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Hichem Hajaiej
California State University Los Angeles
Tianhao LIU
liuth980325@163.com

Tsinghua University
Wenming Zou
Tsinghua University

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Positive solution for an elliptic system with critical exponent and logarithmic terms: the higher dimensional cases *

Hichem Hajaiej[a,], Tianhao Liu[b,], Wenming Zou[b,]

[a] Department of Mathematics, California State University at Los Angeles, Los Angeles, CA 90032, USA
[b] Department of Mathematical Sciences, Tsinghua University, Beijing 100084

Abstract

In this paper, we consider the coupled elliptic system with critical exponent and logarithmic terms:

$$\begin{cases} -\Delta u = \lambda_1 u + \mu_1 |u|^{2p-2}u + \beta |u|^{p-2}|v|^p u + \theta_1 u \log u^2, & x \in \Omega, \\ -\Delta v = \lambda_2 v + \mu_2 |v|^{2p-2}v + \beta |u|^{p}|v|^{p-2} v + \theta_2 v \log v^2, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $2p = 2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent. When $N \geq 5$, for different ranges of $\beta, \lambda_i, \mu_i, \theta_i$, $i = 1, 2$, we obtain existence and nonexistence results of positive solutions via variational methods. The special case $N = 4$ was studied by the authors in (arXiv:2304.13822). Note that for $N \geq 5$, the critical exponent is given by $2p \in (2, 4)$, whereas for $N = 4$, it is $2p = 4$. In the higher dimensional cases $N \geq 5$ brings new difficulties, and requires new ideas. Besides, we also study the Brézis-Nirenberg problem with logarithmic perturbation

$$-\Delta u = \lambda u + \mu |u|^{2p-2}u + \theta u \log u^2 \quad \text{in } \Omega,$$

where $\mu > 0, \theta < 0$, $\lambda \in \mathbb{R}$, and obtain the existence of positive local minima and least energy solution under some certain assumptions.

Key words: Schrödinger system; Brézis-Nirenberg problem; Critical exponent; Logarithmic perturbation; Positive solution;

*Supported NSFC(No.12171265). E-mail addresses: hhajaie@calstatela.edu (H. Hajaiej), liuth19@mails.tsinghua.edu.cn (T. H. Liu), zou-wm@mail.tsinghua.edu.cn (W. M. Zou).

1 Introduction

In this paper, we consider the solitary wave solutions of the following time-dependent nonlinear logarithmic-type Schrödinger system

$$\begin{cases} \imath\partial_t\Psi_1 = \Delta\Psi_1 + \mu_1|\Psi_1|^{2p-2}\Psi_1 + \beta|\Psi_1|^{p-2}|\Psi_2|^p\Psi_1 + \theta_1\Psi_1\log\Psi_1^2, & x\in\Omega,\ t>0,\\ \imath\partial_t\Psi_2 = \Delta\Psi_2 + \ \mu_2|\Psi_2|^{2p-2}\Psi_2 + \beta|\Psi_1|^{p}|\Psi_2|^{p-2}\Psi_2 + \theta_2\Psi_2\log\Psi_2^2, & x\in\Omega,\ t>0,\\ \Psi_i = \Psi_i(x,t)\in\mathbb{C}, \quad i=1,2 \\ \Psi_i(x,t)=0, \quad x\in\partial\Omega, \quad t>0, \quad i=1,2, \end{cases} \tag{1.1}$$

where $\imath$ is the imaginary unit, $\Omega\subset\mathbb{R}^N$ is a bounded smooth domain, $N\geq 4$, $2p = 2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $\lambda_1,\lambda_2,\theta_1,\theta_2\in\mathbb{R}$, $\mu_1,\mu_2>0$ and $\beta\neq 0$ is a coupling constant. System (1.1) appears in many physical fields, such as quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose-Einstein condensation. We refer the readers to the papers [1, 3, 4, 9, 10, 11, 17, 18] for a survey on the related physical backgrounds.

To study the solitary wave solutions of the system (1.1), we set $\Psi_1(x,t) = e^{\imath\lambda_1 t}u(x)$ and $\Psi_2(x,t) = e^{\imath\lambda_2 t}v(x)$, then system (1.1) is reduced to the following coupled elliptic system with logarithmic terms

$$\begin{cases} -\Delta u = \lambda_1 u + \mu_1|u|^{2p-2}u + \beta|u|^{p-2}|v|^p u + \theta_1 u\log u^2, & x\in\Omega,\\ -\Delta v = \lambda_2 v + \mu_2|v|^{2p-2}v + \beta|u|^{p}|v|^{p-2} v + \theta_2 v\log v^2, & x\in\Omega,\\ u=v=0, & x\in\partial\Omega. \end{cases} \tag{1.2}$$

The logarithmic terms present in themselves many mathematical interests and difficulties. It is easy to see that $u = o(u\log u^2)$ for u very close to 0. Compared to the critical term $|u|^{2p-2}u$, the logarithmic term $u\log u^2$ has a lower-order term at infinity. Additionally, the logarithmic term has indefinite sign and makes the structure of the corresponding functional complicated.

Problem (1.2) is connected to the following Bose-Einstein condensates coming from the Gross-Pitaevskii coupled equations.

$$\begin{cases} -\Delta u = \lambda_1 u + \mu_1|u|^{2p-2}u + \beta|u|^{p-2}|v|^p u, & x\in\Omega,\\ -\Delta v = \lambda_2 v + \mu_2|v|^{2p-2}v + \beta|u|^{p}|v|^{p-2} v, & x\in\Omega,\\ u=v=0, & x\in\partial\Omega. \end{cases}$$

Starting from the cerebrated work by Brézis and Nirenberg [5], this critical system has received great attention in the past thirty years, in particular for the existence of positive least energy solutions, we refer to [7, 8, 16, 22] and references therein.

For the single equation setting of (1.2), Deng et al.[12] investigated the existence of positive least energy solutions for the following related single equation

$$-\Delta u = \lambda u + |u|^{2p-2}u + \theta u\log u^2,\ u\in H_0^1(\Omega),\ \Omega\subset\mathbb{R}^N,\ i=1,2, \tag{1.3}$$

In [12], the authors proved that equation (1.3) has a positive least energy solution if $\lambda\in\mathbb{R}$, $\theta>0$ for all $N\geq 4$. Also, they obtained some existence and nonexistence results under other conditions, we refer the readers to [12] for details. Recently, the authors studied the particular case $p=2$ and $N=4$ in [13], and proved various existence and nonexistence results for the system (1.2).

In this paper, we continue our previous work [13] to study the existence and nonexistence of positive solutions for system (1.2) in the *higher dimensional case*. That is, we always work under the assumptions

$$N \geq 5 \quad \text{and} \quad 2p = 2^*.$$

The higher dimensional case introduces different phenomena and challenges compared to the specific case where $N = 4$, since the critical exponent is given by $2p \in (2, 4)$ for $N \geq 5$, whereas for $N = 4$, it is $2p = 4$. This brings some difficulties and requires us to develop some new ideas.

Define $\mathcal{H} := H_0^1(\Omega) \times H_0^1(\Omega)$. To find a positive solution to the system (1.2), we borrow the ideas from [12] to define a modified functional $\mathcal{L} : \mathcal{H} \to \mathbb{R}$

$$\begin{aligned}\mathcal{L}(u,v) = &\frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda_1}{2}\int_\Omega |u^+|^2 - \frac{\mu_1}{2p}\int_\Omega |u^+|^{2p} - \frac{\theta_1}{2}\int_\Omega (u^+)^2(\log(u^+)^2 - 1)\\ &+ \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\lambda_2}{2}\int_\Omega |v^+|^2 - \frac{\mu_2}{2p}\int_\Omega |v^+|^{2p} - \frac{\theta_2}{2}\int_\Omega (v^+)^2(\log(v^+)^2 - 1) - \frac{\beta}{p}\int_\Omega |u^+|^p|v^+|^p,\end{aligned}$$

where $u^+ := \max\{u, 0\}$, $u^- := -\max\{-u, 0\}$. We can see that the functional $\mathcal{L}$ is well-defined in $\mathcal{H}$. Moreover, any nonnegative critical point of $\mathcal{L}$ corresponds to a solution of the system (1.2).

We call a solution (u, v) fully nontrivial if both $u \not\equiv 0$ and $v \not\equiv 0$; we call a solution (u, v) semi-trivial if $u \equiv 0$, $v \not\equiv 0$ or $u \not\equiv 0$, $v \equiv 0$; we call a solution (u, v) positive (resp. nonnegative) if both $u > 0$ and $v > 0$ (resp. both $u \geq 0$ and $v \geq 0$). We say a solution (u, v) of (1.2) is a *least energy solution* if (u, v) is fully nontrivial and $\mathcal{L}(u, v) \leq \mathcal{L}(\varphi, \psi)$ for any other fully nontrivial solution (φ, ψ) of system (1.2). As in [14], we consider

$$\mathcal{N} = \{(u,v) \in \mathcal{H} : u \not\equiv 0, v \not\equiv 0, \mathcal{L}'(u,v)(u,0) = 0 \text{ and } \mathcal{L}'(u,v)(0,v) = 0\}.$$

It is easy to check that $\mathcal{N} \neq \emptyset$. Then we set

$$\mathcal{C}_\mathcal{N} := \inf_{(u,v)\in\mathcal{N}} \mathcal{L}(u,v).$$

Define

$$\varSigma_1 := \{(\lambda, \mu, \theta) : \lambda \in \mathbb{R}, \mu > 0, \theta > 0\}.$$

Theorem 1.1. *Assume that $N \geq 5$ and $(\lambda_i, \mu_i, \theta_i) \in \varSigma_1$ for $i = 1, 2$, then there exists $\beta_0 \in (0, \min\{\mu_1, \mu_2\}]$ such that the system (1.2) has a positive least energy solution for all $\beta \in (-\beta_0, 0) \cup (0, +\infty)$.*

Remark 1.1. (1) For the special case $N = 4$ and $2p = 2^*$, when $(\lambda_1, \mu_1, \theta_1) \in \varSigma_1$ and $(\lambda_2, \mu_2, \theta_2) \in \varSigma_1$, [13, Theorem 1.1] proved that system (1.2) has a positive least energy solution, provided that $\beta \in \left(-\tilde{\beta}_0, 0\right) \cup \left(0, \tilde{\beta}_1\right) \cup \left(\tilde{\beta}_2, +\infty\right)$, where $\tilde{\beta}_i$, $i = 0, 1, 2$, are some specific constants satisfying

$$0 < \tilde{\beta}_0, \tilde{\beta}_1 \leq \min\{\mu_1, \mu_2\} \leq \max\{\mu_1, \mu_2\} \leq \tilde{\beta}_2.$$

However, we do not know whether a least energy solution for (1.2) exists or not if $\tilde{\beta}_1 \leq \beta \leq \tilde{\beta}_2$. Therefore, comparing this with Theorem 1.1, the general case $N \geq 5$ behaves differently from the special case $N = 4$. The *reason* is that $p = 2$ if $N = 4$, whereas $1 < p < 2$ if $N \geq 5$. The fact that $1 < p < 2$ introduces significant differences for $\beta > 0$, which means that the method used in [13] cannot apply directly to this paper, and so we require some new ideas and techniques. As we will see in Proposition 2.4, the approach to establish energy estimate for $\beta > 0$ is completely different from that in the case $N = 4$ [13].

(2) In the particular case $\theta = 0$, [8, Theorem 1.3] said that the system (1.2) has a positive least energy solution for any $\beta \neq 0$ and $\lambda_1, \lambda_2 \in (0, \lambda_1(\Omega))$. Comparing this with Theorem 1.1, the logarithmic term

$\theta_i u \log u^2$ has a significant impact on the existence of solutions, and introduces different and more challenging situations. One of the main difficulties comes from the uncertain sign of the logarithmic term, which implies that the Nehari set $\mathcal{N}$ may not be a C^1-manifold for all $\beta < 0$. As a result, $\mathcal{N}$ cannot be a natural constraint. So we have to restrict the range of β to $\beta \in (-\beta_0, 0)$ and demonstrate that we can find the free critical points on a special set, see Proposition 2.1 and 2.2. Another difficulty is do to the presence of logarithmic terms, which makes the structure of functional complicated by using variational method. This requires us to make more careful calculations and develop new ideas and innovative techniques when establishing the energy estimates, we refer to Step 1 and 2 in the proof of Proposition 2.4 for details.

Now we focus on the existence when $\theta_1, \theta_2 < 0$ and consider the following special sets,

$$\begin{aligned} A_1 := \{(\lambda_1, \mu_1, \theta_1; \lambda_2, \mu_2, \theta_2) :&\lambda_1, \lambda_2 \in [0, \lambda_1(\Omega)), \mu_1, \mu_2 > 0, \theta_1, \theta_2 < 0, \\ &(2p-2)K_1^{\frac{1}{p-1}} K_2^{\frac{p}{p-1}} + p^{\frac{p}{p-1}}(\theta_1 + \theta_2)|\Omega| > 0\}, \end{aligned}$$

$$\begin{aligned} A_2 := \{(\lambda_1, \mu_1, \theta_1; \lambda_2, \mu_2, \theta_2) :&\lambda_1 \in [0, \lambda_1(\Omega)), \lambda_2 \in \mathbb{R}, \mu_1, \mu_2 > 0, \theta_1, \theta_2 < 0, \\ &(2p-2)K_1^{\frac{1}{p-1}} K_3^{\frac{p}{p-1}} + p^{\frac{p}{p-1}}(\theta_1 + \theta_2 e^{-\frac{\lambda_2}{\theta_2}})|\Omega| > 0\}, \end{aligned}$$

$$\begin{aligned} A_3 := \{(\lambda_1, \mu_1, \theta_1; \lambda_2, \mu_2, \theta_2) :&\lambda_1, \lambda_2 \in \mathbb{R}, \mu_1, \mu_2 > 0, \theta_1, \theta_2 < 0, \\ &(p-1)2^{-\frac{1}{p-1}} K_1^{\frac{1}{p-1}} + p^{\frac{p}{p-1}}(\theta_1 e^{-\frac{\lambda_1}{\theta_1}} + \theta_2 e^{-\frac{\lambda_2}{\theta_2}})|\Omega| > 0\}. \end{aligned}$$

Here,

$$K_1 := \frac{2p\mathcal{S}^p}{\max\{\mu_1, \mu_2\}}, \quad K_2 := \frac{\min\{\lambda_1(\Omega) - \lambda_1, \lambda_1(\Omega) - \lambda_2\}}{2\lambda_1(\Omega)}, \quad K_3 := \frac{\lambda_1(\Omega) - \lambda_1}{2\lambda_1(\Omega)},$$

and $\mathcal{S}$ denotes the Sobolev best constant of $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2p}(\mathbb{R}^N)$,

$$\mathcal{S} = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2p}\right)^{\frac{1}{p}}},$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{u \in L^{2p}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\right\}$ with norm $\|u\|_{\mathcal{D}^{1,2}} := \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{1}{2}}$.

The following two existence results Theorem 1.2 and 1.3 correspond to Theorem 1.2 and 1.3 in [13] (which dealt with the critical case $N = 4$).

