Lubrication-Contact Boundary Conditions for Lubrication Problems

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Abstract

Under sever conditions, asperity contacts take place, surrounded by fluid lubrication. Mathematical descriptions are needed to describe local connection between fluid lubrication and solid contacts. In order to differentiate them from conventional boundary conditions, these conditions are called lubrication-contact boundary conditions (LCBCs), which have not been sufficiently addressed by previous studies. In this work, a set of LCBCs formulations are constructed based on local flow continuity from the continuum mechanics point of view, together with pressure inequalities. Numerical implementations are developed and tested with problems involving simple geometries, and they are expected to be integrated with mixed/boundary EHL solvers, as well as be applied to deterministic sub-models of stochastic models in order to obtain flow factors that consider solid contacts.

Keywords: Lubrication; Simulation; Boundary conditions; Asperity contact;

1. Introduction

Modelling full-film lubrication problems requires boundary conditions at the inlet, exit, and side edges, or at the interior cavitation boundaries if applicable, for the Reynolds equation to be properly solved. Some of the exit boundary conditions are summarized by the author in [1]. However, in real engineering products, mixed/boundary lubrication can occurs when speeds are low and/or loads are heavy. As a result, fluid lubrication co-exists with asperity contacts, which causes wear and reduces service lives of these products. When simulating such lubrication problems using the Reynolds equation, additional boundary conditions that reflect the local physical reality of interfaces with solid contacts on the one side and fluid lubrication on the other side become necessary. In fact, these lubrication-contact boundary conditions (LCBCs) complete the mathematical description of lubrication problems involving solid contacts.

Zhu and Wang [2] reviewed the history and progress of elastohydrodynamic lubrication (EHL) simulations, including recent significant accomplishments for relatively thin-film deterministic solutions considering real measured roughness. There are two types of rough-surface EHL models: stochastic [3-9] and deterministic [10-21]. Because stochastic models need separate deterministic sub-models to determine flow factors, deterministic models play a key role in EHL analyses.

In deterministic EHL models, there are two approaches to deal with solid contacts, both under the finite difference methodology: (1) lubrication and solid-contact regions are treated separately [10-12], in which the fluid-film pressure and solid-contact pressure variables are defined, and they are updated in sequential sweeps of an iteration process; (2) lubrication and solid-contact regions are treated with a unified equation system [13-16], where pressure variables, i.e., old and new pressure arrays during iterations, cover both lubrication and solid-contact regions, and they are updated within the same sweep in an iteration process. In the following, several representative treatments are briefly described.

Chang [10] studied transient line-contact mixed EHL problems with a deterministic model and used the treatment of “Check the film thickness at every grid point. If the film thickness is smaller than one-thousandth of the central film thickness for perfect smooth contact, h_{os} , set it to $0.001h_{os}$ ” to obtain the fluid-film pressure from the Reynolds equation for the entire domain, therefore avoided the LCBCs. Jiang et al. [11] presented a transient mixed-EHL model for point-contact problems, and their approach related to the LCBCs can be summarized as follows: when the film thickness at a node (i, j) is less than or equal to zero, (a) the film thickness is set to zero; (b) two coefficients, which are for the upstream and downstream nodes, need to be updated with zero film thickness; and (c) the fluid pressure is set to the solid-contact pressure. Zhao et al. [12] simulated circular-contact start up without explicit LCBCs, but used these constraints: (a) the fluid-film lubrication regions have positive film thicknesses, zero solid-contact pressures, and non-negative fluid-film pressure and (b) the solid-contact regions have zero film thicknesses, positive solid-contact pressures, and zero fluid-film pressures. Zhu and Hu [13] introduced a unified equation system, which uses: (a) full Reynolds equation for the lubrication regions and (b) truncated Reynolds equations (without the Poiseuille and transient terms) for solid-contact regions. Dimensionless film thickness is compared with a fixed small value to decide the truncation in (b).

This approach is practical and has been successfully applied in their subsequent work [14-18]. Holmes et al. [19] used full Reynolds equation and the elastic deflection equation in a differential form to update unknown nodal values of pressure and film thickness. Once a nodal value of film thickness is negative, it is set to zero and the nodal pressure value is updated with the elastic deflection equation only. Li and Kahraman [20] utilized the unified equation system of Zhu and Hu [13], but discretized it by means of an asymmetric integrated control volume. However, LCBCs are ignored in these work [10, 12, 14-18]. Deolalikar et al. [21] treated lubrication and solid-contact regions separately, using film thickness comparison against a fixed small value to differentiate these two kinds of regions. They explicitly applied a no-flow boundary condition to boundaries where the fluid regions are upstream of the solid-contact regions. When a control volume has this boundary as its east wall, flow through the control volume is modified. LCBCs are dealt with to some extent in these works [11, 21]. It should be noted that references [10-21] have successfully used surface gap values to determine whether a node is under fluid-film lubrication or is a solid contact, and thus temporary boundaries between film lubrication and solid contacts can be obtained before each iteration. Once the iteration process is completed, this partition becomes final.

Wang and Zhu [22] summarized their contributions to the mixed lubrication simulations using the finite difference method in a monograph. Venner and Lubrecht [23] and Habchi [24] in their monographs describe in detail how to apply the multigrid method and the finite element method, respectively, to solve EHL problems. However, so far, existing simulation work has either failed to address or has only partially dealt with the LCBCs as mentioned above.

Furthermore, Zhou et al. [25] applied EHL simulation to study lubrication between case-hardened gears, and AL-Mayali et al. [26] studied micropitting initiation with experiments and numerical simulations. Hansen et al. [27] proposed a new updated film parameter. Mohammadpour et al. [28] investigated exterior boundary conditions of point EHL contacts with a combined numerical-experimental analysis, however, the author is not aware of any experimental studies about lubrication-contact boundaries of EHL problems. Such a situation motivates a series of explorations about the LCBCs through theoretical analyses. Liu et al. [26] studied the flow continuity of line-contact EHL results and revealed a flow-continuity view of the discrete Reynolds equation. Liu [27] derived analytical solutions to several lubrication problems with flow blockages. Based on these insights and engineering principles, a set of LCBCs are

proposed in this work and numerical implementations are developed and verified with problems involving simple geometries, without the complication of elastic deformation/cavitation/wall-slip. These LCBCs are expected to be integrated with mixed/boundary EHL solvers, as well as be applied to deterministic sub-models of stochastic models in order to obtain flow factors considering solid contacts.

Fig. 1 Side view illustration of a 1D problem with a lubrication-contact boundary

2. Boundary conditions

From the continuum point of view, the boundary conditions between fluid and solid contacts are discussed in this section, and a set of boundary conditions is explicitly defined to describe the lubrication-contact interaction in a typical mixed lubrication problem. Numerical implementations of these boundary conditions will be presented next in a discrete format. This set of LCBCs includes two parts: pressure and flow. Figure 1 illustrates such boundaries where both lubrication and solid contact co-exist in a one-dimensional (1D) problem. The coordinate of the boundary is x_l , and on the two sides of it there are fluid pressure, p_f , and contact pressure, p_d , where x_l^- is the location to the left but very close to the boundary and x_l^+ is the location to the right but very close to the boundary. Engineering principles suggest that at the boundary,

$$p_d(x_l^+) \geq p_f(x_l^-) \quad (1)$$

In other words, this condition does not permit the solid pressure on the solid side of the boundary to be lower than the fluid lubrication pressure on the other side of the boundary because the lubricant with a higher pressure can penetrate and open the contact junction until the pressures are balanced. Note that when the gap approaches zero, the local situation is at the nanoscale and further

investigation might be conducted from the molecular dynamics point of view, but that is out of the scope of this work.

