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Expectation Identity of the Discrete Uniform Distribution and Its Application in the Calculations of Higher-Order Origin Moments

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Abstract

We provide a novel method to analytically calculate the high-order origin moments of a Discrete Uniform (DU) random variable, that is, the expectation identity method. First, the expectation identity of the DU distribution is discovered and summarized in a theorem. After that, we analytically calculate the first four origin moments and the general k th ($k = 1, 2, \dots$) origin moment of the DU distribution by the expectation identity method. After comparing the corresponding coefficients on both sides of an equation, we obtain a nonhomogeneous linear equations of first degree in $k + 1$ variables. Furthermore, we have provided two ways to solve the nonhomogeneous linear equations. The first way is by matrix inversion, and the second way is by iterative solving. Moreover, the coefficients of the first ten origin moments of the DU distribution are summarized in a table. Finally, we have a proposition for special summations.

Keywords: expectation identity, discrete uniform distribution, high-order origin moments, nonhomogeneous linear equations, special summations

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1. Introduction

A Discrete Uniform (DU) random variable takes n distinct values $1, 2, \dots, n$. Moreover, the DU distribution has attracted continuous interest in the literature. Sandelius (1967) discussed the variance of a DU distribution. Connor (1969) discussed sampling distribution of range from DU finite populations and a range test for homogeneity. Sundt (1988) proposed a recursive algorithm for convolutions of DU distributions. Best and Rayner (1997) carried out goodness-of-fit for the ordered categories DU distribution. Sundt (1999) revisited the recursive algorithm for convolutions of DU distributions. Wu et al. (2002) presented bit-parallel random number generation for DU distributions. Calik and Gungor (2004) derived the expected values of the sample maximum of order statistics from a DU distribution. Li and Shi (2010) presented an adaptive load balancing algorithm based on DU distribution. Belbachir (2011) determined the mode for convolution powers of DU distribution. Zhigljavsky et al. (2016) considered the problem of deconvolution of a DU distribution. Stepanauskas and Zvinyte (2017) discussed DU distribution for a sum of additive functions. Calik and Bugatekin (2018) obtained the m th raw moments of sample extremes of order statistics from DU distribution. Roychowdhury (2019) considered the center of mass and the optimal quantizers for some continuous and DU distributions. Pepelyshev and Zhigljavsky (2020) studied DU and binomial distributions with infinite support.

Exponential families include the continuous families – normal, gamma, and beta, and the discrete families – binomial, Poisson, and negative binomial. For the continuous exponential families, there is an entire class of identities that rely on integration by parts. Stein’s Lemma (Lemma 3.6.5 in Casella and Berger (2002)) gives an expectation identity for the normal family. Moreover, the expectation identities of the gamma and beta families are given in Exercise 3.49 of Casella and Berger (2002). Note that these expectation identities are useful in the calculations of high-order origin moments of the corresponding families. The discrete analogs of the expectation identities of the continuous exponential

families are given in Theorem 3.6.8 of Casella and Berger (2002), which give the expectation identities for the Poisson and negative binomial families (see also Hwang (1982)). Note that the two families take countable infinite values. For the binomial family, which takes finite values, similar to the deviations of the Poisson and negative binomial expectation identities, Zhang et al. (2019) discovered an expectation identity for the binomial family. Moreover, they obtain a closed-form formula of the high-order origin moments for the binomial family by exploiting the binomial expectation identity.

The DU distribution is probably the simplest discrete distribution. However, the analytical calculations of the high-order origin moments of a DU random variable are quite challenging. There are two potential methods to calculate the high-order origin moments of a DU random variable. That is, the definition method and the moment generating function (mgf) method. But they both failed to analytically calculate the high-order origin moments of a DU random variable. The analytical calculations of the high-order origin moments by the definition method are hindered when the order is greater than or equal to 3, because it is difficult to analytically obtain the summations. Moreover, the mgf method also fails to analytically calculate the high-order origin moments, due to the calculation reduces to the definition of the high-order origin moment.

In this paper, we provide a novel method to analytically calculate the high-order origin moments of a DU random variable, that is, the expectation identity method. First, the expectation identity of the DU distribution is discovered and summarized in a theorem. After that, we analytically calculate the first four origin moments and the general k th ($k = 1, 2, \dots$) origin moment of the DU distribution by the expectation identity method. After comparing the corresponding coefficients on both sides of an equation, we obtain a nonhomogeneous linear equations of first degree in $k + 1$ variables. Furthermore, we have provided two ways to solve the nonhomogeneous linear equations. The first way is by matrix inversion, and the second way is by iterative solving.

The rest of the paper is organized as follows. In the next Section 2, we provide some preliminary. Section 3 gives an expectation identity of the DU

distribution. The analytical calculations of the first four origin moments of the DU distribution by the expectation identity method are provided in Section 4. The analytical calculations of the k th origin moment of the DU distribution by the expectation identity method are provided in Section 5. In Section 6, we provide two ways to solve the nonhomogeneous linear equations obtained in Section 5. The first way is by matrix inversion, and the second way is by iterative solving. Some conclusions and discussions are provided in Section 7.

2. Preliminary

Let $Y \sim DU(n)$ be a DU random variable with n distinct values $1, 2, \dots, n$, and its probability mass function (pmf) is given by

$$P(Y = y|n) = \frac{1}{n}, \quad y = 1, 2, \dots, n, \quad n = 1, 2, \dots$$

It is known that (see Casella and Berger (2002))

$$EY = \frac{n+1}{2} \text{ and } \text{Var}(Y) = \frac{(n+1)(n-1)}{12}.$$

Similarly, let $X \sim DU(n-1)$ be a DU random variable with $n-1$ distinct values $1, 2, \dots, n-1$, and its pmf is given by

$$P(X = x|n-1) = \frac{1}{n-1}, \quad x = 1, 2, \dots, n-1, \quad n-1 = 1, 2, \dots$$

Hence,

$$EX = \frac{(n-1)+1}{2} = \frac{n}{2},$$

by replacing n with $n-1$ in the expression of EY .

In this paper, we will analytically calculate the k th origin moment of $Y \sim DU(n)$, EY^k , for $k = 1, 2, \dots$. There are two potential methods to calculate EY^k . That is, the definition method and the mgf method.

The first method is by definition, or the definition method. By definition, we have

$$EY^k = \sum_{i=1}^n i^k \frac{1}{n} = \frac{1}{n} \sum_{i=1}^n i^k, \quad (1)$$

which requires analytical calculations of the summations

$$\sum_{i=1}^n i^k \quad (2)$$

for different $k = 1, 2, \dots$. When $k = 1$, it is easy to see that

$$\sum_{i=1}^n i^1 = 1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

Hence,

$$EY = \frac{1}{n} \sum_{i=1}^n i^1 = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2} = \frac{1}{2}n + \frac{1}{2},$$

which is a polynomial of n of order 1. When $k = 2$, it is known that

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + \dots + n^2 = \frac{n}{6} (n+1) (2n+1).$$

Therefore,

$$\begin{aligned} EY^2 &= \frac{1}{n} \sum_{i=1}^n i^2 \\ &= \frac{1}{n} \frac{n}{6} (n+1) (2n+1) \\ &= \frac{1}{6} (n+1) (2n+1) \\ &= \frac{1}{3}n^2 + \frac{1}{2}n + \frac{1}{6}, \end{aligned}$$

which is a polynomial of n of order 2. However, when $k \geq 3$, it is difficult to analytically obtain the summations (2), and thus the analytical calculations of EY^k by the definition method are hindered.