Theorem 1.2. *Define*

$$\mathcal{C}_\rho := \inf_{(u,v) \in B_\rho} \mathcal{L}(u, v),$$

where $B_r := \{(u, v) \in \mathcal{H} : \sqrt{|\nabla u|_2^2 + |\nabla v|_2^2} < r\}$ *and* ρ *will be given by Lemma 3.1. When* $\beta < 0$*, we assume that one of the following holds:*

(i) $(\lambda_1, \mu_1, \theta_1; \lambda_2, \mu_2, \theta_2) \in A_1$,

(ii) $(\lambda_1, \mu_1, \theta_1; \lambda_2, \mu_2, \theta_2) \in A_2$,

(iii) $(\lambda_1, \mu_1, \theta_1; \lambda_2, \mu_2, \theta_2) \in A_3$;

when $\beta > 0$*, we assume that there exists* $\epsilon > 0$ *such that one of the following holds:*

(iv) $(\lambda_1, \mu_1 + \beta\epsilon, \theta_1; \lambda_2, \mu_2 + \frac{\beta}{\epsilon}, \theta_2) \in A_1$,

(v) $(\lambda_1, \mu_1 + \beta\epsilon, \theta_1; \lambda_2, \mu_2 + \frac{\beta}{\epsilon}, \theta_2) \in A_2$,

(vi) $(\lambda_1, \mu_1 + \beta\epsilon, \theta_1; \lambda_2, \mu_2 + \frac{\beta}{\epsilon}, \theta_2) \in A_3$.
Then system (1.2) *has a positive solution* $(\tilde{u}, \tilde{v})$ *such that* $\mathcal{L}(\tilde{u}, \tilde{v}) = \mathcal{C}_\rho < 0$*, which is a local minima.*

Remark 1.2. Comparing with the case $\theta_1, \theta_2 > 0$, the presence of negative θ_1 and θ_2 changes the geometry of $\mathcal{L}$, and makes it not straightforward to obtain the Palais-Smale sequence in $\mathcal{N}$. The assumptions (i)-(vi) establish the existence of a local minimum $\mathcal{C}_\rho$ for the functional $\mathcal{L}$, thus we can obtain the corresponding Palais-Smale sequence by taking minimizing sequence for $\mathcal{C}_\rho$, and we prove that the weak limit of Palais-Smale sequence is a positive solution of system (1.2). Unfortunately, we do not know whether it is a positive least energy solution.

Theorem 1.3. *Define*

$$\mathcal{C}_K := \inf_{(u,v)\in K} \mathcal{L}(u, v),$$

where

$$K = \{(u, v) \in \mathcal{H} : \mathcal{L}'(u, v) = 0\}.$$

Assume that the conditions stated in Lemma 3.1 hold, and

$$\min\{\theta_1, \theta_2\} \geq -\frac{2}{N}\lambda_1(\Omega) \quad \text{or} \quad \beta \in (-\sqrt{\mu_1\mu_2}, 0) \cup (0, \infty)\,.$$

Then the system (1.2) *possesses a nonnegative solution* $(\hat{u}, \hat{v}) \neq (0, 0)$ *such that* $\mathcal{L}(\hat{u}, \hat{v}) = \mathcal{C}_K < 0$.

Remark 1.3. In Theorem 1.3, we obtain a nonnegative solution $(\hat{u}, \hat{v})$. However, we do not know whether it is a positive least energy solution since we cannot prove that $(\hat{u}, \hat{v})$ is fully nontrivial.

For the nonexistence of positive solutions for the system (1.2), we have the following results.

Theorem 1.4. *Assume that* $N \geq 5$, $(\lambda_1, \mu_1, \theta_1) \in \Sigma_2$ *or* $(\lambda_2, \mu_2, \theta_2) \in \Sigma_2$ *, where*

$$\Sigma_2 := \left\{(\lambda, \mu, \theta) : \theta < 0 \text{ and } \frac{(N-2)|\theta|}{2} + \frac{(N-2)\theta}{2}\log\left(\frac{(N-2)|\theta|}{2\mu}\right) + \lambda - \lambda_1(\Omega) \geq 0\right\}.$$

If $\beta > 0$*, then the system* (1.2) *has no positive solutions.*

Now we consider the Brézis-Nirenberg problem with logarithmic perturbation

$$-\Delta u = \lambda u + \mu|u|^{2p-2}u + \theta u \log u^2 \quad \text{ in } \Omega, \tag{1.4}$$

where $\lambda \in \mathbb{R}$, $\mu > 0, \theta < 0$. We define the associated modified energy functional by

$$\mathcal{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u^+|^2 - \frac{\mu}{2p}\int_\Omega |u^+|^{2p} - \frac{\theta}{2}\int_\Omega (u^+)^2\left(\log(u^+)^2 - 1\right).$$

Set

$$\Sigma_3 := \left\{(\lambda, \mu, \theta) : \lambda \in [0, \lambda_1(\Omega)), \mu > 0, \theta < 0, \frac{(\lambda_1(\Omega) - \lambda)^2}{\lambda_1(\Omega)^2\mu}S^2 + 2\theta|\Omega| > 0\right\},$$

$$\Sigma_4 := \left\{(\lambda, \mu, \theta) : \lambda \in \mathbb{R}, \mu > 0, \theta < 0, \mu^{-1}S^2 + 2\theta e^{-\frac{\lambda}{\theta}}|\Omega| > 0\right\}.$$

Then we have the following result:

Theorem 1.5 (Existence of a local minima)**.** *Assume that* $N \geq 5$ *and* $(\lambda, \mu, \theta) \in \Sigma_3 \cup \Sigma_4$*. Define*

$$\widetilde{\mathcal{C}}_\rho := \inf_{|\nabla u|_2 < \rho} J(u),$$

where ρ *will be given by Lemma 4.1. Then equation* (1.4) *has a positive solution* u *such that* $J(u) = \widetilde{\mathcal{C}}_\rho < 0$.

Theorem 1.6 (Existence of the least energy solution)**.** *Assume that $N \geq 5$ and $(\lambda, \mu, \theta) \in \Sigma_3 \cup \Sigma_4$. Define*

$$\widetilde{\mathcal{C}}_{\mathcal{K}} := \inf_{u \in \mathcal{K}} J(u)$$

where

$$\mathcal{K} = \{u \in H_0^1(\Omega) : J'(u) = 0\}.$$

Then equation (1.4) *has a positive least energy solution u such that $J(u) = \widetilde{\mathcal{C}}_{\mathcal{K}} < 0$.*

Remark 1.4. In [12], the authors obtained the existence of positive solution for (1.4) only in the case $N = 3, 4$ and $\theta < 0$ under certain additional conditions. But they do not give the existence result of positive solutions for the general case $N \geq 5$ and $\theta < 0$. Here, we give a positive answer to this question in Theorem 1.5, 1.6, and improve the results in [12]. Furthermore, we give the type of the positive solution (a local minimum or a least energy solution) and show that its energy level is negative.

Before closing the introduction, we give the outline of our paper and introduce some notations. In Section 2, we will prove Theorem 1.1. In Section 3, we will prove Theorem 1.2, 1.3 and 1.4. In Sections 4, we will prove Theorem 1.5 and 1.6.

Throughout this paper, we denote the norm of $L^p(\Omega)$ by $|\cdot|_p$ for $1 \leq p \leq \infty$. We use "$\to$" and "$\rightharpoonup$" to denote the strong convergence and weak convergence in corresponding space respectively. The capital letter C will appear as a constant which may vary from line to line, and C_1, C_2, C_3 are prescribed constants.

2 Proof of Theorem 1.1

In this section, we always assume that $N \geq 5$ and $(\lambda_i, \mu_i, \theta_i) \in \Sigma_1$ for $i = 1, 2$. Now we establish both lower and upper uniform estimates on the L^{2p}-norms of elements in the Nehari set that fall below a certain energy level.

Lemma 2.1. *Let $-\sqrt{\mu_1 \mu_2} < \beta < 0$. Then there exist $C_2 > C_1 > 0$, such that for any $(u, v) \in \mathcal{N}$ with $\mathcal{L}(u, v) \leq \frac{2}{N} \left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}} \right) \mathcal{S}^{\frac{N}{2}}$, there holds*

$$C_1 \leq |u^+|_{2p}^{2p}, |v^+|_{2p}^{2p} \leq C_2.$$

Here, C_1, C_2 depend only on λ_i, μ_i, θ_i for $i = 1, 2$.

Proof. Take any $(u, v) \in \mathcal{N}$ with $\mathcal{L}(u, v) \leq \frac{2}{N} \left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}} \right) \mathcal{S}^{\frac{N}{2}}$. Since we have the inequality

$$s \log s \leq (p-1)^{-1} e^{-1} s^p \ \text{ for any } \ s > 0,$$

then by the Sobolev inequality and the fact that $\beta < 0$, we have

$$\begin{aligned} \mathcal{S}|u^+|_{2p}^2 \leq |\nabla u|_2^2 &= \lambda_1 |u^+|_2^2 + \mu_1 |u^+|_{2p}^{2p} + \theta_1 \int_\Omega (u^+)^2 \log(u^+)^2 + \beta |u^+ v^+|_p^p \\ &\leq \mu_1 |u^+|_{2p}^{2p} + \theta_1 \int_\Omega (u^+)^2 \log(e^{\frac{\lambda_1}{\theta_1}} (u^+)^2) \\ &\leq \left(\mu_1 + \frac{\theta_1}{(p-1)} e^{(p-1)\frac{\lambda_1}{\theta_1} - 1} \right) |u^+|_{2p}^{2p}. \end{aligned}$$

Therefore, $|u^+|_{2p}^{2p} \geq \mathcal{S}^{\frac{N}{2}} \left(\mu_1 + \frac{\theta_1}{(p-1)} e^{(p-1)\frac{\lambda_1}{\theta_1}-1}\right)^{-\frac{N}{2}}$. Similarly, we have $|v^+|_{2p}^{2p} \geq \mathcal{S}^{\frac{N}{2}} \left(\mu_2 + \frac{\theta_2}{(p-1)} e^{(p-1)\frac{\lambda_2}{\theta_2}-1}\right)^{-\frac{N}{2}}$.

On the other hand, we have

$$\begin{aligned}\frac{2}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\mathcal{S}^{\frac{N}{2}} \geq \mathcal{L}(u,v) &= \mathcal{L}(u,v) - \frac{1}{2p}\mathcal{L}'(u,v)(u,v) \\ &= \frac{1}{N}|\nabla u|_2^2 - \frac{\theta_1}{N}\int_\Omega (u^+)^2 \log\left(e^{\frac{\lambda_1}{\theta_1}}(u^+)^2\right) + \frac{\theta_1}{2}|u^+|_2^2 \\ &+ \frac{1}{N}|\nabla v|_2^2 - \frac{\theta_2}{N}\int_\Omega (v^+)^2 \log\left(e^{\frac{\lambda_2}{\theta_2}}(v^+)^2\right) + \frac{\theta_2}{2}|v^+|_2^2.\end{aligned} \tag{2.1}$$

Moreover,

$$\begin{aligned}\frac{2}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\mathcal{S}^{\frac{N}{2}} \geq \mathcal{L}(u,v) &= \mathcal{L}(u,v) - \frac{1}{2}\mathcal{L}'(u,v)(u,v) \\ &= \frac{1}{N}\left(\mu_1|u^+|_{2p}^{2p} + \mu_2|v^+|_{2p}^{2p} + 2\beta|u^+v^+|_p^p\right) + \frac{\theta_1}{2}|u^+|_2^2 + \frac{\theta_2}{2}|v^+|_2^2.\end{aligned} \tag{2.2}$$

Since $\beta > -\sqrt{\mu_1\mu_2}$, there exists some positive constants $c_0, C_0 > 0$ such that

$$c_0\left(|u^+|_{2p}^{2p} + |v^+|_{2p}^{2p}\right) \leq \mu_1|u^+|_{2p}^{2p} + \mu_2|v^+|_{2p}^{2p} + 2\beta|u^+v^+|_p^p \leq C_0\left(|u^+|_{2p}^{2p} + |v^+|_{2p}^{2p}\right).$$

Therefore, we have

$$|u^+|_2^2 \leq \frac{4}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\theta_1^{-1}\mathcal{S}^{\frac{N}{2}} \text{ and } |v^+|_2^2 \leq \frac{4}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\theta_2^{-1}\mathcal{S}^{\frac{N}{2}}. \tag{2.3}$$

Recalling the following useful inequality (see [19] or [15, Theorem 8.14])

$$\int_\Omega u^2 \log u^2 \leq \frac{a}{\pi}|\nabla u|_2^2 + \left(\log|u|_2^2 - N(1+\log a)\right)|u|_2^2 \quad \text{for } u \in H_0^1(\Omega) \text{ and } a > 0.$$

Let $w^+ = e^{\frac{\lambda_1}{2\theta_1}}u^+$ and $z^+ = e^{\frac{\lambda_2}{2\theta_2}}v^+$. Since $|s^2\log s^2| \leq Cs^{2-\tau} + Cs^{2+\tau}$ for any $\tau \in (0,1)$, we have

$$\begin{aligned}\frac{1}{N}\left(|\nabla u|_2^2 + |\nabla v|_2^2\right) &\leq \frac{2}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\mathcal{S}^{\frac{N}{2}} + \frac{\theta_1}{N}e^{-\frac{\lambda_1}{\theta_1}}\int_\Omega (w^+)^2\log(w^+)^2 + \frac{\theta_2}{N}e^{-\frac{\lambda_2}{\theta_2}}\int_\Omega (z^+)^2\log(z^+)^2 \\ &\leq \frac{2}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\mathcal{S}^{\frac{N}{2}} + \frac{\theta_1}{N}e^{-\frac{\lambda_1}{\theta_1}}\left[\frac{a}{\pi}|\nabla w|_2^2 + |w^+|_2^2\log|w^+|_2^2 - N(1+\log a)|w^+|_2^2\right] \\ &+ \frac{\theta_2}{N}e^{-\frac{\lambda_2}{\theta_2}}\left[\frac{a}{\pi}|\nabla z|_2^2 + |z^+|_2^2\log|z^+|_2^2 - N(1+\log a)|z^+|_2^2\right] \\ &\leq \frac{2}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\mathcal{S}^{\frac{N}{2}} + \frac{1}{2N}|\nabla u|_2^2 + C|u^+|_2^{2-\tau} + C|u^+|_2^{2+\tau} + C|u^+|_2^2 \\ &+ \frac{1}{2N}|\nabla v|_2^2 + C|v^+|_2^{2-\tau} + C|v^+|_2^{2+\tau} + C|v^+|_2^2,\end{aligned} \tag{2.4}$$

where we fix $a > 0$ with $\frac{a}{\pi}\theta_1 < \frac{1}{2}$ and $\frac{a}{\pi}\theta_2 < \frac{1}{2}$. Therefore, combining this with (2.3), we can see that there exists $C_2 > 0$, such that

$$|u^+|_{2p}^{2p} \leq S^{-p}|\nabla u|_2^{2p} \leq C_2.$$

Similarly, we can prove that $|v^+|_{2p}^{2p} \leq C_2$. □

Consider the matrix $M(u,v)=(M_{ij}(u,v))_{2\times 2}$ with

$$\begin{aligned}
M_{11}(u,v) &:= (2p-2)\mu_1|u^+|_{2p}^{2p}+(p-2)\beta|u^+v^+|_p^p+2\theta_1|u^+|_2^2,\\
M_{22}(u,v) &:= (2p-2)\mu_2|v^+|_{2p}^{2p}+(p-2)\beta|u^+v^+|_p^p+2\theta_2|v^+|_2^2,\\
M_{12}(u,v) &= M_{21}(u,v):=p\beta|u^+v^+|_p^p,
\end{aligned}$$

and the set

$$\begin{aligned}
\mathcal{Q} &:= \{(u,v)\in\mathcal{H}:\ \text{the matrix } M(u,v) \text{ is strictly diagonally dominant}\}\\
&= \{(u,v)\in\mathcal{H}: M_{11}(u,v)-|M_{12}(u,v)|>0, M_{22}(u,v)-|M_{21}(u,v)|>0\}.
\end{aligned}$$

Then we show that the set $\mathcal{N}\cap\mathcal{Q}$ is a natural constraint when $\beta<0$.

