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Large deflection analysis of nanocomposite cylindrical shell subjected to internal pressure

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Abstract

In this work, large deflection behavior of a functionally graded carbon nanotube reinforced composite (FG-CNTRC) cylindrical shell under internal pressure is studied. The composite cylindrical shell reinforced along the longitudinal direction and made from a polymeric matrix. Based on first-order shear deformation shell theory (FSDT) and von Kármán geometrical nonlinearity, the set of governing equations are derived. Using the dynamic relaxation (DR) technique, these systems of equations are solved for various boundary conditions and the roles of volume fraction of CNTs, CNTs distributions and geometrical ratios are examined on the responses.

Keywords

Nano-composite shell, Carbon nanotubes, Large deflection, Longitudinal direction, DR method.

1-Introduction

Carbon nanotubes (CNTs) have been perceived as flawless strengthening materials of advanced composites because of extraordinary mechanical, thermodynamic and electrical features. The initial idea of CNTRC was proposed by Ajayan et al. [1] who cut and examined thin slices of a polymer resin-nanotube composite. Later, many complementary pieces of research were performed by other scientists which lead to vast applications in micro-electro-mechanical and micro-optical systems including microscopes, micro-sensors and actuators [2].

Functionally graded (FG) materials can be utilized to tackle the problems of laminated composites' delamination since the initial advent of it in the 1980s (by Japanese researchers). In other words, these continuous spatial gradient materials have been tailored to amend composite defects [3]. FG-CNTRC can be named a novel sub-branch of the composite which is fabricated and brought into play in many engineering performances in recent years. These innovative materials are the outcome of reinforcing the nanofibers with carbon in the ultra-small scale, distributed smoothly and continuously in a polymeric matrix [4]. Shen [5] can be called one of the first researcher who analyzed the mechanical features of FGCNTRC then published an article in 2009 which examined the large deflection of FG-CNTRC in thermal environment. The static examination of an ultra-small size composite beam strengthen by a SWNT via Airy stress-function approach was studied by Vodenitcharova and Zhang [6] both experimentally and theoretically. In another work, Shen [7] perused post-buckling of reinforced nanocomposite cylindrical shells exposed to axial compression in thermal surroundings. Seidel and his colleague Lagoud [8] scrutinized micromechanical elastic properties of reinforced structures by CNTs for recognizing the mechanical properties of nanotubes. With the aid of molecular dynamic simulation, Zhang and Shen [9] carried out a research on CNTs mechanical properties including temperature-dependent properties. Vibrational of FG carbon nanotube–reinforced cylindrical panels via Eshelby–Mori–Tanaka technique and quadrature method was investigated in Ref. [10]. The researchers of Ref. [11] investigated deflection analyzes of FG-CNTRC rectangular shell with simply supported edges exposed to thermo-mechanical loads on the basis of the 3D theory of elasticity. Moradi et al. [12] conduct a research on the dynamic investigation of the reinforced nanocomposite cylinders subjected to an impact load with the aid of a mesh-free technique. The authors of Ref. [13] studied both statistic and free vibration analyzes of single-walled carbon nanotubes composite shells considering the variable thickness on the basis of FSDT via the finite element procedure. Kwon et al. [14] can be called pioneers of fabricating FG-CNTRC. Later, many researches were conducted about the static and dynamic examination of FG-CNTRC, which in the following some of them are discussed. In Ref. [15] FG-CNTRC plates were studied analytically and in Ref. [16] the nonlinear vibration response of FG-CNTRC beams were implemented. Shen et al. [17] carried out the nonlinear dynamic investigation of FG-CNTRC rectangular plates exposed to blast loads using Reddy's higher-order shear deformation theory (HSDT) by means of the weak form quadrature element approach. Jiao and his coworkers [18] analyzed the buckling of thin rectangular FG-

CNTRC plate subjected to compression loads by means of the differential quadrature technique (DQM) with the work equivalent technique. Ansari et al. [19] perused the free vibration of diverse forms of FG-CNTRC plates using HSDT with the aid of the generalized differential quadrature method (GDQM). The modified FSDT was implemented by Mellouli [20] to conduct free vibration of FG-CNTRC shell by means of a method named mesh free radial point interpolation. Civaleka and Jalaeib [21] scrutinized the shear buckling of the FG-CNTRC asymmetric plate with various boundary conditions. Large deflection analysis of the FG-CNTRC annular variable thickness plate on the Pasternak elastic foundation was implemented by Keleshteri and his colleagues [22] using the third-order shear deformation model. Zhong and his assistants [23] studied the vibration of FG-CNTRC circular/annular as well as sector plates utilizing FSDT with the aid of the semi-analytical method. The dynamic response of aero-thermoelastic FG-CNTRC panels was investigated in Ref. [24]. Hajlaoui and Chebbi [25] analyzed the buckling of FG-CNTRC shells on the basis of modified first-order enhanced solid-shell element formulation. Nguyena et al. [26] perused the nonlinear postbuckling of FG-CNTRC shells based on nonuniform rational B-Spline basis functions and FSDT by means of modified Riks numerical operators. The nonlinear buckling of FG-CNTRC cylindrical shells was studied in Ref. [27]. Zhou and Song [28] investigated the nonlinear deflection analysis of FG-CNTRC plates based on the Strain-Rotation (S-R) decomposition using the 3-D element-free Galerkin method. Soni et al. [29] investigated the bending of FG-CNTRC plates on the basis of inverse hyperbolicshear deformation principle by means of Navier method and they inferred that the analysis of FG-CNTRC depends markedly on some items including loading conditions, the volume fraction of CNT, sort of distribution of CNT and span-thickness ratio. Golmakani and Zeighami [30] perused the nonlinear thermo-elastic of the static examination of FG-CNTRC plates on elastic foundations with the aid of FSDT and by means of dynamic relaxation method.