Figure 2 illustrates two boundaries between fluid and solid contacts in a two-dimensional (2D) problem, viewed from the top of the junction. In general, the boundary, Γ , has an arbitrary shape, and at a location, Γ_0 , there is a normal $n_{\Gamma 0}$. Boundary conditions can be expressed along the normal. However, for the purpose of a numerical simulation where the boundaries are along discrete meshes, this normal is either horizontal or vertical. Similar to the 1D problem, the solid-contact pressure in the vicinity of the boundary should not be lower than the fluid lubrication pressure on the other side of the boundary. Therefore, this kind of constraint can be expressed in the following four directions — **w**est, **e**ast, **n**orth, and **s**outh,

Fig. 2 Top view illustration of a 2D problem with a lubrication-contact boundary

$$p_d(x_{\Gamma_w}^+, y_{\Gamma_w}) \geq p_f(x_{\Gamma_w}^-, y_{\Gamma_w}) \text{ at } \Gamma_w \quad (2)$$

$$p_d(x_{\Gamma_e}^-, y_{\Gamma_e}) \geq p_f(x_{\Gamma_e}^+, y_{\Gamma_e}) \text{ at } \Gamma_e$$

$$p_d(x_{\Gamma_n}, y_{\Gamma_n}^-) \geq p_f(x_{\Gamma_n}, y_{\Gamma_n}^+) \text{ at } \Gamma_n$$

$$p_d(x_{\Gamma_s}, y_{\Gamma_s}^+) \geq p_f(x_{\Gamma_s}, y_{\Gamma_s}^-) \text{ at } \Gamma_s$$

where “+” and “-” are for slightly larger and smaller values than the corresponding coordinates defining the boundary.

Furthermore, lubrication-contact boundaries should permit no physical flows to pass through, but they allow relative surface motion. Understanding the effects of blocked fluid flow around interior contact spot(s) is the focus of this paper. In other words, there is no net flow entering into the contact junctions, nor net flow coming out from these junctions. The 1D case only has the Poiseuille and the Couette flows in the x direction, and this zero net flow constraint is expressed as

$$q(x_f^-) \equiv -\frac{h^3}{12\eta} \frac{dp_f}{dx} + \frac{uh}{2} \Big|_{x_f^-} = 0 \quad (3)$$

In fact, due to the flow continuity, zero net flow is true everywhere, and

$$\frac{dp_f}{dx} = \frac{6u\eta}{h^2} \quad (3a)$$

If the gap is continuous around the boundary of $h(x_f) = 0$, $h(x_f^-)$ can be infinitely small. Note that the pressure gradient in Eq. (3a) can be infinitely large. In reality, many more features of the lubricant itself (non-Newtonian behavior, for example) and/or interactions between the lubricant and the bonding surface (slip boundary, for example) may be considered, but those are beyond the scope of this paper.

The lubricant velocity direction in 2D problems can be arbitrary, but in this paper, the entraining-velocity direction is simply along the x axis. One can use the same treatment to handle the velocity component along the y axis. The 2D LCBCs related to flows can be expressed in the following four scenarios,

$$q(x_{\Gamma_w}^-, y_{\Gamma_w}) \equiv -\frac{h^3}{12\eta} \frac{dp_f}{dx} + \frac{uh}{2} \Big|_{x_{\Gamma_w}^-, y_{\Gamma_w}} = 0 \quad \text{at } \Gamma_w \quad (4)$$

$$q(x_{\Gamma_e}^+, y_{\Gamma_e}) \equiv -\frac{h^3}{12\eta} \frac{dp_f}{dx} + \frac{uh}{2} \Big|_{x_{\Gamma_e}^+, y_{\Gamma_e}} = 0 \quad \text{at } \Gamma_e$$

$$q(x_{\Gamma_n}, y_{\Gamma_n}^+) \equiv -\frac{h^3}{12\eta} \frac{dp_f}{dy} \Big|_{x_{\Gamma_n}, y_{\Gamma_n}^+} = 0 \quad \text{at } \Gamma_n$$

$$q(x_{\Gamma_s}, y_{\Gamma_s}^-) \equiv -\frac{h^3}{12\eta} \frac{dp_f}{dy} \Big|_{x_{\Gamma_s}, y_{\Gamma_s}^-} = 0 \quad \text{at } \Gamma_s$$

If there is an entraining-velocity component of v along the y axis, at Γ_n and Γ_s boundaries, $\frac{vh}{2}$ should be added to the last two equations. These are all the necessary LCBCs from the continuum point of view to describe lubrication problems involving lubrication-contact boundaries.

3. Numerical Implementation

In this section, the newly proposed LCBCs for lubrication problems involving contacts are implemented in a discrete format so that they can be incorporated into a numerical algorithm.

3.1 1D problems

The dimensionless Reynolds equation can be expressed as

$$\frac{d}{dX} \left(\varepsilon \frac{dP_f}{dX} \right) = \frac{d(\rho^* H)}{dX} \quad (5)$$

and the dimensionless mass flow can be written as

$$Q \equiv \frac{qR}{\rho_0 u b^2} = \rho^* H - \varepsilon \frac{dP_f}{dX} \quad (6)$$

Which has the Couette flow

$$Q^C = \rho^* H \quad (6a)$$

and the Poiseuille flow

$$Q^P = -\varepsilon \frac{dP_f}{dX} \quad (6b)$$

For the derivative of the Poiseuille flow—the left-hand side term of the Reynolds equation, the second-order central differential scheme is usually applied,

$$\frac{\varepsilon_{i-0.5} P_{f,i-1} - (\varepsilon_{i-0.5} + \varepsilon_{i+0.5}) P_{f,i} + \varepsilon_{i+0.5} P_{f,i+1}}{\Delta X^2} \quad (7)$$

For the derivative of the Couette flow—the right-hand side of the Reynolds equation, one can choose from three basic schemes: the 1st-order backward (1B), 2nd-order central (2C), or 2nd-order backward (2B) [22-23, 31]:

(a) 1B scheme

$$\frac{\rho_i^* H_i - \rho_{i-1}^* H_{i-1}}{\Delta X} \quad (8)$$

(b) 2B scheme

$$\frac{3\rho_i^* H_i - 4\rho_{i-1}^* H_{i-1} + \rho_{i-2}^* H_{i-2}}{2\Delta X} \quad (9)$$

(c) 2C scheme

$$\frac{\rho_{i+1}^* H_{i+1} - \rho_{i-1}^* H_{i-1}}{2\Delta X} \quad (10)$$

Moreover, the derivative of the Couette flow can also be expressed in a separate form [22, 31, 32],

$$\frac{d(\rho^* H)}{dX} = \rho^* \frac{dH}{dX} + H \frac{d\rho^*}{dX} \quad (11)$$

which can be discretized with the following options (“S” in the notation stands for “separate”):

i. S1B scheme: If the 1B scheme is used for the derivatives and ρ^* and H take values at i ,

$$\begin{aligned} & H_i \frac{\rho_i^* - \rho_{i-1}^*}{\Delta X} + \rho_i^* \frac{H_i - H_{i-1}}{\Delta X} \\ &= \frac{\rho_i^* H_i - \rho_{i-1}^* H_{i-1}}{\Delta X} + \frac{(\rho_i^* - \rho_{i-1}^*)(H_i - H_{i-1})}{\Delta X} \end{aligned} \quad (12)$$

which has one more term than what is in Eq. (8). This scheme is used in Eq. (11) of [32].