Proposition 2.1. *Assume that $\beta<0$ and the energy level $\mathcal{C}_{\mathcal{N}}$ is achieved by $(u,v)\in\mathcal{N}\cap\mathcal{Q}$. Then (u,v) is a critical point of the functional $\mathcal{L}$.*

Proof. Let

$$\mathcal{G}_1(u,v):=\mathcal{L}'(u,v)(u,0) \text{ and } \mathcal{G}_2(u,v):=\mathcal{L}'(u,v)(0,v).$$

Take $(u,v)\in\mathcal{N}\cap\mathcal{Q}$. Since the matrix $M(u,v)$ is strictly diagonally dominant and $\beta<0$, we have

$$\mathcal{G}_1'(u,v)(u,v)=-\left[(2p-2)\mu_1\int_\Omega|u^+|^{2p}+(2p-2)\int_\Omega\beta|u^+|^p|v^+|^p+2\theta_1\int_\Omega|u^+|^2\right]<0.$$

Similarly, we have $\mathcal{G}_2'(u,v)(u,v)<0$. Therefore, it follows that $\mathcal{G}_i(u,v)$ defines, locally, a C^1-manifold of codimension 1 in $\mathcal{H}$ for any $(u,v)\in\mathcal{N}\cap\mathcal{Q}$ and $i=1,2$.

Now we claim that in a neighborhood of $(u,v)\in\mathcal{N}\cap\mathcal{Q}$, the set $\mathcal{N}$ is a C^1-manifold of codimension 2 in $\mathcal{H}$. For that purpose, we only need to show that $(\mathcal{G}_1'(u,v),\mathcal{G}_2'(u,v))$ is a surjective as a linear operator $\mathcal{H}\to\mathbb{R}^2$. Notice that

$$\begin{aligned}
\mathcal{G}_1'(u,v)(t_1u,t_2v) &= -M_{11}(u,v)t_1-M_{12}(u,v)t_2,\\
\mathcal{G}_2'(u,v)(t_1u,t_2v) &= -M_{21}(u,v)t_1-M_{22}(u,v)t_2,
\end{aligned}$$

Since $(u,v)\in\mathcal{Q}$ and $\theta_1,\theta_2,\mu_1,\mu_2>0$, the matrix $M(u,v)$ is positive definite. Then for any $s_1,s_2\in\mathbb{R}$, there exist $t_1,t_2\in\mathbb{R}$ such that

$$(\mathcal{G}_1'(w,z),\mathcal{G}_2'(w,z))=(s_1,s_2),$$

where $(w,z)=(t_1u,t_2v)$. Therefore, the claim is true.

Now we suppose that $\mathcal{C}_{\mathcal{N}}$ is achieved by $(u,v)\in\mathcal{N}\cap\mathcal{Q}$. By the Sobolev embedding theorem, the set $\mathcal{N}\cap\mathcal{Q}$ is an open set of $\mathcal{N}$ in the topology of $\mathcal{H}$. Thus (u,v) is an inner critical point of $\mathcal{L}$ in an open subset of $\mathcal{N}$, and in particular it is a constrained critical point of $\mathcal{L}$ on $\mathcal{N}$. Since the set $\mathcal{N}$ is a C^1-manifold of codimension 2 in $\mathcal{H}$ in a neighborhood of $(u,v)\in\mathcal{N}\cap\mathcal{Q}$. Then by the Lagrange multipliers rule, there exist $L_1,L_2\in\mathbb{R}$, such that

$$\mathcal{L}'(u,v)-L_1\mathcal{G}_1'(u,v)-L_2\mathcal{G}_2'(u,v)=0.$$

Multiplying the above equation with $(u,0)$ and $(0,v)$, we deduce from $\mathcal{L}'(u,v)(u,0)=\mathcal{L}'(u,v)(0,v)=0$ that

$$M_{11}(u,v)L_1+M_{12}(u,v)L_2=0, \tag{2.5}$$

$$M_{21}(u,v)L_1+M_{22}(u,v)L_2=0. \tag{2.6}$$

Since the system (2.5)-(2.6) has a strictly positive determinant, by the Cramer's rule, the system has a unique solution $L_1=L_2=0$, which implies that $\mathcal{L}'(u,v)=0$ and the Nehari manifold $\mathcal{N}\cap\mathcal{Q}$ is a natural constraint in $\mathcal{H}$. □

Let

$$\beta_0 := \min\left\{\sqrt{\mu_1\mu_2}, \mu_1\sqrt{\frac{C_1}{C_2}}, \mu_2\sqrt{\frac{C_1}{C_2}}\right\} \leq \min\{\mu_1, \mu_2\},$$

where C_1, C_2 are given by Lemma 2.1. Then the following result, together with Proposition 2.1, show that the constrained critical points of $\mathcal{L}$ on $\mathcal{N} \cap \left\{(u,v) \in \mathcal{H} : \mathcal{L}(u,v) \leq \frac{2}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\mathcal{S}^{\frac{N}{2}}\right\}$ are in fact the free critical points of $\mathcal{L}$.

Proposition 2.2. *Let $\beta \in (-\beta_0, 0)$, then we have*

$$\mathcal{N} \cap \left\{(u,v) \in \mathcal{H} : \mathcal{L}(u,v) \leq \frac{2}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\mathcal{S}^{\frac{N}{2}}\right\} \subset \mathcal{N} \cap \mathcal{Q}.$$

Proof. Take

$$(u,v) \in \mathcal{N} \cap \left\{(u,v) \in \mathcal{H} : \mathcal{L}(u,v) \leq \frac{2}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\mathcal{S}^{\frac{N}{2}}\right\}.$$

Since $\beta \in (-\beta_0, 0)$ and $\theta_1 > 0$, we deduce from Lemma 2.1 and the Hölder's inequality that

$$\begin{aligned} M_{11}(u,v) - |M_{12}(u,v)| &= (2p-2)\mu_1 \int_\Omega |u^+|^{2p} + (2p-2)\int_\Omega \beta|u^+|^p|v^+|^p + 2\theta_1 \int_\Omega |u^+|^2 \\ &\geq (2p-2)\left[\mu_1|u^+|_{2p}^{2p} + \beta\left(|u^+|_{2p}^{2p}\right)^{\frac{1}{2}}\left(|v^+|_{2p}^{2p}\right)^{\frac{1}{2}}\right] \\ &\geq (2p-2)C_1^{\frac{1}{2}}\left[\mu_1 (C_1)^{\frac{1}{2}} + \beta (C_2)^{\frac{1}{2}}\right] > 0. \end{aligned}$$

Similarly, we can prove that $M_{22}(u,v) - |M_{21}(u,v)| > 0$. Therefore, $(u,v) \in \mathcal{N} \cap \mathcal{Q}$. This completes the proof. □

Consider the Brézis-Nirenberg problem with logarithmic perturbation

$$-\Delta u = \lambda_i u + \mu_i |u|^{2p-2}u + \theta_i u \log u^2 \quad \text{in } \Omega,\ i = 1,2,$$

where $\lambda_i \in \mathbb{R}$, $\mu_i, \theta_i > 0$. As in [12], we define the associated modified energy functional

$$\mathcal{L}_i(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda_i}{2}\int_\Omega |u^+|^2 - \frac{\mu_i}{2p}\int_\Omega |u^+|^{2p} - \frac{\theta_i}{2}\int_\Omega (u^+)^2\left(\log(u^+)^2 - 1\right),$$

and the level

$$\mathcal{C}_{\theta_i} = \inf_{u \in \mathcal{N}_i} \mathcal{L}_i(u),$$

where

$$\mathcal{N}_i = \left\{u \in H_0^1(\Omega) \setminus \{0\} : \mathcal{L}_i'(u)u = 0\right\}.$$

Then we have the following proposition, which plays crucial role in the proof of Theorem 1.1.

Proposition 2.3. *Let $\beta \in (-\beta_0, 0)$, then*

$$\mathcal{C}_{\mathcal{N}} < \min\left\{\mathcal{C}_{\theta_1} + \frac{1}{N}\mu_2^{-\frac{N-2}{2}}\mathcal{S}^{\frac{N}{2}},\ \mathcal{C}_{\theta_2} + \frac{1}{N}\mu_1^{-\frac{N-2}{2}}\mathcal{S}^{\frac{N}{2}},\ \frac{1}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\mathcal{S}^{\frac{N}{2}}\right\}.$$

Proof. The main idea of the proof is similar to the proof of [13, Proposition 2.4] in the case $N = 4$, here we give the details for the sake of clarity and completeness. Without loss of generality, we prove that

$$\mathcal{C}_{\mathcal{N}} < \mathcal{C}_{\theta_1} + \frac{1}{N}\mu_2^{-\frac{N-2}{2}}\mathcal{S}^{\frac{N}{2}}.$$

By [12], the energy level $\mathcal{C}_{\theta_1}$ can be achieved by a positive solution u_{θ_1}. Moreover, we know that $u_{\theta_1} \in C^2(\Omega)$ and $u_{\theta_1} \equiv 0$ on $\partial\Omega$. Then, there exists a ball

$$B_{2R_0}(y_0) := \{x \in \Omega : |x - y_0| \leq 2R\} \subset \Omega,$$

satisfying

$$\Pi^p := \max_{B_{2R_0}(y_0)} |u_{\theta_1}|^2 \leq \frac{\theta_2 p}{2^p|\beta|}. \tag{2.7}$$

Let $\xi \in C_0^\infty(\Omega)$ be the radial function, such that $\xi(x) \equiv 1$ for $0 \leq |x - y_0| \leq R$, $0 \leq \xi(x) \leq 1$ for $R \leq |x - y_0| \leq 2R$, $\xi(x) \equiv 0$ for $|x - y_0| \geq 2R$, where we take arbitrary $R < R_0$ such that $B_{2R}(y_0) \subset B_{2R_0}(y_0)$. Take $w_\varepsilon(x) = \xi(x)W_{\varepsilon,y_0}(x)$, where

$$W_{\varepsilon,y_0}(x) = \frac{\left[N(N-2))\varepsilon^2\right]^{\frac{N-2}{4}}}{(\varepsilon^2 + |x - y_0|^2)^{\frac{N-2}{2}}}.$$

Then by [5] or [21, Lemma 1.46], we obtain the following

$$\begin{aligned}&\int_\Omega |\nabla w_\varepsilon|^2 = \mathcal{S}^{\frac{N}{2}} + O(\varepsilon^{N-2}), \quad \int_\Omega |w_\varepsilon|^{2^*} = \mathcal{S}^{\frac{N}{2}} + O(\varepsilon^N),\\ &\int_\Omega |w_\varepsilon|^2 = \widetilde{C}_0\varepsilon^2 + O(\varepsilon^{N-2}).\end{aligned} \tag{2.8}$$

Also, by [12, Lemma 3.4], we have the following

$$\int_\Omega w_\varepsilon^2 \log w_\varepsilon^2 = \widetilde{C}_1\varepsilon^2|\log\varepsilon| + O(\varepsilon^2). \tag{2.9}$$

Moreover, one has

$$\begin{aligned}\int_\Omega |w_\varepsilon|^p &\leq \int_{B_{2R}(y_0)} |W_{\varepsilon,y_0}|^p \, dx = C\int_{B_{2R}(0)} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{N}{2}} \, dx\\ &\leq C\varepsilon^{\frac{N}{2}}\left(\log\frac{2R}{\varepsilon} + 1\right) = o(\varepsilon^2).\end{aligned} \tag{2.10}$$

Then by (2.7), we infer that

$$|\beta|\int_\Omega |u_{\theta_1}|^p|w_\varepsilon|^p = |\beta|\int_{B_{2R}(y_0)} |u_{\theta_1}|^p|w_\varepsilon|^p \leq |\beta|\Pi^p\int_\Omega |w_\varepsilon|^p \leq \frac{\theta_2}{2}\int_\Omega |w_\varepsilon|^p = o(\varepsilon^2). \tag{2.11}$$

Now we claim that there exists $s_{1,\varepsilon}, s_{2,\varepsilon} > 0$ such that

$$(s_{1,\varepsilon}u_{\theta_1}, s_{2,\varepsilon}w_\varepsilon) \in \mathcal{N}.$$

For that purpose, we consider

$$
\begin{aligned}
F(t_1,t_2) &:= \mathcal{L}(t_1u_{\theta_1}, t_2w_\varepsilon)\\
&= \frac{1}{2}t_1^2\int_\Omega |\nabla u_{\theta_1}|^2 - \frac{\lambda_1}{2}t_1^2\int_\Omega |u_{\theta_1}|^2 - \frac{\mu_1}{2p}t_1^{2p}\int_\Omega |u_{\theta_1}|^{2p} - \frac{\theta_1}{2}\int_\Omega (t_1u_{\theta_1})^2\left(\log(t_1u_{\theta_1})^2-1\right)\\
&+ \frac{1}{2}t_2^2\int_\Omega |\nabla w_\varepsilon|^2 - \frac{\lambda_2}{2}t_2^2\int_\Omega |w_\varepsilon|^2 - \frac{\mu_2}{2p}t_2^{2p}\int_\Omega |w_\varepsilon|^{2p} - \frac{\theta_2}{2}\int_\Omega (t_2w_\varepsilon)^2(\log(t_2w_\varepsilon)^2-1)\\
&- \frac{\beta}{p}t_1^pt_2^p\int_\Omega |u_{\theta_1}|^p|w_\varepsilon|^p.
\end{aligned}
$$

For ε small enough, we can see from (2.8) and (2.11) that the matrix

$$
\begin{pmatrix} \mu_1|u_{\theta_1}|_{2p}^{2p} & \beta|u_{\theta_1}w_\varepsilon|_p^p \\ \beta|u_{\theta_1}w_\varepsilon|_p^p & \mu_2|w_\varepsilon|_{2p}^{2p} \end{pmatrix}
$$

is strictly diagonally dominant, so it is positive definite. Therefore, there exists a constant $C>0$ such that

$$
\frac{\mu_1}{2p}t_1^{2p}\int_\Omega |u_{\theta_1}|^{2p} + \frac{\beta}{p}t_1^pt_2^p\int_\Omega |u_{\theta_1}|^2|w_\varepsilon|^2 + \frac{\mu_2}{2p}t_2^{2p}\int_\Omega |w_\varepsilon|^{2p} \geq C(t_1^{2p}+t_2^{2p}).
$$

Then

$$
\begin{aligned}
F(t_1,t_2) \leq{}& \frac{1}{2}t_1^2\int_\Omega |\nabla u_{\theta_1}|^2 - \frac{\lambda_1}{2}t_1^2\int_\Omega |u_{\theta_1}|^2 - Ct_1^{2p} - \frac{\theta_1}{2}\int_\Omega (t_1u_{\theta_1})^2\left(\log(t_1u_{\theta_1})^2-1\right)\\
&+ \frac{1}{2}t_2^2\int_\Omega |\nabla w_\varepsilon|^2 - \frac{\lambda_2}{2}t_2^2\int_\Omega |w_\varepsilon|^2 - Ct_2^{2p} - \frac{\theta_2}{2}\int_\Omega (t_2w_\varepsilon)^2(\log(t_2w_\varepsilon)^2-1).
\end{aligned}
$$

It follows from $2p\geq 2$ and $\lim_{s\to+\infty}\frac{s^{2p}}{s^2\log s^2}=+\infty$ that $F(t_1,t_2)\to-\infty$, as $|(t_1,t_2)|\to+\infty$, where $|(t_1,t_2)|=\sqrt{t_1^2+t_2^2}$. This implies that there exists a global maximum point $(s_{1,\varepsilon},s_{2,\varepsilon})\in\overline{(\mathbb{R}^+)^2}$.