According to this literature and available contexts, the large deflection of nanocomposite cylindrical shell reinforced by carbon nanotubes in the longitudinal direction under internal pressure has not been published on the basis of FSDT via DR technique. The material properties of FG-CNTRC shells are assumed to vary smoothly along thickness and gained through a micromechanical model. Eventually, the roles of effective factors including volume fraction of CNTs, boundary conditions, thickness-to-radius and length-to-radius ratios and CNTs distributions on the responses are examined.

2- Mathematical formulation and theoretical frameworks

In this part, the theoretical frameworks on the properties of carbon nanotube materials are discussed and the equations that set their distribution are expressed initially. Then, the governing equations based on axial symmetry and strain-displacement are expressed.

The distributions of the nanotubes in the longitudinal direction of the cylindrical shell with uniform direction (UD) and functional gradients including FG-O, FG-X and FG-V are illustrated in Fig 1. More specifically, in the longitudinal direction, 1 and 2 signify that the fibers are in x (longitudinal axis) and y (or θ) directions, respectively.

Fig 1. Schematic of carbon nanotubes' distributions along the longitudinal direction

Fig 2. Schematic of axial symmetric cylindrical shell

In Fig. 1, UD-CNTRC illustrates the uniform distribution, and FG-O, FG-V and FG-X signify the other sorts of FG distributions of carbon nanotubes along the thickness of the composite shell. Assuming the extended rule of the mixture and by presenting the CNT efficiency coefficients, the effective material features including elastic modules and Poisson ratios of CNTRC shell can be introduced as [31]:

$$\begin{aligned}
E_{11} &= \eta_1 V_{CNT} E_{11}^{CNT} + V_m E^m \\
\frac{\eta_2}{E_{22}} &= \frac{V_{CNT}}{E_{22}^{CNT}} + \frac{V_m}{G^m} \\
\frac{\eta_3}{G_{12}} &= \frac{V_{CNT}}{G_{12}^{CNT}} + \frac{V_m}{G^m} \\
\nu_{12} &= V_{CNT}^* \nu_{12}^{CNT} + V_m \nu^m
\end{aligned} \tag{1}$$

where E_{22}^{CN} , E_{11}^{CN} and G_{12}^{CN} express the Young's and shear moduli of the CNTs, also E^m and G^m belong to the related properties of the matrix, respectively. Additionally, $\eta_i (i=1,2,3)$ express the CNT efficiency coefficients. The interfacial interactions between size-dependent reinforcement and the matrix can be assumed by means of the SWCNTs efficiency parameters $\eta_i (i=1,2,3)$ obtained from Ref. [31].

Furthermore, V_m and V_{CN} signify the volume fractions of the matrix and CNT, respectively ($V_{CN} + V_m = 1$). The uniform and the other FG distributions of the CNT can be presented as [31]:

$$\begin{aligned}
V_{CNT(z)} &= V_{CNT}^* && \text{UD} \\
V_{CNT(z)} &= 2 \left(2 \frac{|z|}{h} \right) V_{CNT}^* && \text{FG-X} \\
V_{CNT(z)} &= \left(2 \frac{z}{h} + 1 \right) V_{CNT}^* && \text{FG-V} \\
V_{CNT(z)} &= 2 \left(1 - 2 \frac{z}{h} \right) V_{CNT}^* && \text{FG-O}
\end{aligned} \tag{2}$$

where V_{CNT}^* denotes the CNT volume fractions and can be expressed as follows:

$$V_{CNT}^* = \frac{w_{CNT}}{w_{CNT} + \left(\frac{\rho^{CNT}}{\rho^m} \right) - \left(\frac{\rho^{CNT}}{\rho^m} \right) w_{CNT}} \quad (3)$$

More specifically, in the mathematical formulation (3) w_{CNT} introduces the mass fraction of CNTs in the composite shell, also ρ^m and ρ^{CNT} imply the densities of the matrix and CNTs, respectively. The four types of carbon nanotube reinforced composite (CNTRC) shells possess the same mass fraction (w_{CNT}) and volume of CNTs. The impacts of the thickness on the stress analysis are considered with the aid of FSDT. Hence, the displacement field can be presented as equation (1):

$$\begin{aligned} u(x, z) &= u_0(x) + z\varphi_x \\ w(x, z) &= w_0(x) \end{aligned} \quad (4)$$

where in the above relations, u and w are displacement fields, also u_0 and w_0 define the displacement components of the mid-plane along x and z axis, respectively. Additionally, φ_x signifies rotations of a transverse normal about x direction. In the case of axisymmetric cylindrical shell, the strain-displacement relations can be obtained as follows [32]:

$$\begin{aligned} \varepsilon_x &= \frac{du_0}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 + z \left(\frac{d\varphi_x}{dx} \right) \\ \varepsilon_\theta &= \frac{w}{R} \quad \varepsilon_z = 0 \quad \gamma_{xz} = \frac{dw}{dx} + \varphi_x \\ \gamma_{r\theta} &= \gamma_{\theta z} = 0 \end{aligned} \quad (5)$$