The second method to analytically calculate EY^k is the mgf method. We have

$$EY^k = \left. \frac{d^k}{dt^k} M_Y(t) \right|_{t=0},$$

where

$$M_Y(t) = Ee^{tY} = \frac{1}{n} \sum_{i=1}^n e^{it}$$

is the mgf of Y . It is easy to obtain

$$\frac{d^k}{dt^k} e^{it} = i^k e^{it}$$

for $k = 1, 2, \dots$. Therefore,

$$\begin{aligned}
EY^k &= \frac{d^k}{dt^k} \left(\frac{1}{n} \sum_{i=1}^n e^{it} \right) \Big|_{t=0} \\
&= \frac{1}{n} \sum_{i=1}^n \frac{d^k}{dt^k} e^{it} \Big|_{t=0} \\
&= \frac{1}{n} \sum_{i=1}^n i^k e^{it} \Big|_{t=0} \\
&= \frac{1}{n} \sum_{i=1}^n i^k,
\end{aligned}$$

which reduces to the definition of EY^k . Consequently, the mgf method fails to analytically calculate EY^k .

In this paper, we provide a novel method to analytically calculate EY^k , that is, the expectation identity method.

3. The Expectation Identity of the DU Distribution

The expectation identity of the DU distribution is discovered and summarized in the following theorem and the proof is straightforward and simple as with many important results.

Theorem 1. *Let $g(x)$ be a function with $-\infty < E[g(x)] < \infty$ and $-\infty < g(0) < \infty$. Let $X \sim DU(n-1)$ and $Y \sim DU(n)$. Then we have the expectation identity of the DU distribution:*

$$E[g(X)] = \frac{n}{n-1} E[g(Y-1)] - \frac{g(0)}{n-1}. \quad (3)$$

Proof. We have

$$\begin{aligned}
E[g(X)] &= \sum_{x=1}^{n-1} g(x) P(X=x) \\
&= \sum_{x=1}^{n-1} \frac{g(x)}{n-1} \quad (\text{let } y = x+1) \\
&= \sum_{y=2}^n \frac{g(y-1)}{n-1} \\
&= \sum_{y=1}^n \frac{n}{n-1} \frac{g(y-1)}{n} - \frac{g(0)}{n-1} \\
&= \frac{n}{n-1} E[g(Y-1)] - \frac{g(0)}{n-1}.
\end{aligned}$$

The proof is complete. $\square$

4. The Analytical Calculations of the First Four Origin Moments of the DU Distribution by the Expectation Identity Method

In this section, we will analytically calculate the first four origin moments of the DU distribution $Y \sim DU(n)$ by the expectation identity method.

In the following, we will use the fact that EY^k ($k = 1, 2, 3, 4$) can be written as a polynomial of n of order k , and the detailed and technical proof can be found in the supplement.

First, let us calculate EY . Take $g(x) = x$, then $g(0) = 0$. By the expectation identity (3), we have

$$EX = \frac{n}{n-1} E(Y-1) - \frac{0}{n-1}.$$

Rearranging, we obtain

$$(n-1)EX = nEY - n. \quad (4)$$

Since EY can be written as a polynomial of n of order 1, it is assumed that

$$EY = c_1 n + c_0 \quad (5)$$

and

$$EX = c_1(n-1) + c_0. \quad (6)$$

Substituting (5) and (6) into (4) and simplifying, we obtain

$$c_1(n-1)^2 + c_0(n-1) = c_1n^2 + (c_0-1)n.$$

The left side of the above equation simplifies to

$$\begin{aligned} & c_1(n^2 - 2n + 1) + c_0(n-1) \\ &= c_1n^2 + (-2c_1 + c_0)n + (c_1 - c_0). \end{aligned}$$

Hence, (4) reduces to

$$c_1n^2 + (-2c_1 + c_0)n + (c_1 - c_0) = c_1n^2 + (c_0 - 1)n.$$

Comparing the corresponding coefficients on both sides of the above equation, we obtain a nonhomogeneous linear equations of first degree in two variables:

$$\begin{cases} 2c_1 = 1 \\ c_1 - c_0 = 0. \end{cases}$$

Solving the above linear equations, we have

$$c_1 = c_0 = \frac{1}{2}.$$

Therefore,

$$EY = \frac{1}{2}n + \frac{1}{2}. \quad (7)$$

Second, let us calculate EY^2 . Take $g(x) = x^2$, then $g(0) = 0$. By the expectation identity (3), we have

$$EX^2 = \frac{n}{n-1}E[(Y-1)^2] - \frac{0}{n-1}.$$

Rearranging, we obtain

$$(n-1)EX^2 = nE(Y^2 - 2Y + 1). \quad (8)$$

Since EY^2 can be written as a polynomial of n of order 2, it is assumed that

$$EY^2 = c_2n^2 + c_1n + c_0 \quad (9)$$

and

$$EX^2 = c_2(n-1)^2 + c_1(n-1) + c_0. \quad (10)$$

Substituting (9), (10), and (7) into (8) and simplifying, we obtain

$$c_2(n-1)^3 + c_1(n-1)^2 + c_0(n-1) = c_2n^3 + (c_1-1)n^2 + c_0n.$$

The left side of the above equation simplifies to

$$\begin{aligned} & c_2(n^3 - 3n^2 + 3n - 1) + c_1(n^2 - 2n + 1) + c_0(n-1) \\ &= c_2n^3 + (-3c_2 + c_1)n^2 + (3c_2 - 2c_1 + c_0)n + (-c_2 + c_1 - c_0). \end{aligned}$$

Hence, (8) reduces to

$$c_2n^3 + (-3c_2 + c_1)n^2 + (3c_2 - 2c_1 + c_0)n + (-c_2 + c_1 - c_0) = c_2n^3 + (c_1 - 1)n^2 + c_0n.$$

Comparing the corresponding coefficients on both sides of the above equation, we obtain a nonhomogeneous linear equations of first degree in three variables:

$$\begin{cases} 3c_2 = 1 \\ 3c_2 - 2c_1 = 0 \\ -c_2 + c_1 - c_0 = 0. \end{cases}$$

Solving the above linear equations, we have

$$c_2 = \frac{1}{3}, c_1 = \frac{1}{2}, c_0 = \frac{1}{6}.$$

Therefore,

$$EY^2 = \frac{1}{3}n^2 + \frac{1}{2}n + \frac{1}{6}. \quad (11)$$

Third, let us calculate EY^3 . Take $g(x) = x^3$, then $g(0) = 0$. By the expectation identity (3), we have

$$EX^3 = \frac{n}{n-1}E[(Y-1)^3] - \frac{0}{n-1}.$$

Rearranging, we obtain

$$(n-1)EX^3 = nE(Y^3 - 3Y^2 + 3Y - 1). \quad (12)$$

Since EY^3 can be written as a polynomial of n of order 3, it is assumed that

$$EY^3 = c_3n^3 + c_2n^2 + c_1n + c_0 \quad (13)$$

and

$$EX^3 = c_3(n-1)^3 + c_2(n-1)^2 + c_1(n-1) + c_0. \quad (14)$$

Substituting (13), (14), (7), and (11) into (12) and simplifying, we obtain

$$c_3(n-1)^4 + c_2(n-1)^3 + c_1(n-1)^2 + c_0(n-1) = c_3n^4 + (c_2-1)n^3 + c_1n^2 + c_0n.$$

The left side of the above equation simplifies to

$$\begin{aligned} & c_3(n^4 - 4n^3 + 6n^2 - 4n + 1) + c_2(n^3 - 3n^2 + 3n - 1) + c_1(n^2 - 2n + 1) + c_0(n - 1) \\ &= c_3n^4 + (-4c_3 + c_2)n^3 + (6c_3 - 3c_2 + c_1)n^2 + (-4c_3 + 3c_2 - 2c_1 + c_0)n + (c_3 - c_2 + c_1 - c_0). \end{aligned}$$

Hence, (12) reduces to

$$\begin{aligned} & c_3n^4 + (-4c_3 + c_2)n^3 + (6c_3 - 3c_2 + c_1)n^2 + (-4c_3 + 3c_2 - 2c_1 + c_0)n + (c_3 - c_2 + c_1 - c_0) \\ &= c_3n^4 + (c_2 - 1)n^3 + c_1n^2 + c_0n. \end{aligned}$$

Comparing the corresponding coefficients on both sides of the above equation, we obtain a nonhomogeneous linear equations of first degree in four variables:

$$\begin{cases} 4c_3 = 1 \\ 2c_3 - c_2 = 0 \\ 4c_3 - 3c_2 + 2c_1 = 0 \\ c_3 - c_2 + c_1 - c_0 = 0. \end{cases}$$

