Fractional-Order Delayed Ross-Macdonald Model for Malaria Transmission

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Fractional-order delayed Ross-Macdonald model for malaria transmission

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Abstract This paper proposes a novel fractional-order delayed Ross-Macdonald model for malaria transmission. This paper aims to systematically investigate the effect of both the incubation periods of Plasmodium and the order on the dynamic behavior of diseases. Utilizing inequality techniques, contraction mapping theory, fractional linear stability theorem and bifurcation theory, several sufficient conditions for the existence, uniqueness, local stability of the positive equilibrium point, and the existence of fractional-order Hopf bifurcation are obtained under different time delays cases. The results show that time delay can change the stability of the system. The system becomes unstable and generates a Hopf bifurcation when the delay increases to a certain value. Besides, the value of order influences the stability interval size. Thus, incubation periods and the order have a major effect on the dynamic behavior of the model. The effectiveness of the theoretical results is shown through numerical simulations.

Keywords Malaria · Fractional-order · Incubation periods · Local stability · Hopf bifurcation

1 Introduction

Malaria is a serious life-threatening disease prevalent near the equator, especially in Sub-Saharan Africa. It is an insect-borne disease without transmitted directly from one person to another, the most common spread way through which is infected female Anopheles mosquitos. Malaria causes illness and weakens the routine immunization systems, further contributed to the risk of severe disease for infected persons than others. The World Health Organization (WHO) reported 229 million malaria cases in 87 malaria-endemic countries, accounting for an estimated 409 000 deaths in 2019 [1]. This year, the number of malaria deaths may increase due to the COVID-19 outbreak occupying medical care service resources. Up to now, malaria remains one of the major global public health emergencies [2].

At present, most of the extensive research on malaria transmission is underpinned by mathematical models reflecting the process, law, and trend of malaria transmission. The earliest use of a mathematical model to explain malaria transmission was Ross [3], using two nonlinear differential equations to describe the cross-infection of malaria between humans and mosquitoes. Later, Macdonald [4, 5] extended the basic model of Ross and proposed the concept of basic

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reproduction number, representing the beginning of malaria transmission models. However, the classical Ross-Macdonald model has not considered the incubation periods of Plasmodium within mosquitos and humans. To some extent, this model cannot explain the natural behavior of malaria transmission adequately. In 2010, Saker [3] proposed the following delayed Ross-Macdonald model to describe malaria transmission.

$$\begin{cases} \dot{I}_h = abkI_m(t-\tau)[1-I_h(t-\tau)] - (\mu + \gamma)I_h(t), \\ \dot{I}_m = acI_h(t-\sigma)[1-I_m(t-\sigma)] - \delta I_m(t), \end{cases} \quad (1)$$

where I_h and I_m represent the number of infected humans and mosquitos at time t , respectively. The parameter a is the biting rate of female mosquitoes to people per unit time, k is the ratio of mosquitos to humans, b is the proportion of infections after bite, c is the effective transmission rate of the humans to mosquitoes, γ is the recovery rate of the infective humans, μ is the rate of death from malaria infection, δ is per capita natural death rate of mosquitoes. τ and σ denote the incubation period of the parasite within the humans and mosquitos, respectively.

This mathematical model is regarded as the simplest, straightforward, and effective one in describing the level of malaria transmission. For this model, the author studied the boundedness, persistence and stability of the system, and proved the existence of Hopf bifurcation under two special cases. Nevertheless, this discussion has some certain limitations.

Fractional calculus, as an extension of traditional calculus, has attracted remarkable attention in recent years since it can better depict memory and hereditary properties of various materials and processes [6]. Various studies have already demonstrated that a fractional-order model better depicted the dynamic behavior of the system than the integer-order counterpart. Most biological systems have memory and hereditary attributes. Accordingly, it is accurate to use fractional differential to model biological systems. At present, Caputo fractional derivatives have become a preferred method to study and model biological systems instead of first-order derivatives, such as the fractional model of HIV transmission [7], fractional prey-predator system [8], fractional happiness model [9], fractional birhythmic biological system [10], fractional SIR epidemic model for dengue transmission [11], etc.