In which, ε_x and ε_θ express normal strains, and γ_{xz} signifies the transverse shear strains. Also, moment and shear stress resultants of nonlocal elasticity can be expressed as follows:

$$\begin{aligned} (N_x, N_\theta, Q_x) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_\theta, \tau_{rz}) dz \\ (M_x, M_\theta) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_\theta) z dz \end{aligned} \quad (6)$$

The stress resultants-strain relationships of an axis-symmetric shell can be presents as follows:

$$\begin{aligned}
\begin{bmatrix} N_x \\ N_\theta \\ N_{x\theta} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_\theta^0 \\ \varepsilon_{x\theta}^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{21} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} k_x^0 \\ k_\theta^0 \\ k_{x\theta}^0 \end{Bmatrix} \\
\begin{bmatrix} M_x \\ M_\theta \\ M_{x\theta} \end{bmatrix} &= \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{21} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_\theta^0 \\ \varepsilon_{x\theta}^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{21} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} k_x^0 \\ k_\theta^0 \\ k_{x\theta}^0 \end{Bmatrix} \\
\begin{bmatrix} Q_x \\ Q_\theta \end{bmatrix} &= \begin{bmatrix} KA_{44} & KA_{45} \\ KA_{45} & KA_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xz}^0 \\ \varepsilon_{\theta z}^0 \end{Bmatrix}
\end{aligned} \tag{7}$$

Where $k=5/6$ signifies the transverse shear correction parameter. In equation (7), $A_{ij}(i, j=1,2,6)$ and $D_{ij}(i, j=1,2,6)$ introduce the extensional and bending stiffness matrices, respectively, which can be written as follows:

$$\begin{aligned}
\begin{bmatrix} A_{ij} \end{bmatrix} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij} dz, \quad \begin{bmatrix} B_{ij} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij} z dz \\
\begin{bmatrix} D_{ij} \end{bmatrix} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij} z^2 dz, \quad i, j = 1, 2 \\
\begin{bmatrix} F_{44} \end{bmatrix} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{44} dz
\end{aligned} \tag{8}$$

More specifically, in the equation (8), $Q_{ij}(i, j=1,2,6)$ can be obtained by:

$$Q_{11} = \frac{E_{11}}{1 - \nu_{12}\nu_{21}}, Q_{22} = \frac{E_{22}}{1 - \nu_{12}\nu_{21}}, Q_{12} = \frac{E_{11}\nu_{21}}{1 - \nu_{12}\nu_{21}}$$

$$Q_{66} = G_{12} \quad Q_{44} = G_{23} \quad Q_{55} = G_{13} \quad (9)$$

The stress resultants of nonlocal elasticity can be obtained by utilizing Eq. (7) on the basis of the displacements, as follows:

$$N_x = A_{11} \left(\frac{du_0}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right) + A_{12} \left(\frac{w}{R} \right) + B_{11} \frac{d\varphi_x}{dx}$$

$$N_\theta = A_{12} \left(\frac{du_0}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right) + A_{22} \left(\frac{w}{R} \right) + B_{12} \frac{d\varphi_x}{dx}$$

$$M_x = B_{11} \left(\frac{du_0}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right) + B_{12} \left(\frac{w}{R} \right) + D_{11} \frac{d\varphi_x}{dx}$$

$$Q_x = F_{44} \left(\frac{dw}{dx} + \varphi_x \right) \quad (10)$$

The equations of equilibrium can be achieved by minimizing energy with the aid of FSDT. Utilizing principle of minimum energy, the total potential energy is the total of strain energy and the potential energy from external loads. Hence, the equation (11) is applied to gain the equations of equilibrium:

$$V = U + \Omega \quad (11)$$

where on the basis of the principle of minimum energy, the variations of Eq. (11) are as follows:

$$\delta^* V = \delta^* U + \delta^* \Omega = 0$$

$$\delta^* \Omega = \frac{1}{2} \int_0^L \int_0^{2\pi} \frac{q}{2\pi R} \frac{\partial w}{\partial x} R d\theta dx$$

$$\delta^* U = \frac{1}{2} \int_0^L \int_0^{2\pi} \int_{-h/2}^{h/2} (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{\theta\theta} \delta \varepsilon_{\theta\theta} + \tau_{x\theta} \delta \gamma_{x\theta} + \tau_{xz} \delta \gamma_{xz} + \tau_{z\theta} \delta \gamma_{z\theta}) \quad (12)$$

By inserting Eq. (5) in Eq. (12), the Eq. (13) is gained:

$$\begin{aligned} \delta^* U = & \frac{1}{2} \int_0^L \int_0^{2\pi} (N_x \delta^* (\frac{\partial \delta u}{\partial X} + \frac{1}{2} (\frac{\partial \delta w}{\partial X})^2 + \frac{\partial w}{\partial X} \frac{\partial \delta w}{\partial X}) + M_x \delta^* \frac{\partial \delta \varphi_x}{\partial X} + N_x \delta^* (\frac{\delta w}{R} \\ & + Q_{xz} \delta^* (\frac{\partial \delta \varphi_x}{\partial X} + \frac{\partial \delta w}{\partial X})) R d\theta dx \end{aligned} \quad (13)$$

Hence, the governing equations of the cylindrical shell under internal pressure are extracted by the aid of the principle of minimum energy. The equations of equilibrium can be written on the basis of FSDT and utilizing Eq. (7), as [13, 33]:

$$\begin{aligned} R \frac{dN_x}{dx} &= 0 \\ R \frac{dQ_x}{dX} - N_\theta - Rq &= 0 \\ R \frac{dM_x}{dx} + RQ_x &= 0 \end{aligned} \quad (14)$$

