Analysis of Two-Dimensional Nonlinear Sloshing in a Rectangular Tank by Using a Concentrated Mass Model

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Analysis of two-dimensional nonlinear sloshing in a rectangular tank by using a concentrated mass model

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Abstract

Major problems can occur when liquid sloshes in a tank, such as occurs in liquid storage tanks during an earthquake, and this is an important engineering problem to address. To analyze this phenomenon, semi-analytical methods such as the perturbation method, the multimodal method, and the finite-element method are generally used. However, semi-analytical methods involve quite complicated equations, and the finite-element method involves many degrees of freedom when the tank is large. In this paper, a nonlinear numerical model with relatively few degrees of freedom is established for vertical and horizontal two-dimensional nonlinear sloshing in a rectangular tank excited horizontally. The model comprises concentrated masses of liquid connected by nonlinear springs and dampers. The connecting springs have characteristics based on the static and dynamic pressures of the liquid. In addition, a method is proposed for reducing the number of degrees of freedom in the two-dimensional model. The natural frequencies, modes, and frequency responses are then compared among the concentrated-mass model, theoretical calculations, and experimental results. Good agreement was achieved among them, thus demonstrating the validity of the model.

Keywords: Vibration analysis, Modeling, Sloshing, Storage tank, Concentrated mass model

1. Introduction

Major problems can occur when liquid sloshes in a tank. For instance, the 2003 Tokachi-oki earthquake in Japan damaged a large tank storing oil at an industrial facility, causing a fire [1][2]. Numerical simulations have been attempted by many researchers to understand sloshing behavior.

In analytical research, Moiseev [3] presented a generalized theory for determining nonlinear sloshing behavior in a tank of arbitrary shape by using a perturbation method, and Faltinsen [4] applied Moiseev's theory to sloshing phenomena in a rectangular tank. Hayama et al. [5] derived a similar theory that considers nonlinear free-surface conditions not considered by Moiseev, and they compared numerical results from their theory with the results of experiments involving a rectangular tank. With regard to the multimodal method, Faltinsen et al. [6] and Faltinsen and Timokha [7][8] derived a multimodal method based on two-dimensional potential theory and used it to simulate sloshing phenomena in a rectangular tank at shallow and intermediate water depths. However, the perturbation method and the multimodal method involve quite complicated equations and cannot be applied to tanks with complex shapes.

Regarding numerical analysis methods based on potential flow theory, Ikegawa [9] derived a functional considering nonlinear free-surface boundary conditions and applied it to a finite-element method to solve nonlinear sloshing problems in a two-dimensional rectangular tank excited horizontally. Nakayama and Washizu [10] extended Ikegawa's method and analyzed the problem of a rectangular tank subjected to forced pitching oscillation. Wu and Taylor [11] and Wu et al. [12] developed a finite-element model by using the Galerkin method and used it to analyze sloshing in two- and three-dimensional rectangular tanks. Nagashima [13] applied the level-set extended finite-element method to sloshing in a rigid tank; in that approach, a finite-element mesh covers the entire domain in the tank including the air domain, and a signed distance function is defined to represent the distance from the free surface. In another approach, Faltinsen [14] and Nakatama and Washizu [15] developed a boundary-element method based on potential flow theory to solve nonlinear sloshing problems in a two-dimensional rectangular tank. However, finite-element and boundary-element methods based on potential flow theory cannot consider the damping that arises from viscosity and increase computational costs in the case of deep-water depths, and such methods make it complicated to analyze the coupled vibrations of a sloshing liquid and a structure, such as the walls of a semi-rigid tank or the floating lid of an oil tank [16].

To calculate the effects of viscous damping, analyses have been performed by subjecting the Navier–Stokes equations to finite-difference methods. Arai et al. [17] used the marker-and-cell method for sloshing phenomena in marine vehicles, Kim [18] simulated sloshing in two- and three-dimensional liquid containers by using the SOLA (synchronized overlap and add) scheme, and Celebi and Akyildiz [19] investigated sloshing in a rectangular tank by using the volume-of-fluid technique. Other than finite-difference methods, Okamoto and Kawahara [20] developed a finite-element method based on the Lagrangian description of the Navier–Stokes equations, Mandal and Maity [21] used the finite-element method to provide a Eulerian description of sloshing in a rectangular tank, and Kim [22] and Cao et al. [23] used smoothed-particle hydrodynamics to simulate violent sloshing flows in ship cargo. However, analyses of the Navier–Stokes equations require more computation time than do those based on potential flow theory.

To overcome the difficulty of using potential flow theory for the coupled vibrations of a sloshing liquid and a structure, Lagrangian finite-element methods are used instead, where the variable is displacement in fluid space. The advantage of those methods is that the compatibility and equilibrium conditions along the boundary between the fluid region and the structure are easily satisfied [24]. Using a finite-element method in which the fluid-space elements are elastic ones that neglect shear stress, Kim and Yun [24] analyzed the problem of coupled sloshing and wall vibration. Using a similar method, Yoshida [25] and Nagashima and Tsukuda [26] analyzed the problem of sloshing coupled with the floating roof of an oil storage tank. However, Lagrangian models have far more degrees of freedom (DOFs) than those based on potential flow theory and can result in physically meaningless spurious modes [24][25]. Other than the Lagrangian finite-element method for coupled problems, there is a

method that involves coupling a finite-element method based on potential flow theory and a finite-element method for the structure [16], and another that involves coupling a boundary-element method based on potential flow theory and a finite-element method for the structure [27]. However, those methods consider complicated interactions between the fluid and the structure.

The aim of the present study is to establish a practical analytical model for analyzing nonlinear sloshing phenomena in a tank. In previous studies, we proposed a one-dimensional concentrated-mass model to analyze large-amplitude nonlinear sloshing phenomena in a rectangular tank containing shallow liquid [28], where shallow is defined as a water depth that is less than one-fifth of the wavelength. Furthermore, we proposed a two-dimensional concentrated mass model to analyze small-amplitude linear sloshing in the case of deep liquid [29][30], where deep is defined as a water depth that is more than one-fifth of the wavelength. The natural frequencies in the linear two-dimensional model were compared with theoretical natural frequencies, but the numerical results have not been compared with experimental results. The linear two-dimensional model cannot be used to analyze nonlinear sloshing because the nonlinear effects cause numerical divergence.

In the present study, a two-dimensional concentrated mass model that consists of masses and nonlinear connecting springs is proposed to enable analysis of not only a nonlinear shallow-water wave, but also a nonlinear deep-water wave. In this model, a method to transform the two-dimensional model into a one-dimensional model is proposed to reduce the number of DOFs. The advantage of the proposed method is that DOFs of the one-dimensional model are less than that of two-dimensional finite-element method, boundary element method, and finite-difference method, which reduces the computation time dramatically. Furthermore, despite being similar to a Lagrangian finite-element model, the proposed model does not produce spurious modes, does not increase DOFs, and can analyze coupled problems easily. Regarding mass–spring models for sloshing phenomena, such as the proposed model, Housner [31] proposed an equivalent model composed of masses and springs to find the force acting on a tank, but that model cannot calculate the variation of the water level accurately [32], nor can it calculate the shapes of the natural modes.