ii. If the 1B scheme is used for the derivatives but ρ^* and H take averages values between i and $i-1$, one obtains the same equation as Eq. (8).

iii. S2C scheme: If the 2C scheme is used for the derivatives and ρ^* and H take values at i ,

$$\begin{aligned} & H_i \frac{\rho_{i+1}^* - \rho_{i-1}^*}{2\Delta X} + \rho_i^* \frac{H_{i+1} - H_{i-1}}{2\Delta X} \\ &= \frac{\rho_{i+1}^* H_{i+1} - \rho_{i-1}^* H_{i-1}}{2\Delta X} + \frac{(\rho_i^* - \rho_{i-1}^*)(H_i - H_{i-1})}{2\Delta X} - \frac{(\rho_{i+1}^* - \rho_i^*)(H_{i+1} - H_i)}{2\Delta X} \end{aligned} \quad (13)$$

which has two more terms than what is in Eq. (10).

iv. If the 2C scheme is used for the derivatives with ρ^* and H taking the averages values between $i+1$ and $i-1$, one obtains the same equation as Eq. (10).

$$\begin{aligned} & \frac{H_{i+1} + H_{i-1}}{2} \frac{\rho_{i+1}^* - \rho_{i-1}^*}{2\Delta X} + \frac{\rho_{i+1}^* + \rho_{i-1}^*}{2} \frac{H_{i+1} - H_{i-1}}{2\Delta X} \\ &= \frac{\rho_{i+1}^* H_{i+1} + \rho_{i+1}^* H_{i-1} - \rho_{i-1}^* H_{i+1} - \rho_{i-1}^* H_{i-1} + \rho_{i+1}^* H_{i+1} + \rho_{i-1}^* H_{i+1} - \rho_{i+1}^* H_{i-1} - \rho_{i-1}^* H_{i-1}}{4\Delta X} \\ &= \frac{\rho_{i+1}^* H_{i+1} - \rho_{i-1}^* H_{i-1}}{2\Delta X} \end{aligned} \quad (14)$$

v. Using the 2B scheme for the derivatives with ρ^* and H taking averages values between $i+1$ and $i-1$, one obtains,

$$\frac{H_{i+1} + H_{i-1}}{2} \frac{3\rho_i^* - 4\rho_{i-1}^* + \rho_{i-2}^*}{2\Delta X} + \frac{\rho_{i+1}^* + \rho_{i-1}^*}{2} \frac{3H_i - 4H_{i-1} + H_{i-2}}{2\Delta X}$$

vi. Using the 2B scheme for the derivatives with ρ^* and H take values at i , one has,

$$\mathbf{H}_i \frac{3\rho_i^* - 4\rho_{i-1}^* + \rho_{i-2}^*}{2\Delta X} + \rho_i^* \frac{3H_i - 4H_{i-1} + H_{i-2}}{2\Delta X}$$

Liu et al. [29] presented a flow-continuity view of the discrete Reynolds equation, which is extended in this work to handle the LCBC. Regardless of which differential scheme is used for the derivative of the Couette flow, the discrete Reynolds equation can be expressed in four flow terms as follows,

$$Q_{i-0.5}^P + Q_{i-0.5}^C = Q_{i+0.5}^P + Q_{i+0.5}^C \quad (15)$$

where the two terms of flow leaving the node of i are,

$$Q_{i+0.5}^P = -\frac{\varepsilon_{i+0.5}(P_{f,i+1} - P_{f,i})}{\Delta X} \quad (16)$$

$$Q_{i+0.5}^C \equiv \begin{cases} \rho_i^* H_i, & 1B \\ \frac{3\rho_i^* H_i - \rho_{i-1}^* H_{i-1}}{2}, & 2B \\ \frac{\rho_i^* H_i + \rho_{i+1}^* H_{i+1}}{2}, & 2C \\ \rho_i^* H_i + (\rho_i^* - \rho_{i-1}^*) H_i, & S1B \\ \frac{\rho_i^* H_i + \rho_{i+1}^* H_{i+1}}{2} - \frac{(\rho_{i+1}^* - \rho_i^*)(H_{i+1} - H_i)}{2}, & S2C \end{cases} \quad (17)$$

and the two terms of flow entering the node of i , $Q_{i-0.5}^P$ and $Q_{i-0.5}^C$, are obtained by replacing i with $i-1$, except no such a replacement for the subscripts of $\rho_i^* - \rho_{i-1}^*$ in the S1B, e.g.,

$$Q_{i-0.5}^C \equiv \begin{cases} \rho_{i-1}^* H_{i-1}, & 1B \\ \frac{3\rho_{i-1}^* H_{i-1} - \rho_{i-2}^* H_{i-2}}{2}, & 2B \\ \frac{\rho_i^* H_i + \rho_{i-1}^* H_{i-1}}{2}, & 2C \\ \rho_{i-1}^* H_{i-1} + (\rho_i^* - \rho_{i-1}^*) H_{i-1}, & S1B \\ \frac{\rho_i^* H_i + \rho_{i-1}^* H_{i-1}}{2} - \frac{(\rho_i^* - \rho_{i-1}^*)(H_i - H_{i-1})}{2}, & S2C \end{cases} \quad (18)$$

Furthermore, for the S1B scheme, one could use $\rho_i^* H_i + \rho_i^* (H_i - H_{i-1})$ in Eq. (17) and $\rho_{i-1}^* H_{i-1} + \rho_{i-1}^* (H_i - H_{i-1})$ in Eq. (18). Unlike options i-iv, options v and vi can not be further simplified, so if chosen, they can be treated as two terms with the 2B schemes.

Note that for the Couette flow Q^c , the 2C and S2C schemes of Eq. (17) have subscripts of i and $i+1$ on the right-hand side, and the subscript of $i + 0.5$ on the left-hand side is their center. Similarly, the 2C and S2C schemes of Eq. (18) have subscripts of i and $i-1$ on the right-hand side,

and the subscript of $i - 0.5$ on the left-hand side is their center. But for the other schemes, subscripts on the right-hand sides, except those of the bold font are smaller than the subscripts in the left-hand side. For example, the 1B scheme of Eq. (17) has a subscript of i on the right-hand side, which is smaller than the subscript of $i + 0.5$ on the left-hand side. Therefore, if subscript values increase from the left to the right in the discretization (Fig. 3), the differential schemes of 1B, 2B, or S1B use Couette flow values slightly to the left to approximate the Couette flow terms needed at the two ends of the element (In Fig. 3, the thick line represents one element and these Couette flow terms are illustrated with hollow arrows).

The discrete 1D lubrication problem involving contacts is shown in Fig. 3. Each node represents a length that is one half intervals in front of and behind the node, represented by the thick line on the horizontal axis in Fig. 3. Note that if the boundary location, X_r , is known, one could arrange two nodes with X_r right at the center. However, X_r could be unknown at the beginning and change during iterations. In a discrete mesh, if X_r is located between two neighboring nodes, the center of them is used to approximate X_r . Of course, a smaller interval between nodes (finer mesh in other words) could reduce error of such approximation. A dashed line is used in Fig. 3 to represent the local geometry. i_r is used to represent the fluid lubrication node closest to the discrete boundary.