The governing relations would be written in terms of the displacement as the set of equation (15):

$$\begin{aligned} A_{11} \left(\frac{d^2 u}{dx^2} \right) + A_{11} \left(\frac{dw}{dx} \right) \left(\frac{d^2 w}{dx^2} \right) + \frac{A_{12}}{R} \left(\frac{dw}{dx} \right) + B_{11} \left(\frac{d^2 \varphi_x}{dx^2} \right) &= 0 \\ A_{55} \left(\frac{d^2 w}{dx^2} + \frac{d\varphi_x}{dx} \right) - \frac{A_{12}}{R} \left(\frac{1}{2} \right) \left(\frac{dw}{dx} \right)^2 - \frac{A_{12}}{R} \left(\frac{d^2 u}{dx^2} \right) - \frac{A_{22}}{R^2} w - \frac{B_{12}}{R} \left(\frac{d\varphi_x}{dx} \right) - q &= 0 \\ B_{11} \left(\frac{d^2 u}{dx^2} \right) + B_{11} \left(\frac{dw}{dx} \right) \left(\frac{d^2 w}{dx^2} \right) + \frac{B_{12}}{R} \left(\frac{dw}{dx} \right) + \left(\frac{d^2 \varphi_x}{dx^2} \right) + A_{55} \left(\frac{dw}{dx} \right) + A_{55} \varphi_x &= 0 \end{aligned} \quad (15)$$

A set of boundary conditions should be applied in the above equations to obtain the results. In this article, the boundary conditions are as follows:

(a) SS boundary condition:

$$\begin{aligned} w = u = M_x &= 0 & (x = 0) \\ w = u = M_x &= 0 & (x = L) \end{aligned} \quad (16)$$

(b) CC boundary condition:

$$\begin{aligned}
u = w = \phi_x &= 0 & (x = 0) \\
u = w = \phi_x &= 0 & (x = L)
\end{aligned} \tag{17}$$

(c) SC boundary condition:

$$\begin{aligned}
u = w = M_x &= 0 & (x = 0) \\
u = w = \phi_x &= 0 & (x = L)
\end{aligned} \tag{18}$$

(d) CS boundary condition:

$$\begin{aligned}
u = w = \phi_x &= 0 & (x = 0) \\
u = w = M_x &= 0 & (x = L)
\end{aligned} \tag{19}$$

3. Solution procedure

In this context, the DR technique with the central finite-difference procedure is utilized to solve nonlinear problems including the nonlinear differential relationships of the CNTRC shell [34-36].

DR is an iterative approach which its basic aim is to gain a steady-state solution with the aid of converting the static problem into a dynamic one [37-39]. For solving the shell equations by means of this technique, damping and fictitious inertia terms are added to the right sides of the shell equations of equilibrium. Therefore, they would convert from the boundary value problem to an initial value format. Subsequently, the equations of equilibrium can be expressed as:

$$\begin{aligned}
R \frac{dN_x}{dx} &= m_u \frac{d^2 u}{dt^2} + C_u \frac{du}{dt} \\
R \frac{dQ_x}{dx} - N_\theta - Rq &= m_w \frac{d^2 w}{dt^2} + C_w \frac{dw}{dt} \\
R \frac{dM_x}{dx} + RQ_x &= m_{\phi_x} \frac{d^2 \phi_x}{dt^2} + C_{\phi_x} \frac{d\phi_x}{dt}
\end{aligned} \tag{20}$$

where in the equation (20) m_u, m_w, m_{ϕ_x} and c_u, c_w, c_{ϕ_x} signify elements of the diagonal fictitious mass (M) and damping matrices (C), respectively. Hence, the mass matrix as well as nodal damping factors need to be determined to make the iterative procedure stable and converged. The most prevalent method of achieving $m_{ii}^l[l : u, w, \phi_x]$ at node i in the n^{th} iteration is to apply the Gershgorin principle. By means of this theory, the equation (21) needs to be satisfied in order to assure the convergence of iterations [38]:

$$m_{ii}^l = 0.25(\tau^n)^2 \sum_{j=1}^N |k_{ij}^l| \quad (21)$$

In equation (21), the symbols n and τ signify the n^{th} iteration step and an increment of fictitious time, respectively. The element k_{ij} of the stiffness matrix K can be obtained by computing equation (28) as follows:

$$k_{ij} = \frac{\partial P_i}{\partial X_j} \quad (22)$$

where $X_i = u_i, w_i, \phi_{xi}$ expresses a rough solution vector. Furthermore, internal forces P_i is the left side of the equilibrium relationships (Eqs. (15)) and for instance, it is expressed on the basis of stress and moment resultants in Eq. (21). Here, the instant critical damping factor c_i^n for node i at the n^{th} iteration can be achieved as follows [39]:

$$c_i^n = 2 \left\{ \frac{(X_i^n)^T P_i^n}{(X_i^n)^T m_{ii}^n X_i^n} \right\}^{1/2} \quad (23)$$