Herein, we propose a two-dimensional concentrated mass model to analyze the nonlinear sloshing phenomena. In this model, the average displacement is introduced to avoid numerical divergence and a method is proposed to transform the two-dimensional model into one-dimensional model. To validate the two dimensional and reduced-DOF models, we compare the linear natural frequencies of the concentrated mass model with the corresponding theoretical values for a rectangular tank. To validate the model in a nonlinear region, we report on an experiment involving a rectangular tank and compare its results with the numerical ones.

2. Concentrated mass model

We consider two-dimensional nonlinear sloshing of liquid in a rigid rectangular tank excited horizontally, as shown in Fig. 1(a). We assume that the liquid is incompressible with no surface tension and oscillates at amplitudes that are insufficient to cause wave breaking. The coordinate system is as

shown in Fig. 1. The liquid has horizontal length L_x and equilibrium depth L_z and is divided into N_x and N_z uniform elements in the x and z directions, respectively (Figs. 1(b) and 2(a)); the element lengths are l_x and l_z in the x and z directions, respectively. The width of the tank in the y direction is taken as the unit length. Mass is concentrated on the centers of the right and left sides of each element, and the mass points are indexed as $(i, j)_x$. Considering the mass of liquid in the shaded area in Fig. 2(b), the mass m at each mass point is

$$m = \rho l_x l_z, \quad (1)$$

Fig. 1. Sloshing phenomena in rectangular tank

(a) Displacements of mass points and z nodal points

(b) Number of elements, mass points, z nodal points

Fig. 2. Elements, mass points in x direction, and z nodal points.

where ρ is the density of the liquid. The mass points can move only in the x direction, and the displacement of mass point $(i, j)_x$ is $\delta x_{i,j}$, as shown in Fig. 2(a). Nodal points that can move vertically are set at the top and bottom sides of each element; called z nodal points, these are indexed as $(i, j)_z$ and have displacement $\delta z_{i,j}$. At the liquid surface, we have $i = 0$.

Fig. 3. Deformation of element $[i, j]$.

Fig. 4. Deformation of element from bottom

Fig. 5. Overlap and gap among elements

Fig. 6. Average displacement

2.1. Connecting nonlinear springs

The nonlinear restoring forces acting on the mass points are derived from the hydrostatic pressure caused by the vertical displacement of the liquid surface and the hydrodynamic pressure caused by the vertical acceleration of the liquid. First, we derive the relationship between the displacement $\delta x_{i,j}$ of the mass points and the displacement $\delta z_{i,j}$ of the z nodal points. Because the liquid is incompressible, we assume that the z nodal points move while keeping the volume of each element in an arbitrary

column j constant when the masses move, as shown Fig. 3. From these relations, we obtain

$$(\delta z_{i-1,j} - \delta z_{i,j})(l_x + \delta x_{i,j} - \delta x_{i,j-1}) = -l_z(\delta x_{i,j} - \delta x_{i,j-1}), \quad (2)$$

the left-hand side of which is the extruded volume in the z direction and the right-hand side of which is the volume change by the displacements $\delta x_{i,j}$ and $\delta x_{i,j-1}$. From Eq. (2), the relative displacement of an arbitrary z nodal point is

$$\delta z_{i-1,j} - \delta z_{i,j} = -\frac{l_z(\delta x_{i,j} - \delta x_{i,j-1})}{l_x + \delta x_{i,j} - \delta x_{i,j-1}}. \quad (3)$$

As the boundary condition at the bottom of the tank in Fig. 4, the displacement in the z direction at the bottom is assumed as

$$\delta z_{N_z,j} = 0. \quad (4)$$

Assuming that the displacements in the z direction are stacked from the bottom, from Eqs. (3) and (4) the displacement at z nodal point $(i,j)_z$ is

$$\delta z_{i,j} = -\sum_{k=i+1}^{N_z} \frac{l_z(\delta x_{k,j} - \delta x_{k,j-1})}{l_x + \delta x_{k,j} - \delta x_{k,j-1}} \quad (0 \leq i \leq N_z - 1). \quad (5)$$

However, upon using Eq. (5) and considering the elements in column $j+1$, overlaps or gaps occur as shown in Fig. 5, and these change the volume of the whole liquid despite it being incompressible. To avoid this problem, we introduce $\delta \bar{x}_j$ as the average displacement of the mass points in column j , as shown in Fig. 6. The average displacement is given by

$$\delta \bar{x}_j = \frac{1}{N_z} \sum_{i=1}^{N_z} \delta x_{i,j}. \quad (6)$$

We use $\delta \bar{x}_j$ when considering the extruded volume in the z direction. Replacing $\delta x_{i,j} - \delta x_{i,j-1}$ on the left-hand side in Eq. (2) with $\delta \bar{x}_j - \delta \bar{x}_{j-1}$, the relative displacement of an arbitrary z nodal point becomes

$$\delta z_{i-1,j} - \delta z_{i,j} = -\frac{l_z(\delta \bar{x}_j - \delta \bar{x}_{j-1})}{l_x + \delta \bar{x}_j - \delta \bar{x}_{j-1}}, \quad (7)$$

and the displacement at z nodal point $(i,j)_z$ becomes

$$\delta z_{i,j} = -\frac{l_z}{l_x + \delta \bar{x}_j - \delta \bar{x}_{j-1}} \sum_{k=i+1}^{N_z} (\delta \bar{x}_j - \delta \bar{x}_{j-1}) \quad (0 \leq i \leq N_z - 1). \quad (8)$$

Considering Eqs. (6) and (8), the displacement $\delta z_{0,j}$ of the z nodal points at the top matches the displacement η_j of the liquid surface:

$$\eta_j = \delta z_{0,j} = -\frac{l_z N_z (\delta \bar{x}_j - \delta \bar{x}_{j-1})}{l_x + \delta \bar{x}_j - \delta \bar{x}_{j-1}} = -\frac{L_z (\delta \bar{x}_j - \delta \bar{x}_{j-1})}{l_x + \delta \bar{x}_j - \delta \bar{x}_{j-1}}. \quad (9)$$

Equations (8) and (9) are nonlinear relationships, which is caused by nonlinearity of the element deformation.

Fig. 7. Control volume and interval of integration

Second, we derive the pressure $p_{i,j}$ in element $[i, j]$. In the area of column j in Fig. 7(a), Euler's equation in the vertical direction is written as

$$\frac{\partial w_j(z, t)}{\partial t} = -\frac{1}{\rho} \frac{\partial p_j(z, t)}{\partial z} - g, \quad (10)$$

where $w_j(z, t)$ is the z -directional velocity at the z coordinate in column j , $p_j(z, t)$ is the pressure in column j , and g is the acceleration due to gravity. The z coordinate of the center position in element $[i, j]$ is $z_i = -(i - 1/2)l_z$ ($i \geq 2$). Integrating Eq. (10) with respect to z from z_i to η_j , as shown in Fig. 7(b), the pressure $p_{i,j}$ at z_i becomes

$$\left. \begin{aligned} p_{1,j} &= \rho \int_{(\eta_j - l_z)/2}^{\eta_j} \frac{\partial w_j}{\partial t} dz + \frac{1}{2} \rho g (\eta_j + l_z) & (i=1) \\ p_{i,j} &= \rho \int_{z_i}^{\eta_j} \frac{\partial w_j}{\partial t} dz + \rho g \left\{ \eta_j + (i-1)l_z + \frac{l_z}{2} \right\} & (i \geq 2) \end{aligned} \right\}. \quad (11)$$

Herein, the pressure at the liquid surface is zero. We assume that the pressure $p_{i,j}$ of element $[i, j]$ is represented by the pressure at the center position z_i . The top element $[1, j]$ contains the region protruding above the equilibrium free surface when the liquid level changes, so the z coordinate of the center position of the top element is $z_1 = (\eta_j - l_z)/2$, as shown in Fig. 8(a). On the right-hand sides in Eq. (11), the first terms are related to the hydrodynamic pressure and the second terms are related to the hydrostatic pressure.