Solving the above linear equations, we have

$$c_3 = \frac{1}{4}, c_2 = \frac{1}{2}, c_1 = \frac{1}{4}, c_0 = 0.$$

Therefore,

$$EY^3 = \frac{1}{4}n^3 + \frac{1}{2}n^2 + \frac{1}{4}n. \quad (15)$$

Fourth, let us calculate EY^4 . Take $g(x) = x^4$, then $g(0) = 0$. By the expectation identity (3), we have

$$EX^4 = \frac{n}{n-1}E[(Y-1)^4] - \frac{0}{n-1}.$$

Rearranging, we obtain

$$(n-1)EX^4 = nE(Y^4 - 4Y^3 + 6Y^2 - 4Y + 1). \quad (16)$$

Since EY^4 can be written as a polynomial of n of order 4, it is assumed that

$$EY^4 = c_4n^4 + c_3n^3 + c_2n^2 + c_1n + c_0 \quad (17)$$

and

$$EX^4 = c_4(n-1)^4 + c_3(n-1)^3 + c_2(n-1)^2 + c_1(n-1) + c_0. \quad (18)$$

Substituting (17), (18), (7), (11), and (15) into (16) and simplifying, we obtain

$$c_4(n-1)^5 + c_3(n-1)^4 + c_2(n-1)^3 + c_1(n-1)^2 + c_0(n-1) = c_4n^5 + (c_3 - 1)n^4 + c_2n^3 + c_1n^2 + c_0n.$$

The left side of the above equation simplifies to

$$\begin{aligned} & c_4(n^5 - 5n^4 + 10n^3 - 10n^2 + 5n - 1) + c_3(n^4 - 4n^3 + 6n^2 - 4n + 1) \\ & + c_2(n^3 - 3n^2 + 3n - 1) + c_1(n^2 - 2n + 1) + c_0(n - 1) \\ & = c_4n^5 + (-5c_4 + c_3)n^4 + (10c_4 - 4c_3 + c_2)n^3 + (-10c_4 + 6c_3 - 3c_2 + c_1)n^2 \\ & + (5c_4 - 4c_3 + 3c_2 - 2c_1 + c_0)n + (-c_4 + c_3 - c_2 + c_1 - c_0). \end{aligned}$$

Hence, (16) reduces to

$$\begin{aligned} & c_4n^5 + (-5c_4 + c_3)n^4 + (10c_4 - 4c_3 + c_2)n^3 + (-10c_4 + 6c_3 - 3c_2 + c_1)n^2 \\ & + (5c_4 - 4c_3 + 3c_2 - 2c_1 + c_0)n + (-c_4 + c_3 - c_2 + c_1 - c_0) \\ & = c_4n^5 + (c_3 - 1)n^4 + c_2n^3 + c_1n^2 + c_0n. \end{aligned}$$

Comparing the corresponding coefficients on both sides of the above equation, we obtain a nonhomogeneous linear equations of first degree in five variables:

$$\left\{ \begin{array}{l} 5c_4 = 1 \\ 5c_4 - 2c_3 = 0 \\ 10c_4 - 6c_3 + 3c_2 = 0 \\ 5c_4 - 4c_3 + 3c_2 - 2c_1 = 0 \\ c_4 - c_3 + c_2 - c_1 + c_0 = 0. \end{array} \right.$$

Solving the above linear equations, we have

$$c_4 = \frac{1}{5}, c_3 = \frac{1}{2}, c_2 = \frac{1}{3}, c_1 = 0, c_0 = -\frac{1}{30}.$$

Therefore,

$$EY^4 = \frac{1}{5}n^4 + \frac{1}{2}n^3 + \frac{1}{3}n^2 - \frac{1}{30}.$$

5. The Analytical Calculations of the k th Origin Moment of the DU Distribution by the Expectation Identity Method

In this section, we will analytically calculate the k th origin moment of the DU distribution $Y \sim DU(n)$ by the expectation identity method.

Take $g(x) = x^k$, then $g(0) = 0$. By the expectation identity (3), we have

$$EX^k = \frac{n}{n-1}E[(Y-1)^k] - \frac{0}{n-1}.$$

Rearranging, we obtain

$$(n-1)EX^k = nE[(Y-1)^k]. \quad (19)$$

Since EY^k can be written as a polynomial of n of order k , it is assumed that

$$EY^k = c_k n^k + c_{k-1} n^{k-1} + \cdots + c_2 n^2 + c_1 n + c_0 \quad (20)$$

and

$$EX^k = c_k (n-1)^k + c_{k-1} (n-1)^{k-1} + \cdots + c_2 (n-1)^2 + c_1 (n-1) + c_0, \quad (21)$$

where $c_k \neq 0$. The detailed and technical proof that EY^k ($k = 1, 2, \dots$) can be written as a polynomial of n of order k can be found in the supplement, where we have mainly used the mathematical induction and the expectation identity (3) of the DU distribution. By (21), the left side of (19) reduces to

$$c_k (n-1)^{k+1} + c_{k-1} (n-1)^k + c_{k-2} (n-1)^{k-1} + c_{k-3} (n-1)^{k-2} + \cdots + c_1 (n-1)^2 + c_0 (n-1).$$

Now let us calculate the right side of (19). Note that

$$Y-1 \sim \begin{pmatrix} 0 & 1 & 2 & \cdots & n-1 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}.$$

Hence,

$$nEY^k = 1^k + 2^k + \cdots + (n-1)^k + n^k$$

and

$$nE \left[(Y-1)^k \right] = 1^k + 2^k + \cdots + (n-1)^k.$$

Therefore, the right side of (19) becomes

$$\begin{aligned} nE \left[(Y-1)^k \right] &= nEY^k - n^k \\ &= c_k n^{k+1} + c_{k-1} n^k + c_{k-2} n^{k-1} + c_{k-3} n^{k-2} + \cdots + c_1 n^2 + c_0 n - n^k \\ &= c_k n^{k+1} + (c_{k-1} - 1) n^k + c_{k-2} n^{k-1} + c_{k-3} n^{k-2} + \cdots + c_1 n^2 + c_0 n. \end{aligned}$$

Because the left and right sides of the equation (19) are equal, we have

$$\begin{aligned} &c_k (n-1)^{k+1} + c_{k-1} (n-1)^k + c_{k-2} (n-1)^{k-1} + c_{k-3} (n-1)^{k-2} + \cdots + c_1 (n-1)^2 + c_0 (n-1) \\ &= c_k n^{k+1} + (c_{k-1} - 1) n^k + c_{k-2} n^{k-1} + c_{k-3} n^{k-2} + \cdots + c_1 n^2 + c_0 n. \end{aligned} \quad (22)$$

Note that

$$\begin{aligned} c_k (n-1)^{k+1} &= c_k \sum_{i=0}^{k+1} C_{k+1}^i (-1)^i n^{k+1-i} \\ &= c_k C_{k+1}^0 (-1)^0 n^{k+1} + c_k C_{k+1}^1 (-1)^1 n^k + c_k C_{k+1}^2 (-1)^2 n^{k-1} + c_k C_{k+1}^3 (-1)^3 n^{k-2} + \cdots \\ &\quad + c_k C_{k+1}^k (-1)^k n^1 + c_k C_{k+1}^{k+1} (-1)^{k+1} n^0, \\ c_{k-1} (n-1)^k &= c_{k-1} \sum_{i=0}^k C_k^i (-1)^i n^{k-i} \\ &= c_{k-1} C_k^0 (-1)^0 n^k + c_{k-1} C_k^1 (-1)^1 n^{k-1} + c_{k-1} C_k^2 (-1)^2 n^{k-2} + \cdots \\ &\quad + c_{k-1} C_k^{k-1} (-1)^{k-1} n^1 + c_{k-1} C_k^k (-1)^k n^0, \end{aligned}$$