Various values of c for diverse nodes in each direction $I = u, \psi, w$ can be considered for each element of diagonal fictitious damping matrices $C (N \times N)$ in a discrete system connected through N nodes to achieve the form used in the DR method as follows:

$$c_{ii} = c_i m_{ii} \quad i = 1, \dots, N \quad (24)$$

Replacing the velocity and the acceleration based on Eq. (21) with the equivalent central finite-difference equations, the converting process is achieved. Therefore, the equations of equilibrium can be written into an initial value format as equation (25) (for more details refer to [39]):

$$\begin{aligned}
\dot{u}_i^{n+\frac{1}{2}} &= \frac{(2\Delta t^n)}{(2+\Delta t^n c_i)} (m_{ii}^n)^{-1} \left(R \frac{dN_x}{dx} \right)_i^n + \frac{(2-\Delta t^n c_i^n)}{(2+\Delta t^n c_i^n)} u_i^{n-\frac{1}{2}} \\
\dot{w}_i^{n+\frac{1}{2}} &= \frac{(2\Delta t^n)}{(2+\Delta t^n c_i)} (m_{ii}^n)^{-1} \left(R \frac{dQ_x}{dx} - N_\theta - Rq \right)_i^n + \frac{(2-\Delta t^n c_i^n)}{(2+\Delta t^n c_i^n)} w_i^{n-\frac{1}{2}} \\
\dot{\phi}_{xi}^{n+\frac{1}{2}} &= \frac{(2\Delta t^n)}{(2+\Delta t^n c_i)} (m_{ii}^n)^{-1} \left(R \frac{dM_x}{dx} + RQ_x \right)_i^n + \frac{(2-\Delta t^n c_i^n)}{(2+\Delta t^n c_i^n)} \phi_{xi}^{n-\frac{1}{2}}
\end{aligned} \tag{25}$$

After integrating the velocities of each time step, the displacements can be computed by:

$$X^{n+1} = X^n + \tau^{n+1} \dot{X}^{n+\frac{1}{2}} \tag{26}$$

Utilizing the above equations together with the certain boundary conditions in their finite-difference formats create the set of equations for the sequential DR technique [35].

4-Results

In this section, numeric outcomes of the bending examination of the FG-CNTRC cylindrical shell in the longitudinal direction exposed to internal pressure are discussed. The CNTRC is supposed to be consisted of the Poly methyl methacrylate (referred to as PMMA) with CNT which fibers aligned in the longitudinal direction of the shell.

In order to verify the accuracy of the numeric outcomes of the mechanical analysis, two examples are considered and comparative studies with similar articles (in this field) or Abaqus finite element software [40] are conducted. Also, parametric studies have been shown to examine effective items such as the distribution of carbon nanotubes, the thickness-to- radius, length -to- radius, boundary conditions, and the volume fraction of CNTs. Due to the unavailability of a reliable published article in the present subject, the results were first compared with the same article for the isotropic state or with those gained from the Abaqus finite element software [40]. In the following, two examples of them are discussed.

Example 1: As can be seen in figure 3, in order to verify the accuracy of the linear bending examination of the isotropic cylindrical shell, the results of the present study are obtained with those published by Ref. [41] in CC boundary condition. The research is concerned with FGM and

the comparison is made for $n = 0$ (which represents pure metal). It can be monitored that the maximum values achieved by the present procedure are roughly similar to those of Ref. [41].

Fig.3. Comparison of results between DR technique and reference [41]

Example 2: In this case, the linear results of the present study are compared with those obtained by Abaqus finite element software [40]. The results are obtained for a composite shell reinforced with carbon nanotubes in the longitudinal direction under internal pressure loading with CC boundary conditions, in two cases of uniform and FG distributions for two volume fractions of 0.12 and 0.17. Plate dimensions and properties of the composite material are considered as follows: $r = 50cm$, $h = 1cm$, $L = 100cm$, $E^m = 2.5GPa$, $\nu^m = 0.34$, $G^m = 0.933Gpa$, $E_{11}^{CN} = 5.6466TPa$, $E_{22}^{CN} = 7.0800TPa$, $G_{12}^{CN} = 1.944TPa$, $G_{12} = G_{23} = G_{13}$, $\eta_2 = \eta_3$. The young's moduli for PMMA/CNT composites are strengthened by (10, 10) tube and other properties utilized as shown in table 1.

Table 1. Results of the material properties obtained from molecular dynamics and rules of mixtures.

V_{CNT}^*	molecular dynamics[42]			rules of mixtures[43]		
	E_{11}	η_1	E_{22}	E_{11}	η_2	E_{22}
0.12	94.6	0.137	2.9	94.78	1.022	2.9
0.17	138.9	0.142	4.9	138.68	1.626	4.9
0.28	224.2	0.141	5.5	224.5	1.585	5.5

The maximum radial dimensionless deflection results of the linear bending obtained by the DR method are compared with those of results gained by Abaqus finite element software [40] and gathered in tables 2. and 3. It can be deduced that the maximum value of non-dimension deflection obtained from the DR technique is almost similar to those of results obtained from Abaqus finite element software [40]. Also, it can be comprehended that the maximum values of the deflection increase in FG state under different loads are more than those of values with uniform distribution. Furthermore, the maximum dimensionless deflections in the two states of FG-X and FG-V are very close to each other.

Table 2. Comparison between the maximum dimensionless deflections obtained from linear analysis and Abaqus software [40] for CC boundary condition with $L/R = 1.5$ and $V_{CNT}^* = 0.12$.

	<i>FG-X</i>		<i>FG-V</i>		<i>UD</i>	
$\bar{q}$	Present study	[40]	Present study	[40]	Present study	[40]
2000	0.157	0.151	0.156	0.150	0.0941	0.0978
4000	0.314	0.302	0.314	0.301	0.188	0.194
6000	0.471	0.454	0.470	0.452	0.282	0.290
8000	0.628	0.605	0.628	0.603	0.376	0.387

Table 3. Comparison between the maximum dimensionless deflections obtained from linear analysis and Abaqus software [40] for CC boundary condition with $L/R = 1.5$ and $V_{CNT}^* = 0.17$.