In this paper, Caputo fractional derivatives are introduced into model (1), so the non-linear fractional-order malaria transmission model with delays can be expressed as:

$$\begin{cases} \mathcal{D}^q I_h(t) = abkI_m(t-\tau)[1-I_h(t-\tau)] - (\mu + \gamma)I_h(t), \\ \mathcal{D}^q I_m(t) = acI_h(t-\sigma)[1-I_m(t-\sigma)] - \delta I_m(t), \end{cases} \quad (2)$$

where $q \in (0, 1)$ is the order of fractional derivative, the notation $\mathcal{D}^q$ denotes Caputo fractional derivative operator ${}_0^C \mathcal{D}_t^q$. Based on biological and epidemiologic significance, the following initial conditions are considered

$$\begin{aligned} I_h(\theta) &= \phi_1(\theta), I_m(\theta) = \phi_2(\theta), \theta \in [-\vartheta, 0], \vartheta = \max\{\sigma, \tau\}, \\ I_h(0) &= \phi_1(0) > 0, I_m(0) = \phi_2(0) > 0, \end{aligned}$$

where $\phi = (\phi_1, \phi_2)^T \in \mathcal{C}([-\vartheta, 0], R_+^2)$, $\phi_i(\theta) > 0, i = 1, 2$, $\mathcal{C}$ denotes the Banach space of all continuous functions.

The structure of this paper is organized as follows: Section 2 formulates some basic definitions and lemmas of fractional calculus. Section 3 discusses the existence and uniqueness of the positive equilibrium employing contraction mapping theory and inequality techniques. When σ and τ are given different values, by analyzing the associated characteristic equation, some sufficient conditions are obtained to ensure the local stability of the positive equilibrium and the existence of Hopf bifurcation in Section 4. The numerical simulations are presented in Section 5. Finally, conclusions are given in Section 6.

2 Preliminaries

Definition 1 The Caputo fractional-order derivative of order $\alpha > 0$ for a continuous differentiable function $y(t) : [0, +\infty) \rightarrow R$ is defined as following form:

$${}^C_{t_0}\mathcal{D}_t^q y(t) = \frac{1}{\Gamma(m-q)} \int_{t_0}^t \frac{y^{(m)}(\tau)}{(t-\tau)^{q+1-m}} d\tau,$$

where $\Gamma(\cdot)$ denotes the Euler's Gamma function, m is the smallest positive integer not less than q , i.e. $m = \lceil q \rceil$. In particular, when $0 < q < 1$, such that

$${}^C_{t_0}\mathcal{D}_t^q y(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^t \frac{y'(\tau)}{(t-\tau)^q} d\tau.$$

The Laplace transform of the Caputo fractional-order derivative is given as

$$\mathcal{L}\{ {}^C_{t_0}\mathcal{D}_t^q y(t); s \} = s^q Y(s) - \sum_{k=0}^{m-1} s^{q-k-1} y^{(k)}(t_0), m = \lceil q \rceil.$$

where $Y(s) = \mathcal{L}(y(t))$.

Definition 2 The Riemann-Liouville fractional-order integral of order $q > 0$ for a continuous function $y(t) : [0, +\infty) \rightarrow R$ is defined as:

$${}^{RL}_{t_0}\mathcal{D}_t^{-q} y(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t \frac{y(\tau)}{(t-\tau)^{1-q}} d\tau.$$

Property 1 For a continuously differentiable function $y(t)$, the following equation holds

$${}^{RL}_{t_0}\mathcal{D}_t^{-q} [{}^C_{t_0}\mathcal{D}_t^q y(t)] = y(t) - \sum_{k=0}^{n-1} y^{(k)}(t_0) \frac{(t-t_0)^k}{k!}.$$

Property 2 Fractional calculus is linear, that is to say, for any constant a, b satisfies

$${}_{t_0}\mathcal{D}_t^q (af(t) + bg(t)) = a {}_{t_0}\mathcal{D}_t^q f(t) + b {}_{t_0}\mathcal{D}_t^q g(t).$$

Lemma 1 (Contraction mapping theory) Let (M, ρ) and $F : M \rightarrow M$ denote a non-empty complete metric space and a contracting map, respectively. If there exists $0 \leq w < 1$ such that

$$\forall x, y \in M : \rho(Fx, Fy) \leq w\rho(x, y),$$

then F has a unique fixed point in M .

Given an n -dimension linear fractional-order continuous dynamical system with delay

$$\begin{cases} {}^C_{t_0}\mathcal{D}_t^q Z(t) = AZ(t) + MZ(t-\tau), \\ Z(t) = \phi(t), t \in [t_0 - \tau, t_0], \end{cases} \quad (3)$$

where $Z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T$, $Z(t-\tau) = (z_1(t-\tau), z_2(t-\tau), \dots, z_n(t-\tau))^T$, $q = (0, 1)$, $A = (a_{ij})_{n \times n}$, $M = (m_{ij})_{n \times n}$. The corresponding characteristic matrix is

$$\Delta(s) = \begin{pmatrix} s^q - m_{11}e^{-s\tau} - a_{11} & -m_{12}e^{-s\tau} - a_{12} & \cdots & -m_{1n}e^{-s\tau} - a_{1n} \\ -m_{21}e^{-s\tau} - a_{21} & s^q - m_{22}e^{-s\tau} - a_{22} & \cdots & -m_{2n}e^{-s\tau} - a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -m_{n1}e^{-s\tau} - a_{n1} & -m_{n2}e^{-s\tau} - a_{n2} & \cdots & s^q - m_{nn}e^{-s\tau} - a_{nn} \end{pmatrix}.$$