Fig. 8. Acceleration of each element

Third, we derive the restoring force acting on mass point $(i, j)_x$ from pressure $p_{i,j}$. Assuming that $\delta \ddot{z}_{i,j}$ is the average vertical acceleration of z nodal points $(i-1, j)_z$ and $(i, j)_z$ (Fig. 8), $\delta \ddot{z}_{i,j}$ is

$$\delta \ddot{z}_{i,j} = \frac{\delta \ddot{z}_{i-1,j} + \delta \ddot{z}_{i,j}}{2}. \quad (12)$$

We assume that the vertical acceleration in element $[i, j]$ is represented by the pressure at the center position $\delta \ddot{z}_{i,j}$. In the integration in Eq. (11), $\partial w_j / \partial t$ is replaced by the vertical average of vertical accelerations $\delta \ddot{z}_{i,j}$. The integration in Eq. (11) is replaced by the summation using $\ddot{z}_{i,j}$ from element $[1, j]$ to element $[i, j]$ as

$$\left. \begin{aligned} \int_{(\eta_j - l_z)/2}^{\eta_j} \frac{\partial w_j}{\partial t} dz &= \frac{\delta \ddot{z}_{1,j} (\eta_j + l_z)}{2} & (i=1) \\ \int_{z_i}^{\eta_j} \frac{\partial w_j}{\partial t} dz &= \delta \ddot{z}_{1,j} (l_z + \eta_j) + \sum_{k=2}^{i-1} \delta \ddot{z}_{k,j} l_z + \delta \ddot{z}_{i,j} \frac{l_z}{2} & (2 \leq i < N_z) \end{aligned} \right\}. \quad (13)$$

Multiplying $p_{i,j}$ in Eq. (11) by the vertical element length l_z and using Eq. (13), the force $f_{i,j}$ acting on mass point $(i, j)_x$ becomes

$$\left. \begin{aligned} f_{1,j} &= \frac{\rho}{2} \delta \ddot{z}_{1,j} (\eta_j + l_z)^2 + \frac{\rho}{2} g (\eta_j + l_z)^2 & (i=1) \\ f_{i,j} &= \rho \delta \ddot{z}_{1,j} l_z (\eta_j + l_z) + \rho \sum_{k=2}^{i-1} \delta \ddot{z}_{k,j} l_z^2 + \rho \delta \ddot{z}_{i,j} \frac{l_z^2}{2} + \rho g l_z \left\{ \eta_j + (i-1)l_z + \frac{l_z}{2} \right\} & (2 \leq i \leq N_z) \end{aligned} \right\}. \quad (14)$$

Herein, we assume that the vertical length of the top elements ($i=1$) is $(\eta_j + l_z)$ when the liquid level changes. In the equilibrium state, the vertical displacement and the acceleration of the liquid surface are zero, therefore the force $f_{i,j}^0$ acting on an arbitrary mass point in the equilibrium state is

$$f_{i,j}^0 = \rho g l_z^2 \left(i - \frac{1}{2} \right). \quad (15)$$

Therefore, the restoring force $\delta f_{i,j}^k$ from the pressure is written as

$$\delta f_{i,j}^k = f_{i,j} - f_{i,j}^0 = \alpha_{i,j} + \beta_{i,j}, \quad (16)$$

$$\left. \begin{aligned} \alpha_{1,j} &= \rho g l_z \eta_j + \frac{1}{2} \rho g \eta_j^2 \quad (i=1) \\ \alpha_{i,j} &= \rho g l_z \eta_j \quad (2 \leq i \leq N_z) \end{aligned} \right\}, \quad (17)$$

$$\left. \begin{aligned} \beta_{1,j} &= \frac{1}{2} \rho \delta \ddot{z}_{1,j} (\eta_j + l_z)^2 \quad (i=1) \\ \beta_{i,j} &= \rho \left\{ \delta \ddot{z}_{1,j} l_z (\eta_j + l_z) + \sum_{k=2}^{i-1} \delta \ddot{z}_{k,j} l_z^2 + \frac{1}{2} \delta \ddot{z}_{i,j} l_z^2 \right\} \quad (2 \leq i \leq N_z) \end{aligned} \right\}, \quad (18)$$

where $\alpha_{i,j}$ is the restoring force derived from the static pressure caused by the vertical displacement of the liquid surface, and $\beta_{i,j}$ is the restoring force derived from the dynamic pressure that is caused by vertical acceleration of the liquid. Note that Eqs. (17) and (18) include nonlinear terms, which are classified as follows:

- (i) $\rho g l_z \eta_j$ in Eq. (17) is due to the nonlinearity of element deformation under static pressure, which is the nonlinearity of $\delta z_{i,j}$ in Eq. (8);
- (ii) $\rho g \eta_j^2 / 2$ in Eq. (17) is due to the nonlinearity of the displacement of the liquid surface under static pressure;
- (iii) terms other than $l_z \delta \ddot{z}_{1,j} \eta_j$ in the second equation in Eq. (18) are due to the nonlinearity of element deformation under dynamic pressure;
- (iv) $l_z \delta \ddot{z}_{1,j} \eta_j$ in the first equation in Eq. (18) is due to the nonlinearity of the displacement of the liquid surface under dynamic pressure.

Linearizing Eqs. (7), (9), (17), and (18), we obtain

$$z_{i,j}^L = -\frac{l_z}{l_x} \sum_{k=i+1}^{N_z} (\delta x_{k,j} - \delta x_{k,j-1}), \quad (19)$$

$$\eta_j^L = \delta z_{0,j} = -\frac{l_z}{l_x} \sum_{k=1}^{N_z} (\delta x_{k,j} - \delta x_{k,j-1}), \quad (20)$$

$$\alpha_{i,j}^L = \rho g l_z \eta_j^L, \quad (21)$$

$$\left. \begin{aligned} \beta_{1,j}^L &= \frac{1}{2} \rho \delta \ddot{z}_{1,j} l_z^2 \quad (i=1) \\ \beta_{i,j}^L &= \rho \left(\delta \ddot{z}_{1,j} l_z^2 + \sum_{k=1}^{i-1} \delta \ddot{z}_{k,j} l_z^2 + \frac{1}{2} \delta \ddot{z}_{i,j} l_z^2 \right) \quad (i \geq 2) \end{aligned} \right\}. \quad (22)$$

Note that the average displacement $\delta \bar{x}_j$ in Eq. (6) is not necessary in the linearized Eqs. (19) and (20).