	<i>FG-X</i>		<i>FG-V</i>		<i>UD</i>	
$\bar{q}$	Present study	[40]	Present study	[40]	Present study	[40]
2000	0.0987	0.101	0.0988	0.0983	0.0591	0.0642
4000	0.197	0.201	0.197	0.198	0.118	0.122
6000	0.296	0.302	0.296	0.298	0.177	0.189
8000	0.395	0.403	0.392	0.397	0.236	0.252

Figure 4. demonstrates the convergence of solving governing equations in the DR technique (with the aid of Fortran software). In this figure, the variation of the maximum dimensionless deflection (W_{max}/h) is shown in terms of the change in the number of nodes, subjected to the internal pressure $q = 20\text{Mpa}$ for the two distributions including uniform and X. According to the figure, the best convergence choice is $M = 60$.

Figure 4. Convergence study for uniform and FG-X distributions.

Figure 5 shows changes in the maximum nondimensional deflection versus variations of the nondimensional internal pressure for the shell with $r = 50cm$, $h = 2cm$, $L = 100cm$. According to this figure, for both the CC and SS boundary conditions, the highest maximum dimensionless deflection is related to the FG-O distribution but the lowest of those is in the UD one. It is also observed that with increasing internal pressure, the difference of maximum non-dimension deflections between the FG-X and FG-V distributions increases (for both boundary conditions).

Figure 5. Maximum dimensionless deflection in terms of dimensionless load change with $V_{CN}^* = 0.17$ in a) CC and b) SS boundary conditions.

Figures 6 and 7 illustrates roles of the dimensionless thickness on the radial displacement of the shell in terms of the dimensionless length for both CC and SS boundary conditions, respectively. The dimensions of the shell are $r = 50cm$, $L = 100cm$ with the volume fraction of 0.17 and the

shell is assumed under internal pressure $q = 10\text{Mpa}$. The w/h is obtained for the case where the shell radius (r) is constant and only the thickness (h) changes and the results are gained for different ratios ($h/r = 0.03$, $h/r = 0.04$, $h/r = 0.05$). It can be mentioned that by increasing the thickness in the SS boundary condition, the uniform distribution has the highest drop in dimensionless deflection but the lowest that of the drop is related to FG-X mode. Furthermore, due to the increase of the thickness, the highest drop in dimensionless deflection is related to the FG-O distribution but the lowest that of the drop is in the FG-V mode at the CC boundary condition. In addition, for all distributions, with the increase of the non-dimension thickness from $h/r = 0.03$ to $h/r = 0.04$, the decrease rate of non-dimension deflection is about 3 times higher than the case which the dimensionless thickness increases from $h/r = 0.04$ to $h/r = 0.05$.

Figure 6. The effect of dimensionless thickness on changes of dimensionless deflection versus non-dimensional length for 4 types of nanotube distributions including a)UD, b) FG-X, c)FG-V and d)FG-O in CC boundary condition with $V_{CN}^* = 0.17$.

Figure 7. The effect of dimensionless thickness on changes of dimensionless deflection versus non-dimensional length for 4 types of nanotube distributions including a)UD, b) FG-X, c)FG-V and d)FG-O in SS boundary condition with $V_{CN}^* = 0.17$.

Tables 4, 5, 6, and 7 compare the maximum dimensionless deflection obtained from the linear and nonlinear examination of the axial symmetric shell with four types of distributions with $r = 50cm$, $h = 2cm$ in CC and SS boundary conditions. According to the tables, by increasing dimensionless load, the difference between linear and nonlinear results increases (in all distributions and boundary conditions). More specifically, that of difference in the CC boundary condition is less than SS condition. Also, it can be deduced that for both boundary conditions and both length-to-radius ratios, the highest percentage difference between linear and nonlinear dimensionless deflection is related to FG-X distribution and the lowest that of difference is in FG-O mode. It should be noted that as the length-to-radius ratio increases, the difference of non-dimensional deflections between the two linear and nonlinear states decreases.

Table 4. Comparison between dimensionless maximum deflection obtained from linear and nonlinear examinations for SS boundary condition with $V_{CN}^* = 0.17$ and $L/r = 2.5$.

$0.006 \times \bar{q}$	UD		FG-X		FG-V		FG-O	
	linear	nonlinear	linear	nonlinear	linear	nonlinear	linear	nonlinear
500	0.244	0.097	0.380	0.300	0.458	0.263	0.344	0.320
1000	0.487	0.144	0.770	0.256	0.911	0.482	0.69	0.562
1500	0.735	0.292	1.160	0.432	1.386	0.681	1.051	0.821
2000	0.975	0.389	1.550	0.462	1.684	0.873	1.408	1.025
2500	1.222	0.488	1.940	0.592	1.860	0.042	1.76	1.201

Table 5. Comparison between W_{max}/h obtained from linear and nonlinear examinations for SS boundary condition with $V_{CN}^* = 0.17$ and $L/r = 1.5$.