Fig. 3 Illustration of the lubrication-contact boundary in a 1D case, filled circles--lubrication; empty circles--solid contact, with p_f and p_d for the fluid and contact pressures, respectively.

In the following, the no-flow constraint is integrated into the discrete Reynolds equation in order to determine the pressure at i_r . If the film thickness at H_{i+1} is the first zero value tracing from inlet, the discrete boundary is between node i and node $i + 1$. The no flow boundary condition at the **$i+0.5$ node** is $Q_{i+0.5}^P + Q_{i+0.5}^C = 0$, reducing Eq. (15) to,

$$Q_{i-0.5}^P + Q_{i-0.5}^C = 0 \quad (19)$$

Therefore, based on the differential schemes for the Couette flow, the following can be obtained

$$(a) \text{ 1B: } P_{f,i} = P_{f,i-1} + \frac{\rho_{i-1}^* H_{i-1} \Delta X}{\varepsilon_{i-0.5}} \quad (20)$$

$$(b) \text{ 2B: } P_{f,i} = P_{f,i-1} + \frac{(3\rho_{i-1}^* H_{i-1} - \rho_{i-2}^* H_{i-2}) \Delta X}{2\varepsilon_{i-0.5}} \quad (21)$$

$$(c) \text{ 2C: } P_{f,i} = P_{f,i-1} + \frac{(\rho_i^* H_i + \rho_{i-1}^* H_{i-1}) \Delta X}{2\varepsilon_{i-0.5}} \quad (22)$$

$$(d) \text{ S1B: } P_{f,i} = P_{f,i-1} + \frac{\rho_i^* H_{i-1} \Delta X}{\varepsilon_{i-0.5}} \quad (23)$$

$$(e) \text{ S2C: } P_{f,i} = P_{f,i-1} + \frac{(\rho_i^* H_{i-1} + \rho_{i-1}^* H_i) \Delta X}{2\varepsilon_{i-0.5}} \quad (24)$$

Since the net flow is zero everywhere, these recursion formulae are also valid for other nodes in the lubrication region. Once the lubrication and contact have been solved, at that **i_r node**, the following inequality needs to be true,

$$P_{f,i_r} \leq P_{d,i_r+1} \quad (25)$$

Otherwise, the location of the boundary has to be adjusted.

3.2 2D problems

For 2D problems, assuming that the motion is along the horizontal axis for convenience, Fig. 4 shows a discrete mixed lubrication example, where the grey box is a representative boundary in the continuous space. In the discrete space, once a node is in contact, the material inside a rectangle area around it is in contact and the same is true for the nodes in fluid lubrication. The dashed-line rectangle around node (i, j) is a control volume, and the solid black-line box is the lubrication-contact boundary which connects the middle points between fluid and solid-contact nodes.

3.2.1 Nodes in the normal lubrication region

The 2D dimensionless Reynolds equation can be expressed as

$$\frac{\partial}{\partial X} \left(\varepsilon \frac{\partial P_f}{\partial X} \right) + \frac{\partial}{\partial Y} \left(\varepsilon \frac{\partial P_f}{\partial Y} \right) = \frac{\partial(\rho^* H)}{\partial X} \quad (26)$$

Fig. 4 (a) 2D lubrication-contact boundaries, filled circles—lubrication nodes; empty circles—solid contact nodes. Five cases are illustrated, which are a fluid case and four boundary cases at the east, west, south and north sides of the contact spot marked by the black-line box. (b) flow terms with various schemes. Solid lines are used to guide the additions of the Couette flow to the Poiseuille flow on the west and east sides of the control volumes.

and the dimensionless mass flows in the X and Y directions can be written as

$$Q_X \equiv \frac{q_X R}{\rho_0 u b^2} = \rho^* H - \varepsilon \frac{\partial P_f}{\partial X} = Q_X^C + Q_X^P \quad (27)$$

$$Q_Y \equiv \frac{q_Y R}{\rho_0 u b^2} = -\varepsilon \frac{\partial P_f}{\partial Y} = Q_Y^P \quad (28)$$

Descriptions of the differential and finite difference schemes in 1D problems can be expanded. If node (i, j) and its surrounding four nodes are all under fluid lubrication (see the illustration in the top-right of Fig. 4), the discrete Reynolds equation for such a node has a standard form. Equation (7) is still valid for the Poiseuille flow in the X direction, and the Poiseuille flow in the Y direction has a similar expression,

$$\frac{\varepsilon_{i,j-0.5} P_{f,i,j-1} - (\varepsilon_{i,j-0.5} + \varepsilon_{i,j+0.5}) P_{f,i,j} + \varepsilon_{i,j+0.5} P_{f,i,j+1}}{\Delta Y^2} \quad (29)$$

where $P_{f,i,j}$ is the fluid pressure at the node of (i, j) . Totally five nodes are involved in these discrete equations. The Poiseuille flows in both directions can be expressed as

$$Q_{X,i-0.5,j}^P = -\frac{\varepsilon_{i-0.5,j} (P_{f,i,j} - P_{f,i-1,j})}{\Delta X} \quad (30)$$

$$Q_{Y,i,j-0.5}^P = -\frac{\varepsilon_{i,j-0.5} (P_{f,i,j} - P_{f,i,j-1})}{\Delta Y}$$

Those different differential schemes for the Couette flow in Eqs. (8-14) can be readily applied here with an additional subscript of j . Similar to Eqs. (17-18), the following definitions of the Couette flow can be made for different differential schemes,

(a) Couette flow leaving the control volume

$$Q_{X,i+0.5,j}^C \equiv \begin{cases} \rho_{i,j}^* H_{i,j} & , & 1B \\ (3\rho_{i,j}^* H_{i,j} - \rho_{i-1,j}^* H_{i-1,j})/2 & , & 2B \\ (\rho_{i,j}^* H_{i,j} + \rho_{i+1,j}^* H_{i+1,j})/2 & , & 2C \\ \rho_{i,j}^* H_{i,j} + (\rho_{i,j}^* - \rho_{i-1,j}^*) H_{i,j} & , & S1B \\ \frac{\rho_{i,j}^* H_{i,j} + \rho_{i+1,j}^* H_{i+1,j}}{2} - \frac{(\rho_{i+1,j}^* - \rho_{i,j}^*)(H_{i+1,j} - H_{i,j})}{2} & , & S2C \end{cases} \quad (31)$$

(b) Couette flow entering the control volume

$$Q_{X,i-0.5,j}^C \equiv \begin{cases} \rho_{i-1,j}^* H_{i-1,j} & , & 1B \\ (3\rho_{i-1,j}^* H_{i-1,j} - \rho_{i-2,j}^* H_{i-2,j})/2 & , & 2B \\ (\rho_{i,j}^* H_{i,j} + \rho_{i-1,j}^* H_{i-1,j})/2 & , & 2C \\ \rho_{i-1,j}^* H_{i-1,j} + (\rho_{i,j}^* - \rho_{i-1,j}^*) H_{i-1,j} & , & S1B \\ \frac{\rho_{i,j}^* H_{i,j} + \rho_{i-1,j}^* H_{i-1,j}}{2} - \frac{(\rho_{i,j}^* - \rho_{i-1,j}^*)(H_{i,j} - H_{i-1,j})}{2} & , & S2C \end{cases} \quad (32)$$