Assume that $(s_{1,\varepsilon},s_{2,\varepsilon})\in\partial\overline{(\mathbb{R}^+)^2}$. Without loss of generality, we assume that $s_{1,\varepsilon}=0$ and $s_{2,\varepsilon}\neq 0$. Note that $1<p<2$ and $\lim_{s\to 0+}\frac{s^2\log s^2}{s^p}=0$, thanks to $\beta<0$, we have

$$
\begin{aligned}
F(t_1,s_{2,\varepsilon})-F(s_{1,\varepsilon},s_{2,\varepsilon}) ={}& \frac{1}{2}t_1^2\int_\Omega |\nabla u_{\theta_1}|^2 - \frac{\lambda_1}{2}t_1^2\int_\Omega |u_{\theta_1}|^2 - \frac{\mu_1}{2p}t_1^{2p}\int_\Omega |u_{\theta_1}|^{2p} - \frac{\theta_1}{2}t_1^2\log t_1^2\int_\Omega |u_{\theta_1}|^2\\
&- \frac{\theta_1}{2}t_1^2\int_\Omega u_{\theta_1}^2(\log u_{\theta_1}^2-1) - \frac{\beta}{p}t_1^ps_{2,\varepsilon}^p\int_\Omega |u_{\theta_1}|^p|w_\varepsilon|^p > 0
\end{aligned}
$$

for t_1 small enough. This contradicts to the fact that $(s_{1,\varepsilon},s_{2,\varepsilon})$ is a global maximum point in $\overline{(\mathbb{R}^+)^2}$. Therefore, $(s_{1,\varepsilon},s_{2,\varepsilon})\notin\partial\overline{(\mathbb{R}^+)^2}$ and it is a critical point of $F(t_1,t_2)$. Also, we can see that $s_{i,\varepsilon}(i=1,2)$ are bounded from above and below for ε small enough. Then we have

$$
\frac{\partial F}{\partial t_1}(s_{1,\varepsilon},s_{2,\varepsilon}) = \frac{\partial F}{\partial t_2}(s_{1,\varepsilon},s_{2,\varepsilon}) = 0,
$$

which is equivalent to

$$
(s_{1,\varepsilon}u_{\theta_1}, s_{2,\varepsilon}w_\varepsilon)\in\mathcal{N}.
$$

For ε small enough, since $s_{i,\varepsilon}(i=1,2)$ are bounded from above and below, then by (2.8), we have

$$\theta_2 \log s_{2,\varepsilon}^2 \int_\Omega |w_\varepsilon|^2 = O(\varepsilon^2).$$

By $(s_{1,\varepsilon}u_{\theta_1}, s_{2,\varepsilon}w_\varepsilon) \in \mathcal{N}$, we have

$$\begin{aligned} s_{2,\varepsilon}^{2p-2} &= \frac{\int_\Omega |\nabla w_\varepsilon|^2 - \lambda_2 \int_\Omega |w_\varepsilon|^2 - \theta_2 \int_\Omega (w_\varepsilon)^2 \log (w_\varepsilon)^2 - \theta_2 \log s_{2,\varepsilon}^2 \int_\Omega |w_\varepsilon|^2 - s_{1,\varepsilon}^p s_{2,\varepsilon}^{p-2} \beta \int_\Omega |u_{\theta_1}|^p |w_\varepsilon|^p}{\mu_2 \int_\Omega |w_\varepsilon|^{2p}} \\ &= \frac{\mathcal{S}^{\frac{N}{2}} + O(\varepsilon^2 |\log \varepsilon|)}{\mu_2 \mathcal{S}^{\frac{N}{2}} + O(\varepsilon^N)} \to \frac{1}{\mu_2} \text{ as } \varepsilon \to 0^+, \end{aligned}$$

and

$$\begin{aligned} 0 &= \int_\Omega |\nabla u_{\theta_1}|^2 - \lambda_1 \int_\Omega |u_{\theta_1}|^2 - s_{1,\varepsilon}^{2p-2} \mu_1 \int_\Omega |u_{\theta_1}|^{2p} - \theta_1 \int_\Omega u_{\theta_1}^2 \log u_{\theta_1}^2 - \theta_1 \log s_{1,\varepsilon}^2 \int_\Omega |u_{\theta_1}|^2 - s_{1,\varepsilon}^{p-2} s_{2,\varepsilon}^p \beta \int_\Omega |u_{\theta_1}|^p |w_\varepsilon|^p \\ &= \left(1 - s_{1,\varepsilon}^{2p-2}\right) \mu_1 \int_\Omega |u_{\theta_1}|^{2p} - \theta_1 \log s_{1,\varepsilon}^2 \int_\Omega |u_{\theta_1}|^2 + o(\varepsilon^2). \end{aligned}$$

Then we can see that $s_{1,\varepsilon} \to 1$ as $\varepsilon \to 0^+$. Moreover, for ε small enough, we have

$$\frac{1}{2} \leq s_{1,\varepsilon} \leq 2 \quad \text{and} \quad \frac{1}{2\mu_2} \leq s_{2,\varepsilon}^{2p-2} \leq \frac{2}{\mu_2}. \tag{2.12}$$

Since $(s_{1,\varepsilon}u_{\theta_1}, s_{2,\varepsilon}w_\varepsilon) \in \mathcal{N}$, there holds

$$\mathcal{C}_\mathcal{N} \leq \mathcal{L}(s_{1,\varepsilon}u_{\theta_1}, s_{2,\varepsilon}w_\varepsilon) =: f_1(s_{1,\varepsilon}) + f_2(s_{2,\varepsilon}) - \frac{\beta}{p} s_{1,\varepsilon}^p s_{2,\varepsilon}^p \int_\Omega |u_{\theta_1}|^p |w_\varepsilon|^p, \tag{2.13}$$

where

$$f_1(s_1) := \frac{1}{2} s_1^2 \int_\Omega |\nabla u_{\theta_1}|^2 - \frac{\lambda_1}{2} s_1^2 \int_\Omega |u_{\theta_1}|^2 - \frac{\mu_1}{2p} s_1^{2p} \int_\Omega |u_{\theta_1}|^{2p} - \frac{\theta_1}{2} \int_\Omega (s_1 u_{\theta_1})^2 \left(\log(s_1 u_{\theta_1})^2 - 1\right),$$

and

$$f_2(s_2) := \frac{1}{2} s_2^2 \int_\Omega |\nabla w_\varepsilon|^2 - \frac{\lambda_2}{2} s_2^2 \int_\Omega |w_\varepsilon|^2 - \frac{\mu_2}{2p} s_2^{2p} \int_\Omega |w_\varepsilon|^{2p} - \frac{\theta_2}{2} \int_\Omega (s_2 w_\varepsilon)^2 (\log(s_2 w_\varepsilon)^2 - 1).$$

Recalling that u_{θ_1} is a positive least energy solution of $-\Delta u = \lambda_1 u + \mu_1 |u|^{2p-2} u + \theta_1 u \log u^2$. Then, we have

$$\int_\Omega |\nabla u_{\theta_1}|^2 = \lambda_1 \int_\Omega |u_{\theta_1}|^2 + \mu_1 \int_\Omega |u_{\theta_1}|^{2p} + \theta_1 \int_\Omega u_{\theta_1}^2 \log u_{\theta_1}^2, \tag{2.14}$$

and

$$\mathcal{C}_{\theta_1} = \frac{1}{2} \int_\Omega |\nabla u_{\theta_1}|^2 - \frac{\lambda_1}{2} \int_\Omega |u_{\theta_1}|^2 - \frac{\mu_1}{2p} \int_\Omega |u_{\theta_1}|^{2p} - \frac{\theta_1}{2} \int_\Omega u_{\theta_1}^2 (\log u_{\theta_1}^2 - 1). \tag{2.15}$$

By a direct calculation, we can see from (2.14) that

$$\begin{aligned} f'(s_1) &= s_1 \int_\Omega |\nabla u_{\theta_1}|^2 - s_1 \lambda_1 \int_\Omega |u_{\theta_1}|^2 - s_1^{2p-1} \mu_1 \int_\Omega |u_{\theta_1}|^{2p} - \theta_1 \int_\Omega s_1 u_{\theta_1} \log(s_1 u_{\theta_1})^2 \\ &= (s_1 - s_1^{2p-1}) \mu_1 \int_\Omega |u_{\theta_1}|^{2p} - (s_1 \log s_1^2) \theta_1 \int_\Omega |u_{\theta_1}|^2. \end{aligned} \tag{2.16}$$

Thanks to $\theta_1 > 0$, one can see that $f'(s_1) > 0$ for $0 < s_1 < 1$ and $f'(s_1) < 0$ for $s_1 > 1$. So, by (2.15),

$$f_1(s_{1,\varepsilon}) \leq f_1(1) = \mathcal{C}_{\theta_1}.$$

On the other hand, we deduce from (2.10),(2.12) that

$$\frac{|\beta|}{p} s_{1,\varepsilon}^p s_{2,\varepsilon}^p \int_\Omega |u_{\theta_1}|^p |w_\varepsilon|^p \leq \frac{\theta_2}{2} \left(\frac{2}{\mu_2}\right)^{\frac{p}{2p-2}} \int_\Omega |w_\varepsilon|^p = o(\varepsilon^2).$$

If follows from (2.8) and (2.9) that

$$\begin{aligned}
&f_2(s_{2,\varepsilon}) - \frac{\beta}{p} s_{1,\varepsilon}^p s_{2,\varepsilon}^p \int_\Omega |u_{\theta_1}|^p |w_\varepsilon|^p \\
&\leq \frac{1}{2} s_{2,\varepsilon}^2 \int_\Omega |\nabla w_\varepsilon|^2 - \frac{\lambda_2}{2} s_{2,\varepsilon}^2 \int_\Omega |w_\varepsilon|^2 - \frac{\mu_2}{2p} s_{2,\varepsilon}^{2p} \int_\Omega |w_\varepsilon|^{2p} - \frac{\theta_2}{2} \int_\Omega (s_{2,\varepsilon} w_\varepsilon)^2 (\log(s_{2,\varepsilon} w_\varepsilon)^2 - 1) \\
&\leq \left(\frac{1}{2} s_{2,\varepsilon}^2 - \frac{\mu_2}{2p} s_{2,\varepsilon}^{2p}\right) \mathcal{S}^{\frac{N}{2}} + O(\varepsilon^{N-2}) - \frac{\lambda_2 - \theta_2}{2} s_{2,\varepsilon}^2 \int_\Omega |w_\varepsilon|^2 - \frac{\theta_2}{2} s_{2,\varepsilon}^2 \log s_{2,\varepsilon}^2 \int_\Omega |w_\varepsilon|^2 - \frac{\theta_2}{2} s_{2,\varepsilon}^2 \int_\Omega w_\varepsilon^2 \log w_\varepsilon^2 + o(\varepsilon^2) \\
&\leq \frac{1}{N} \mu_2^{-\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}} - C\theta_2 \varepsilon^2 |\log \varepsilon| + O(\varepsilon^2) \\
&< \frac{1}{N} \mu_2^{-\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}}, \text{ for } \varepsilon \text{ small enough.}
\end{aligned}$$

Combining this with (2.13) and (2.16), we have

$$\mathcal{C}_\mathcal{N} < \mathcal{C}_{\theta_1} + \frac{1}{N} \mu_2^{-\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}}.$$

Similarly, we can also prove that $\mathcal{C}_\mathcal{N} < \mathcal{C}_{\theta_2} + \frac{1}{N} \mu_1^{-\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}}$. By [12, Lemma 3.3], we can easily see that $\mathcal{C}_{\theta_i} < \frac{1}{N} \mu_i^{-\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}}$ for $i = 1, 2$. Therefore, we have

$$\mathcal{C}_\mathcal{N} < \min\left\{\mathcal{C}_{\theta_1} + \frac{1}{N} \mu_2^{-\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}},\ \mathcal{C}_{\theta_2} + \frac{1}{N} \mu_1^{-\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}},\ \frac{1}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right) \mathcal{S}^{\frac{N}{2}}\right\}.$$

The proof is completed. □

Proof of Theorem 1.1 for the case $\beta < 0$. Repeating the proof of Theorem 1.3 for the case $\beta < 0$ in [8] with some slight modifications, we can construct a Palais-Smale sequence at the level $\mathcal{C}_\mathcal{N}$. Then there exists a sequence $\{(u_n, v_n)\} \subset \mathcal{N}$ satisfying

$$\lim_{n\to\infty} \mathcal{L}(u_n, v_n) = \mathcal{C}_\mathcal{N}, \quad \lim_{n\to\infty} \mathcal{L}'(u_n, v_n) = 0. \tag{2.17}$$

By Proposition 2.3, we can see that $\mathcal{L}(u_n, v_n) \leq \frac{2}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right) \mathcal{S}^{\frac{N}{2}}$ for n large enough. Then by Lemma 2.1 and (2.4), we can see that $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. Hence, we may assume that

$$(u_n, v_n) \rightharpoonup (u, v) \text{ weakly in } \mathcal{H}.$$

Passing to subsequence, we may also assume that

$$\begin{aligned}
&u_n \rightharpoonup u, \quad v_n \rightharpoonup v \ \text{ weakly in } L^{2p}(\Omega), \\
&u_n \to u, \quad v_n \to v \ \text{ strongly in } L^q(\Omega) \text{ for } 2 \leq q < 2p, \\
&u_n \to u, \quad v_n \to v \ \text{ almost everywhere in } \Omega.
\end{aligned}$$

By using the inequality $|s^2 \log s^2| \le Cs^{2-\tau} + Cs^{2+\tau}$, $\tau \in (0,1)$ and the dominated convergence theorem, one gets

$$\lim_{n\to\infty} \int_\Omega u_n^+ \varphi^+ \log(u_n^+)^2 = \int_\Omega u^+ \varphi^+ \log(u^+)^2 \quad \text{for any } \varphi \in C_0^\infty(\Omega).$$

Then by (2.17), we have $\mathcal{L}'(u,v) = 0$. Moreover, by using the weak-lower semicontinuity of norm, we have

$$\mathcal{L}(u,v) \le \frac{2}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\mathcal{S}^{\frac{N}{2}}. \tag{2.18}$$

Let $w_n = u_n - u$ and $z_n = v_n - v$. Then by the Brézis-Lieb Lemma (see [6] and [8, Lemma 3.3]). we have

$$\begin{aligned} &|u_n^+|_{2p}^{2p} = |u^+|_{2p}^{2p} + |w_n^+|_{2p}^{2p} + o_n(1), \quad |v_n^+|_{2p}^{2p} = |v^+|_{2p}^{2p} + |z_n^+|_{2p}^{2p} + o_n(1),\\ &|u_n^+ v_n^+|_p^p = |u^+ v^+|_p^p + |w_n^+ z_n^+|_p^p + o_n(1). \end{aligned} \tag{2.19}$$

Since $(u_n, v_n) \in \mathcal{N}$ and $\mathcal{L}'(u,v) = 0$, by (2.19) and [12, Lemma 2.3], we have

$$|\nabla w_n|_2^2 = \mu_1 |w_n^+|_{2p}^{2p} + \beta |w_n^+ z_n^+|_p^p + o_n(1), \quad |\nabla z_n|_2^2 = \mu_2 |z_n^+|_{2p}^{2p} + \beta |w_n^+ z_n^+|_p^p + o_n(1). \tag{2.20}$$

By a direct calculation, one gets

$$\mathcal{L}(u_n, v_n) = \mathcal{L}(u,v) + \frac{1}{N}\int_\Omega |\nabla w_n|^2 + \frac{1}{N}\int_\Omega |\nabla z_n|^2 + o_n(1). \tag{2.21}$$

Passing to subsequence, we may assume that

$$\int_\Omega |\nabla w_n|^2 = k_1 + o_n(1), \quad \int_\Omega |\nabla z_n|^2 = k_2 + o_n(1).$$

Letting $n \to +\infty$ in (2.21), we have

$$0 \le \mathcal{L}(u,v) \le \mathcal{L}(u,v) + \frac{1}{N}k_1 + \frac{1}{N}k_2 = \lim_{n\to\infty} \mathcal{L}(u_n, v_n) = \mathcal{C}_\mathcal{N}. \tag{2.22}$$

Now we claim that $u \not\equiv 0$ and $v \not\equiv 0$.