$0.006 \times \bar{q}$	UD		FG-X		FG-V		FG-O	
	linear	nonlinear	linear	nonlinear	linear	nonlinear	linear	nonlinear
500	0.197	0.063	0.322	0.08	0.391	0.153	0.264	0.211
1000	0.395	0.126	0.642	0.154	0.810	0.289	0.542	0.387
1500	0.601	0.190	0.96	0.222	1.236	0.410	0.82	0.541
2000	0.805	0.259	1.291	0.288	1.544	0.523	1.098	0.68
2500	1.002	0.318	1.620	0.342	1.782	0.642	1.375	0.811

Table 6. Comparison between W_{max}/h obtained from linear and nonlinear examinations for CC boundary condition with $V_{CN}^* = 0.17$ and $L/r = 2.5$.

$0.006 \times \bar{q}$	UD		FG-X		FG-V		FG-O	
	linear	nonlinear	linear	nonlinear	linear	nonlinear	linear	nonlinear
500	0.117	0.053	0.182	0.075	0.261	0.138	0.184	0.191
1000	0.228	0.107	0.374	0.147	0.520	0.279	0.425	0.379
1500	0.332	0.161	0.576	0.216	0.786	0.393	0.641	0.532
2000	0.442	0.215	0.751	0.282	1.04	0.513	0.850	0.682
2500	0.563	0.269	0.940	0.342	1.313	0.635	1.071	0.822

Table 7. Comparison between W_{max}/h obtained from linear and nonlinear examinations for CC boundary condition with $V_{CN}^* = 0.17$ and $L/r = 1.5$.

$0.006 \times \bar{q}$	UD		FG-X		FG-V		FG-O	
	linear	nonlinear	linear	nonlinear	linear	nonlinear	linear	nonlinear
500	0.073	0.026	0.121	0.039	0.183	0.069	0.142	0.101
1000	0.147	0.052	0.242	0.077	0.366	0.137	0.282	0.194
1500	0.222	0.079	0.375	0.112	0.556	0.210	0.432	0.285
2000	0.296	0.105	0.496	0.151	0.734	0.266	0.578	0.372
2500	0.360	0.131	0.620	0.182	0.902	0.333	0.729	0.451

Figure 8 demonstrates changes in the dimensionless deflection versus non-dimension length with various volume fractions for uniform and FG distributions of the shell with CC and SS boundary conditions and considering $r = 50cm$, $L = 100cm$, $h = 1cm$, under internal pressure $q = 10Mpa$. It can be comprehended that in the CC boundary condition, by increasing the volume fraction from 0.12 to 0.17, the highest percentage decrease of non-dimension deflection is in the V distribution but the lowest that of decrease is in the UD condition. Moreover, in the SS boundary condition the highest percentage decrease of the dimensionless deflection is related to the uniform and V distributions but the lowest that of decrease is in the X mode. By increasing the volume fraction from 0.17 to 0.28, the highest percentage decrease of the dimensionless deflection is related to the X distribution but the lowest that of reduction is in the UD for the CC boundary condition. Furthermore, in the SS boundary condition, the highest and lowest percentages decrease of dimensionless deflection are in the O and UD distributions, respectively.

Figure 8. Changes of dimensionless deflection in terms of length for four types of nanotube distributions including a) UD, b) FG-X, c) FG-V and d) FG-O in diverse volume fractions of nanotubes at CC and SS boundary conditions.

Figure 9 illustrates the maximum dimensionless deflection in terms of length-to-radius for diverse volume fractions and FG distributions in CC and SS boundary conditions in the axial-symmetric shell with $r = 50\text{cm}$, $L = 100\text{cm}$, $h = 3\text{cm}$ and exposed to internal pressure $q = 10\text{Mpa}$. As can be seen, by the increase of the length-to-radius ratio the maximum dimensionless deflection enhances. Also, (in both boundary conditions and all four distributions) with $L/r > 1$, the effect of volume fraction change on the maximum deflection increases. Also, by increasing the ratio of length-to-radius and increasing the volume fraction (for all four distributions and both boundary conditions), the increase of maximum dimensionless deflection reduces by approximately 2.8 times. According to the figures, by increasing the ratio of length-to-radius for uniform, X, V and O distributions, the highest drop in non-dimensional deflection is related to the

volume fraction of 0.12 with SS boundary condition, but the lowest that of drop is in the volume fraction of 0.28 with CC boundary condition.

Figure 9. Maximum non-dimension deflection versus dimensionless length for diverse volume fractions.

Tables 8 to 11 compares the maximum non-dimension deflection versus length-to-radius change at different boundary conditions in the axial-symmetric shell with $r = 50cm$, $h = 1cm$, and $\bar{q} = 25000$. It can be concluded that for all distributions, the highest percentage of deflection increase is related to the CC boundary condition but the lowest that of increase is in the SS

boundary condition. Moreover, in $L/r > 3$, the rate of non-dimension maximum deflection growth will reduce significantly and will be less than 1%, for all distributions.

Table 8. Non-dimension maximum deflection in terms of increasing the ratio of length-to-radius for UD distribution ($V_{CN}^* = 0.17, R = 50cm, h = 1cm$) and CC boundary condition.

L/r	1	2	3	4	5
CC	0.0309	0.172	0.258	0.264	0.254
SC	0.051	0.222	0.269	0.271	0.268
SS	0.09	0.259	0.271	0.270	0.268

Table 9. Non-dimension maximum deflection in terms of increasing the ratio of length-to-radius for FG-X distribution ($V_{CN}^* = 0.17, R = 50cm, h = 1cm$) and CC boundary condition.