2.2. Equations of motion

The equation of motion of mass point $(i, j)_x$ is expressed as follows by considering the restoring forces acting from elements $[i, j]$ and $[i, j+1]$ (Fig. 2(b)):

$$m\ddot{x}_{i,j} = \alpha_{i,j} + \beta_{i,j} - \alpha_{i,j+1} - \beta_{i,j+1} - f_{i,j}^c + f_{i,j+1}^c + f_{i,j}^e(t). \quad (23)$$

Here, $f_{i,j}^c$ is the linear viscous damping force and $f_{i,j}^e(t)$ is an external force acting on mass point $(i, j)_x$. Equation (23) becomes an equation involving only displacements $x_{i,j}$ by substituting Eq. (9) into Eqs. (17) and (18), and Eq. (8) into Eq. (18). The mass points at the wall of the container are assumed to be fixed, so the displacements $x_{i,0}$ and x_{i,N_x} are zero. Writing the equations of motion for all the mass points and arranging them in matrix form, we obtain

$$\left. \begin{aligned} \mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} &= \mathbf{G} \\ \mathbf{G} &= \frac{1}{2}\rho g \begin{bmatrix} \boldsymbol{\sigma}^T & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}^T \\ \boldsymbol{\sigma} &= \begin{bmatrix} \eta_1^2 - \eta_2^2 & \eta_2^2 - \eta_3^2 & \cdots & \eta_{N-1}^2 - \eta_N^2 \end{bmatrix}^T \end{aligned} \right\}, \quad (24)$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the mass, damping, and stiffness matrices, respectively, of size $N_z(N_x - 1) \times N_z(N_x - 1)$. Pertaining to the nonlinearity of the displacement of the liquid surface, matrix $\mathbf{G}$ is derived from the second term of the first equation of Eq. (17) and is of size $N_z(N_x - 1) \times 1$. If $\mathbf{x}_i$ is a displacement vector containing the displacements of the mass points in line i and $\mathbf{x}$ is a vector containing each $\mathbf{x}_i$, then $\mathbf{x}$ and $\mathbf{x}_i$ are expressed as

$$\left. \begin{aligned} \mathbf{x} &= \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T & \cdots & \mathbf{x}_{N_z}^T \end{bmatrix}^T \\ \mathbf{x}_i &= \begin{bmatrix} x_{i,1} & x_{i,2} & \cdots & x_{i,N_x-1} \end{bmatrix}^T \end{aligned} \right\}. \quad (25)$$

Because $\alpha_{i,j}$ in Eq. (17) is a function of the displacements of the mass points and $\beta_{i,j}$ in Eq. (18) includes their accelerations, the former constitutes the stiffness matrix and the latter the mass matrix. The components of the mass matrix are written as follows:

$$\mathbf{M} = \mathbf{M}^m + \mathbf{M}^d = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \cdots & \mathbf{M}_{1N_z} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \cdots & \mathbf{M}_{2N_z} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{N_z 1} & \mathbf{M}_{N_z 2} & \cdots & \mathbf{M}_{N_z N_z} \end{bmatrix}, \quad (26)$$

where $\mathbf{M}^m$ and $\mathbf{M}^d$ are

$$\mathbf{M}^m = \begin{bmatrix} m & 0 & \cdots & 0 \\ 0 & m & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m \end{bmatrix}, \mathbf{M}^d = \begin{bmatrix} \mathbf{M}_1^1/2 & \mathbf{M}_2^1 & \mathbf{M}_3^1 & \cdots & \mathbf{M}_{N_z}^1 \\ \mathbf{M}_{2,1}^b & \mathbf{M}_{2,2}^a & \mathbf{M}_{2,3}^b & \cdots & \mathbf{M}_{2,N_z}^b \\ \mathbf{M}_{3,1}^b & \mathbf{M}_{3,2}^b & \mathbf{M}_{3,3}^a & \cdots & \mathbf{M}_{3,N_z}^b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{N_z,1}^b & \mathbf{M}_{N_z,2}^b & \mathbf{M}_{N_z,3}^b & \cdots & \mathbf{M}_{N_z,N_z}^a \end{bmatrix}. \quad (27)$$

$\mathbf{M}^m$ and $\mathbf{M}^d$ represent the mass of liquid and the dynamic pressure, respectively. $\mathbf{M}_i^1$, which means the dynamic pressure acting on the masses in the first row, is written as

$$\mathbf{M}_i^1 = \left[\begin{array}{ccccccc} m_{i,1}^{z1} + m_{i,2}^{z1} & -m_{i,2}^{z1} & & & & & \\ m_{i,2}^{z1} & m_{i,2}^{z1} + m_{i,3}^{z1} & -m_{i,3}^{z1} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -m_{i,N_x-1}^{z1} & m_{i,N_x-1}^{z1} + m_{i,N_x}^{z1} & & & \end{array} \right] \left\{ \begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array} \right. \quad (28)$$

$$m_{i,j}^{z1} = \frac{\rho(l_z + \eta_j)^2 l_x l_z}{2(l_x + \delta x_{i,j} - \delta x_{i,j-1})^2}$$

$\mathbf{M}_{i,j}^a$ and $\mathbf{M}_{i,j}^b$, which mean the dynamic pressure acting on the masses in the rows other than the first one, are written as

$$\mathbf{M}_{i,j}^a = \left[\begin{array}{ccccccc} m_{j,1}^{z1} m_{i,1}^{z2} + m_{j,2}^{z1} m_{i,2}^{z2} & -m_{j,2}^{z1} m_{i,2}^{z2} & & & & & \\ -m_{j,2}^{z1} m_{i,2}^{z2} & m_{j,2}^{z1} m_{i,2}^{z2} + m_{j,3}^{z1} m_{i,3}^{z2} & -m_{j,3}^{z1} m_{i,3}^{z2} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -m_{j,N_x-1}^{z1} m_{i,N_x-1}^{z2} & m_{j,N_x-1}^{z1} m_{i,N_x-1}^{z2} + m_{j,N_x}^{z1} m_{i,N_x}^{z2} & & & \end{array} \right], \quad (29)$$

$$\mathbf{M}_{i,j}^b = \left[\begin{array}{ccccccc} m_{j,1}^{z1} m_{i,1}^{z3} + m_{j,2}^{z1} m_{i,2}^{z3} & -m_{j,2}^{z1} m_{i,2}^{z3} & & & & & \\ -m_{j,2}^{z1} m_{i,2}^{z3} & m_{j,2}^{z1} m_{i,2}^{z3} + m_{j,3}^{z1} m_{i,3}^{z3} & -m_{j,3}^{z1} m_{i,3}^{z3} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -m_{j,N_x-1}^{z1} m_{i,N_x-1}^{z3} & m_{j,N_x-1}^{z1} m_{i,N_x-1}^{z3} + m_{j,N_x}^{z1} m_{i,N_x}^{z3} & & & \end{array} \right], \quad (30)$$

where

$$m_{i,k}^{z2} = (l_z + \eta_k) l_z + \left(i - \frac{7}{4}\right) l_z^2, \quad m_{i,k}^{z3} = (l_z + \eta_k) l_z + \left(i - \frac{3}{2}\right) l_z^2. \quad (31)$$