Case 1. $u \equiv 0$ and $v \equiv 0$.

Firstly, we prove that $k_1 > 0$ and $k_2 > 0$. Without loss of generality, we assume by contradiction that $k_1 = 0$, then we can see that $w_n \to 0$ strongly in $H_0^1(\Omega)$ and $u_n \to 0$ strongly in $H_0^1(\Omega)$, then by the Sobolev inequality, we can see that $u_n \to 0$ strongly in $L^{2p}(\Omega)$, which is impossible by Lemma 2.1. Therefore, we have that $k_1 > 0$ and $k_2 > 0$. Since (2.20) holds, it follows from the proof of [8, Theorem 1.3] that there exists $t_n, s_n > 0$ such that $(t_n w_n, s_n z_n) \in \widetilde{\mathcal{N}}$, which is given by (2.26). Moreover, $t_n = 1 + o_n(1), s_n = 1 + o_n(1)$. Therefore, by (2.27) we have

$$\mathcal{C}_\mathcal{N} = \frac{1}{N}k_1 + \frac{1}{N}k_2 = \lim_{n\to\infty} \mathcal{E}(w_n, z_n) = \lim_{n\to\infty} \mathcal{E}(t_n w_n, s_n z_n) \ge \mathcal{A} = \frac{1}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\mathcal{S}^{\frac{N}{2}},$$

a contradiction with Proposition 2.3. Therefore, Case 1 is impossible.

Case 2. $u \equiv 0$, $v \not\equiv 0$ or $u \not\equiv 0$, $v \equiv 0$.

Without loss of generality, we may assume that $u \equiv 0$, $v \not\equiv 0$. Then by Case 1, we have that $k_1 > 0$, and we may assume that $k_2 = 0$. Then we know that $|w_n^+ z_n^+|_p^p = o_n(1)$. By (2.20), we have

$$\int_\Omega |\nabla w_n|^2 = \mu_1 |w_n^+|_{2p}^{2p} + o_n(1) \le \mu_1 \mathcal{S}^{-p}\left(\int_\Omega |\nabla w_n|^2\right)^p.$$

Thus, letting $n \to \infty$, we have $k_1 \geq \mu_1^{-\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}}$. Notice that v is a solution of $-\Delta w = \lambda_2 w + \mu_2 |w|^2 w + \theta_2 w \log w^2$, we have $\mathcal{L}(0, v) \geq \mathcal{C}_{\theta_2}$. Therefore, by (2.22) we have that

$$\mathcal{C} \geq \mathcal{C}_{\theta_2} + \frac{1}{N} \mu_1^{-\frac{N-2}{2}} \mathcal{S}^{\frac{N}{2}},$$

which is a contradiction with Proposition 2.3. Therefore, Case 2 is impossible.

Since Case 1 and 2 are both impossible, we get that $u \not\equiv 0$ and $v \not\equiv 0$. Therefore, $(u, v) \in \mathcal{N}$ and by (2.22) we have that $\mathcal{L}(u, v) = \mathcal{C}_{\mathcal{N}}$. Then combining (2.18) with Proposition 2.1 and 2.2, (u, v) is a solution of system (1.2). Since $\mathcal{L}'(u, v) = 0$, we can see that

$$0 = \mathcal{L}'(u, v)(u^-, 0) = \int_\Omega |\nabla u^-|^2, \quad 0 = \mathcal{L}'(u, v)(0, v^-) = \int_\Omega |\nabla v^-|^2.$$

which implies that $u \geq 0$, $v \geq 0$. By the Morse's iteration, the solutions u, v belong to $L^\infty(\Omega)$. Then the Hölder estimate implies that $u, v \in C^{0,\gamma}(\Omega)$ for any $0 < \gamma < 1$. Define $g_i : [0, +\infty) \to \mathbb{R}$, $i = 1, 2$ by

$$g_i(s) := \begin{cases} 2\theta_i |s \log s^2|, & s > 0, \\ 0, & s = 0. \end{cases}$$

Then we follow the arguments in [12, 20], and get that $u, v \in C^2(\Omega)$ and $u, v > 0$ in Ω. This completes the proof. □

It remains to prove the Theorem 1.1 for the case $\beta > 0$. For that purpose, we first introduce some definitions and lemmas. From now on, we assume that $\beta > 0$. Let

$$\mathcal{B} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{L}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], \mathcal{H}) : \gamma(0) = 0, \mathcal{L}(\gamma(1)) < 0\}$. By a similar argument as used in Proposition 2.3, we can see that for any $(u, v) \in \mathcal{H}$ with $(u, v) \neq (0, 0)$, there exists $s_{u,v} > 0$ such that

$$\max_{t > 0} \mathcal{L}(tu, tv) = \mathcal{L}(s_{u,v} u, s_{u,v} v).$$

Moreover, we have $(s_{u,v} u, s_{u,v} v) \in \mathcal{M}$, where

$$\mathcal{M} = \{(u, v) \in \mathcal{H} \setminus \{(0, 0)\} : \mathcal{L}'(u, v)(u, v) = 0\}.$$

Note that $\theta_i > 0$ for $i = 1, 2$, it is easy to check that

$$\mathcal{B} = \inf_{(u,v) \in \mathcal{H} \setminus \{(0,0)\}} \max_{t > 0} \mathcal{L}(tu, tv) = \inf_{(u,v) \in \mathcal{M}} \mathcal{L}(u, v). \tag{2.23}$$

Since $\mathcal{N} \subset \mathcal{M}$, one has that

$$\mathcal{B} \leq \mathcal{C}_{\mathcal{N}}. \tag{2.24}$$

Lemma 2.2. *The functional $\mathcal{L}$ has a mountain pass geometry structure, that is,*

(i) there exists $\alpha, \zeta > 0$, such that $\mathcal{L}(u, v) \geq \alpha > 0$ for all $\|(u, v)\|_{\mathcal{H}} = \zeta$;

(ii) there exists $(w, z) \in \mathcal{H}$, such that $\|(w, z)\|_{\mathcal{H}} \geq \zeta$ and $\mathcal{L}(w, z) < 0$,

where $\|(u,v)\|_{\mathcal{H}}^2 := \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)$.

Proof. Note that $\theta_i > 0$, it follows from the inequality $s^2 \log s^2 \le (p-1)^{-1} e^{-1} s^{2p}$ for all $s > 0$ that

$$\begin{aligned}\frac{\lambda_1}{2}\int_\Omega |u^+|^2 + \frac{\theta_1}{2}\int_\Omega (u^+)^2(\log(u^+)^2 - 1) &\le \frac{\theta_1}{2}\int_\Omega (u^+)^2 \log(e^{\frac{\lambda_1}{\theta_1}}(u^+)^2)\\ &\le \frac{\theta_1}{2(p-1)} e^{(p-1)\frac{\lambda_1}{\theta_1}-1}\int_\Omega |u^+|^{2p}\\ &\le \frac{\theta_1}{2(p-1)} e^{(p-1)\frac{\lambda_1}{\theta_1}-1}\mathcal{S}^{-p}\left(\int_\Omega |\nabla u|^2\right)^p.\end{aligned}$$

Similarly, we have

$$\frac{\lambda_2}{2}\int_\Omega |v^+|^2 + \frac{\theta_2}{2}\int_\Omega (v^+)^2(\log(v^+)^2 - 1) \le \frac{\theta_2}{2(p-1)} e^{(p-1)\frac{\lambda_2}{\theta_2}-1}\mathcal{S}^{-p}\left(\int_\Omega |\nabla v|^2\right)^p.$$

Notice that $\frac{\beta}{p}\int_\Omega |u^+|^p|v^+|^p \le \frac{\beta}{2p}\left(\int_\Omega |u^+|^{2p} + |v^+|^{2p}\right)$, we have

$$\begin{aligned}\mathcal{L}(u,v) &\ge \frac{1}{2}\int_\Omega |\nabla u|^2 - \left(\frac{\mu_1+\beta}{2p} + \frac{\theta_1}{2(p-1)}e^{(p-1)\frac{\lambda_1}{\theta_1}-1}\right)\mathcal{S}^{-p}\left(\int_\Omega |\nabla u|^2\right)^p\\ &\quad + \frac{1}{2}\int_\Omega |\nabla v|^2 - \left(\frac{\mu_2+\beta}{2p} + \frac{\theta_2}{2(p-1)}e^{(p-1)\frac{\lambda_2}{\theta_2}-1}\right)\mathcal{S}^{-p}\left(\int_\Omega |\nabla v|^2\right)^p\\ &\ge \frac{1}{2}\|(u,v)\|_{\mathcal{H}} - C\|(u,v)\|_{\mathcal{H}}^{2p},\end{aligned}$$

which implies that there exists $\alpha > 0$ and $\zeta > 0$ such that $\mathcal{L}(u,v) \ge \alpha > 0$ for all $\|(u,v)\|_{\mathcal{H}} = \zeta$.

On the other hand, let $\varphi_1, \varphi_2 \in H_0^1(\Omega)\setminus\{0\}$ be fixed positive functions, then for any $t > 0$, we have

$$\begin{aligned}\mathcal{L}(t\varphi_1, t\varphi_2) &= \frac{t^2}{2}\int_\Omega \left(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2\right) - \frac{t^{2p}}{2p}\int_\Omega \left(\mu_1|\varphi_1|^{2p} + 2\beta|\varphi_1|^p|\varphi_2|^p + \mu_2|\varphi_2|^{2p}\right)\\ &\quad - \frac{t^2\log t^2}{2}\int_\Omega \left(\theta_1\varphi_1^2 + \theta_2\varphi_2^2\right) - \frac{t^2}{2}\int_\Omega \left(\theta_1\varphi_1^2\log\left(e^{\frac{\lambda_1}{\theta_1}-1}\varphi_1^2\right) + \theta_2\varphi_2^2\log\left(e^{\frac{\lambda_2}{\theta_2}-1}\varphi_2^2\right)\right)\\ &\quad \to -\infty \quad \text{as } t \to +\infty,\end{aligned}$$

which is guaranteed by $2 \le 2p < 4$ and $\lim_{t\to+\infty}\frac{t^{2p}}{t^2\log t^2} = +\infty$. Therefore, we can choose $t_0 > 0$ large enough such that

$$\mathcal{L}(t_0\varphi_1, t_0\varphi_2) < 0, \quad \text{and} \quad \|(t_0\varphi_1, t_0\varphi_2)\|_{\mathcal{H}} > \zeta.$$

This completes the proof. □

Now we consider the following limit system

$$\begin{cases}-\Delta u = \mu_1|u|^{2p-2}u + \beta|u|^{p-2}|v|^p u, & x \in \mathbb{R}^N\\ -\Delta v = \mu_2|v|^{2p-2}v + \beta|u|^p|v|^{p-2}v, & x \in \mathbb{R}^N\\ u, v \in \mathcal{D}^{1,2}(\mathbb{R}^N).\end{cases} \tag{2.25}$$

Define $\mathcal{D} = \mathcal{D}^{1,2}(\mathbb{R}^N)\times\mathcal{D}^{1,2}(\mathbb{R}^N)$ and a C^2-functional $\mathcal{E}: \mathcal{D} \to \mathbb{R}$ given by

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2) - \frac{1}{4}\int_{\mathbb{R}^N}\left(\mu_1|u|^{2p} + 2\beta|u|^p|v|^p + \mu_2|v|^{2p}\right).$$

We consider the level

$$\mathcal{A} = \inf_{(u,v)\in\widetilde{\mathcal{N}}} \mathcal{E}(u,v), = \inf_{(u,v)\in\widetilde{\mathcal{N}}} \frac{1}{N}\int_{\mathbb{R}^4} (|\nabla u|^2 + |\nabla v|^2).$$

with

$$\widetilde{\mathcal{N}} = \left\{(u,v)\in\mathcal{D} : u\not\equiv 0, v\not\equiv 0, \mathcal{E}'(u,v)(u,0)=0, \mathcal{E}'(u,v)(0,v)=0\right\}. \tag{2.26}$$

From [8, Theorem 1.6], we know that when $\beta < 0$,

$$\mathcal{A} = \frac{1}{N}\left(\mu_1^{-\frac{N-2}{2}} + \mu_2^{-\frac{N-2}{2}}\right)\mathcal{S}^{\frac{N}{2}}. \tag{2.27}$$

Proposition 2.4. *For any $\beta > 0$, we have*

$$\mathcal{B} < \min\left\{\mathcal{C}_{\theta_1}, \mathcal{C}_{\theta_2}, \mathcal{A}\right\}.$$

Proof. The proof is inspired by [8, Lemma 3.4], but the logarithmic terms in (1.2) make the proof much more delicate, and we require some new ideas. To prove this proposition, we divide the proof into two steps.