$$\begin{aligned}
c_{k-2} (n-1)^{k-1} &= c_{k-2} \sum_{i=0}^{k-1} C_{k-1}^i (-1)^i n^{k-1-i} \\
&= c_{k-2} C_{k-1}^0 (-1)^0 n^{k-1} + c_{k-2} C_{k-1}^1 (-1)^1 n^{k-2} + \dots \\
&\quad + c_{k-2} C_{k-1}^{k-2} (-1)^{k-2} n^1 + c_{k-2} C_{k-1}^{k-1} (-1)^{k-1} n^0, \\
c_{k-3} (n-1)^{k-2} &= c_{k-3} \sum_{i=0}^{k-2} C_{k-2}^i (-1)^i n^{k-2-i} \\
&= c_{k-3} C_{k-2}^0 (-1)^0 n^{k-2} + \dots \\
&\quad + c_{k-3} C_{k-2}^{k-3} (-1)^{k-3} n^1 + c_{k-3} C_{k-2}^{k-2} (-1)^{k-2} n^0, \\
&\vdots \\
c_1 (n-1)^2 &= c_1 \sum_{i=0}^2 C_2^i (-1)^i n^{2-i} \\
&= c_1 C_2^0 (-1)^0 n^2 + c_1 C_2^1 (-1)^1 n^1 + c_1 C_2^2 (-1)^2 n^0, \\
c_0 (n-1) &= c_0 \sum_{i=0}^1 C_1^i (-1)^i n^{1-i} \\
&= c_0 C_1^0 (-1)^0 n^1 + c_0 C_1^1 (-1)^1 n^0,
\end{aligned}$$

where

$$C_m^i = \binom{m}{i} = \frac{m!}{i! (m-i)!}$$

is the binomial coefficient for $m \geq i \geq 0$. Therefore, the left side of (22) reduces to

$$\begin{aligned}
&c_k C_{k+1}^0 (-1)^0 n^{k+1} \\
&+ \left(c_k C_{k+1}^1 (-1)^1 + c_{k-1} C_k^0 (-1)^0 \right) n^k \\
&+ \left(c_k C_{k+1}^2 (-1)^2 + c_{k-1} C_k^1 (-1)^1 + c_{k-2} C_{k-1}^0 (-1)^0 \right) n^{k-1} \\
&+ \left(c_k C_{k+1}^3 (-1)^3 + c_{k-1} C_k^2 (-1)^2 + c_{k-2} C_{k-1}^1 (-1)^1 + c_{k-3} C_{k-2}^0 (-1)^0 \right) n^{k-2} \\
&+ \dots \\
&+ \left(c_k C_{k+1}^k (-1)^k + c_{k-1} C_k^{k-1} (-1)^{k-1} + c_{k-2} C_{k-1}^{k-2} (-1)^{k-2} + \dots + c_1 C_2^1 (-1)^1 + c_0 C_1^0 (-1)^0 \right) n^1 \\
&+ \left(c_k C_{k+1}^{k+1} (-1)^{k+1} + c_{k-1} C_k^k (-1)^k + c_{k-2} C_{k-1}^{k-1} (-1)^{k-1} + \dots + c_1 C_2^2 (-1)^2 + c_0 C_1^1 (-1)^1 \right) n^0.
\end{aligned} \tag{23}$$

Substituting (23) into (22) and comparing the corresponding coefficients on both sides of the equation (22), we obtain a nonhomogeneous linear equations of first degree in $k + 1$ variables:

$$\begin{cases} c_k C_{k+1}^1 (-1)^1 + c_{k-1} C_k^0 (-1)^0 = c_{k-1} - 1 \\ c_k C_{k+1}^2 (-1)^2 + c_{k-1} C_k^1 (-1)^1 + c_{k-2} C_{k-1}^0 (-1)^0 = c_{k-2} \\ c_k C_{k+1}^3 (-1)^3 + c_{k-1} C_k^2 (-1)^2 + c_{k-2} C_{k-1}^1 (-1)^1 + c_{k-3} C_{k-2}^0 (-1)^0 = c_{k-3} \\ \vdots \\ c_k C_{k+1}^k (-1)^k + c_{k-1} C_k^{k-1} (-1)^{k-1} + c_{k-2} C_{k-1}^{k-2} (-1)^{k-2} + \cdots + c_1 C_2^1 (-1)^1 + c_0 C_1^0 (-1)^0 = c_0 \\ c_k C_{k+1}^{k+1} (-1)^{k+1} + c_{k-1} C_k^k (-1)^k + c_{k-2} C_{k-1}^{k-1} (-1)^{k-1} + \cdots + c_1 C_2^2 (-1)^2 + c_0 C_1^1 (-1)^1 = 0. \end{cases}$$

After simplifications, the above nonhomogeneous linear equations reduce to

$$\begin{cases} c_k C_{k+1}^1 (-1)^1 = -1 \\ c_k C_{k+1}^2 (-1)^2 + c_{k-1} C_k^1 (-1)^1 = 0 \\ c_k C_{k+1}^3 (-1)^3 + c_{k-1} C_k^2 (-1)^2 + c_{k-2} C_{k-1}^1 (-1)^1 = 0 \\ \vdots \\ c_k C_{k+1}^k (-1)^k + c_{k-1} C_k^{k-1} (-1)^{k-1} + c_{k-2} C_{k-1}^{k-2} (-1)^{k-2} + \cdots + c_1 C_2^1 (-1)^1 = 0 \\ c_k C_{k+1}^{k+1} (-1)^{k+1} + c_{k-1} C_k^k (-1)^k + c_{k-2} C_{k-1}^{k-1} (-1)^{k-1} + \cdots + c_1 C_2^2 (-1)^2 + c_0 C_1^1 (-1)^1 = 0. \end{cases} \quad (24)$$

The unknown coefficients vector $\mathbf{c}$ can be obtained by solving the nonhomogeneous linear equations

$$\mathbf{L}\mathbf{c} = \mathbf{b}, \quad (25)$$

where

$$\mathbf{L} = \begin{pmatrix} C_{k+1}^1 (-1)^1 & & & & & \\ C_{k+1}^2 (-1)^2 & C_k^1 (-1)^1 & & & & \\ C_{k+1}^3 (-1)^3 & C_k^2 (-1)^2 & C_{k-1}^1 (-1)^1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ C_{k+1}^k (-1)^k & C_k^{k-1} (-1)^{k-1} & C_{k-1}^{k-2} (-1)^{k-2} & \cdots & C_2^1 (-1)^1 & \\ C_{k+1}^{k+1} (-1)^{k+1} & C_k^k (-1)^k & C_{k-1}^{k-1} (-1)^{k-1} & \cdots & C_2^2 (-1)^2 & C_1^1 (-1)^1 \end{pmatrix},$$

$$\mathbf{c} = \begin{pmatrix} c_k & c_{k-1} & c_{k-2} & \cdots & c_1 & c_0 \end{pmatrix}^T,$$

and

$$\mathbf{b} = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}^T.$$

6. Solving the Nonhomogeneous Linear Equations

We provide two ways to solve the nonhomogeneous linear equations (25). The first way is by matrix inversion, and the second way is by iterative solving.

6.1. Matrix inversion

The first way to solve the nonhomogeneous linear equations (25) is by matrix inversion. We first give the theoretical derivations to obtain the iterative expressions of the components of the solution vector $\mathbf{c} = \left(c_i^{(k)} \right)_{i=k, k-1, k-2, \dots, 1, 0}$ by matrix inversion. Then we verify the first four origin moments from the iterative expressions of the components of $\mathbf{c} = \left(c_i^{(k)} \right)_{i=k, k-1, k-2, \dots, 1, 0}$ by matrix inversion.

6.1.1. Theoretical derivations

In this section, we will give the theoretical derivations to obtain the iterative expressions of the components of the solution vector $\mathbf{c} = \left(c_i^{(k)} \right)_{i=k, k-1, k-2, \dots, 1, 0}$ by matrix inversion.