From Eqs. (26) and (27), $\mathbf{M}_{i,j}$ becomes

$$\left. \begin{array}{l} \mathbf{M}_{11} = \text{diag}[m \ m \ \cdots \ m] + \frac{\mathbf{M}_1^1}{2} \\ \mathbf{M}_{1j} = \mathbf{M}_j^1 \quad (j \geq 2) \\ \mathbf{M}_{ii} = \text{diag}[m \ m \ \cdots \ m] + \mathbf{M}_{ii}^a \\ \mathbf{M}_{ij} = \mathbf{M}_{ij}^b \quad (i \neq j) \end{array} \right\}. \quad (32)$$

Regarding the linearized mass matrix $\mathbf{M}^L$, $m_{i,j}^{z1}$ in Eq. (28) and $m_{i,k}^{z2}$ and $m_{i,k}^{z3}$ in Eq. (31) are treated as

$$m_{i,j}^{z1,L} = \frac{\rho l_z^3}{l_x}, m_{i,k}^{z2,L} = l_z^2 + \left(i - \frac{7}{4}\right) l_z^2, m_{i,k}^{z3,L} = l_z^2 + \left(i - \frac{3}{2}\right) l_z^2. \quad (33)$$

The components of the stiffness matrix are written as follows:

$$\mathbf{K} = \left\{ \begin{array}{c} \begin{bmatrix} \hat{\mathbf{K}} & \hat{\mathbf{K}} & \dots & \hat{\mathbf{K}} \\ \hat{\mathbf{K}} & \hat{\mathbf{K}} & \dots & \hat{\mathbf{K}} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\mathbf{K}} & \hat{\mathbf{K}} & \dots & \hat{\mathbf{K}} \end{bmatrix} \\ \hat{\mathbf{K}} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \dots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -k_{N-2} & k_{N-2} + k_{N-1} & -k_{N-1} \\ 0 & \dots & 0 & -k_{N-1} & k_{N-1} + k_N \end{bmatrix} \end{array} \right\}, k_j = \frac{\rho g l_z^2}{l_x + \delta \bar{x}_j - \delta \bar{x}_{j-1}} \quad (34)$$

From Eqs. (28), (31), and (34), the mass and stiffness matrices contain displacements.

In addition, the damping matrix $\mathbf{C}$ is represented as a proportion of the linearized stiffness matrix $\mathbf{K}^L$:

$$\mathbf{C} = \gamma \mathbf{K}^L$$

$$\mathbf{K}^L = \left\{ \begin{array}{c} \begin{bmatrix} \tilde{\mathbf{K}} & \tilde{\mathbf{K}} & \dots & \tilde{\mathbf{K}} \\ \tilde{\mathbf{K}} & \tilde{\mathbf{K}} & \dots & \tilde{\mathbf{K}} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{K}} & \tilde{\mathbf{K}} & \dots & \tilde{\mathbf{K}} \end{bmatrix} \\ \tilde{\mathbf{K}} = \begin{bmatrix} 2\tilde{k} & -\tilde{k} & 0 & \dots & 0 \\ -\tilde{k} & 2\tilde{k} & -\tilde{k} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\tilde{k} & 2\tilde{k} & -\tilde{k} \\ 0 & \dots & 0 & -\tilde{k} & 2\tilde{k} \end{bmatrix} \end{array} \right\}, \tilde{k} = \frac{\rho g l_z^2}{l_x} \quad (35)$$

where γ is a constant of the proportional viscous damping whose optimal value is determined experimentally.

Fig. 9. Reducing the number of degrees of freedom (DOFs)

3. Reducing the number of degrees of freedom

In the model proposed in Section 2, the elements are divided vertically to consider the vertical velocity distribution, but consequently the model has an unnecessarily high number of DOFs, with zero-frequency eigenvalues being generated. In this section, we propose a way to reduce the number of DOFs of the concentrated mass model, as shown in Fig. 9.

Using Eqs. (26), (34), and (35), the equations of motion in Eq. (24) are rewritten as

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \cdots & \mathbf{M}_{1N_z} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \cdots & \mathbf{M}_{2N_z} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{N_z1} & \mathbf{M}_{N_z2} & \cdots & \mathbf{M}_{N_zN_z} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}}_1 \\ \ddot{\mathbf{x}}_2 \\ \vdots \\ \ddot{\mathbf{x}}_{N_z} \end{bmatrix} + \gamma \begin{bmatrix} \tilde{\mathbf{K}} & \tilde{\mathbf{K}} & \cdots & \tilde{\mathbf{K}} \\ \tilde{\mathbf{K}} & \tilde{\mathbf{K}} & \cdots & \tilde{\mathbf{K}} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{K}} & \tilde{\mathbf{K}} & \cdots & \tilde{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \\ \vdots \\ \dot{\mathbf{x}}_{N_z} \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{K}} & \hat{\mathbf{K}} & \cdots & \hat{\mathbf{K}} \\ \hat{\mathbf{K}} & \hat{\mathbf{K}} & \cdots & \hat{\mathbf{K}} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\mathbf{K}} & \hat{\mathbf{K}} & \cdots & \hat{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_{N_z} \end{bmatrix} = \frac{1}{2} \rho g \begin{bmatrix} \boldsymbol{\sigma} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}. \quad (36)$$

Here, the matrices $\mathbf{M}_{i,j}$, $\tilde{\mathbf{K}}$, and $\hat{\mathbf{K}}$ are of size $(N_x - 1) \times (N_x - 1)$. The damping matrix $\mathbf{C}$ includes the block matrix $\tilde{\mathbf{K}}$ overall and the stiffness matrix $\mathbf{K}$ includes the block matrix $\hat{\mathbf{K}}$. Subtracting the equation of row 1 from those of rows 2 to N_z in Eq. (36), we obtain

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \cdots & \mathbf{M}_{1N_z} \\ \tilde{\mathbf{M}}_{21} & \tilde{\mathbf{M}}_{22} & \cdots & \tilde{\mathbf{M}}_{2N_z} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{M}}_{N_z1} & \tilde{\mathbf{M}}_{N_z2} & \cdots & \tilde{\mathbf{M}}_{N_zN_z} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}}_1 \\ \ddot{\mathbf{x}}_2 \\ \vdots \\ \ddot{\mathbf{x}}_{N_z} \end{bmatrix} + \gamma \begin{bmatrix} \tilde{\mathbf{K}} & \tilde{\mathbf{K}} & \cdots & \tilde{\mathbf{K}} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \\ \vdots \\ \dot{\mathbf{x}}_{N_z} \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{K}} & \hat{\mathbf{K}} & \cdots & \hat{\mathbf{K}} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_{N_z} \end{bmatrix} = \frac{1}{2} \rho g \begin{bmatrix} \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} \\ \vdots \\ -\boldsymbol{\sigma} \end{bmatrix}. \quad (37)$$

Here, $\tilde{\mathbf{M}}_{i,j}$ is

$$\tilde{\mathbf{M}}_{i,j} = \mathbf{M}_{i,j} - \mathbf{M}_{1,j}. \quad (38)$$

In addition, we introduce the average displacement vector $\bar{\mathbf{x}}$, whose elements are the average displacements in Eq. (6). The average displacement vector can be written as the summation of $\mathbf{x}_1$ to $\mathbf{x}_{N_z}$:

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_{N_x-1} \end{bmatrix} = \frac{1}{N_z} \begin{bmatrix} \mathbf{I}_{N_x-1} & \mathbf{I}_{N_x-1} & \cdots & \mathbf{I}_{N_x-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_{N_z} \end{bmatrix}, \quad (39)$$

where $\mathbf{I}$ is the unit matrix. Taking the equations of rows 2 to N_z in Eq. (37) and inserting Eq. (39) into the top row, we obtain

$$\begin{bmatrix} \mathbf{I}_{N_x-1} & \mathbf{I}_{N_x-1} & \cdots & \mathbf{I}_{N_x-1} \\ \tilde{\mathbf{M}}_{21} & \tilde{\mathbf{M}}_{22} & \cdots & \tilde{\mathbf{M}}_{2N_z} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{M}}_{N_z1} & \tilde{\mathbf{M}}_{N_z2} & \cdots & \tilde{\mathbf{M}}_{N_zN_z} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}}_1 \\ \ddot{\mathbf{x}}_2 \\ \vdots \\ \ddot{\mathbf{x}}_{N_z} \end{bmatrix} = N_z \begin{bmatrix} \mathbf{I}_{N_x-1} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \ddot{\bar{\mathbf{x}}} - \frac{1}{2} \rho g \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\sigma} \\ \vdots \\ \boldsymbol{\sigma} \end{bmatrix}, \quad (40)$$

which can be written as

$$\mathbf{A}_1 \ddot{\mathbf{x}} = \mathbf{A}_2 \ddot{\bar{\mathbf{x}}} - \tilde{\mathbf{S}} \quad \left. \begin{aligned} \mathbf{A}_1 = \begin{bmatrix} \mathbf{I}_{N_x-1} & \mathbf{I}_{N_x-1} & \cdots & \mathbf{I}_{N_x-1} \\ \tilde{\mathbf{M}}_{21} & \tilde{\mathbf{M}}_{22} & \cdots & \tilde{\mathbf{M}}_{2N_z} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{M}}_{N_z1} & \tilde{\mathbf{M}}_{N_z2} & \cdots & \tilde{\mathbf{M}}_{N_zN_z} \end{bmatrix}, \mathbf{A}_2 = N_z \begin{bmatrix} \mathbf{I}_{N_x-1} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \tilde{\mathbf{S}} = \frac{1}{2} \rho g \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\sigma} \\ \vdots \\ \boldsymbol{\sigma} \end{bmatrix} \end{aligned} \right\}. \quad (41)$$

Multiplying $\mathbf{A}_1^{-1}$ by Eq. (41) from the left-hand side, we obtain

$$\left. \begin{aligned} \ddot{\mathbf{x}} &= \mathbf{T} \ddot{\bar{\mathbf{x}}} - \mathbf{S} \\ \mathbf{T} &= \mathbf{A}_1^{-1} \mathbf{A}_2, \mathbf{S} = \mathbf{A}_1^{-1} \tilde{\mathbf{S}} \end{aligned} \right\}, \quad (42)$$

which represents the relationship between the two-dimensional acceleration $\ddot{\mathbf{x}}$ and the one-dimensional acceleration $\ddot{\bar{\mathbf{x}}}$. $\mathbf{T}$ is the transformation matrix from the two-dimensional model to the one-dimensional model. We assume that the displacement vectors $\mathbf{x}$ and $\bar{\mathbf{x}}$ and the velocity vectors $\dot{\mathbf{x}}$ and $\dot{\bar{\mathbf{x}}}$ are also related by $\mathbf{T}$:

$$\left. \begin{aligned} \mathbf{x} &= \mathbf{T} \bar{\mathbf{x}} \\ \dot{\mathbf{x}} &= \mathbf{T} \dot{\bar{\mathbf{x}}} \end{aligned} \right\}. \quad (43)$$

Substituting Eqs. (42) and (43) into Eq. (24) and left-multiplying by $\mathbf{T}^T$, we obtain

$$\left. \begin{aligned} \bar{\mathbf{M}} \ddot{\bar{\mathbf{x}}} + \bar{\mathbf{C}} \dot{\bar{\mathbf{x}}} + \bar{\mathbf{K}} \bar{\mathbf{x}} &= \bar{\mathbf{G}} + \mathbf{T}^T \mathbf{M} \mathbf{S} \\ \bar{\mathbf{M}} &= \mathbf{T}^T \mathbf{M} \mathbf{T}, \bar{\mathbf{C}} = \mathbf{T}^T \mathbf{C} \mathbf{T}, \bar{\mathbf{K}} = \mathbf{T}^T \mathbf{K} \mathbf{T}, \bar{\mathbf{G}} = \mathbf{T}^T \mathbf{G} \end{aligned} \right\}. \quad (44)$$

Equation (44) is the equation of motion of the reduced-DOF model using the average displacement. In Eq. (44), there are $N_x - 1$ DOFs, which is fewer than the $N_z \times (N_x - 1)$ DOFs in Eq. (24). Having solved for $\bar{\mathbf{x}}$ in Eq. (44), the displacements η_j of the liquid surface can be calculated by using Eq. (9).

However, the transformation matrix $\mathbf{T}$ in Eqs. (42) and (43) contains the displacements $\tilde{\alpha}_{i,j}$ because it contains the mass matrix $\tilde{\mathbf{M}}_{i,j}$ whose elements are a function of the displacements, as shown in Eqs. (28) and (31). Therefore, $\mathbf{T}$ changes at each time step in the numerical calculation, as does the equation of motion Eq. (44). In the case of linear analysis, the transformation matrix $\mathbf{T}^L$ becomes

$$\left. \begin{aligned} \mathbf{T}^L &= (\mathbf{A}_1^L)^{-1} \mathbf{A}_2 \\ \mathbf{A}_1^L &= \begin{bmatrix} \mathbf{I}_{N_x-1} & \mathbf{I}_{N_x-1} & \cdots & \mathbf{I}_{N_x-1} \\ \tilde{\mathbf{M}}_{21}^L & \tilde{\mathbf{M}}_{22}^L & \cdots & \tilde{\mathbf{M}}_{2N_z}^L \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{M}}_{N_z1}^L & \tilde{\mathbf{M}}_{N_z2}^L & \cdots & \tilde{\mathbf{M}}_{N_zN_z}^L \end{bmatrix}, \tilde{\mathbf{M}}_{i,j}^L = \mathbf{M}_{i,j}^L - \mathbf{M}_{1,j}^L \end{aligned} \right\}. \quad (45)$$

4. Comparison of natural modes in rectangular tank

We consider the natural frequencies and modal shapes of the liquid in the rectangular container shown in Fig. 1. To validate the proposed model in the linear region, we compare the numerical results calculated using the concentrated mass model with the corresponding theoretical values. To validate the method for reducing the number of DOFs, we compare the numerical results calculated using the reduced-DOF model with those calculated using the original two-dimensional model.