For the S1B scheme, one can also use $\rho_{i,j}^* H_{i,j} + \rho_{i,j}^* (\mathbf{H}_{i,j} - \mathbf{H}_{i-1,j})$ in Eq. (31) and $\rho_{i-1,j}^* H_{i-1,j} + \rho_{i-1,j}^* (\mathbf{H}_{i,j} - \mathbf{H}_{i-1,j})$ in Eq. (32). Note that subscripts in the right hand side of Eqs. (31-32) are also slightly shifted from the subscript of Q^c (see Fig. 4b). In other words, the right-hand side of the expressions in Eq. (31) are:

- (i) independent of the node $(i+1, j)$ when the 1B, 2B, or S1B scheme is selected. When node $(i+1, j)$ is in solid contact ($H_{i+1,j} = 0$), these expressions are for the Couette flow leaving the control volume, which are often overlooked in literatures about the mixed lubrication.
- (ii) related to the node $(i+1, j)$ when the 2C or S2C scheme is used. It should be noted that the conventional 2C scheme uses $Q_{X,i+0.5,j}^C - Q_{X,i-0.5,j}^C = (\rho_{i+1,j}^* H_{i+1,j} - \rho_{i-1,j}^* H_{i-1,j})/2$ without differentiating which part is on the left/right boundary. When node $(i+1, j)$ is in solid contact ($H_{i+1,j} = 0$), both the conventional way and Eq. (31) gives the same change of the Couette flow, but Eq. (31) is able to show there is a partial Couette flow leaving the control volume.

On the other hand, for the right-hand side of Eq. (32), if node $(i-1, j)$ is in solid contact:

- (a) the 1B/S1B scheme can naturally shut off the Couette flow entering the control volume.
- (b) the 2B scheme has node $(i-2, j)$ involved, and if this node is in lubrication, the Couette flow entering the control volume cannot be naturally shut off; otherwise, the 2B scheme can naturally shut off the Couette flow entering the control volume.
- (c) If the 2C/S2C scheme is used, similar to the case (ii) above, Eq. (31) is able to show there is a partial Couette flow to enter the control volume.

Therefore, Eqs. (17-18, 31-32) are critical to the discrete Reynolds equation for the lubrication problems involving contacts, since they properly identify the Couette flow terms entering and leaving control volumes, thus enabling treatments of the Couette flow around the lubrication-contact boundaries.

The discrete Reynolds equation can be expressed in the form of flow continuity as follows,

$$Q_{X,i-0.5,j}^P + Q_{X,i-0.5,j}^C + \kappa Q_{Y,i,j-0.5}^P = Q_{X,i+0.5,j}^P + Q_{X,i+0.5,j}^C + \kappa Q_{Y,i,j+0.5}^P \quad (33)$$

where $\kappa = \frac{\Delta X}{\Delta Y}$. After combining the Poiseuille and the Couette flow, the above can be concisely written as

$$Q_{X,i-0.5,j} + \kappa Q_{Y,i,j-0.5} = Q_{X,i+0.5,j} + \kappa Q_{Y,i,j+0.5} \quad (34)$$

In total, four flows are involved in these discrete equations. When one or more of these four surrounding nodes is under solid contact, these equations should be adjusted according to the LCBCs mentioned in section 2. In the following, different cases are discussed in detail.

3.2.2 Control volumes with one side on the boundary

Figure 4 depicts four cases with nodes whose control volumes have one side on the boundary in the 2D problem. For simplicity, each representative node is labelled as (i, j) . No flow penetrates these boundary sides, so that the corresponding flow related to that side of such a control volume should be eliminated in the discrete Reynolds equation, which is detailed as follows:

(1) West: $Q_{X,i+0.5,j} = 0$

(2) East: $Q_{X,i-0.5,j} = 0$

(3) North: $Q_{Y,i,j-0.5} = 0$

(4) South: $Q_{Y,i,j+0.5} = 0$

3.2.3 Control volumes with two connecting sides on the boundary

Figure 5 shows four cases where the control volumes have two sides on the boundary. No flow penetrates these boundary sides. These cases are a combination of two cases in Section 3.2.2. According to their orientation, they are labeled as northwest, northeast, southwest, southeast, as shown below:

(i) Northwest--combination of north and west sides

$$Q_{X,i+0.5,j} = 0 \text{ and } Q_{Y,i,j-0.5} = 0$$

(ii) Northeast--combination of north and east sides

$$Q_{X,i-0.5,j} = 0 \text{ and } Q_{Y,i,j-0.5} = 0,$$

(iii) Southwest--combination of south and west sides

$$Q_{X,i+0.5,j} = 0 \text{ and } Q_{Y,i,j+0.5} = 0$$

(iv) Southeast--combination of south and east sides

$$Q_{X,i-0.5,j} = 0 \text{ and } Q_{Y,i,j+0.5} = 0$$

Fig. 5 2D cases for two sides on the boundary, corners and parallel sides. Filled circles--lubrication; empty circles--solid contact.

3.2.4 Control volumes with two parallel sides on the boundary

Figure 5 also shows two cases where the control volumes have two parallel sides on the boundary. No flow penetrates these boundary sides. Each of these cases is also a combination of two cases in Section 3.2.2. According to their orientations, they are labelled as the horizontal and the vertical channels.

(v) If this node is in the *horizontal channel*, $Q_{Y,i,j-0.5} = 0$ and $Q_{Y,i,j+0.5} = 0$

(vi) If this node is in the *vertical channel*, $Q_{X,i-0.5,j} = 0$ and $Q_{X,i+0.5,j} = 0$

3.2.5 Control volumes with three sides on the boundary

The control volume may have three sides on the boundary. No flow penetrates these boundary sides, and as a result, no flow can go through the non-boundary side either. These cases are combinations of three cases in Section 3.2.2. According to their orientations, they are labeled as west open, north open, east open, and south open. If this control volume has only the following side open to the flow (which is, of course, also zero),

(1) West open: $Q_{X,i+0.5,j} = 0$, $Q_{Y,i,j-0.5} = 0$ and $Q_{Y,i,j+0.5} = 0$

(2) East open: $Q_{X,i-0.5,j} = 0$, $Q_{Y,i,j-0.5} = 0$ and $Q_{Y,i,j+0.5} = 0$

(3) North open: $P_{f,i,j+1} = P_{f,i,j}$

(4) South open: $P_{f,i,j-1} = P_{f,i,j}$

3.2.6 Simplified treatments

Sections 3.2.2 to 3.2.5 described the 2D Reynolds equations modified by the LCBCs, when at least one of the surrounding nodes is in solid contact. There are fourteen cases in total. One could directly code them into a computer program. Alternatively, one could handle them in simplified ways. The discrete Reynolds equation is written as,

$$-\frac{\varepsilon_{i-0.5,j}(P_{f,i,j}-P_{f,i-1,j})}{\Delta X} + \frac{\varepsilon_{i+0.5,j}(P_{f,i+1,j}-P_{f,i,j})}{\Delta X} - \frac{\varepsilon_{i,j-0.5}(P_{f,i,j}-P_{f,i,j-1})}{\Delta Y} + \frac{\varepsilon_{i,j+0.5}(P_{f,i,j+1}-P_{f,i,j})}{\Delta Y} = Q_{X,i+0.5,j}^C - Q_{X,i-0.5,j}^C \quad (35)$$

where the definitions of the Couette flow terms, $Q_{X,i+0.5,j}^C$ and $Q_{X,i-0.5,j}^C$, are given in Eqs. (31-32). $P_{f,i,j}$ is used for fluid and $P_{d,i,j}$ for solid contact pressures.