If $\tau = 0$, the system (3) can be written as

$${}^C_{t_0} \mathcal{D}_t^q Z(t) = (A + M) Z(t), \quad (4)$$

It is generally known that the stability of the system is determined by the roots of characteristic polynomials. Then, the stability results of the above two systems are shown as follows.

Lemma 2 ([6]) For $q \in (0, 1)$, the zero solution of system (3) is Lyapunov asymptotically stable if and only if all the roots of $\det(\Delta(s)) = 0$ have negative real parts.

Remark 1 Based on Lemma 2, we can draw conclusion that the stability boundary of fractional-order delayed systems is the imaginary axis.

Lemma 3 The zero solution of system (4) is Lyapunov asymptotically globally stable if and only if all the eigenvalue satisfy $|\arg(\lambda)| > \frac{q\pi}{2}$ with $\lambda = s^q$.

Lemma 4 ([6]) For $q \in (0, 1)$, if all the eigenvalues of λs satisfy $|\arg(\lambda)| > \frac{q\pi}{2}$ for $\tau = 0$ and $\det(\Delta(s)) = 0$ has no purely imaginary roots for any $\tau > 0$, then the zero solution of system is Lyapunov asymptotically stable.

Remark 2 The stability analysis for the non-linear delayed fractional-order system is more complicated. At present, there is no mature theory to provide useful methods on non-linear delayed fractional-order dynamical systems. Thus, in general, the local stability of non-linear delayed fractional-order systems is researched with the stability theory of fractional-order delayed linear systems.

3 Existence and uniqueness of the positive solution

The existence and uniqueness of solution of system (2) is researched in the region $X \in \Omega$, where $\Omega = \{X \in \mathbb{R}^n : \max |x_i| \leq \kappa\}$. For $X \in \Omega$, the norm is defined by $\|X\| = \sum_{i=1}^n \sup_{t \in [0, t]} \{e^{-Nt} |x_i(t)|\}$, $N > 0$. For any $x, y, \bar{x}, \bar{y} \in \Omega$ satisfy the initial value condition of the system (2), the following inequalities hold.

$$(i) \quad e^{-Nt} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |x(s-\tau) - y(s-\tau)| ds \leq \frac{e^{-N\tau}}{N^q} \|x - y\|.$$

Obviously,

$$e^{-Nt} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |x(s) - y(s)| ds \leq \frac{1}{N^q} \|x - y\|.$$

$$(ii) \quad e^{-Nt} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |x(s-\tau)y(s-\tau) - \bar{x}(s-\tau)\bar{y}(s-\tau)| ds \\ \leq \frac{\kappa e^{-N\tau}}{N^q} (\|x - \bar{x}\| + \|y - \bar{y}\|).$$

The detailed proof process is shown in Appendix A.

Theorem 1 For any initial value $X(0) = (I_h(0), I_m(0)) \in \mathcal{C}([-\vartheta, 0], R_+^2)$, there exists a unique positive solution $X = (I_h, I_m)$ to system (2) for all $t > 0$.

Proof Using the properties of fractional calculus, the solution of the system (2) satisfies

$$\begin{aligned} I_h(t) &= \phi_1(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \{abkI_m(s-\tau)[1-I_h(s-\tau)] - (\mu+\gamma)I_h(s)\} ds, \\ I_m(t) &= \phi_2(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \{acI_h(s-\sigma)[1-I_m(s-\sigma)] - \delta I_m(s)\} ds. \end{aligned}$$