Step 1. We prove that $\mathcal{B} < \mathcal{A}$. By [8, Theorem 1.6], $\mathcal{A}$ can be achieved by a positive least energy solution (U,V) of the system (2.25), which is radially symmetric decreasing. Moreover, there exists a constant $C > 0$, such that

$$U(x) + V(x) \le C(1+|x|)^{2-N}, \quad |\nabla U(x)| + |\nabla V(x)| \le C(1+|x|)^{1-N}. \tag{2.28}$$

Define

$$(U_\varepsilon(x), V_\varepsilon(x)) := \left(\varepsilon^{-\frac{N-2}{2}} U\left(\frac{x}{\varepsilon}\right), \varepsilon^{-\frac{N-2}{2}} V\left(\frac{x}{\varepsilon}\right)\right).$$

Without loss of generality, we assume that $0\in\Omega$. Then there exist a ball $B_{2R}(0) := \{x\in\Omega : |x|\le 2R\}\subset\Omega$. Let $\xi\in C_0^\infty(\Omega)$ be the radial function, such that $\xi(x)\equiv 1$ for $0\le |x|\le R$, $0\le\xi(x)\le 1$ for $R\le|x|\le 2R$, and $\xi(x)\equiv 0$ for $|x|\ge 2R$. We define

$$(u_\varepsilon, v_\varepsilon) := (\xi U_\varepsilon, \xi V_\varepsilon).$$

It follows from [8, Lemma 3.4] and the choice of ξ that

$$\begin{aligned}
&\int_\Omega |\nabla u_\varepsilon|^2 \le \int_{\mathbb{R}^N} |\nabla U|^2 + O(\varepsilon^{N-2}), \quad \int_\Omega |\nabla v_\varepsilon|^2 \le \int_{\mathbb{R}^N} |\nabla V|^2 + O(\varepsilon^{N-2}),\\
&\int_\Omega |u_\varepsilon|^{2p} = \int_{\mathbb{R}^N} |U|^{2p} + O(\varepsilon^N), \qquad \int_\Omega |v_\varepsilon|^{2p} = \int_{\mathbb{R}^N} |V|^{2p} + O(\varepsilon^N),\\
&\int_\Omega |u_\varepsilon|^2 = C\varepsilon^2 + O(\varepsilon^{N-2}), \quad \int_\Omega |v_\varepsilon|^2 = C\varepsilon^2 + O(\varepsilon^{N-2}),\\
&\int_\Omega |u_\varepsilon|^p |v_\varepsilon|^p = \int_{\mathbb{R}^N} |U|^p |V|^p + O(\varepsilon^N).
\end{aligned} \tag{2.29}$$

Moreover, we claim that

$$\int_\Omega |\nabla u_\varepsilon|^2 \ge \int_{\mathbb{R}^N} |\nabla U|^2 + O(\varepsilon^{N-2}), \quad \int_\Omega |\nabla v_\varepsilon|^2 \ge \int_{\mathbb{R}^N} |\nabla V|^2 + O(\varepsilon^{N-2}). \tag{2.30}$$

In fact, by a similar argument as used in that of [8, Lemma 3.4], we have

$$\begin{aligned}
\int_\Omega |\nabla u_\varepsilon|^2 &= \int_\Omega |\nabla U_\varepsilon|^2 |\xi|^2 + \int_\Omega |\nabla \xi|^2 |U_\varepsilon|^2 + 2\int_\Omega \xi U_\varepsilon \nabla\xi\nabla U_\varepsilon\\
&\ge \int_{B_R(0)} |\nabla U_\varepsilon|^2 + O(\varepsilon^{N-2})\\
&= \int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 - \int_{B_R^c(0)} |\nabla U_\varepsilon|^2 + O(\varepsilon^{N-2}).
\end{aligned}$$

Note that

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 = \int_{\mathbb{R}^N} |\nabla U|^2,$$

and

$$\begin{aligned}\int_{B_R^c(0)} |\nabla U_\varepsilon|^2 &= \varepsilon^{-N} \int_{B_R^c(0)} |\nabla_x U(x/\varepsilon)|^2 \, dx \\ &\le \int_{B_{R/\varepsilon}^c(0)} |\nabla U(x)|^2 \, dx \\ &\le \int_{B_{R/\varepsilon}^c(0)} |x|^{2(1-N)} dx = O(\varepsilon^{N-2}).\end{aligned}$$

Therefore, we have

$$\int_\Omega |\nabla u_\varepsilon|^2 \ge \int_{\mathbb{R}^N} |\nabla U|^2 + O(\varepsilon^{N-2}).$$

Similarly, we can prove that

$$\int_\Omega |\nabla v_\varepsilon|^2 \ge \int_{\mathbb{R}^N} |\nabla V|^2 + O(\varepsilon^{N-2}).$$

Because of the presence of logarithmic terms in system (1.2), we also need the following new inequalities,

$$\int_\Omega u_\varepsilon^2 \log u_\varepsilon^2 = C\varepsilon^2 |\log \varepsilon| + O(\varepsilon^2), \quad \int_\Omega v_\varepsilon^2 \log v_\varepsilon^2 = C\varepsilon^2 |\log \varepsilon| + O(\varepsilon^2). \tag{2.31}$$

Note that

$$\begin{aligned}\int_\Omega u_\varepsilon^2 \log u_\varepsilon^2 &= \int_\Omega \left(\xi^2 \log \xi^2\right) U_\varepsilon^2 + \int_\Omega \xi^2 U_\varepsilon^2 \log U_\varepsilon^2 \\ &=: I_1 + I_2,\end{aligned}$$

Since $|s^2 \log s^2| \le C$ for $0 \le s \le 1$, we have

$$I_1 = \int_\Omega \left(\xi^2 \log \xi^2\right) U_\varepsilon^2 \le C \int_\Omega U_\varepsilon^2 = O(\varepsilon^2),$$

and

$$\begin{aligned}I_2 &= \varepsilon^{2-N} \log \varepsilon^{2-N} \int_\Omega \xi^2 U^2 \left(\frac{x}{\varepsilon}\right) \, dx + \varepsilon^{2-N} \int_\Omega \xi^2 U^2 \left(\frac{x}{\varepsilon}\right) \log \left(U^2 \left(\frac{x}{\varepsilon}\right)\right) \, dx \\ &=: I_{21} + I_{22}.\end{aligned}$$

Note that

$$I_{21} = \log \varepsilon^{2-N} \int_\Omega \xi^2 U_\varepsilon^2 = \log \varepsilon^{2-N} \int_\Omega |u_\varepsilon|^2 = C\varepsilon^2 |\log \varepsilon| + O(\varepsilon^2),$$

and

$$\begin{aligned}I_{22} &\le \varepsilon^{2-N} \int_{B_{2R}(0)} U^2 \left(\frac{x}{\varepsilon}\right) \log \left(U^2 \left(\frac{x}{\varepsilon}\right)\right) \, dx \\ &= \varepsilon^2 \int_{B_{\frac{2R}{\varepsilon}}(0)} U^2(y) \log \left(U^2(y)\right) \, dy \\ &\le \varepsilon^2 \int_{\mathbb{R}^N} U^2 \log U^2.\end{aligned}$$

By (2.28), there exists $R_0 > 0$ such that

$$U(x) \le C(1+|x|)^{2-N} \le e^{-1} \text{ for all } |x| \ge R_0, \tag{2.32}$$

then we have

$$\int_{\mathbb{R}^N} |U^2 \log U^2| = \int_{B_{R_0}(0)} |U^2 \log U^2| + \int_{B_{R_0}^c(0)} |U^2 \log U^2|,$$

it is easy to see that there exists a constant $C > 0$ such that $\int_{B_{R_0}(0)} |U^2 \log U^2| \le C$. On the other hand, since $f(s) = |s \log s|$ is increasing for $0 \le s \le e^{-1}$, we can see from (2.32) that

$$\begin{aligned}
\int_{B_{R_0}^c(0)} |U^2 \log U^2| &\le \int_{B_{R_0}^c(0)} \left| \frac{C}{(1+|x|)^{2(N-2)}} \log \frac{C}{(1+|x|)^{2(N-2)}} \right| \, dx \\
&\le C \int_{\mathbb{R}^N} \frac{1}{(1+|x|)^{2(N-2)-\frac{1}{2}}} \, dx \le C.
\end{aligned}$$

Therefore, $I_{22} \le \varepsilon^2 \int_{\mathbb{R}^N} U^2 \log U^2 = O(\varepsilon^2)$ and

$$\int_\Omega u_\varepsilon^2 \log u_\varepsilon^2 = I_1 + I_{21} + I_{22} = C\varepsilon^2 |\log \varepsilon| + O(\varepsilon^2).$$

Similarly, we can prove that

$$\int_\Omega v_\varepsilon^2 \log v_\varepsilon^2 = C\varepsilon^2 |\log \varepsilon| + O(\varepsilon^2).$$

Let $h(t) := \mathcal{L}(tu_\varepsilon, tv_\varepsilon)$. By Lemma 2.2, $h(0) = 0$ and $\lim_{t\to+\infty} h(t) = -\infty$, we can find $t_\varepsilon \in (0, \infty)$, such that

$$\mathcal{L}(t_\varepsilon u_\varepsilon, t_\varepsilon v_\varepsilon) = h(t_\varepsilon) = \sup_{t\ge 0} h(t) = \sup_{t \ge 0} \mathcal{L}(tu_\varepsilon, tv_\varepsilon).$$

and

$$(t_\varepsilon u_\varepsilon, t_\varepsilon v_\varepsilon) \in \mathcal{M},$$

which is equivalent to the following

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right) - \int_\Omega \left(\lambda_1 u_\varepsilon^2 + \lambda_2 v_\varepsilon^2\right) - \int_\Omega \left(\theta_1 u_\varepsilon^2 \log u_\varepsilon^2 + \theta_2 v_\varepsilon^2 \log v_\varepsilon^2\right) \\
&= t_\varepsilon^{2p-2} \int_\Omega \left(\mu_1 |u_\varepsilon|^{2p} + 2\beta |u_\varepsilon|^p |v_\varepsilon|^p + \mu_2 |v_\varepsilon|^{2p}\right) + \log t_\varepsilon^2 \int_\Omega \left(\theta_1 u_\varepsilon^2 + \theta_2 v_\varepsilon^2\right).
\end{aligned} \tag{2.33}$$

Recall that $\mathcal{E}(U, V) = \mathcal{A}$, we have

$$N\mathcal{A} = \int_{\mathbb{R}^N} |\nabla U|^2 + |\nabla V|^2 = \int_{\mathbb{R}^N} \left(\mu_1 |U|^{2p} + 2\beta |U|^p |V|^p + \mu_2 |V|^{2p}\right). \tag{2.34}$$

Combining this with (2.29) and (2.31), we deduce from (2.33) that, as $\varepsilon \to 0^+$,

$$\begin{aligned}
2N\mathcal{A} &\ge \int_\Omega \left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right) - \int_\Omega \left(\lambda_1 u_\varepsilon^2 + \lambda_2 v_\varepsilon^2\right) - \int_\Omega \left(\theta_1 u_\varepsilon^2 \log u_\varepsilon^2 + \theta_2 v_\varepsilon^2 \log v_\varepsilon^2\right) \\
&= t_\varepsilon^{2p-2} \int_\Omega \left(\mu_1 |u_\varepsilon|^{2p} + 2\beta |u_\varepsilon|^p |v_\varepsilon|^p + \mu_2 |v_\varepsilon|^{2p}\right) + \log t_\varepsilon^2 \int_\Omega \left(\theta_1 u_\varepsilon^2 + \theta_2 v_\varepsilon^2\right) \\
&\ge t_\varepsilon^{2p-2} \left(\frac{1}{2} N\mathcal{A}\right) - C|\log t_\varepsilon^2|,
\end{aligned}$$

which implies that there exists $c_1 > 0$ such that $t_\varepsilon < c_1$ for ε small enough.

On the other hand, we can see from (2.30) that, as $\varepsilon \to 0^+$,

$$\begin{aligned}\frac{1}{2}N\mathcal{A} &\le \int_\Omega \left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right) - \int_\Omega \left(\lambda_1 u_\varepsilon^2 + \lambda_2 v_\varepsilon^2\right) - \int_\Omega \left(\theta_1 u_\varepsilon^2 \log u_\varepsilon^2 + \theta_2 v_\varepsilon^2 \log v_\varepsilon^2\right) \\ &= t_\varepsilon^{2p-2} \int_\Omega \left(\mu_1 |u_\varepsilon|^{2p} + 2\beta |u_\varepsilon|^p |v_\varepsilon|^p + \mu_2 |v_\varepsilon|^{2p}\right) + \log t_\varepsilon^2 \int_\Omega \left(\theta_1 u_\varepsilon^2 + \theta_2 v_\varepsilon^2\right) \\ &\le t_\varepsilon^{2p-2}\left(2N\mathcal{A}\right) + C t_\varepsilon^{2p-2},\end{aligned}$$

So, there exists $c_2 > 0$ such that $t_\varepsilon > c_2$ for ε small enough. Therefore, t_ε is bounded from below and above for ε small.

Combining this with (2.29), (2.31) and (2.34), we have

$$\begin{aligned}&\mathcal{L}(t_\varepsilon u_\varepsilon, t_\varepsilon v_\varepsilon) \\ &= \frac{t_\varepsilon^2}{2}\int_\Omega \left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right) - \frac{t_\varepsilon^{2p}}{2p}\int_\Omega \left(\mu_1 |u_\varepsilon|^{2p} + 2\beta |u_\varepsilon|^p |v_\varepsilon|^p + \mu_2 |v_\varepsilon|^{2p}\right) - \frac{t_\varepsilon^2 \log t_\varepsilon^2}{2}\int_\Omega \left(\theta_1 u_\varepsilon^2 + \theta_2 v_\varepsilon^2\right) \\ &- \frac{t_\varepsilon^2}{2}\int_\Omega \left[\left(\lambda_1 - \theta_1\right) u_\varepsilon^2 + \left(\lambda_2 - \theta_2\right) v_\varepsilon^2\right] - \frac{t_\varepsilon^2}{2}\int_\Omega \left(\theta_1 u_\varepsilon^2 \log u_\varepsilon^2 + \theta_2 v_\varepsilon^2 \log v_\varepsilon^2\right) \\ &\le \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} \left(|\nabla U|^2 + |\nabla V|^2\right) - \frac{t_\varepsilon^{2p}}{2p}\int_{\mathbb{R}^N} \left(\mu_1 |U|^{2p} + 2\beta |U|^p |V|^p + \mu_2 |V|^{2p}\right) - C\varepsilon^2 |\log \varepsilon| + O(\varepsilon^2) \\ &\le \mathcal{A} - C\varepsilon^2 |\log\varepsilon| + O(\varepsilon^2) < \mathcal{A} \quad \text{ for } \varepsilon > 0 \text{ small enough.}\end{aligned}$$

Hence, for $\varepsilon > 0$ small enough, there holds

$$\mathcal{B} \le \max_{t>0} \mathcal{L}(t u_\varepsilon, t v_\varepsilon) = \mathcal{L}(t_\varepsilon u_\varepsilon, t_\varepsilon v_\varepsilon) < \mathcal{A}.$$

Step 2. We prove that $\mathcal{B} < \mathcal{C}_{\theta_1}$. It follows from [12, Theorem 1.2] that the energy level $\mathcal{C}_{\theta_1}$ can be achieved by a positive solution u_{θ_1}. Then by an argument similar to Step 1, there exists $t(s) > 0$ such that $(t(s)u_{\theta_1}, t(s)su_{\theta_1}) \in \mathcal{M}$ for any $s \in \mathbb{R}$. More precisely,

$$\begin{aligned}&t(s)^{2p-2}\left(\mu_1 + |s|^{2p}\mu_2 + 2\beta|s|^p\right)\int_\Omega |u_{\theta_1}|^{2p} + \left(\theta_1 \int_\Omega u_{\theta_1}^2 + s^2\theta_2 \int_\Omega u_{\theta_1}^2\right)\log t(s)^2 \\ &= \int_\Omega |\nabla u_{\theta_1}|^2 - \lambda_1 \int_\Omega |u_{\theta_1}|^2 - \theta_1 \int_\Omega u_{\theta_1}^2 \log u_{\theta_1}^2 - \left(\theta_2 \int_\Omega u_{\theta_1}^2\right) s^2 \log s^2 \\ &+ s^2\left(\int_\Omega |\nabla u_{\theta_1}|^2 - \lambda_2 \int_\Omega |u_{\theta_1}|^2 - \theta_2 \int_\Omega u_{\theta_1}^2 \log u_{\theta_1}^2\right).\end{aligned}$$