In a point-by-point pressure evaluation process, if all four surrounding nodes are in lubrication, no treatment is needed and

$$P_{f,i,j} = [Q_{X,i-0.5,j}^C - Q_{X,i+0.5,j}^C + \frac{\varepsilon_{i-0.5,j}P_{f,i-1,j} + \varepsilon_{i+0.5,j}P_{f,i+1,j}}{\Delta X} + \kappa \frac{\varepsilon_{i,j-0.5}P_{f,i,j-1} + \varepsilon_{i,j+0.5}P_{f,i,j+1}}{\Delta Y}] / [\frac{\varepsilon_{i-0.5,j} + \varepsilon_{i+0.5,j}}{\Delta X} + \kappa \frac{\varepsilon_{i,j-0.5} + \varepsilon_{i,j+0.5}}{\Delta Y}] \quad (36)$$

One could re-arrange the terms in the Couette flow related to $P_{f,i,j}$, and increase the denominator thus make iteration more stable [23, 31]. Otherwise, i.e., if any one of the surrounding nodes is in a solid contact region (zero gap), Eq. (35) needs to be modified to nullify the Poiseuille flow in that direction as follows:

- method A: values of ε are directly adjusted. For example, if node $(i-1, j)$ is under a solid contact, $\varepsilon_{i-0.5,j} = 0$;
- method B: the pressure variable(s) at those surrounding nodes are set to be the central pressure value at the last iteration. For example, if node $(i-1, j)$ is under a solid contact, $P_{f,i-1,j}$ equals the value of $P_{f,i,j}$ at the last iteration.

Furthermore, the Couette flow terms in Eq. (35) have to be adjusted: If one of surrounding contact nodes is on the west side, $Q_{X,i-0.5,j}^C$ is set to zero. Similarly, if one of surrounding contact nodes is on the east side, $Q_{X,i+0.5,j}^C$ is set to zero. After these necessary treatments, Eq. (36) can then be used to obtain the updated pressure at the current node of the control volume.

If a line-by-line pressure evaluation is executed, one can use this format [22, 31]

$$\alpha_i P_{f,i-1,j} + \beta_i P_{f,i,j} + \gamma_i P_{f,i+1,j} - \varphi_i = 0 \quad (37)$$

where α_i , β_i , γ_i , and φ_i are coefficients. These coefficients need to be modified accordingly when one or more surrounding nodes are in a solid contact region to make sure both the Poiseuille and Couette flow terms are properly removed.

When a single array of $P_{i,j}$ is used for both fluid and solid contact pressures and a point-by-point pressure evaluation process is used, one may need to define four temporary variables P_e , P_s , P_w , P_n to replace $P_{f,...,j}$ in Eqs. (35-36). Thus,

$$P_{i,j} = [Q_{X,i-0.5,j}^C - Q_{X,i+0.5,j}^C + \frac{\varepsilon_{i-0.5,j} P_w + \varepsilon_{i+0.5,j} P_e}{\Delta X} + \kappa \frac{\varepsilon_{i,j-0.5} P_s + \varepsilon_{i,j+0.5} P_n}{\Delta Y}] / [\frac{\varepsilon_{i-0.5,j} + \varepsilon_{i+0.5,j}}{\Delta X} + \kappa \frac{\varepsilon_{i,j-0.5} + \varepsilon_{i,j+0.5}}{\Delta Y}] \quad (38)$$

3.2.7 Pressure constraints

During iteration, a node can change between film lubrication and solid contact. If a node in solid contact has one, two, three, or even four surrounding nodes in lubrication (in Fig. 4 and 5, there are various cases illustrated), its solid contact pressure value should be larger or equal to these fluid-lubrication pressure values. Otherwise, this node may very likely become a node in lubrication in the subsequent iteration. In a final result, these pressure constraints have to be satisfied.

In summary, various modifications, detailed in sections 3.2.2 to 3.2.5, have been derived to satisfy the lubrication-contact boundary conditions and the discrete Reynolds equation. These modifications are applied in the following section to obtain numerical solutions.

4. Numerical results and discussion

Steady-state lubrication problems are used to explain the numerical implementation and validate the proposed lubrication-contact boundary conditions in term of flow constraints without evaluating solid-contact pressures. The pressure constraints are not implemented in this section, but can be activated for EHL problems once solid-contact pressures are available. Due to their simplicity, 1D problems are not included here.

A fixed incline pad bearing is shown in Figure 6 with a rigid block. It is assumed that (a) the bottom plate is flat and moving along the X axis with a constant velocity of u , and the top pad is stationary; (b) no cavitation around the rigid block; (c) no leakage occurs between the rigid block and the moving plate, and (d) there is no wall slip between lubricant and the plate/pad surfaces. Lubricant density ρ and viscosity η are constant and solid bodies are rigid in this work. This is a lubrication problem governed by a two-dimensional Reynolds equation,

$$\frac{\partial}{\partial X} \left(\frac{h^3}{\eta} \frac{\partial p}{\partial X} \right) + \frac{\partial}{\partial Y} \left(\frac{h^3}{\eta} \frac{\partial p}{\partial Y} \right) = -6u \frac{\partial h}{\partial X} \quad (40)$$

where $h = mX$ and m is the slope. A modified pressure is defined as,

$$P = \frac{m^2 p}{\eta u} \quad (41)$$

which makes Eq. (40) independent of m . Because the 1B scheme performs better than the other schemes in the mixed lubrication simulation [31], it is adopted in this Section with a mesh of 128 by 128 nodes, so the nodal coordinates are

$$n = 128, \Delta X = \frac{l}{n-1.5}, X_i = X_0 + \frac{l(2i-3)}{2n-3}$$

$$\Delta Y = \frac{b}{n-1}, Y_i = -\frac{b}{2} + \frac{b(i-1)}{n-1}$$

The first node is deliberately located outside of the plate so that the middle point between the first and second nodes in the x direction is the edge of the plate, x_0 . The convergence criterion of

$$\zeta^p = \frac{\sum |P^{new} - P^{old}|}{\sum |P^{new}|} \leq e^p \quad (48)$$

is applied to judge convergence, and the error of pressure, e^p , is 4E-5.

4.1 Blockage with a rigid square column

Fig. 6 Rigid blockage inside a fixed incline pad bearing

A rigid square column with side lengths of 8mm and a center at (35, 0) locates inside the fixed inclined pad bearing with all sides at the ambient pressure (Fig. 6). This column is an approximation of a large debris caught in the bearing. The width and length of the plate are 20 and 30 mm, respectively. The coordinates of X_0 and X_1 are 20 and 50 mm, respectively. There is no analytical solution for this problem. With LCBCs, numerical results of the modified fluid pressure are shown in Fig. 7 over half of the plate due to symmetry.

Fig. 7 Modified fluid pressures when a square column (gray region) is the fixed incline pad bearing

4.2 Blockage with a rigid cylinder pillar

In this example, the square column in Fig. 6 is replaced with a cylinder pillar, which has a radius of 4mm, and its center is at (35, 0). The modified fluid pressure result is depicted in Fig. 8 with the same color map as in Fig. 7.