When the matrix $\mathbf{L}$ is invertible, the solution vector $\mathbf{c}$ can be represented as

$\mathbf{c} = \mathbf{L}^{-1}\mathbf{b}$. Define

$$\mathbf{L} = \begin{pmatrix} C_{k+1}^1 (-1)^1 & & & & & \\ C_{k+1}^2 (-1)^2 & C_k^1 (-1)^1 & & & & \\ C_{k+1}^3 (-1)^3 & C_k^2 (-1)^2 & C_{k-1}^1 (-1)^1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ C_{k+1}^k (-1)^k & C_k^{k-1} (-1)^{k-1} & C_{k-1}^{k-2} (-1)^{k-2} & \cdots & C_2^1 (-1)^1 & \\ C_{k+1}^{k+1} (-1)^{k+1} & C_k^k (-1)^k & C_{k-1}^{k-1} (-1)^{k-1} & \cdots & C_2^2 (-1)^2 & C_1^1 (-1)^1 \end{pmatrix}$$

$$= \begin{pmatrix} L_{11} & & & & & \\ L_{21} & L_{22} & & & & \\ L_{31} & L_{32} & L_{33} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ L_{k1} & L_{k2} & L_{k3} & \cdots & L_{kk} & \\ L_{k+1,1} & L_{k+1,2} & L_{k+1,3} & \cdots & L_{k+1,k} & L_{k+1,k+1} \end{pmatrix},$$

which is a lower-triangular matrix, where

$$L_{ij} = C_{k+2-j}^{i-(j-1)} (-1)^{i-(j-1)} = C_{k+2-j}^{i-j+1} (-1)^{i-j+1}, \quad i, j = 1, 2, \dots, k+1; \quad i \geq j.$$

Because the determinant of the matrix $\mathbf{L}$ satisfies

$$|\mathbf{L}| = C_{k+1}^1 C_k^1 C_{k-1}^1 \cdots C_2^1 C_1^1 (-1)^{k+1} \neq 0,$$

so $\mathbf{L}$ must be invertible (see Zhan (2008); Golub and Van Loan (2013); Jin et al. (2015)). Define the inverse matrix of $\mathbf{L}$ by

$$\mathbf{L}^{-1} = \begin{pmatrix} a_{11} & & & & & \\ a_{21} & a_{22} & & & & \\ a_{31} & a_{32} & a_{33} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kk} & \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,k} & a_{k+1,k+1} \end{pmatrix}.$$

Hence,

$$\begin{aligned} \mathbf{c} &= \mathbf{L}^{-1}\mathbf{b} \\ &= \begin{pmatrix} a_{11} & & & & & \\ a_{21} & a_{22} & & & & \\ a_{31} & a_{32} & a_{33} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kk} & \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,k} & a_{k+1,k+1} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -a_{11} \\ -a_{21} \\ -a_{31} \\ \vdots \\ -a_{k1} \\ -a_{k+1,1} \end{pmatrix}. \end{aligned}$$

Therefore, the solution vector $\mathbf{c}$ requires only the first column elements of the inverse matrix $\mathbf{L}^{-1}$. According to the definition of the inverse matrix, we have

$\mathbf{L}\mathbf{L}^{-1} = I$, that is,

$$\begin{aligned}
& \mathbf{L}\mathbf{L}^{-1} \\
&= \begin{pmatrix} L_{11} & & & & & \\ L_{21} & L_{22} & & & & \\ L_{31} & L_{32} & L_{33} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ L_{k1} & L_{k2} & L_{k3} & \cdots & L_{kk} & \\ L_{k+1,1} & L_{k+1,2} & L_{k+1,3} & \cdots & L_{k+1,k} & L_{k+1,k+1} \end{pmatrix} \begin{pmatrix} a_{11} & & & & & \\ a_{21} & a_{22} & & & & \\ a_{31} & a_{32} & a_{33} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kk} & \\ a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,k} & a_{k+1,k+1} \end{pmatrix} \\
&= \begin{pmatrix} L_{11}a_{11} & & & & & \\ L_{21}a_{11} + L_{22}a_{21} & L_{22}a_{22} & & & & \\ L_{31}a_{11} + L_{32}a_{21} + L_{33}a_{31} & L_{32}a_{22} + L_{33}a_{32} & L_{33}a_{33} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \sum_{j=1}^k L_{kj}a_{j1} & \sum_{j=2}^k L_{kj}a_{j2} & \sum_{j=3}^k L_{kj}a_{j3} & \cdots & L_{kk}a_{kk} & \\ \sum_{j=1}^{k+1} L_{k+1,j}a_{j1} & \sum_{j=2}^{k+1} L_{k+1,j}a_{j2} & \sum_{j=3}^{k+1} L_{k+1,j}a_{j3} & \cdots & \sum_{j=k}^{k+1} L_{k+1,j}a_{jk} & L_{k+1,k+1}a_{k+1,k+1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \\
&= I.
\end{aligned}$$

Taking the first column elements of the matrices on both sides of the above

matrix equation, we have

$$\left\{ \begin{array}{l} L_{11}a_{11} = 1 \\ L_{21}a_{11} + L_{22}a_{21} = 0 \\ L_{31}a_{11} + L_{32}a_{21} + L_{33}a_{31} = 0 \\ \vdots \\ \sum_{j=1}^k L_{kj}a_{j1} = 0 \\ \sum_{j=1}^{k+1} L_{k+1,j}a_{j1} = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} a_{11} = L_{11}^{-1} \\ a_{21} = -L_{22}^{-1}L_{21}a_{11} \\ a_{31} = -L_{33}^{-1}(L_{31}a_{11} + L_{32}a_{21}) \\ \vdots \\ a_{k1} = -L_{kk}^{-1} \left(\sum_{j=1}^{k-1} L_{kj}a_{j1} \right) \\ a_{k+1,1} = -L_{k+1,k+1}^{-1} \left(\sum_{j=1}^k L_{k+1,j}a_{j1} \right). \end{array} \right.$$

Consequently,

$$\mathbf{c} = \mathbf{L}^{-1}\mathbf{b} = \begin{pmatrix} c_k \\ c_{k-1} \\ c_{k-2} \\ \vdots \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} -a_{11} \\ -a_{21} \\ -a_{31} \\ \vdots \\ -a_{k1} \\ -a_{k+1,1} \end{pmatrix} = \begin{pmatrix} -L_{11}^{-1} \\ \sum_{j=1}^1 L_{22}^{-1}L_{2j}a_{j1} \\ \sum_{j=1}^2 L_{33}^{-1}L_{3j}a_{j1} \\ \vdots \\ \sum_{j=1}^{k-1} L_{kk}^{-1}L_{kj}a_{j1} \\ \sum_{j=1}^k L_{k+1,k+1}^{-1}L_{k+1,j}a_{j1} \end{pmatrix}.$$

Define $L_{10}a_{01} = -2$, then we have

$$c_i^{(k)} = -a_{k+1-i,1} = \sum_{j=1}^{k-i} L_{k-i+1,k-i+1}^{-1} L_{k-i+1,j}a_{j1}, \quad i = k, k-1, k-2, \dots, 1, 0. \quad (26)$$

Note that in $c_i^{(k)}$, we have added the superscript (k) to indicate that $c_i^{(k)}$ is the coefficient of EY^k . When $i \leq k-1$, the expressions in (26) are clearly true.

When $i = k$, we have

$$c_k^{(k)} = \sum_{j=1}^0 L_{11}^{-1}L_{1j}a_{j1} = L_{11}^{-1}L_{11}a_{11} + L_{11}^{-1}L_{10}a_{01} = a_{11} - 2L_{11}^{-1} = a_{11} - 2a_{11} = -a_{11}.$$

Therefore, the expressions in (26) are true for all $i = k, k-1, k-2, \dots, 1, 0$. As a result,

$$EY^k = \sum_{i=0}^k c_i^{(k)} n^i = \sum_{i=0}^k \sum_{j=1}^{k-i} L_{k-i+1,k-i+1}^{-1} L_{k-i+1,j}a_{j1} n^i.$$

6.1.2. Verifications of the first four origin moments

In this section, we will verify the first four origin moments from the iterative expressions (26) of the components of $\mathbf{c} = (c_i^{(k)})_{i=k, k-1, k-2, \dots, 1, 0}$ by matrix inversion.