Similar to (2.16), we can prove that $t(0) = 1$. Since $N \ge 5$, $2 < 2p < 4$, a direct computation shows that

$$\lim_{s\to 0} \frac{t'(s)}{|s|^{p-2}s} = \frac{-2p\beta |u_{\theta_1}|_{2p}^{2p}}{(2p-2)\mu_1 |u_{\theta_1}|_{2p}^{2p} + 2\theta_1 |u_{\theta_1}|_2^2} =: -A < 0, \tag{2.35}$$

that is,

$$t'(s) = -A|s|^{p-2}s(1 + o(1)), \quad \text{ as } s \to 0.$$

Then as $s \to 0$, we have

$$
\begin{aligned}
t(s) &= 1 - \frac{A}{p}|s|^p(1+o(1)),\\
t(s)^2 &= 1 - \frac{2A}{p}|s|^p(1+o(1)),\\
t(s)^{2p} &= 1 - 2A|s|^p(1+o(1)).
\end{aligned} \tag{2.36}
$$

By (2.14) and (2.15), we have

$$
\mathcal{C}_{\theta_1} = \frac{\mu_1}{N}\int_\Omega |u_{\theta_1}|^{2p} + \frac{\theta_1}{2}\int_\Omega |u_{\theta_1}|^2.
$$

Combining this with (2.23), (2.35), (2.36) and recalling that $2p = \frac{2N}{N-2} \in (2,4)$, $N \geq 5$, we have

$$
\begin{aligned}
\mathcal{B} &\leq \mathcal{L}\left(t(s)u_{\theta_1}, t(s)su_{\theta_1}\right)\\
&= \frac{t(s)^{2p}}{N}\left(\mu_1 + |s|^{2p}\mu_2 + 2\beta|s|^p\right)\int_\Omega |u_{\theta_1}|^{2p} + \frac{\theta_1}{2}t(s)^2\int_\Omega |u_{\theta_1}|^2 + \frac{\theta_2}{2}t(s)^2s^2\int_\Omega |u_{\theta_1}|^2\\
&= \mathcal{C}_{\theta_1} + \frac{t(s)^2-1}{N}\mu_1\int_\Omega |u_{\theta_1}|^{2p} + \frac{t(s)^{2p}}{N}2\beta\int_\Omega |u_{\theta_1}|^{2p} + \frac{\theta_1}{2}\left(t(s)^2-1\right)\int_\Omega |u_{\theta_1}|^2 + o(|s|^p)\\
&= \mathcal{C}_{\theta_1} + \left(\frac{2\beta}{N}\int_\Omega |u_{\theta_1}|^{2p} - \frac{2A\mu_1}{N}\int_\Omega |u_{\theta_1}|^{2p} - \frac{A\theta_1}{p}\int_\Omega |u_{\theta_1}|^2\right)|s|^p + o(|s|^p)\\
&= \mathcal{C}_{\theta_1} - 2\beta\left(\frac{1}{2} - \frac{1}{N}\right)|s|^p\int_\Omega |u_{\theta_1}|^{2p} + o(|s|^p)\\
&< \mathcal{C}_{\theta_1} \text{ for } |s| > 0 \text{ small enough,}
\end{aligned}
$$

that is, $\mathcal{B} < \mathcal{C}_{\theta_1}$. Similarly, we can prove that $\mathcal{B} < \mathcal{C}_{\theta_2}$. This completes the proof. □

Lemma 2.3. *Assume that $\beta > 0$, then there exist a constant $C_3 > 0$, such that for any $(u,v) \in \mathcal{M}$, there holds*

$$
\int_\Omega (|u^+|^{2p} + |v^+|^{2p}) \geq C_3.
$$

Here, C_3 depends on β, μ_i, λ_i, $i = 1,2$.

Proof. Since $(u,v) \in \mathcal{M}$ and $s\log s \leq (p-1)^{-1}e^{-1}s^p$ for any $s > 0$, we have

$$
\begin{aligned}
\mathcal{S}(|u^+|_{2p}^2 + |v^+|_{2p}^2) &\leq |\nabla u|_2^2 + |\nabla v|_2^2\\
&= \mu_1|u^+|_{2p}^{2p} + 2\beta|u^+v^+|_p^p + \mu_2|v^+|_{2p}^{2p} + \theta_1\int_\Omega (u^+)^2\log(e^{\frac{\lambda_1}{\theta_1}}(u^+)^2) + \theta_2\int_\Omega (v^+)^2\log(e^{\frac{\lambda_2}{\theta_2}}(v^+)^2)\\
&\leq \left(\mu_1 + \beta + \frac{\theta_1}{p-1}e^{(p-1)\frac{\lambda_1}{\theta_1}-1}\right)|u^+|_{2p}^{2p} + \left(\mu_2 + \beta + \frac{\theta_2}{p-1}e^{(p-1)\frac{\lambda_2}{\theta_2}-1}\right)|v^+|_{2p}^{2p}\\
&\leq C\left(|u^+|_{2p}^{2p} + |v^+|_{2p}^{2p}\right).
\end{aligned}
$$

Therefore, there exists a constant $C_3 > 0$ such that $\int_\Omega (|u^+|^{2p} + |v^+|^{2p}) \geq C_3$. This completes the proof. □

Proof of Theorem 1.1 for the case $\beta > 0$. Assume that $\beta > 0$. By Lemma 2.2 and the mountain pass theorem (see [2, 21]), there exists a sequence $\{(u_n,v_n)\} \subset \mathcal{H}$ such that

$$
\lim_{n\to\infty}\mathcal{L}(u_n,v_n) = \mathcal{B}, \quad \lim_{n\to\infty}\mathcal{L}'(u_n,v_n) = 0.
$$

By Proposition 2.4, we know that $\mathcal{L}(u_n, v_n) \le 2\mathcal{A}$ for n large enough. Then we can see from (2.1) and (2.4) that $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. So, we may assume that

$$(u_n, v_n) \rightharpoonup (u, v) \text{ weakly in } \mathcal{H}.$$

Let $w_n = u_n - u$ and $z_n = v_n - v$. For simplicity, we still use the same symbol as in the proof of Theorem 1.1 for the case $\beta < 0$. So,

$$|\nabla w_n|_2^2 = \mu_1 |w_n^+|_{2p}^{2p} + \beta |w_n^+ z_n^+|_p^p + o_n(1), \quad |\nabla z_n|_2^2 = \mu_2 |z_n^+|_{2p}^{2p} + \beta |w_n^+ z_n^+|_p^p + o_n(1). \tag{2.37}$$

Also, we have $\mathcal{L}'(u, v) = 0$ and

$$0 \le \mathcal{L}(u, v) \le \mathcal{L}(u, v) + \frac{1}{N}(k_1 + k_2) = \lim_{n \to \infty} \mathcal{L}(u_n, v_n) = \mathcal{B}. \tag{2.38}$$

Next, we will show that $u \not\equiv 0$ and $v \not\equiv 0$.

Case 1. $u \equiv 0$ and $v \equiv 0$.

Firstly, we deduce from (2.2), (2.23) and Lemma 2.3 that $\mathcal{B} > 0$. Then by (2.38), we have $k_1 + k_2 = N\mathcal{B} > 0$. Since (2.37) holds, from an argument similar to the proof of Theorem 1.3 for the case $\beta > 0$ in [8], we have

$$\mathcal{B} = \frac{1}{N}(k_1 + k_2) = \lim_{n \to \infty} \mathcal{E}(w_n, v_n) \ge \mathcal{A},$$

Here, $\mathcal{E} : \mathcal{D} \to \mathbb{R}$ is the functional corresponding to the system (2.25). So, we get a contradiction with Proposition 2.4, which implies that Case 1 is impossible.

Case 2. $u \equiv 0$, $v \not\equiv 0$ or $u \not\equiv 0$, $v \equiv 0$.

Without loss of generality, we may assume that $u \equiv 0$, $v \not\equiv 0$. Notice that v is a solution of $-\Delta w = \lambda_2 w + \mu_2 |w|^{2p-2} w + \theta_2 w \log w^2$, we have $\mathcal{L}(0, v) \ge \mathcal{C}_{\theta_2}$. Therefore, by (2.38) we have that $\mathcal{B} \ge \mathcal{L}(0, v) \ge \mathcal{C}_{\theta_2}$, which is a contradiction with Proposition 2.4. Therefore, Case 2 is impossible.

Since Case 1 and Case 2 are both impossible, we get that $u \not\equiv 0$ and $v \not\equiv 0$. Since $\mathcal{L}'(u, v) = 0$, we can see that $(u, v) \in \mathcal{N}$. Combining with this fact, we deduce from (2.24) and (2.38) that $\mathcal{B} \le \mathcal{C}_{\mathcal{N}} \le \mathcal{L}(u, v) \le \mathcal{B}$. Hence, $\mathcal{L}(u, v) = \mathcal{B} = \mathcal{C}_{\mathcal{N}}$. By $\mathcal{L}'(u, v) = 0$, we know that (u, v) is a least energy solution of (1.2). By a similar argument as used in the proof of Theorem 1.1 for the case $\beta < 0$, we can show that $u, v > 0$ and $u, v \in C^2(\Omega)$. This completes the proof. □

3 Proof of Theorem 1.2, 1.3 and 1.4

Lemma 3.1. *When $\beta < 0$, we assume that one of the following holds:*

(i) $(\lambda_1, \mu_1, \theta_1; \lambda_2, \mu_2, \theta_2) \in A_1$,

(ii) $(\lambda_1, \mu_1, \theta_1; \lambda_2, \mu_2, \theta_2) \in A_2$,

(iii) $(\lambda_1, \mu_1, \theta_1; \lambda_2, \mu_2, \theta_2) \in A_3$;

when $\beta > 0$, we assume that there exists $\epsilon > 0$ such that one of the following holds:

(iv) $(\lambda_1, \mu_1 + \beta\epsilon, \theta_1; \lambda_2, \mu_2 + \frac{\beta}{\epsilon}, \theta_2) \in A_1$,

(v) $(\lambda_1, \mu_1 + \beta\epsilon, \theta_1; \lambda_2, \mu_2 + \frac{\beta}{\epsilon}, \theta_2) \in A_2$,

(vi) $(\lambda_1, \mu_1 + \beta\epsilon, \theta_1; \lambda_2, \mu_2 + \frac{\beta}{\epsilon}, \theta_2) \in A_3$.

Then there exist $\delta, \rho > 0$ such that $\mathcal{L}(u, v) \ge \delta$ for all $\sqrt{|\nabla u|_2^2 + |\nabla v|_2^2} = \rho$.

Proof. By the choice of A_i, $i = 1, 2, 3$, the proof follows word by word the one of Lemma 4.1 in [13]. □

Lemma 3.2. *Assume that the conditions stated in Lemma 3.1 hold, then we have $-\infty < \mathcal{C}_\rho < 0$, where $\mathcal{C}_\rho$ is given by Theorem 1.2.*

Proof. If $|\nabla u|_2^2 + |\nabla v|_2^2 < \rho^2$, then we can easily verify that $\mathcal{L}(u,v) > -\infty$, which implies that $\mathcal{C}_\rho > -\infty$. Now we show that $\mathcal{C}_\rho < 0$. Since $\theta_1 < 0$, we fix $(u,0)$ satisfying $|\nabla u|_2 < \rho$, and consider $\mathcal{L}(tu,0)$ with $t < 1$:

$$\begin{aligned}&\mathcal{L}(tu,0)\\ =&t^2\left(\frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda_1}{2}\int_\Omega |u^+|^2 - \frac{\mu_1}{2p}t^{2p-2}\int_\Omega |u^+|^{2p} - \frac{\theta_1}{2}\int_\Omega (u^+)^2(\log(u^+)^2 - 1) - \theta_1 \log t \int_\Omega (u^+)^2\right).\end{aligned}$$

Choose u such that $\int_\Omega (u^+)^2 > 0$. Then for small $t > 0$ we have $\mathcal{L}(tu,0) < 0$. This completes the proof. □

Lemma 3.3 (Boundedness of (PS) sequence)**.** *Let $\{(u_n,v_n)\}$ be a (PS)$_c$ sequence and $\theta_i < 0, i = 1,2$. Then $\{(u_n,v_n)\}$ is bounded in $\mathcal{H}$.*

Proof. Since $\mathcal{L}'(u_n,v_n) \to 0$, we have $\mathcal{L}'(u_n,v_n)(u_n,v_n) = o_n(|\nabla u_n|_2 + |\nabla v_n|_2)$. Then for large n, we have

$$\begin{aligned}&c + |\nabla u_n|_2 + |\nabla v_n|_2 + 1\\ &\geq \mathcal{L}(u_n,v_n) - \frac{1}{2p}\mathcal{L}'(u_n,v_n)(u_n,v_n)\\ &= \frac{1}{N}|\nabla u_n|_2^2 - \frac{\theta_1}{N}\int_\Omega (u_n^+)^2 \log\left(e^{\frac{\lambda_1}{\theta_1} - \frac{N}{2}}(u_n^+)^2\right) + \frac{1}{N}|\nabla v_n|_2^2 - \frac{\theta_2}{N}\int_\Omega (v_n^+)^2 \log\left(e^{\frac{\lambda_2}{\theta_2} - \frac{N}{2}}(v_n^+)^2\right).\end{aligned}$$

Since $\theta_1 < 0$, using the inequality $t\log t \geq -e^{-1}$, one gets

$$\begin{aligned}\frac{\theta_1}{N}\int_\Omega (u_n^+)^2 \log\left(e^{\frac{\lambda_1}{\theta_1} - \frac{N}{2}}(u_n^+)^2\right) &\leq \frac{\theta_1}{N}\int_{e^{\frac{\lambda_1}{\theta_1} - \frac{N}{2}}(u_n^+)^2 \leq 1} (u_n^+)^2 \log\left(e^{\frac{\lambda_1}{\theta_1} - \frac{N}{2}}(u_n^+)^2\right)\\ &\leq -\frac{\theta_1}{N}e^{\frac{N}{2} - \frac{\lambda_1}{\theta_1} - 1}|\Omega|.\end{aligned}$$

Similarly, we have

$$\frac{\theta_2}{N}\int_\Omega (v_n^+)^2 \log\left(e^{\frac{\lambda_2}{\theta_2} - \frac{N}{2}}(v_n^+)^2\right) \leq -\frac{\theta_2}{N}e^{\frac{N}{2} - \frac{\lambda_2}{\theta_2} - 1}|\Omega|$$

since $\theta_2 < 0$. For n large enough, we have

$$c + |\nabla u_n|_2 + |\nabla v_n|_2 + 1 \geq \frac{1}{N}|\nabla u_n|_2^2 + \frac{1}{N}|\nabla v_n|_2^2 + \frac{\theta_1}{N}e^{\frac{N}{2} - \frac{\lambda_1}{\theta_1} - 1}|\Omega| + \frac{\theta_2}{N}e^{\frac{N}{2} - \frac{\lambda_2}{\theta_2} - 1}|\Omega|,$$

yielding to the boundedness of $\{(u_n,v_n)\}$ in $\mathcal{H}$. □

Lemma 3.4. *Assume that the conditions stated in Lemma 3.1 hold and*

$$\min\{\theta_1,\theta_2\} \geq -\frac{2}{N}\lambda_1(\Omega) \quad \textit{or} \quad \beta > 0 \quad \textit{or} \quad \beta \in (-\sqrt{\mu_1\mu_2}, 0).$$

Then there holds $-\infty < \mathcal{C}_K < 0$.