Constructing a mapping $F(X) = (F_1(X), F_2(X))$,

$$\begin{aligned} F_1(X) &= \phi_1(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \{abkI_m(s-\tau)[1-I_h(s-\tau)] - (\mu+\gamma)I_h(s)\} ds, \\ F_2(X) &= \phi_2(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \{acI_h(s-\sigma)[1-I_m(s-\sigma)] - \delta I_m(s)\} ds. \end{aligned}$$

for any $X, \tilde{X} \in \Omega$,

$$\begin{aligned} e^{-Nt}|F(X) - F(\tilde{X})| &= e^{-Nt} \left| F_1(X) - F_1(\tilde{X}) + F_2(X) - F_2(\tilde{X}) \right| \\ &\leq abke^{-Nt} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left| I_m(s-\tau) - \tilde{I}_m(s-\tau) \right| ds \\ &\quad + abke^{-Nt} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left| I_m(s-\tau)I_h(s-\tau) - \tilde{I}_m(s-\tau)\tilde{I}_h(s-\tau) \right| ds \\ &\quad + (\mu+\gamma)e^{-Nt} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left| I_h(s) - \tilde{I}_h(s) \right| ds \\ &\quad + ace^{-Nt} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left| I_h(s-\sigma) - \tilde{I}_h(s-\sigma) \right| ds \\ &\quad + ace^{-Nt} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left| I_m(s-\sigma)I_h(s-\sigma) - \tilde{I}_m(s-\sigma)\tilde{I}_h(s-\sigma) \right| ds \\ &\quad + \delta e^{-Nt} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left| I_m(s) - \tilde{I}_m(s) \right| ds \\ &\leq \frac{abke^{-N\tau}}{N^q} \|I_m - \tilde{I}_m\| + \frac{abk\kappa e^{-N\tau}}{N^q} \left(\|I_m - \tilde{I}_m\| + \|I_h - \tilde{I}_h\| \right) + \frac{\mu+\gamma}{N^q} \|I_h - \tilde{I}_h\| \\ &\quad + \frac{ace^{-N\sigma}}{N^q} \|I_h - \tilde{I}_h\| + \frac{ac\kappa e^{-N\sigma}}{N^q} \left(\|I_m - \tilde{I}_m\| + \|I_h - \tilde{I}_h\| \right) + \frac{\delta}{N^q} \|I_m - \tilde{I}_m\| \\ &= \frac{(1+\kappa)abke^{-N\tau} + ac\kappa e^{-N\sigma} + \delta}{N^q} \|I_m - \tilde{I}_m\| + \frac{(1+\kappa)ace^{-N\sigma} + abk\kappa e^{-N\tau} + \mu + \gamma}{N^q} \|I_h - \tilde{I}_h\| \\ &\leq \mathcal{M} \|X - \tilde{X}\| \end{aligned}$$

where $\mathcal{M} = \max\left\{ \frac{(1+\kappa)abke^{-N\tau} + ac\kappa e^{-N\sigma} + \delta}{N^q}, \frac{(1+\kappa)ace^{-N\sigma} + abk\kappa e^{-N\tau} + \mu + \gamma}{N^q} \right\}$. When N is large enough, it can make $\mathcal{M} < 1$. Thus it can be inferred

$$\|F(X) - F(\tilde{X})\| < \|X - \tilde{X}\|,$$

this implies that the mapping $F : X \rightarrow X$ is a contraction mapping. Based on Lemma 1, the system (2) has a unique positive solution.

The solution of the system (2) satisfies the following equation:

$$\begin{cases} abkI_m^*[1-I_h^*] - (\mu+\gamma)I_h^* = 0, \\ acI_h^*[1-I_m^*] - \delta I_m^* = 0, \end{cases}$$

The system always has a trivial equilibrium point $E_0 = (0, 0)$. If the basic reproduction number $\mathcal{R}_0 = \frac{a^2 bck}{(\mu+\gamma)\delta} > 1$, the system admits a unique positive equilibrium point E^* satisfying initial conditions, where $E^* = (I_h^*, I_m^*) = \left(\frac{a^2 bck - (\mu+\gamma)\delta}{ac(abk + \mu + \gamma)}, \frac{a^2 bck - (\mu+\gamma)\delta}{abk(ac + \delta)} \right)$. In the following, we will only consider the positive equilibrium point in accordance with the real situation.

4 Stability and the Hopf bifurcation

Assume that $u_1(t) = I_h(t) - I_h^*$, $u_2(t) = I_m(t) - I_m^*$, the linearized of the system (2) at the origin takes the form:

$$\begin{cases} D^q u_1(t) = -(\gamma + \mu)u_1(t) - abkI_m^* u_1(t - \tau) + abk(1 - I_h^*)u_2(t - \tau), \\ D^q u_2(t) = -\delta u_2(t) - acI_h^* u_2(t - \sigma) + ac(1 - I_m^*)u_1(t - \sigma). \end{cases} \quad (5)$$