When $k = 0$, we have

$$\begin{aligned} \mathbf{L} &= L_{11} = C_1^1(-1) = -1, \\ c_0^{(0)} &= -a_{11} = -L_{11}^{-1} = 1, \\ \mathbf{EY}^0 &= \sum_{i=0}^0 c_i^{(0)} n^i = c_0^{(0)} n^0 = 1. \end{aligned}$$

When $k = 1$, we have

$$\begin{aligned} \mathbf{L} &= \begin{pmatrix} L_{11} & \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} C_2^1(-1)^1 & 0 \\ C_2^2(-1)^2 & C_1^1(-1)^1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}, \\ c_1^{(1)} &= -a_{11} = -L_{11}^{-1} = \frac{1}{2} \quad \Rightarrow \quad a_{11} = -\frac{1}{2}, \\ c_0^{(1)} &= -a_{21} = \sum_{j=1}^1 L_{22}^{-1} L_{2j} a_{j1} = L_{22}^{-1} L_{21} a_{11} = \frac{1}{2} \quad \Rightarrow \quad a_{21} = -\frac{1}{2}, \\ \mathbf{EY}^1 &= \sum_{i=0}^1 c_i^{(1)} n^i = c_1^{(1)} n^1 + c_0^{(1)} n^0 = \frac{1}{2} n + \frac{1}{2}. \end{aligned}$$

When $k = 2$, we have

$$\begin{aligned} \mathbf{L} &= \begin{pmatrix} L_{11} & & \\ L_{21} & L_{22} & \\ L_{31} & L_{32} & L_{33} \end{pmatrix} = \begin{pmatrix} C_3^1(-1)^1 & 0 & 0 \\ C_3^2(-1)^2 & C_2^1(-1)^1 & 0 \\ C_3^3(-1)^3 & C_2^2(-1)^2 & C_1^1(-1)^1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 3 & -2 & 0 \\ -1 & 1 & -1 \end{pmatrix}, \\ c_2^{(2)} &= -a_{11} = -L_{11}^{-1} = \frac{1}{3} \quad \Rightarrow \quad a_{11} = -\frac{1}{3}, \\ c_1^{(2)} &= -a_{21} = \sum_{j=1}^1 L_{22}^{-1} L_{2j} a_{j1} = \frac{1}{2} \quad \Rightarrow \quad a_{21} = -\frac{1}{2}, \\ c_0^{(2)} &= -a_{31} = \sum_{j=1}^2 L_{33}^{-1} L_{3j} a_{j1} = \frac{1}{6} \quad \Rightarrow \quad a_{31} = -\frac{1}{6}, \\ \mathbf{EY}^2 &= \sum_{i=0}^2 c_i^{(2)} n^i = c_2^{(2)} n^2 + c_1^{(2)} n^1 + c_0^{(2)} n^0 = \frac{1}{3} n^2 + \frac{1}{2} n + \frac{1}{6}. \end{aligned}$$

When $k = 3$, we have

$$\begin{aligned} \mathbf{L} &= \begin{pmatrix} L_{11} & & & \\ L_{21} & L_{22} & & \\ L_{31} & L_{32} & L_{33} & \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} = \begin{pmatrix} C_4^1(-1)^1 & 0 & 0 & 0 \\ C_4^2(-1)^2 & C_3^1(-1)^1 & 0 & 0 \\ C_4^3(-1)^3 & C_3^2(-1)^2 & C_2^1(-1)^1 & 0 \\ C_4^4(-1)^4 & C_3^3(-1)^3 & C_2^2(-1)^2 & C_1^1(-1)^1 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 0 & 0 & 0 \\ 6 & -3 & 0 & 0 \\ -4 & 3 & -2 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \end{aligned}$$

$$c_3^{(3)} = -a_{11} = -L_{11}^{-1} = \frac{1}{4} \quad \Rightarrow \quad a_{11} = -\frac{1}{4},$$

$$c_2^{(3)} = -a_{21} = \sum_{j=1}^1 L_{22}^{-1} L_{2j} a_{j1} = \frac{1}{2} \quad \Rightarrow \quad a_{21} = -\frac{1}{2},$$

$$c_1^{(3)} = -a_{31} = \sum_{j=1}^2 L_{33}^{-1} L_{3j} a_{j1} = \frac{1}{4} \quad \Rightarrow \quad a_{31} = -\frac{1}{4},$$

$$c_0^{(3)} = -a_{41} = \sum_{j=1}^3 L_{33}^{-1} L_{3j} a_{j1} = 0 \quad \Rightarrow \quad a_{41} = 0,$$

$$\text{EY}^3 = \sum_{i=0}^3 c_i^{(3)} n^i = c_3^{(3)} n^3 + c_2^{(3)} n^2 + c_1^{(3)} n^1 + c_0^{(3)} n^0 = \frac{1}{4} n^3 + \frac{1}{2} n^2 + \frac{1}{4} n.$$

When $k = 4$, we have

$$\begin{aligned} \mathbf{L} &= \begin{pmatrix} L_{11} & & & & \\ L_{21} & L_{22} & & & \\ L_{31} & L_{32} & L_{33} & & \\ L_{41} & L_{42} & L_{43} & L_{44} & \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{pmatrix} = \begin{pmatrix} C_5^1(-1)^1 & 0 & 0 & 0 & 0 \\ C_5^2(-1)^2 & C_4^1(-1)^1 & 0 & 0 & 0 \\ C_5^3(-1)^3 & C_4^2(-1)^2 & C_3^1(-1)^1 & 0 & 0 \\ C_5^4(-1)^4 & C_4^3(-1)^3 & C_3^2(-1)^2 & C_2^1(-1)^1 & 0 \\ C_5^5(-1)^5 & C_4^4(-1)^4 & C_3^3(-1)^3 & C_2^2(-1)^2 & C_1^1(-1)^1 \end{pmatrix} \\ &= \begin{pmatrix} -5 & 0 & 0 & 0 & 0 \\ 10 & -4 & 0 & 0 & 0 \\ -10 & 6 & -3 & 0 & 0 \\ 5 & -4 & 3 & -2 & 0 \\ -1 & 1 & -1 & 1 & -1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
c_4^{(4)} &= -a_{11} = -L_{11}^{-1} = \frac{1}{5} & \Rightarrow & a_{11} = -\frac{1}{5}, \\
c_3^{(4)} &= -a_{21} = \sum_{j=1}^1 L_{22}^{-1} L_{2j} a_{j1} = \frac{1}{2} & \Rightarrow & a_{21} = -\frac{1}{2}, \\
c_2^{(4)} &= -a_{31} = \sum_{j=1}^2 L_{33}^{-1} L_{3j} a_{j1} = \frac{1}{3} & \Rightarrow & a_{31} = -\frac{1}{3}, \\
c_1^{(4)} &= -a_{41} = \sum_{j=1}^3 L_{44}^{-1} L_{4j} a_{j1} = 0 & \Rightarrow & a_{41} = 0, \\
c_0^{(4)} &= -a_{51} = \sum_{j=1}^4 L_{55}^{-1} L_{5j} a_{j1} = -\frac{1}{30} & \Rightarrow & a_{51} = \frac{1}{30},
\end{aligned}$$

$$\begin{aligned}
\mathbf{EY}^4 &= \sum_{i=0}^4 c_i^{(4)} n^i = c_4^{(4)} n^4 + c_3^{(4)} n^3 + c_2^{(4)} n^2 + c_1^{(4)} n^1 + c_0^{(4)} n^0 \\
&= \frac{1}{5} n^4 + \frac{1}{2} n^3 + \frac{1}{3} n^2 - \frac{1}{30}.
\end{aligned}$$

6.2. Iterative solving

The second way to solve the nonhomogeneous linear equations (25) is by iterative solving. We first give the theoretical derivations to obtain the iterative expressions of the components of the solution vector $\mathbf{c} = \left(c_j^{(k)} \right)_{j=k, k-1, k-2, \dots, 1, 0}$ by iterative solving. Then we verify the first four origin moments from the iterative expressions of the components of $\mathbf{c} = \left(c_j^{(k)} \right)_{j=k, k-1, k-2, \dots, 1, 0}$ by iterative solving.