4.1. Analytical method using concentrated-mass model

Using the linearized Eqs. (33) and (35) and assuming $\mathbf{C} = \mathbf{G} = \mathbf{0}$ in Eq. (24), we obtain the linearized equation of motion of the two-dimensional model:

$$\mathbf{M}^L \ddot{\mathbf{x}} + \mathbf{K}^L \mathbf{x} = \mathbf{0}. \quad (46)$$

The components in the linearized matrices $\mathbf{M}^L$ and $\mathbf{K}^L$ are constants.

By solving the eigenvalue problem of Eq. (46), we obtain the order- r natural angular frequency ω_r and modal vector $\mathbf{X}_r$ of size $N_z(N_x - 1) \times 1$. Meanwhile, using the linearized transfer matrix in Eq. (45), we obtain the linearized equation of motion of the reduced-DOF model as follows:

$$\left. \begin{aligned} \bar{\mathbf{M}} \ddot{\bar{\mathbf{x}}} + \bar{\mathbf{K}} \bar{\mathbf{x}} &= \mathbf{0} \\ \bar{\mathbf{M}} &= (\mathbf{T}^L)^T \mathbf{M}^L \mathbf{T}^L, \bar{\mathbf{K}} = (\mathbf{T}^L)^T \mathbf{K}^L \mathbf{T}^L \end{aligned} \right\}, \quad (47)$$

from which we obtain the order- r natural angular frequency ω_r and modal vector $\bar{\mathbf{X}}_r$ of size $N_x - 1$. From Eq. (43), the modal vector $\mathbf{X}_r$ of the two-dimensional model and of size $N_z \times (N_x - 1)$ is reproduced from $\bar{\mathbf{X}}_r$ as

$$\mathbf{X}_r = \mathbf{T}^L \bar{\mathbf{X}}_r. \quad (48)$$

4.2. Validating the concentrated-mass model in linear region

From small-amplitude wave theory, the order- r natural frequency f_r of the rectangular tank and the order- r natural modes u_r and w_r for the horizontal and vertical velocities, respectively, are given by [1]

$$\left. \begin{aligned} f_r &= \frac{1}{2\pi} \sqrt{g \kappa_r \tanh(\kappa_r L_z)} \\ \kappa_r &= \frac{r\pi}{L_x} \end{aligned} \right\}, \quad (49)$$

$$\left. \begin{aligned} u_r(x, z) &= A \sin \kappa_r x \cosh \kappa_r (z + L_z) \\ w_r(x, z) &= -A \cos \kappa_r x \sinh \kappa_r (z + L_z) \end{aligned} \right\}. \quad (50)$$

Table 1 lists the parameter values used in the numerical calculation for the concentrated mass model. The natural frequencies and natural modes of the two-dimensional model are obtained by solving the eigenvalue problem of Eq. (46), and those of the reduced-DOF model are obtained by solving the

eigenvalue problem of Eq. (47).

Table 1. Parameter values

L_x [m]	0.5	L_z [m]	0.25
N_x	100	N_z	50
ρ [kg/m ³]	1000	g [m/s ²]	9.81

Table 2. Comparison of natural frequencies

Order	Theoretical value [Hz]	Two-dimensional model [Hz]	Reduced-DOF model [Hz]
-	-	0 (4851 eigenvalues)	
1	1.1966	1.1966	1.1966
2	1.7638	1.7637	1.7637
3	2.1641	2.1637	2.1637
4	2.4990	2.4982	2.4982
5	2.7940	2.7926	2.7926
6	3.0607	3.0584	3.0584

Fig. 10. Comparison of shapes of natural modes

In Table 2, the natural frequencies given by the concentrated mass model (two-dimensional and reduced-DOF models) are compared with the theoretical values given by Eq. (49). In addition, the theoretical natural modes given numerically and by Eq. (50) are shown in Fig. 10. The numerical

results for the natural frequencies and modal shapes agree well with the theoretical ones. Therefore, the proposed concentrated mass model is valid for sloshing in the linear region.

However, zero-frequency natural frequencies appear in the two-dimensional model (Table 2) because the stiffness matrix is singular. There are 4851 zero-frequency eigenvalues, and that number increases with the number of vertical separations. With a zero-frequency natural mode, the water surface does not move. By contrast, the reduced-DOF model has no zero-frequency natural frequencies, and its other natural frequencies agree exactly with those of the two-dimensional model. In addition, the reduced-DOF model reproduces the natural modal shapes of the two-dimensional model (Fig. 10 (c)). Therefore, our method for reducing the number of DOFs is valid.

5. Comparison with experimental results

In this section, numerical results from the nonlinear concentrated mass model are compared with experimental results to assess the validity of the proposed model in the nonlinear region.

5.1. Experimental apparatus and analysis method

To validate the proposed models, we performed sloshing experiments in a rectangular tank, as shown in Fig. 11. The rectangular tank of height $H = 400$ mm and width 100 mm was formed from 10-mm-thick acrylic boards. It was set on a linear guide and excited horizontally by a vibration exciter. The displacement of the tank was measured using a laser displacement meter (LK-G80; Keyence, Japan), and a capacitance-type water-level gauge (WTS01A0505; CS Tokki, Japan) was installed at a distance of L_s from the left wall. We performed experiments in four configurations, which we refer to as cases 1–4. The values of the horizontal length L_x , water depth L_z , and excitation displacement Δ in each case are given in Table 3. In cases 1 and 2, the water was deeper than in cases 3 and 4; the latter two cases satisfy the shallow-water wave condition (i.e., the water depth is less than one-fifth of the wavelength) [28].

We assume that the inertial force $f^I(t)$ expressed by

$$f^I(t) = -m\Delta\Omega^2 \sin \Omega t \quad (51)$$

acts on each mass point by horizontal excitation, where Ω is the excitation angular frequency. The frequency response is calculated using the harmonic balance method [34]. The analysis considers all nonlinearities from (i) to (iv) in Section 2.1. The calculation parameters are given in Tables 3 and 4. As explained in Section 3, the transformation matrix $\mathbf{T}$ in Eq. (42) contains the displacements $\delta x_{i,j}$ and so must be calculated at each time step. In this section, the linearized transfer matrix $\mathbf{T}^L$ in Eq. (45) is used for the simulations.

Fig. 11. Experimental apparatus

Table 3. Parameter values for each case

	L_x [mm]	L_z [mm]	L_s [mm]	Δ [mm]	γ
Case 1	500	250	30	0.25	0.3
Case 2	500	250	30	0.3	0.25
Case 3	1000	50	50	2.25	0.05
Case 4	1000	50	50	3.5	0.087

Table 4. Other parameter values

N_x	30	N_z	10
ρ [kg/m ³]	1000	g [m/s ²]	9.81
ζ [s]	6.88×10^{-3}

Fig. 12. Frequency responses (cases 1 and 2, reduced-DOF model)

Fig. 13. Frequency responses (cases 3 and 4, reduced-DOF model)

5.2. Experimental and numerical results

For cases 1 and 2 (deep water), Fig. 12 shows the frequency responses around the first-order resonance. The linear first-order natural frequency is 1.20 Hz. The resonance curves tilt slightly to the left in these cases. In general, the resonance curve tilts to the left when $L_z/L_x > 0.337$ [1], and in cases 1 and 2 we have $L_z/L_x = 0.5$. The numerical results agree well with the experimental results, and the numerical results calculated using the reduced-DOF model agree well with those calculated using the two-dimensional model.