Fig. 8 Modified fluid pressure for a cylindrical pillar with a white line indicating its edge

4.3 Comparisons

These fluid pressures along the centerline at $Y = 0$ are shown together in Fig. 9. The long dashed line (classic) is the traditional solution to the problem of a tilted plate. When there are rigid blockages inside the simulation region, comparing to the traditional solution, the corresponding pressures have much higher build-up values between the inlet and the blockage, and much lower values between the blockage and the exit. The square column builds a higher highest pressure right in front of the blockage than the cylindrical pillar, as expected. When the exit side is completely blocked, an analytical solution has been derived by Liu [30] and the corresponding values at selected locations labeled by “B-Exit ana.” in the legend are shown in Fig. 9 with filled square marks. Numerical results obtained by the iteration process and labeled by “B-Exit num.”

are depicted too and they match the analytical solution very well. One can see clearly in the region close to the inlet that the ranking of pressure values from high to low are square-column blockage, cylinder-pillar blockage, exit blockage, and no blockage.

Fig. 9 Comparison of modified pressures at $Y = 0$

In order to demonstrate the effects of the treatment of the Couette flow based on the proposed LCBCs, Fig. 10a depicts some comparisons. The label “NTCF” means “no treatment of the Couette flow” and such results nullify the Poiseuille flow for nodes on the lubrication side, adjacent to the lubrication-contact boundaries, but do not have any necessary treatments about the Couette flows for those nodes. As expected, these results have much higher pressure values close to the exit, indicating some Couette flows are unintentionally allowed from the solid contact zone to the lubrication zone (see the arrow) in the 1B scheme. The extra Couette flows and their effects are significant in these examples because the local gap values are large. Also as expected, the pressure profiles around the inlet are very close, because the Couette flows there are naturally handled by the 1B scheme.

Fig. 10 Comparisons of modified pressures. “NTCF” treatment with the 1B scheme in fact allows the Couette flow exchange (see arrows) when the fluid lubrication region is on the left-hand side of the solid contact region

The same conditions are simulated with a different configuration in Fig. 10b, where the bearing rotates 180 degree with respect to the vertical axis of the blockage. So the inlet is on the left hand side and velocity is from left to right. In this configuration, the pressure build-up values in the inlet region from the “NTCF” treatment is much lower, because the 1B scheme allows some Couette flows unintentionally from the lubrication zone to the solid contact zone (see the arrow). In summary, the “NTCF” treatment with the 1B scheme in fact allows the Couette flow crossing the lubrication-contact boundaries (see arrows) when the fluid lubrication region is on the left hand side of the solid contact region. A denser mesh with 256 by 256 nodes gives very similar pressure values.

4.4 Cone-shaped pad bearing

In order to test how LCBCs affects pressure around asperities in mixed/boundary lubrication, a cone-shaped pad bearing (Fig. 11) is simulated with 2 different meshes: 128 by 128 and 256 by 256. The lower plate has the same dimensions as the plate in Fig. 6. The bottom of the cone has the same radius as in the cylinder pillar (4mm), and the same center at (35, 0). The cone surface is expressed as,

$$h = \begin{cases} m \left[\sqrt{(X - 35)^2 + Y^2} - 4 \right], & \sqrt{(X - 35)^2 + Y^2} > 4 \\ 0, & \text{otherwise} \end{cases} \quad (49)$$

Fig. 11 Cone-shaped pad bearing

Figure 12 depicts comparison of modified pressures of two meshes at two cross sections. When a node is not close to the boundary where the gap is zero, similar numerical results are obtained independent of mesh size. However, for the nodes that are close to the boundary, finer mesh produces much higher pressure values. This could be physically true to some extent, however, when the gap is around several nanometers, lubricant is not a continuum and the Reynolds equation is not valid. For these regions with ultra-small gaps right around the lubrication-contact boundaries, a multi-scale approach together with molecular dynamics at the nanoscale [33], should be developed to understand the true story there.

Fig. 12 Comparison of modified pressures of two meshes at two cross sections:

(a) at $Y = 0$; (b) at X close to 39 (dotted line in fig. 11)

5. Conclusions

Lubrication-contact boundary conditions (LCBCs) between fluid and solid contacts in lubrication problems are investigated because of their importance to numerical simulations. Expressions for Couette flows entering or leaving a control volume have been clearly defined for several differential schemes. This work proposes a set of lubrication-contact boundary conditions, including nullifying flows and pressure inequalities, and completes the mathematical description of lubrication problems involving solid contacts. Numerical implementations are developed and numerical results with the first backward scheme as an example are tested with problems involving simple geometries. Numerical results with this type of boundary conditions, mainly the treatment

of the Couette flow around boundaries, satisfactorily capture the effect of a blockage, while results without such a treatment can artificially allow flow to enter into or exit the blockage, thus giving incorrect pressure distributions. The proposed LCBCs and treatments can eliminate these problems, thus enabling more accurate simulations and further development of mixed/boundary EHL solvers.

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Nomenclature

b = width of the plate (m)

h, h_0, h_i = gaps (m)

p = pressure (Pa)

H = dimensionless gap

$P = \frac{m^2 p}{\eta u}$, modified pressure in section 4, or dimensionless pressure

l = length of the plate (m)

m = slope

q, Q = flows per unit length (kg/m/s)

u = horizontal velocity of the bottom plate along the x axis (m/s)

x, X = coordinate (m)

η = viscosity (Pa·s)

ρ = density (kg/m³)

References

- [1] Liu, S., 2012, “On Boundary Conditions in Lubrication with One Dimensional Analytical Solutions,” *Tribol. Int.*, 48, pp. 182–190.
- [2] Zhu, D., and Wang, Q., 2011, “Elastohydrodynamic Lubrication: A Gateway to Interfacial Mechanics - Review and Prospect,” *ASME. J. Tribol.*, 133(4), p. 041001.
- [3] Patir, N., Cheng, H. S., 1978, “An average flow model for determining effects of three-dimensional roughness on partial hydrodynamic lubrication,” *ASME. J. Tribol.*, 100, pp. 12–17.
- [4] Patir, N., Cheng, H. S., 1979, “Application of average flow model to lubrication between rough sliding surfaces,” *ASME. J. Tribol.*, 101 pp. 220–230.
- [5] Harp, S. R., Salant, R. F., 2001, “An average flow model of rough surface lubrication with inter-asperity cavitation,” *ASME. J. Tribol.*, 1231, pp. 134–143.
- [6] Wang, Q. J., Zhu, D., Cheng, H. S., Yu, T., Jiang, X., Liu, S., 2004, “Mixed lubrication analyses by a macro-micro approach and a full-scale mixed EHL model,” *ASME. J. Tribol.*, 1261, pp. 81–91.
- [7] Bayada, G., Martin, S., Vazquez, C., 2005, “An average flow model of the Reynolds roughness including a mass-flow preserving cavitation model,” *ASME. J. Tribol.*, 1274, pp. 793–802
- [8] Sahlin, F., Larsson, R., Almqvist, A., Lugt, P. M., Marklund, P., 2009, “A mixed lubrication model incorporating measured surface topography. Part 1: Theory of flow factors,” *Proc. IMechE, Part J: J. Engineering Tribology*, 224(4), pp. 335-351.
- [9] Sahlin, F., Larsson, R., Marklund, P., Almqvist, A., Lugt, P. M., 2010, “A mixed lubrication model incorporating measured surface topography. Part 2: Roughness treatment, model validation, and simulation’ *Proc. IMechE, Part J: J. Engineering Tribology* 224(4) pp.353-365.
- [10] Chang, L., 1995, “A Deterministic Model for Line Contact Partial Elastohydrodynamic Lubrication,” *Tribol. Int.*, 28, pp. 75–84.