6.2.1. Theoretical derivations

In this section, we will give the theoretical derivations to obtain the iterative expressions of the components of the solution vector $\mathbf{c} = \left(c_j^{(k)} \right)_{j=k, k-1, k-2, \dots, 1, 0}$ by iterative solving.

Rearranging the linear equations (24), we have

$$\begin{cases} c_k C_{k+1}^1 = 1 \\ c_{k-1} C_k^1 = c_k C_{k+1}^2 (-1)^2 \\ c_{k-2} C_{k-1}^1 = c_k C_{k+1}^3 (-1)^3 + c_{k-1} C_k^2 (-1)^2 \\ \vdots \\ c_1 C_2^1 = c_k C_{k+1}^k (-1)^k + c_{k-1} C_k^{k-1} (-1)^{k-1} + c_{k-2} C_{k-1}^{k-2} (-1)^{k-2} + \dots + c_2 C_3^2 (-1)^2 \\ c_0 C_1^1 = c_k C_{k+1}^{k+1} (-1)^{k+1} + c_{k-1} C_k^k (-1)^k + c_{k-2} C_{k-1}^{k-1} (-1)^{k-1} + \dots + c_1 C_2^2 (-1)^2. \end{cases}$$

By iterative solving, we obtain

$$\begin{cases} c_k = \frac{1}{C_{k+1}^1} \\ c_{k-1} = \frac{c_k C_{k+1}^2 (-1)^2}{C_k^1} \\ c_{k-2} = \frac{c_k C_{k+1}^3 (-1)^3 + c_{k-1} C_k^2 (-1)^2}{C_{k-1}^1} \\ \vdots \\ c_1 = \frac{c_k C_{k+1}^k (-1)^k + c_{k-1} C_k^{k-1} (-1)^{k-1} + c_{k-2} C_{k-1}^{k-2} (-1)^{k-2} + \dots + c_2 C_3^2 (-1)^2}{C_2^1} \\ c_0 = \frac{c_k C_{k+1}^{k+1} (-1)^{k+1} + c_{k-1} C_k^k (-1)^k + c_{k-2} C_{k-1}^{k-1} (-1)^{k-1} + \dots + c_1 C_2^2 (-1)^2}{C_1^1}. \end{cases}$$

Further simplifying the above formulas, the coefficients can be expressed as

$$\begin{cases} c_k^{(k)} = \frac{1}{C_{k+1}^1} \\ c_j^{(k)} = \frac{\sum_{i=j+1}^k c_i^{(k)} C_{i+1}^{i-j+1} (-1)^{i-j+1}}{C_{j+1}^1}, \quad j = k-1, k-2, \dots, 1, 0. \end{cases} \quad (27)$$

Finally,

$$\text{EY}^k = \sum_{j=0}^k c_j^{(k)} n^j = \frac{1}{C_{k+1}^1} n^k + \sum_{j=0}^{k-1} \frac{\sum_{i=j+1}^k c_i^{(k)} C_{i+1}^{i-j+1} (-1)^{i-j+1}}{C_{j+1}^1} n^j.$$

6.2.2. Verifications of the first four origin moments

In this section, we will verify the first four origin moments from the iterative expressions (27) of the components of $\mathbf{c} = (c_j^{(k)})_{j=k, k-1, k-2, \dots, 1, 0}$ by iterative solving.

When $k = 0$, we have

$$c_0^{(0)} = \frac{1}{C_1^1} = 1,$$

$$EY^0 = \sum_{j=0}^0 c_j^{(0)} n^j = c_0^{(0)} n^0 = 1.$$

When $k = 1$, we have

$$c_1^{(1)} = \frac{1}{C_2^1} = \frac{1}{2},$$

$$c_0^{(1)} = \frac{\sum_{i=1}^1 c_i^{(1)} C_{i+1}^{i+1} (-1)^{i+1}}{C_1^1} = \frac{c_1^{(1)} C_2^2 (-1)^2}{C_1^1} = \frac{1}{2},$$

$$EY^1 = \sum_{j=0}^1 c_j^{(1)} n^j = c_1^{(1)} n^1 + c_0^{(1)} n^0 = \frac{1}{2}n + \frac{1}{2}.$$

When $k = 2$, we have

$$c_2^{(2)} = \frac{1}{C_3^1} = \frac{1}{3},$$

$$c_1^{(2)} = \frac{\sum_{i=2}^2 c_i^{(2)} C_{i+1}^i (-1)^i}{C_2^1} = \frac{c_2^{(2)} C_3^2 (-1)^2}{C_2^1} = \frac{1}{2},$$

$$c_0^{(2)} = \frac{\sum_{i=1}^2 c_i^{(2)} C_{i+1}^{i+1} (-1)^{i+1}}{C_1^1} = \frac{c_1^{(2)} C_2^2 (-1)^2 + c_2^{(2)} C_3^3 (-1)^3}{C_1^1} = \frac{1}{6},$$

$$EY^2 = \sum_{j=0}^2 c_j^{(2)} n^j = c_2^{(2)} n^2 + c_1^{(2)} n^1 + c_0^{(2)} n^0 = \frac{1}{3}n^2 + \frac{1}{2}n + \frac{1}{6}.$$

When $k = 3$, we have

$$\begin{aligned}
c_3^{(3)} &= \frac{1}{C_4^1} = \frac{1}{4}, \\
c_2^{(3)} &= \frac{\sum_{i=3}^3 c_i^{(3)} C_{i+1}^{i-1} (-1)^{i-1}}{C_3^1} = \frac{c_3^{(3)} C_4^2 (-1)^2}{C_3^1} = \frac{1}{2}, \\
c_1^{(3)} &= \frac{\sum_{i=2}^3 c_i^{(3)} C_{i+1}^i (-1)^i}{C_2^1} = \frac{c_2^{(3)} C_3^2 (-1)^2 + c_3^{(3)} C_4^3 (-1)^3}{C_2^1} = \frac{1}{4}, \\
c_0^{(3)} &= \frac{\sum_{i=1}^3 c_i^{(3)} C_{i+1}^{i+1} (-1)^{i+1}}{C_1^1} = \frac{c_1^{(3)} C_2^2 (-1)^2 + c_2^{(3)} C_3^3 (-1)^3 + c_3^{(3)} C_4^4 (-1)^4}{C_1^1} = 0, \\
EY^3 &= \sum_{j=0}^3 c_j^{(3)} n^j = c_3^{(3)} n^3 + c_2^{(3)} n^2 + c_1^{(3)} n^1 + c_0^{(3)} n^0 = \frac{1}{4} n^3 + \frac{1}{2} n^2 + \frac{1}{4} n.
\end{aligned}$$

When $k = 4$, we have

$$\begin{aligned}
c_4^{(4)} &= \frac{1}{C_5^1} = \frac{1}{5}, \\
c_3^{(4)} &= \frac{\sum_{i=4}^4 c_i^{(4)} C_{i+1}^{i-2} (-1)^{i-2}}{C_4^1} = \frac{c_4^{(4)} C_5^2 (-1)^2}{C_4^1} = \frac{1}{2}, \\
c_2^{(4)} &= \frac{\sum_{i=3}^4 c_i^{(4)} C_{i+1}^{i-1} (-1)^{i-1}}{C_3^1} = \frac{c_3^{(4)} C_4^2 (-1)^2 + c_4^{(4)} C_5^3 (-1)^3}{C_3^1} = \frac{1}{3}, \\
c_1^{(4)} &= \frac{\sum_{i=2}^4 c_i^{(4)} C_{i+1}^i (-1)^i}{C_2^1} = \frac{c_2^{(4)} C_3^2 (-1)^2 + c_3^{(4)} C_4^3 (-1)^3 + c_4^{(4)} C_5^4 (-1)^4}{C_2^1} = 0, \\
c_0^{(4)} &= \frac{\sum_{i=1}^4 c_i^{(4)} C_{i+1}^{i+1} (-1)^{i+1}}{C_1^1} = \frac{c_1^{(4)} C_2^2 (-1)^2 + c_2^{(4)} C_3^3 (-1)^3 + c_3^{(4)} C_4^4 (-1)^4 + c_4^{(4)} C_5^5 (-1)^5}{C_1^1} = -\frac{1}{30}, \\
EY^4 &= \sum_{j=0}^4 c_j^{(4)} n^j = c_4^{(4)} n^4 + c_3^{(4)} n^3 + c_2^{(4)} n^2 + c_1^{(4)} n^1 + c_0^{(4)} n^0 = \frac{1}{5} n^4 + \frac{1}{2} n^3 + \frac{1}{3} n^2 - \frac{1}{30}.
\end{aligned}$$