Figure 13 shows the frequency responses around the first-order resonance for cases 3 and 4 (shallow water) with the reduced-DOF model. The first-order natural frequency is 0.349 Hz. The resonance curves tilt to the right in the case of shallow water. In general, the resonance curve tilts to the right when $L_z/L_x < 0.337$, and in cases 3 and 4 we have $L_z/L_x = 0.05$. In cases 3 and 4, there are some peaks around the first-order resonance, and at these peak frequencies solitary waves move back and forth in the tank [28]. From Figs. 12 and 13, the numerical results from the proposed model agree well with the experimental results. Therefore, the proposed model is valid for numerical analysis of nonlinear sloshing phenomena.

Table 5 gives the natural frequencies in cases 1–4. For cases 3 and 4, half of the second-order natural frequency is 0.3446 Hz ($= 0.6892/2$), which is close to the frequency of (iii) in Fig. 13(a), therefore second-order superharmonic resonance occurs at (iii). A third of the third-order natural frequency is 0.3380 Hz, which is close to the frequency of (ii), and a quarter of the fourth-order natural frequency is 0.3297 Hz, which is close to the frequency of (i), therefore these peaks are superharmonic resonances. Meanwhile, for cases 1 and 2, half of the second-order natural frequency is 0.8819 Hz and a third of the third-order natural frequency is 0.7214 Hz, but these frequencies are far from the first-order natural frequency (1.1966 Hz), therefore there is only one peak around the first-order natural frequency in the case of deep water, as shown in Fig. 12.

Table 5. Natural frequencies in each case

Order	Cases 1 and 2 [Hz]	Cases 3 and 4 [Hz]
1	1.1966	0.3487
2	1.7638	0.6892
3	2.1641	1.0142
4	2.4990	1.3187
5	2.7940	1.5999
6	3.0607	1.8572

Fig. 14. Effect of each nonlinearity (case 2, reduced-DOF model)

Fig. 15. Effect of two nonlinearities (case 2, reduced-DOF model)

Fig. 16. Effect of each nonlinearity (case 4, reduced-DOF model)

5.3. Effects of nonlinearities

In this section, we consider the effects of nonlinearities (i)–(iv) of the proposed model in Section 2.1. Figure 14 shows the frequency response for case 2 by using each nonlinearity. The black lines show the nonlinear analysis results that consider all nonlinearities, the green lines show the linear analysis results, and the red lines show the results that consider each nonlinearity from (i) to (iv). The solid lines are stable solutions and the dashed lines are unstable solutions. From Fig. 14, the peak curve for all nonlinearities (black line) tilts to the left. The peak curves for nonlinearities (i) and (iv) tilt to the right slightly, and that for nonlinearity (iii) tilts to the left. Figure 15 shows the frequency response for case 2 by using two nonlinearities. The red lines show the results that consider nonlinearities (ii) and (iii) in Fig. 15(a) and nonlinearities (iii) and (iv) in Fig. 15(b). The peak curve for the combination of nonlinearities (iii) and (iv) tilts strongly to the left, therefore the effect of nonlinearities (iii) and (iv) due to the dynamic pressure is large in the case of deep water.

Figure 16 shows the effect of each nonlinearity for case 4. From Fig. 16(c) and (d), the peak curves of nonlinearities (iii) and (iv) agree with the linear analysis results. Considering all nonlinearities, the peak curve of nonlinearity (i) is the closest to the results, so nonlinearity (i) due to the nonlinearity of element deformation under static pressure is dominant in the case of shallow water.

In the case of shallow water (Fig. 16), the most effective nonlinearities are (i) and (ii) derived from the static pressure, but in the case of deep water (Fig. 15), the most effective nonlinearities are (iii) and (iv) derived from the dynamic pressure. The nonlinearity in the dynamic pressure contains the acceleration $\ddot{z}$, as shown in Eq. (18). From Table 5, the first-order natural frequency of the deep water (cases 1 and 2) is larger than that of the shallow water (cases 3 and 4), so the acceleration of Eq. (8) in the case of deep water is larger than that in the case of shallow water. Therefore, nonlinearities (iii) and (iv) are effective in the case of deep water.

6. Conclusions

We proposed a concentrated mass model for analyzing nonlinear two-dimensional sloshing in a rectangular tank, and our conclusions are as follows.

1. A concentrated mass model was proposed for two-dimensional sloshing in a rectangular tank excited horizontally. The model consists of mass points that move horizontally and z nodal points that move vertically. The restoring forces were derived from static and dynamic pressures, the latter of which considers vertical acceleration of the z nodal points.
2. A method for reducing the number of DOFs of the proposed two-dimensional model was proposed, by which the number of DOFs in the vertical direction can be reduced to one.
3. The advantage of the proposed method is that DOFs of the model are less than that of two-dimensional finite-element method, boundary element method, and finite-difference method, thereby reducing the computation time dramatically. Furthermore, despite being similar to a Lagrangian finite-element model, the proposed model does not produce spurious modes, does not

increase DOFs, and can easily analyze coupled problems between sloshing and structural vibration.

4. The natural frequencies in a rectangular tank as obtained from the concentrated-mass model (two-dimensional and reduced-DOF models) agreed well with the theoretical natural frequencies.
5. Experiments were performed on a rectangular tank in the case of shallow water and deep water, and the experimental results were compared with the model results. All the numerical results agreed well with the experimental results both qualitatively and quantitatively.
6. The effect of each nonlinearity was studied. In the case of deep water, the nonlinearities due to the dynamic pressure were dominant. In the case of shallow water, the nonlinearities due to the static pressure were dominant.
7. In summary, the proposed model is suitable for the numerical analysis of nonlinear two-dimensional sloshing in a rectangular tank. Future tasks include proposing a model for a rectangular tank excited vertically and models for tanks of other shapes.

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Declarations

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Conflicts of interest

The authors declare that they have no conflict of interest.

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Figures

Figure 1

Sloshing phenomena in rectangular tank

(a) Displacements of mass points and z nodal points

(b) Number of elements, mass points, z nodal points

Figure 2

Elements, mass points in x direction, and z nodal points.

Figure 3

Deformation of element $[i, j]$.

Figure 4

Deformation of element from bottom

Figure 5

Overlap and gap among elements

Figure 6

Average displacement

(a) Control volume at column j

(b) Interval of integration

Figure 7

Control volume and interval of integration

(a) Around element $[1, j]$

(b) Around element $[i, j]$

Figure 8

Acceleration of each element

Figure 9

Reducing the number of degrees of freedom (DOFs)

Figure 10

Comparison of shapes of natural modes

Figure 11

Experimental apparatus

- : Numerical results (Stable)
- : Numerical results (Unstable)
- : Experimental results

(a) Case 1

(b) Case 2

Figure 12

Frequency responses (cases 1 and 2, reduced-DOF model)

- : Numerical results (Stable)
- - - : Numerical results (Unstable)
- : Experimental results

Figure 13

Frequency responses (cases 3 and 4, reduced-DOF model)

Figure 14

Effect of each nonlinearity (case 2, reduced-DOF model)

Figure 15

Effect of two nonlinearities (case 2, reduced-DOF model)

Figure 16

Effect of each nonlinearity (case 4, reduced-DOF model)