L/r	1	2	3	4	5
CC	0.048	0.252	0.432	0.502	0.511
SC	0.076	0.328	0.479	0.511	0.518
SS	0.119	0.390	0.501	0.520	0.521

Table 10. Non-dimension maximum deflection in terms of increasing the ratio of length-to-radius for FG-V distribution ($V_{CN}^* = 0.17, R = 50cm, h = 1cm$) and CC boundary condition.

L/r	1	2	3	4	5
CC	0.083	0.472	0.800	0.931	0.960
SC	0.121	0.601	0.879	0.963	0.962
SS	0.200	0.739	0.936	0.989	0.990

Table 11. Non-dimension maximum deflection in terms of increasing the ratio of length-to-radius for FG-O distribution ($V_{CN}^* = 0.17, R = 50cm, h = 1cm$) and CC boundary condition.

L/r	1	2	3	4	5
CC	0.105	0.555	0.898	1.057	1.088
SC	0.161	0.670	0.969	1.086	1.103
SS	0.251	0.791	1.024	1.010	1.099

Figure 10 reveals the non-dimension stress resultant versus the dimensionless length for axial-symmetric shell with $r = 50cm, L = 100cm, h = 2cm$ and under internal pressure $p = 10Mpa$ at CC and SS boundary conditions. As can be seen, the change in the value of membrane force for

uniform and FG distributions along the shell is roughly constant. Also, for both boundary conditions, the FG-X distribution will have the highest non-dimension stress resultant but the UD has the lowest value.

Figure 10. Dimensionless stress resultant along the length of the shell in $V_{CN}^* = 0.17$; (a) CC and (b) SS boundary conditions.

Figure 11 depicts the values of the non-dimension moment versus the dimensionless length where in the SS boundary condition, the highest and lowest dimensionless moments are in the FG-V distribution and UD, respectively. Moreover, in CC boundary conditions, since the edges of the shell tolerates the moment, the highest maximum dimensionless moment is for the FG-V distribution and the lowest maximum dimensionless moment is related to the FG-X distribution.

Figure 11. Dimensionless moment resultant along the length of shell and $V_{CN}^* = 0.17$ in a) CC and b) SS boundary conditions.

5-Conclusion

This context scrutinizes the large deflection study of the FG-CNTRC cylindrical shell under internal pressure on the basis of FSDT with the aid of the DR technique with the central finite difference approach. The composite cylindrical shell reinforced along the longitudinal direction and made from a polymeric matrix. For verification, the present results are compared with available references and ABAQUS finite element package which confirm the accuracy of the obtained results. The most noteworthy results are as follows:

- With increasing internal pressure, the difference of maximum non-dimensional deflections between the FG-X and FG-V distributions increases in both CC and SS boundary conditions.
- For all distributions, with the increase of non-dimensional thickness (h/r) from 0.03 to 0.04, the decrease rate of non-dimensional deflection is about 3 times more than the increase of h/r from 0.04 to 0.05.
- As the length-to-radius ratio increases, the difference of non-dimensional deflections between the two linear and nonlinear analyzes decreases.
- By the increase of the length-to-radius ratio the maximum dimensionless deflection goes up.
- With increasing the ratio of length-to-radius and rising the volume fraction (for all four distributions and both SS and CC boundary conditions), the growth of maximum dimensionless deflection reduces by approximately 2.8 times.
- For both CC and SS boundary conditions, the FG-X and UD distributions have the highest and the lowest value of stress resultants, respectively.
- By increasing the ratio of length-to-radius for uniform, X, V and O distributions, the highest drop in non-dimensional maximum deflection is in the volume fraction of 0.12 with SS boundary condition, but the lowest that of drop is in the volume fraction of 0.28 with CC boundary condition.

- The change in the value of membrane force for uniform and FG distributions along the shell is roughly constant.
- In $L/r > 3$, the rate of non-dimensional maximum deflection growth will reduce significantly and will be less than 1%, for all distributions.
- The highest maximum dimensionless moment versus dimensionless length is in the FG-V distribution for both SS and CC boundary conditions.

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Competing interests

The authors declare that they have no competing interests.

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Figures

Figure 1

Schematic of carbon nanotubes' distributions along the longitudinal direction

Figure 2

Schematic of axial symmetric cylindrical shell

Figure 3

Comparison of results between DR technique and reference [41]

Figure 4

Convergence study for uniform and FG-X distributions.

Figure 5

Maximum dimensionless deflection in terms of dimensionless load change with $V_{CN} = 0.17$ in a) CC and b) SS boundary conditions.

Figure 6

The effect of dimensionless thickness on changes of dimensionless deflection versus non-dimensional length for 4 types of nanotube distributions including a)UD, b) FG-X, c)FG-V and d)FG-O in CC boundary condition with $V_{CN} = 0.17$.

Figure 7

The effect of dimensionless thickness on changes of dimensionless deflection versus non-dimensional length for 4 types of nanotube distributions including a)UD, b) FG-X, c)FG-V and d)FG-O in SS boundary condition with $V_{CN} = 0.17$.

Figure 8

Changes of dimensionless deflection in terms of length for four types of nanotube distributions including a) UD, b) FG-X, c) FG-V and d) FG-O in diverse volume fractions of nanotubes at CC and SS boundary conditions.

Figure 9

Maximum non-dimension deflection versus dimensionless length for diverse volume fractions.

Figure 10

Dimensionless stress resultant along the length of the shell in $V_{CN} = 0.17$; (a) CC and (b) SS boundary conditions.

Figure 11

Dimensionless moment resultant along the length of shell and $V_{CN} = 0.17$ in a) CC and b) SS boundary conditions.