According to the Laplace transform of the Caputo fractional-order derivative in Definition 1, yields

$$\begin{aligned} s^q U_1(s) - s^{q-1} \phi_1(0) &= -(\gamma + \mu)U_1(s) - abkI_m^* e^{-s\tau} \left[U_1(s) + \int_{-\tau}^0 e^{-st} \phi_1(t) dt \right] \\ &\quad + abk(1 - I_h^*) e^{-s\tau} \left[U_2(s) + \int_{-\tau}^0 e^{-st} \phi_2(t) dt \right], \\ s^q U_2(s) - s^{q-1} \phi_2(0) &= -\delta U_2(s) - acI_h^* e^{-s\sigma} \left[U_2(s) + \int_{-\sigma}^0 e^{-st} \phi_2(t) dt \right] \\ &\quad + ac(1 - I_m^*) e^{-s\sigma} \left[U_1(s) + \int_{-\sigma}^0 e^{-st} \phi_1(t) dt \right], \end{aligned} \quad (6)$$

where $U_i(s) = \mathcal{L}(u_i(t))$, $i = 1, 2$.

The above Eq. (6) is rewritten as

$$\Delta(s) \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix} = \begin{pmatrix} d_1(s) \\ d_2(s) \end{pmatrix}, \quad (7)$$

in which

$$\begin{aligned} d_1(s) &= s^{q-1} \phi_1(0) - abkI_m^* e^{-s\tau} \int_{-\tau}^0 e^{-st} \phi_1(t) dt + abk(1 - I_h^*) e^{-s\tau} \int_{-\tau}^0 e^{-st} \phi_2(t) dt, \\ d_2(s) &= s^{q-1} \phi_2(0) - acI_h^* e^{-s\sigma} \int_{-\sigma}^0 e^{-st} \phi_2(t) dt + ac(1 - I_m^*) e^{-s\sigma} \int_{-\sigma}^0 e^{-st} \phi_1(t) dt, \\ \Delta(s) &= \begin{pmatrix} s^q + \gamma + \mu + abkI_m^* e^{-s\tau} & -abk(1 - I_h^*) e^{-s\tau} \\ -ac(1 - I_m^*) e^{-s\sigma} & s^q + \delta + acI_h^* e^{-s\sigma} \end{pmatrix} \end{aligned}$$

thus,

$$\det \Delta(s) = s^{2q} + k_1 s^q + k_2 + (k_3 s^q + k_4) e^{-s\tau} + (k_5 s^q + k_6) e^{-s\sigma} + k_7 e^{-s(\tau+\sigma)} = 0, \quad (8)$$

where

$$\begin{aligned} k_1 &= \delta + \mu + \gamma, k_2 = \delta(\mu + \gamma), k_3 = abkI_m^*, k_4 = abk\delta I_m^*, \\ k_5 &= acI_h^*, k_6 = (\mu + \gamma)acI_h^*, k_7 = a^2 bck(I_h^* + I_m^* - 1). \end{aligned}$$

4.1 Case I: The time delays $\tau = \sigma = 0$

In this case, the characteristic equation (8) is rewritten as

$$\det \Delta(s) = (s^q)^2 + (k_1 + k_3 + k_5) s^q + k_2 + k_4 + k_6 + k_7 = 0.$$

where

$$\begin{aligned} k_1 + k_3 + k_5 &= \frac{abk(ac+\delta)^2 + ac(abk+\gamma+\mu)^2}{(ac+\delta)(abk+\gamma+\mu)} > 0; \\ k_2 + k_4 + k_6 + k_7 &= \delta(\mu + \gamma)(\mathcal{R}_0 - 1) > 0. \end{aligned}$$

Obviously, the two roots of the above equation are located in the left half-plane. Thus the following theorem is established.

Theorem 2 If $\mathcal{R}_0 > 1$ holds, $E^* = (I_h^*, I_m^*)$ is Lyapunov asymptotically globally stable for the time delays $\tau = \sigma = 0$.

Remark 3 If $\tau = \sigma = 0$, all characteristic roots of Eq. (8) lie to the left of the imaginary axis. Remarkably, the time delay may affect the stability of the system and even induce oscillations, i.e., the root of Eq. (8) may cross the imaginary axis and enter the right half-plane as time delays increase. For this reason, the purely imaginary root is an important critical case, in which characteristic roots may enter the left half-plane or the right half-plane under a small perturbation.