6.3. The coefficients table of the k th origin moment of the DU distribution

Using the above two methods, namely, the matrix inversion method and the iterative solving method, we can use R software (R Core Team (2021)) to compute the unknown coefficients $\mathbf{c} = \left(c_j^{(k)} \right)_{j=k, k-1, k-2, \dots, 1, 0}$ of the k th origin

moment of $Y \sim DU(n)$. The coefficients of the k th ($k = 0, 1, \dots, 10$) origin moment of $Y \sim DU(n)$ are summarized in Table 1. According to this table, we observe that

$$\begin{aligned} c_k^{(k)} &= \frac{1}{k+1}, \\ c_{k-1}^{(k)} &= \frac{1}{2}, \\ c_{k-2l-1}^{(k)} &= 0, \quad l = 1, 2, 3, \dots, \left\lfloor \frac{k-1}{2} \right\rfloor, \end{aligned}$$

where $\lfloor \cdot \rfloor$ expresses to round down the number.

Table 1: The coefficients of the k th ($k = 0, 1, \dots, 10$) origin moment of $Y \sim DU(n)$.

k	$c_0^{(k)}$	$c_1^{(k)}$	$c_2^{(k)}$	$c_3^{(k)}$	$c_4^{(k)}$	$c_5^{(k)}$	$c_6^{(k)}$	$c_7^{(k)}$	$c_8^{(k)}$	$c_9^{(k)}$	$c_{10}^{(k)}$
0	1										
1	$\frac{1}{2}$	$\frac{1}{2}$									
2	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$								
3	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$							
4	$-\frac{1}{30}$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{5}$						
5	0	$-\frac{1}{12}$	0	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{6}$					
6	$\frac{1}{42}$	0	$-\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{7}$				
7	0	$\frac{1}{12}$	0	$-\frac{7}{24}$	0	$\frac{7}{12}$	$\frac{1}{2}$	$\frac{1}{8}$			
8	$-\frac{1}{30}$	0	$\frac{2}{9}$	0	$-\frac{7}{15}$	0	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{9}$		
9	0	$-\frac{3}{20}$	0	$\frac{1}{2}$	0	$-\frac{7}{10}$	0	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{10}$	
10	$\frac{5}{66}$	0	$-\frac{1}{2}$	0	1	0	-1	0	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{11}$

By (1) and Table 1, we have the following proposition for the summations (2) for $k = 1, 2, \dots, 10$. For $k > 10$, the summations can also be obtained by more computations in R software.

Proposition 1. *The summations $\sum_{i=1}^n i^k$ for $k = 1, 2, \dots, 10$ are given by*

$$\begin{aligned}
\sum_{i=1}^n i^1 &= n \left(\frac{1}{2}n + \frac{1}{2} \right), \\
\sum_{i=1}^n i^2 &= n \left(\frac{1}{3}n^2 + \frac{1}{2}n + \frac{1}{6} \right), \\
\sum_{i=1}^n i^3 &= n \left(\frac{1}{4}n^3 + \frac{1}{2}n^2 + \frac{1}{4}n \right), \\
\sum_{i=1}^n i^4 &= n \left(\frac{1}{5}n^4 + \frac{1}{2}n^3 + \frac{1}{3}n^2 - \frac{1}{30} \right), \\
\sum_{i=1}^n i^5 &= n \left(\frac{1}{6}n^5 + \frac{1}{2}n^4 + \frac{5}{12}n^3 - \frac{1}{12}n \right), \\
\sum_{i=1}^n i^6 &= n \left(\frac{1}{7}n^6 + \frac{1}{2}n^5 + \frac{1}{2}n^4 - \frac{1}{6}n^2 + \frac{1}{42} \right), \\
\sum_{i=1}^n i^7 &= n \left(\frac{1}{8}n^7 + \frac{1}{2}n^6 + \frac{7}{12}n^5 - \frac{7}{24}n^3 + \frac{1}{12}n \right), \\
\sum_{i=1}^n i^8 &= n \left(\frac{1}{9}n^8 + \frac{1}{2}n^7 + \frac{2}{3}n^6 - \frac{7}{15}n^4 + \frac{2}{9}n^2 - \frac{1}{30} \right), \\
\sum_{i=1}^n i^9 &= n \left(\frac{1}{10}n^9 + \frac{1}{2}n^8 + \frac{3}{4}n^7 - \frac{7}{10}n^5 + \frac{1}{2}n^3 - \frac{3}{20}n \right), \\
\sum_{i=1}^n i^{10} &= n \left(\frac{1}{11}n^{10} + \frac{1}{2}n^9 + \frac{5}{6}n^8 - n^6 + n^4 - \frac{1}{2}n^2 + \frac{5}{66} \right).
\end{aligned}$$

7. Conclusions and Discussions

For the calculations of the k th ($k = 1, 2, \dots$) origin moment of a DU random variable $Y \sim DU(n)$, EY^k , there are two common methods. That is, the definition method and the mgf method. The analytical calculations of EY^k ($k = 1, 2, \dots$) by the definition method are hindered when $k \geq 3$, because it is difficult to analytically obtain the summations (2). Moreover, the mgf method also fails to analytically calculate EY^k , due to the calculation reduces to the definition of EY^k . In this paper, we provide a novel method to analytically calculate EY^k , that is, the expectation identity method.

First, the expectation identity of the DU distribution is discovered and summarized in a theorem. After that, we analytically calculate the first four origin moments of the DU distribution $Y \sim DU(n)$ by the expectation identity method. Furthermore, we analytically calculate the k th ($k = 1, 2, \dots$) origin moment of the DU distribution $Y \sim DU(n)$ by the expectation identity method. After comparing the corresponding coefficients on both sides of the equation (22), we obtain a nonhomogeneous linear equations of first degree in $k + 1$ variables. After simplifications, the nonhomogeneous linear equations reduce to (24) or (25).

We have provided two ways to solve the nonhomogeneous linear equations (25). The first way is by matrix inversion, and the second way is by iterative solving. For each way, we first give the theoretical derivations to obtain the iterative expressions of the components of the solution vector $\mathbf{c} = \left(c_i^{(k)} \right)_{i=k, k-1, k-2, \dots, 1, 0}$. Then we verify the first four origin moments from the iterative expressions of the components of $\mathbf{c} = \left(c_i^{(k)} \right)_{i=k, k-1, k-2, \dots, 1, 0}$. Moreover, the coefficients of the k th ($k = 0, 1, \dots, 10$) origin moment of $Y \sim DU(n)$ are summarized in Table 1. Finally, we have a proposition for the summations (2) for $k = 1, 2, \dots, 10$. For $k > 10$, the summations can also be obtained by more computations in R software.

Supporting Information

Additional information for this article is available.

Supplement: Some proofs of the article.

R folder: R codes used in the article. The R folder will be supplied after acceptance of the article.

Declarations

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Conflict of interest

The authors declare that they have no conflict of interest.

Availability of data and material

Not applicable.

Code availability

The R folder will be supplied after acceptance of the article.

Authors' contributions

All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by Jia-Lei Liu, Ying-Ying Zhang, and Yuan-Quan Wang. The first draft of the manuscript was written by Jia-Lei Liu and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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