- [11] Jiang, X., Hua, D. Y., Cheng, H. S., Ai, X., and Lee, S. C., 1999, “A Mixed Elastohydrodynamic Lubrication Model with Asperity Contact,” *ASME J. Tribol.*, 121, pp. 481–491.
- [12] Zhao, J. X., Sadeghi, F., and Hoeprich, M. H., 2001, “Analysis of EHL Circular Contact Start Up: Part I—Mixed Contact Model with Pressure and Film Thickness Results,” *ASME J. Tribol.*, 123, pp. 67–74.
- [13] Zhu, D., and Hu, Y., 1999, “The Study of Transition from Full Film Elastohydrodynamic to Mixed and Boundary Lubrication,” *The Advancing Frontier of Engineering Tribology, Proceedings of the 1999 STLE/ASME H.S. Cheng Tribology Surveillance*, pp. 150–156.
- [14] Hu, Y., and Zhu, D., 2000, “A Full Numerical Solution to the Mixed Lubrication in Point Contacts,” *ASME. J. Tribol.*, 122, pp.1-9.
- [15] Wang, W., Li, S., Shen, D., Zhang, S., Hu, Y., 2012, “A mixed lubrication model with consideration of starvation and interasperity cavitations,” *Proc. IMechE, Part J: J. Engineering Tribology*, 226 (12), pp. 1023-1038.
- [16] Wang, W., Wang, S., Shi, F., Wang, Y., Chen, H., Wang, H., and Hu, Y., 2007, “Simulations and Measurements of Sliding Friction Between Rough Surfaces in Point Contacts: From EHL to Boundary Lubrication,” *ASME. J. Tribol.*, 129(3), pp. 495–501.
- [17] Azam, A., Dorgham, A., Morina, A., Neville, A., Wilson, M., 2019, “A simple deterministic plastoelastohydrodynamic lubrication (PEHL) model in mixed lubrication,” *Tribol. Int.*, 131, pp. 520-529.
- [18] Wang, Y., Dorgham, A., Liu, Y., Wang, C., Wilson, M., Neville, A., Azam, A., 2021, “An Assessment of Quantitative Predictions of Deterministic Mixed Lubrication Solvers,” *ASME. J. Tribol.*, 143(1): 011601.
- [19] Holmes, M. J. A., Evans, H. P., and Snidle, R. W., 2005, “Analysis of Mixed Lubrication Effects in Simulated Gear Tooth Contacts,” *ASME J. Tribol.*, 127, pp. 61–69.
- [20] Li, S., and Kahraman, A., 2009, “A Mixed EHL Model with Asymmetric Integrated Control Volume Discretization,” *Tribol. Int.*, 42, pp. 1163–1172.

- [21] Deolalikar, N., Sadeghi, F, Marble S., 2008, “Numerical Modeling of Mixed Lubrication and Flash Temperature in EHL Elliptical Contacts,” ASME. J. Tribol., 130, 011004-1.
- [22] Wang, Q., and Zhu, D., 2019, *Interfacial Mechanics--Theories and Methods for Contact and Lubrication*, CRC Press.
- [23] Venner, C. H., and Lubrecht, A. A., 2000, *Multi-Level Methods in Lubrication*, Elsevier Science, New York.
- [24] Habchi, W., 2018, *Finite Element Modeling of Elastohydrodynamic Lubrication Problems*, John Wiley & Sons Ltd.
- [25] Zhou, Y., Zhu, C., Liu, H., Song, H., 2019, “Investigation of Contact Performance of Case-Hardened Gears Under Plasto-elastohydrodynamic Lubrication,” Tribol. Lett. 67, 92 (2019).
- [26] AL-Mayali, M.F., Hutt, S., Sharif, K.J., Clarke, A., Evans, H.P., 2018, “Experimental and Numerical Study of Micropitting Initiation in Real Rough Surfaces in a Micro-elastohydrodynamic Lubrication Regime,” Tribol. Lett. 66, 150.
- [27] Hansen, J., Björling, M., Larsson, R., 2021, “A New Film Parameter for Rough Surface EHL Contacts with Anisotropic and Isotropic Structures,” Tribol. Lett. 69, 37.
- [28] Mohammadpour, M., Johns-Rahnejat, P.M., Rahnejat, H., Gohar R, 2014, “Boundary Conditions for Elastohydrodynamics of Circular Point Contacts,” Tribol. Lett., 53, 107–118.
- [29] Liu, S., Qiu, L., Wang, Z., Chen, X., 2021, “Influences of Iteration Details on Flow Continuities of Numerical solutions in Isothermal Elastohydrodynamic Lubrication with Micro-Cavitations,” ASME. J. Tribol., 143(10): 101601.
- [30] Liu, S., 2021, “Analytical Solutions to Blocked Lubrications--Simplified Mixed Lubrication Problems,” in preparation.
- [31] Liu, Y., Wang, Q., Wang, W., Hu, Y., and Zhu, D., 2006, “Effects of Differential Scheme and Mesh Density on EHL Film Thickness in Point Contacts,” ASME. J. Tribol., 128(3), pp. 641–653.

[32] Wang, Y., Dorgham, A., Liu, Y., Wang, C., Wilson, M. C. T., Neville, A., and Azam, A., 2021, “An Assessment of Quantitative Predictions of Deterministic Mixed Lubrication Solvers,” ASME. J. Tribol., 143(1), pp. 1–16.

[33] Fillot, N., Berro, H., Vergne, P., 2011, “From Continuous to Molecular Scale in Modelling Elastohydrodynamic Lubrication: Nanoscale Surface Slip Effects on Film Thickness and Friction,” Tribol. Lett. 43, 257–266.

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Figures

Figure 1

Side view illustration of a 1D problem with a lubrication-contact boundary

Figure 2

Top view illustration of a 2D problem with a lubrication-contact boundary

Figure 3

Illustration of the lubrication-contact boundary in a 1D case, filled circles–lubrication; empty circles–solid contact, with p_f and p_d for the fluid and contact pressures, respectively.

(a)

With Q^c by $1B/S1B$

With Q^c by $2B$

(b)

Figure 4

(a) 2D lubrication-contact boundaries, filled circles—lubrication nodes; empty circles—solid contact nodes. Five cases are illustrated, which are a fluid case and four boundary cases at the east, west, south and north sides of the contact spot marked by the black-line box. (b) flow terms with various schemes. Solid lines are used to guide the additions of the Couette flow to the Poiseuille flow on the west and east sides of the control volumes.

Figure 7

Modified fluid pressures when a square column (gray region) is the fixed incline pad bearing

Figure 8

Modified fluid pressure for a cylindrical pillar with a white line indicating its edge

Figure 9

Comparison of modified pressures at $Y = 0$

Figure 10

Comparisons of modified pressures. “NTCF” treatment with the 1B scheme in fact allows the Couette flow exchange (see arrows) when the fluid lubrication region is on the left-hand side of the solid contact region

Figure 11

Cone-shaped pad bearing

Figure 12

Comparison of modified pressures of two meshes at two cross sections: (a) at $Y = 0$; (b) at X close to 39 (dotted line in fig. 11)