For further analysis, the fourth-order polynomial equation with real coefficients is first introduced.

$$f(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0. \quad (9)$$

Define the matrices

$$N_1 = a_3, N_2 = \begin{bmatrix} a_3 & a_1 \\ 1 & a_2 \end{bmatrix}, N_3 = \begin{bmatrix} a_3 & a_1 & 0 \\ 1 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{bmatrix}, N_4 = \begin{bmatrix} a_3 & a_1 & 0 & 0 \\ 1 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & 1 & a_2 & a_0 \end{bmatrix},$$

with the corresponding determinants

$$\begin{aligned} |N_1| &= a_3, |N_2| = a_2a_3 - a_1, |N_3| = a_3(a_1a_2 - a_0a_3) - a_1^2, \\ |N_4| &= a_0[a_3(a_1a_2 - a_0a_3) - a_1^2]. \end{aligned}$$

Lemma 5 For Eq. (9), the following results hold:

- I) If $a_0 < 0$, then the polynomial $f(x)$ has at least one positive real root.
- II) If $a_i > 0 (i = 0, 1, 2, 3)$, then the polynomial $f(x)$ has no positive real root.
- III) The polynomial $f(x)$ has no positive real root if and only if the following three conditions are satisfied.
 - i) $a_0 > 0$ ii) $a_2 > 0$ or $a_1 > 0, a_3 > 0$
 - iii) $|N_2| > 0, |N_4| > 0$ or $|N_1| > 0, |N_3| > 0$.
- IV) If $a_0 \geq 0$, then the necessary condition for the polynomial $f(x)$ has at least one positive real root is any one of the following conditions.
 - i) $a_1 < 0$ ii) $a_2 < 0$ iii) $a_3 < 0$ iv) $|N_1| < 0$ v) $|N_2| < 0$ vi) $|N_3| < 0$ vii) $|N_4| < 0$.

Proof I) Obviously, $f(0) = a_0 < 0$ and $\lim_{x \rightarrow +\infty} f(x) = +\infty$ means that exists $x_0 \in (0, \infty)$ such that $f(x_0) = 0$.

II) Clearly, $f(0) = a_0 > 0$ and $f'(x) > 0$ for all $x > 0$ means that $f(x)$ is monotonic increasing function for $x > 0$. Hence, Eq. (9) has no positive real root.

III) It is only necessary to prove that $f(x)$ is a Hurwitz polynomial whose zeros are all in the left half-plane. Only one of the cases is proved here, and the method of proof for other cases is similar, that is, to prove if $a_0 > 0, a_2 > 0, |N_2| > 0, |N_4| > 0$, the polynomial $f(x)$ has no positive real root. The detailed derivation process is as follows

$$|N_4| = a_0|N_3| > 0 \Rightarrow |N_3| > 0 \Rightarrow a_1(a_2a_3 - a_1) = a_1|N_2| > a_0a_3^2 > 0 \Rightarrow a_1 > 0,$$

$$|N_2| = a_2a_3 - a_1 > 0 \Rightarrow a_2a_3 > a_1 > 0 \Rightarrow a_3 > 0 \Rightarrow |N_1| > 0.$$

Thus, the polynomial $f(x)$ is a Hurwitz polynomial based on the Hurwitz criterion, that is, $f(x)$ has no positive real root.

IV) The proof is derived from III) above.

4.2 Case II: The time delays $\tau = 0$, $\sigma > 0$

In this case, the characteristic equation (8) can be rewritten as

$$\det \Delta(s) = E_1(s) + E_2(s)e^{-s\sigma} = 0, \quad (10)$$

where $E_1(s) = s^{2q} + (k_1 + k_3)s^q + k_2 + k_4$, $E_2(s) = k_5s^q + k_6 + k_7$. Suppose that $s = iw$ ($w > 0$) is a purely imaginary root of Eq. (10), then it follows that

$$\begin{cases} A_2 \cos(w\sigma) + B_2 \sin(w\sigma) = -A_1, \\ B_2 \cos(w\sigma) - A_2 \sin(w\sigma) = -B_1, \end{cases} \quad (11)$$

where

$$\begin{aligned} A_1 &= \operatorname{Re}(E_1(s)) = w^{2q} \cos q\pi + b_1 w^q \cos\left(\frac{q\pi}{2}\right) + b_2; \\ B_1 &= \operatorname{Im}(E_1(s)) = w^{2q} \sin q\pi + b_1 w^q \sin\left(\frac{q\pi}{2}\right); \\ A_2 &= \operatorname{Re}(E_2(s)) = k_5 w^q \cos\frac{q\pi}{2} + b_3; \\ B_2 &= \operatorname{Im}(E_2(s)) = k_5 w^q \sin\left(\frac{q\pi}{2}\right); \end{aligned}$$