Proof. Notice that $(u,v) \in K$ where (u,v) is the solution given by Theorem 1.2. Hence $\mathcal{C}_K \leq \mathcal{C}_\rho < 0$. Now we show that $\mathcal{C}_K > -\infty$.

Case 1. $\min\{\theta_1,\theta_2\} \geq -\frac{2}{N}\lambda_1(\Omega)$.

For any $(u,v)\in K$ one gets

$$\begin{aligned}\mathcal{L}(u,v) &= \mathcal{L}(u,v) - \frac{1}{2p}\mathcal{L}'(u,v)(u,v)\\ &= \frac{1}{N}|\nabla u|_2^2 - \frac{\theta_1}{N}\int_\Omega (u^+)^2\log\left(e^{\frac{\lambda_1}{\theta_1}}(u^+)^2\right) + \frac{\theta_1}{2}|u^+|_2^2\\ &+ \frac{1}{N}|\nabla v|_2^2 - \frac{\theta_2}{N}\int_\Omega (v^+)^2\log\left(e^{\frac{\lambda_2}{\theta_2}}(v^+)^2\right) + \frac{\theta_2}{2}|v^+|_2^2.\end{aligned}$$

Since $\theta_1<0$, using the inequality $t\log t \geq -e^{-1}$, one gets

$$\begin{aligned}\frac{\theta_1}{N}\int_\Omega (u^+)^2\log\left(e^{\frac{\lambda_1}{\theta_1}}(u^+)^2\right) &\leq \frac{\theta_1}{N}\int_{e^{\frac{\lambda_1}{\theta_1}}(u^+)^2\leq 1}(u^+)^2\log\left(e^{\frac{\lambda_1}{\theta_1}}(u^+)^2\right)\\ &\leq -\frac{\theta_1}{N}e^{-\frac{\lambda_1}{\theta_1}-1}|\Omega|.\end{aligned}$$

Similarly, we have

$$\frac{\theta_2}{N}\int_\Omega (v^+)^2\log\left(e^{\frac{\lambda_2}{\theta_2}-2}(v^+)^2\right) \leq -\frac{\theta_2}{N}e^{-\frac{\lambda_2}{\theta_2}-1}|\Omega|$$

since $\theta_2<0$. Moreover, we have

$$\frac{1}{N}|\nabla u|_2^2 + \frac{\theta_1}{2}|u^+|_2^2 \geq (\frac{1}{N}\lambda_1(\Omega) + \frac{\theta_1}{2})|u^+|_2^2 \geq 0$$

and similarly $\frac{1}{N}|\nabla v|_2^2 + \frac{\theta_2}{2}|v^+|_2^2 \geq 0$. Hence,

$$\mathcal{L}(u,v) \geq \frac{\theta_1}{N}e^{-\frac{\lambda_1}{\theta_1}-1}|\Omega| + \frac{\theta_2}{N}e^{-\frac{\lambda_2}{\theta_2}-1}|\Omega| > -\infty,$$

showing that $\mathcal{C}_K > -\infty$.

Case 2. $\beta>0$ or $\beta\in(-\sqrt{\mu_1\mu_2},0)$.

Since $\beta>0$ or $\beta\in(-\sqrt{\mu_1\mu_2},0)$, there exists $c_0>0$ such that

$$\mu_1|u^+|_{2p}^{2p} + \mu_2|v^+|_{2p}^{2p} + 2\beta|u^+v^+|_p^p \geq c_0\left(|u^+|_{2p}^{2p} + |v^+|_{2p}^{2p}\right).$$

For any $(u,v)\in K$ we have

$$\begin{aligned}&\mathcal{L}(u,v) - \frac{1}{2}\mathcal{L}'(u,v)(u,v)\\ &= \frac{1}{N}\left(\mu_1|u^+|_{2p}^{2p} + \mu_2|v^+|_{2p}^{2p} + 2\beta|u^+v^+|_p^p\right) + \frac{\theta_1}{2}|u^+|_2^2 + \frac{\theta_2}{2}|v^+|_2^2\\ &\geq \frac{c_0}{N}\left(|u^+|_{2p}^{2p} + |v^+|_{2p}^{2p}\right) + \frac{\theta_1}{2}|\Omega|^{\frac{p-1}{p}}|u^+|_{2p}^2 + \frac{\theta_2}{2}|\Omega|^{\frac{p-1}{p}}|v^+|_{2p}^2.\end{aligned}$$

Then we deduce from $2p>2$ that $\mathcal{C}_K > -\infty$. □

Proof of Theorem 1.2 and 1.3. Applying Lemma 3.1- 3.4, the proofs of Theorem 1.2 and 1.3 follow exactly the same steps as the proofs of Theorem 1.2 and 1.3 in [13], respectively. □

Proof of Theorem 1.4. Without loss of generality, we assume that $\beta > 0$ and $(\lambda_1, \mu_1, \theta_1) \in \Sigma_2$. We also suppose that the system (1.2) has a positive solution (u, v). Then we multiply the equation for u in (1.2) by the first eigenfunction $\phi_1(x) > 0$ and integrate over Ω,

$$\int_\Omega \left(\lambda_1 + \mu_1 u^{2p-2} + \beta u^{p-2}v^p + \theta_1 \log u^2\right) u\phi_1 = \int_\Omega (-\Delta u)\phi_1 = \int_\Omega (-\Delta \phi_1)u = \int_\Omega \lambda_1(\Omega)u\phi_1.$$

Therefore,

$$\int_\Omega \left(\lambda_1 - \lambda_1(\Omega) + \mu_1 u^{2p-2} + \theta_1 \log u^2\right) u\phi_1 = -\int_\Omega \beta u^{p-1}v^p\phi_1 < 0. \tag{3.1}$$

Define

$$g(s) := \lambda_1 - \lambda_1(\Omega) + \mu_1 s^{p-1} + \theta_1 \log s, \quad s > 0,$$

then it is easy to see that $g(s)$ is decreasing in $(0, s_0)$, and $g(s)$ is increasing in $(s_0, +\infty)$, where

$$s_0^{p-1} = \frac{|\theta_1|}{(p-1)\mu_1}.$$

So,

$$g(s) \geq g\left(s_0\right) = \frac{|\theta_1|}{p-1} + \frac{\theta_1}{p-1}\log\frac{|\theta|}{(p-1)\mu_1} + \lambda_1 - \lambda_1(\Omega) \geq 0,$$

which is guaranteed by $(\lambda_1, \mu_1, \theta_1) \in \Sigma_2$ and $p - 1 = \frac{2}{N-2}$. Since $u, \phi > 0$, we have

$$\int_\Omega \left(\lambda_1 - \lambda_1(\Omega) + \mu_1 u^{2p-2} + \theta_1 \log u^2\right) u\phi_1 = \int_\Omega g(u)u\phi_1 > 0,$$

which contradicts to (3.1). Therefore, $u = 0$. Indeed, if $\int_\Omega g(u)u\phi_1 = 0$, then $g(u) = 0$ a.e.in Ω and $u = s_0$ a.e.in Ω, which contradicts to $u \in H_0^1(\Omega)$. Hence, the system (1.2) has no positive solutions. □

4 The Brézis-Nirenberg problem with logarithmic perturbation

In this section, we prove Theorem 1.5 and 1.6. Firstly, we have the following basic fact.

Lemma 4.1. *Assume that $(\lambda, \mu, \theta) \in \Sigma_3 \cup \Sigma_4$. Then there exist $\delta, \rho > 0$ such that $J(u) \geq \delta$ for all $|\nabla u|_2 = \rho$.*

Proof. The proof can be found in [12], so we omit it. □

Proof of Theorem 1.5. Similar to Lemma 3.2 we obtain that $-\infty < \widetilde{\mathcal{C}}_\rho < 0$. By Lemma 4.1, we can take a minimizing sequence $\{u_n\}$ for $\widetilde{\mathcal{C}}_\rho$ with $|\nabla u_n|_2 < \rho - \tau$ and $\tau > 0$ small enough. By Ekeland's variational principle, we can assume that $J'(u_n) \to 0$. Similar to Lemma 3.3, we can see that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Hence, we may assume that

$$u_n \rightharpoonup u \text{ weakly in } H_0^1(\Omega).$$

Passing to subsequence, we may also assume that

$$\begin{aligned} u_n &\rightharpoonup u \ \text{ weakly in } L^{2p}(\Omega), \\ u_n &\to u \ \text{ strongly in } L^q(\Omega) \text{ for } 2 \leq q < 2p, \\ u_n &\to u \ \text{ almost everywhere in } \Omega. \end{aligned}$$

By the weak lower semi-continuity of the norm, we see that $|\nabla u|_2 < \rho$. Similar to the proof of Theorem 1.1 for the case $\beta < 0$, we have $J'(u) = 0$ and $u \geq 0$. Let $w_n = u_n - u$. Then similar to the proof of Theorem 1.1 for the case $\beta < 0$, one gets

$$|\nabla w_n|_2^2 = \mu |w_n^+|_{2p}^{2p} + o_n(1).$$

$$J(u_n) = J(u) + \frac{1}{N} \int_\Omega |\nabla w_n|^2 + o_n(1). \tag{4.1}$$

Passing to subsequence, we may assume that

$$\int_\Omega |\nabla w_n|^2 = k + o_n(1).$$

Letting $n \to +\infty$ in (4.1), we have

$$\widetilde{\mathcal{C}}_\rho \leq J(u) \leq J(u) + \frac{1}{N} k = \lim_{n \to \infty} J(u_n) = \widetilde{\mathcal{C}}_\rho,$$

showing that $k = 0$. Hence, up to a subsequence we obtain $u_n \to u$ strongly in $H_0^1(\Omega)$. Since $J(u) = \widetilde{\mathcal{C}}_\rho < 0$ we have $u \neq 0$. Then by a similar argument as used in the proof of Theorem 1.1, we can show that $u > 0$ and $u \in C^2(\Omega)$. This completes the proof. □

Proof of Theorem 1.6. Similar to the case 2 in the proof of Lemma 3.4 we have $-\infty < \widetilde{\mathcal{C}}_\mathcal{K} < 0$. Take a minimizing sequence $\{u_n\} \subset \mathcal{K}$ for $\widetilde{\mathcal{C}}_\mathcal{K}$. Then $J'(u_n) = 0$. Similar to Lemma 3.3, we can see that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Hence, we may assume that

$$u_n \rightharpoonup u \text{ weakly in } H_0^1(\Omega).$$

Passing to subsequence, we may also assume that

$$\begin{aligned} u_n &\rightharpoonup u \ \text{ weakly in } L^{2p}(\Omega), \\ u_n &\to u \ \text{ strongly in } L^p(\Omega) \text{ for } 2 \leq q < 2p, \\ u_n &\to u \ \text{ almost everywhere in } \Omega. \end{aligned}$$

Then by the same arguments as used in proof of Theorem 1.5, we obtain $u_n \to u$ strongly in $H_0^1(\Omega)$. Since $J(u) = \widetilde{\mathcal{C}}_\mathcal{K} < 0$ we have $u \neq 0$. Then by a similar argument as used in the proof of Theorem 1.1 for the case $\beta < 0$, we can show that $u > 0$ and $u \in C^2(\Omega)$. This completes the proof. □

References

[1] M. Alfaro, R. Carles, *Superexponential growth or decay in the heat equation with a logarithmic nonlinearity.* Dyn. Partial Differ. Equ. 14 (2017), 343–358.

[2] A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications.* Journal of Functional Analysis. 14 (1973), 349–381.

[3] I. Bialynicki-Birula, J. Mycielski, *Wave equations with logarithmic nonlinearities.* Bull. Acad. Polon. Sci.23 (1975), 461–466.

[4] I. Bialynicki-Birula, J. Mycielski, *Nonlinear wave mechanics.* Ann. Physics,100 (1976), 62–93.

[5] H. Brézis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents.* Comm. Pure Appl. Math. 36 (1983), 437–477.

[6] H. Brézis, E. H. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*. Proc. Amer. Math. Soc. 88 (1983), 486–490.

[7] Z. J. Chen, W. M. Zou, *Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent.* Arch. Ration. Mech. Anal. 205 (2012), 515–551.

[8] Z. J. Chen, W. M. Zou, *Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent: higher dimensional case.* Calc. Var. Partial Differential Equations. 52 (2015), 423–467.

[9] R. Carles, I. Gallagher, *Universal dynamics for the defocusing logarithmic Schrodinger equation.* Duke Math. J. 167 (2018), 1761–1801.

[10] R. Carles, D. Pelinovsky, *On the orbital stability of Gaussian solitary waves in the log-KdV equation.* Nonlinearity. 27 (2014), 3185–3202.

[11] M. Colin, L. Jeanjean, *Solutions for a quasilinear Schrodinger equation: a dual approach.* Nonlinear Anal. 56 (2004), 213–226.

[12] Y. B. Deng, Q. H. He, Y. Q. Pan, X. X. Zhong, *The existence of positive solution for an elliptic problem with critical growth and logarithmic perturbation.* Adv. Nonlinear Stud. 23 (2023).

[13] H. Hajaiej, T. H. Liu, L. J. Song, W. M. Zou, *Positive solution for an elliptic system with critical exponent and logarithmic terms.* arXiv:2304.13822.

[14] T. C. Lin, J. C. Wei, *Ground State of N Coupled Nonlinear Schrödinger Equations in $\mathbb{R}^n$,$n \leq 3$*. Commun. Math. Phys. 255 (2005), 629–653.

[15] E. Lieb, M. Loss, *Analysis.* Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.

[16] T. H. Liu, S. You, W. M. Zou, *Least energy positive soultions for d-coupled Schrödinger systems with critical exponent in dimension three.* arXiv:2204.00748.

[17] M. Poppenberg, K. Schmitt, Z.-Q. Wang, *On the existence of soliton solutions to quasilinear Schrodinger equations.* Calc. Var. Partial Differential Equations. 14 (2002), 329–344.

[18] Z.-Q. Wang, C. Zhang,*Convergence from power-law to logarithm-law in nonlinear scalar field equations.* Arch. Rational Mech. Anal. 231 (2019), 45–61.

[19] S. Wei, *Multiple solutions for logarithmic Schrödinger equations.* Nonlinearity. 32 (2019), 2201–2225.

[20] J. Vazquez, *A strong maximum principle for some quasilinear elliptic equations.* Appl. Math. Optim., 12 (1984), 191–202.

[21] M. Willem. *Minimax Theorems.* Birkhäuser Boston (1996).

[22] H. Y. Ye, Y. F. Peng, *Positive least energy solutions for a coupled Schrödinger system with critical exponent*. J. Math. Anal. Appl. 417 (2014), 308–326.