy dissipation rate of the NES) and two proposed performance measures (the mechanical energy and the input energy of the main system) are compared. The results show that adopting the energy dissipation rate of the NES may overestimate the vibration mitigation effect of the NES, while the displacement amplitude, the mechanical energy, and the input energy of the main system can better reflect the vibration mitigation effect. In the vicinity of the resonance frequency of the main system, the nonlinear vibration absorber with positive linear stiffness and the bistable NES have the best vibration mitigation effect at small and large excitation amplitudes, respectively. Their optimal response forms are periodic responses rather than strongly modulated responses.

This paper is also an extension of the previous studies on the dynamics and vibration mitigation effect of the polynomial model of the monostable and bistable NESs in harmonic excitation. Using the slow invariant manifold, the generalized harmonic balance method, the Runge-Kutta method, and the fast Fourier transformation, complex dynamic behaviors of the square root model of the NES are shown, such as the modulated response and cross-well chaos. The time and frequency domain features and jump phenomena of these behaviors are analyzed. The effects of geometry and excitation conditions and the two NES models on these behaviors are discussed.

In this paper, the vibration mitigation effect of the NES is optimized in an easy-to-operate way, i.e. by only changing the geometric configuration of the NES, and multiple indexes are used to evaluate the performance of the NES. To achieve a better vibration mitigation effect, all parameters of the NES need to be optimized simultaneously. In future studies, the authors will focus on the multi-parameter and multi-objective optimization of the NES. In addition, the dynamics, the performance measures, and the design criteria of the square root and polynomial models of the NES may also have differences in other excitation conditions or with parameter uncertainty. These aspects are worthy of further study.

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Figures

Figure 1

Model of an LO coupled with an NES: a $L_1 \leq L_2, L_2 \geq 0$; b $L_1 > L_2, L_2 = 0$

Figure 2

Coordinate correction of the dimensionless system

Figure 3

SMR of the monostable unstretched NES: a ε 0-SIM projections; b displacement time histories; c FFT

Figure 4

SMR of the bistable shallow-well NES: a ε 0-SIM projections; b displacement time histories; c FFT

Figure 5

CMR of the bistable deep-well NES: a ε 0-SIM projections; b displacement time histories; c FFT

Figure 6

Chaos of the bistable shallow-well NES: a ε 0-SIM projections; b displacement time histories; c FFT

Figure 7

LPM of the bistable deep-well NES: a ϵ^0 -SIM projections; b displacement time histories; c FFT

Figure 8

GPM of the bistable deep-well NES: a ϵ^0 -SIM projections; b displacement time histories; c FFT

Figure 9

ϵ^0 -SIM curves of the NVA with positive linear stiffness: a $l_1=1.01$; b $l_1=1.02$; c $l_1=1.03$

Figure 10

Frequency responses of the square root model of the NES obtained by three methods with varied geometry parameter l_1 and fixed excitation amplitude ($f_0=0.005$): a ϵ^1 -SIM, $l_1=1$; b G-HBM and RKM, $l_1=1$; c ϵ^1 -SIM, $l_1=0.99$; d G-HBM and RKM, $l_1=0.99$; e ϵ^1 -SIM, $l_1=0.9$; f G-HBM and RKM, $l_1=0.9$

Figure 11

Displacement amplitude of two NES models with varied geometry parameter l_1 , varied excitation amplitude f_0 , and fixed excitation frequency ($\omega=1$): a square root model, $l_1=1$; b polynomial model, $l_1=1$; c square root model, $l_1=0.99$; d polynomial model, $l_1=0.99$; e square root model, $l_1=0.9$; f polynomial model, $l_1=0.9$

Figure 12

Displacement amplitude of the LO with fixed excitation frequency ($\omega=1$) and varied excitation amplitude f_0 and geometry parameter l_1 : a $l_1=0.97\sim 1$; b $l_1=1\sim 1.03$

Figure 13

Frequency response of the LO with varied excitation amplitude f_0 and geometry parameter l_1 : a $f_0=0.005$, $l_1=0.97\sim 1$; b $f_0=0.005$, $l_1=1\sim 1.03$; c $f_0=0.02$, $l_1=0.97\sim 1$; d $f_0=0.02$, $l_1=1\sim 1.03$

Figure 14

Energy dissipation rate of the NES with fixed excitation amplitude ($f_0=0.005$) and varied excitation frequency ω and geometry parameter l_1 : a $l_1=0.97\sim 1$; b $l_1=1\sim 1.03$

Figure 15

Two energy indexes of the LO with fixed excitation amplitude ($f_0=0.005$) and varied excitation frequency ω and geometry parameter l_1 : a and b average mechanical energy of the LO; c and d input energy of the LO

Figure 16

Energy transfer time history of the system with fixed excitation parameters ($f_0=0.005$ and $\omega=0.98$) and varied geometry parameter l_1 : a $l_1=0.99$; b $l_1=1$; c $l_1=1.02$