Let $b_1 = k_1 + k_3$, $b_2 = k_2 + k_4$, $b_3 = k_6 + k_7$ for simplicity. Squaring the both sides of Eq. (11) and adding them yields that

$$w^{4q} + a_3 w^{3q} + a_2 w^{2q} + a_1 w^q + a_0 = 0, \quad (12)$$

where

$$\begin{aligned} a_3 &= 2b_1 \cos\left(\frac{q\pi}{2}\right); \\ a_2 &= b_1^2 + 2b_2 \cos(q\pi) - k_5^2 \geq b_1^2 - 2b_2 - k_5^2 = (\mu + \gamma + abkI_m^*)^2 + \frac{\delta^2(1 - 2I_m^*)}{(1 - I_m^*)^2}; \\ a_1 &= 2(b_1 b_2 - k_5 b_3) \cos\left(\frac{q\pi}{2}\right) \\ &= 2 \left[\delta^2(\mu + \gamma) + \delta(\mu + \gamma + abkI_m^*)^2 + (\mu + \gamma + abk)I_m^* \frac{\delta^2(1 - 2I_m^*)}{(1 - I_m^*)^2} \right] \cos\left(\frac{q\pi}{2}\right); \\ a_0 &= b_2^2 - b_3^2 = \delta^2(\mu + \gamma)^2 (\mathcal{R}_0 - 1) [1 + (1 - 2I_m^*)\mathcal{R}_0]. \end{aligned}$$

It is apparent that the coefficient $a_3 > 0$ for $q \in (0, 1)$. If the condition $a^2 bck \leq abk\delta + 2(\mu + \gamma)\delta$ holds, other the coefficients satisfy

$$a_2 > 0, a_1 > 0, a_0 > 0.$$

This implies that Eq. (12) has no positive real root contradicting this assumption of $s = iw$ ($w > 0$). According to Lemma 4, this allowed us to conclude that $E^* = (I_h^*, I_m^*)$ is Lyapunov asymptotically stable for any $\sigma > 0$, namely independent of the delay.

Note that $a_0 < 0$ if $(\mu + \gamma)\delta^2 + a^2 bck + 3ac(\mu + \gamma)\delta < a^3 bc^2 k$, Eq. (12) has at least one positive real root w_0 according to Lemma 5. Let $s(\sigma) = \varrho(\sigma) + iw(\sigma)$ be the eigenvalue root of Eq. (10) such that $\varrho(\sigma^*) = 0$, $w(\sigma^*) = w_0$. From Eq. (11), we define

$$\sigma^* = \frac{1}{w_0} [\arccos H(w_0) + 2k\pi], k = 0, 1, 2, \dots \quad (13)$$

where

$$H(w_0) = - \frac{k_5 \cos\left(\frac{q\pi}{2}\right) w_0^{3q} + (b_1 k_5 + b_3 \cos(q\pi)) w_0^{2q} + (b_1 b_3 + b_2 k_5) \cos\left(\frac{q\pi}{2}\right) w_0^q + b_2 b_3}{k_5^2 w_0^{2q} + 2k_5 b_3 \cos\left(\frac{q\pi}{2}\right) w_0^q + b_3^2}$$

Then, substituting $s = s(\sigma)$ into Eq. (10) and taking the derivative with respect to σ on two sides yields

$$\frac{ds}{d\sigma} = \frac{(k_5 s^q + b_3) s e^{-s\sigma}}{2qs^{2q-1} + b_1 qs^{q-1} + k_5 qs^{q-1} e^{-s\sigma} - (k_5 s^q + b_3) \sigma e^{-s\sigma}},$$

Applying Eq. (10), it is straightforward to obtain

$$\begin{aligned} \left[\frac{ds}{d\sigma} \right]^{-1} &= \frac{2qs^{2q} + b_1 qs^q + k_5 qs^q e^{-s\sigma}}{s^2 (k_5 s^q + b_3) e^{-s\sigma}} - \frac{\sigma}{s} \\ &= -\frac{q}{s^2} \left[\frac{2s^{2q} + b_1 s^q}{s^{2q} + b_1 s^q + b_2} - \frac{k_5 s^q}{k_5 s^q + b_3} \right] - \frac{\sigma}{s} \\ &= -\frac{q}{s^2} \left[\frac{s^{2q} - b_2}{s^{2q} + b_1 s^q + b_2} + \frac{b_3}{k_5 s^q + b_3} \right] - \frac{\sigma}{s}, \end{aligned}$$

thus,

$$\begin{aligned} \operatorname{Re} \left[\frac{ds}{d\sigma} \right]^{-1} \Big|_{s=iw_0} &= \operatorname{Re} \left[-\frac{q}{s^2} \left(\frac{s^{2q} - b_2}{s^{2q} + b_1 s^q + b_2} + \frac{b_3}{k_5 s^q + b_3} \right) \right] \Big|_{s=iw_0} \\ &= -\frac{q}{s^2} \left[\operatorname{Re} \left(\frac{s^{2q} - b_2}{s^{2q} + b_1 s^q + b_2} \right) + \operatorname{Re} \left(\frac{b_3}{k_5 s^q + b_3} \right) \right] \Big|_{s=iw_0} \\ &= \frac{q}{w_0^2} \left[\frac{w_0^{4q} + b_1 \cos\left(\frac{q\pi}{2}\right) w_0^{3q} + (k_5 b_3 - b_1 b_2) \cos\left(\frac{q\pi}{2}\right) w_0^q + b_3^2 - b_2^2}{k_5^2 w_0^{2q} + 2k_5 b_3 \cos\left(\frac{q\pi}{2}\right) w_0^q + b_3^2} \right]. \end{aligned}$$

By some calculations, it is not difficult to prove that the above expression is always positive. Hence, the transversality condition holds, i.e., the system undergoes a Hopf bifurcation at E^* when $\sigma = \sigma^*$. Summarising the foregoing analysis, the following results can be obtained.

Theorem 3 *For the fractional-order malaria epidemic system (2), considering $\sigma > 0, \tau = 0$, the following results hold:*

- 1) *If the condition $a^2 bck \leq abk\delta + 2(\mu + \gamma)\delta$ holds, E^* is locally asymptotically stable for all $\sigma > 0$.*
- 2) *If the condition $(\mu + \gamma)\delta^2 + a^2 bck + 3ac(\mu + \gamma)\delta < a^3 bc^2 k$ holds, E^* is locally asymptotically stable for $\sigma \in (0, \sigma^*)$ and unstable for $\sigma > \sigma^*$, the system (2) undergoes a Hopf bifurcation when $\sigma = \sigma^*$, i.e., it produces a series of periodic solutions from E^* as σ passes through the critical value σ^* .*

Remark 4 The purely imaginary roots appear in pairs, so we focused only on whether there is root pass through the positive imaginary axis for simplicity (i.e. $s = iw, w > 0$).

Remark 5 When $q = 1$, the integer-order model (1) still satisfies Theorem 3, meaning that the fractional-order delayed malaria epidemic model generalizes the classic integer-order Ross-Macdonald model into an arbitrary order. Moreover, compared with the classic integer-order model, the research of the fractional-order model is more complex.

4.3 Case III: The time delays $\tau > 0$, $\sigma = 0$

In this case, the rewritten characteristic equation (8) satisfies

$$s^{2q} + c_1 s^q + c_2 + (k_3 s^q + c_3) e^{-s\tau} = 0, \quad (14)$$

where $c_1 = k_1 + k_5$, $c_2 = k_2 + k_6$, $c_3 = k_4 + k_7$. The analytical approach is identical to that in Case II, assuming that $s = iw$, $w > 0$ is the pure imaginary root of Eq. (14) and substituting s into Eq. (14) yields

$$\begin{cases} A'_2 \cos(w\tau) + B'_2 \sin(w\tau) = -A'_1 \\ B'_2 \cos(w\tau) - A'_2 \sin(w\tau) = -B'_1 \end{cases} \quad (15)$$

where

$$\begin{aligned} A'_1 &= w^{2q} \cos q\pi + c_1 w^q \cos\left(\frac{q\pi}{2}\right) + c_2; & A'_2 &= k_3 w^q \cos\frac{q\pi}{2} + c_3; \\ B'_1 &= w^{2q} \sin q\pi + c_1 w^q \sin\left(\frac{q\pi}{2}\right); & B'_2 &= k_3 w^q \sin\left(\frac{q\pi}{2}\right). \end{aligned}$$

By squaring and adding the two equations of Eq. (15) to eliminate τ variable, one can derive

$$w^{4q} + a'_3 w^{3q} + a'_2 w^{2q} + a'_1 w^q + a'_0 = 0, \quad (16)$$

where

$$\begin{aligned} a'_3 &= 2c_1 \cos\left(\frac{q\pi}{2}\right); \\ a'_2 &= c_1^2 + 2c_2 \cos(q\pi) - k_3^2 \geq c_1^2 - 2c_2 - k_3^2 = (\delta + acI_h^*)^2 + (\mu + \gamma)^2 \frac{1 - 2I_h^*}{(1 - I_h^*)^2}; \\ a'_1 &= 2(c_1 c_2 - k_3 c_3) \cos\left(\frac{q\pi}{2}\right) \\ &= 2 \left[(\mu + \gamma)^2 \delta + (\mu + \gamma)(\delta + acI_h^*)^2 + (\delta + ac)(\mu + \gamma)^2 I_h^* \frac{1 - 2I_h^*}{(1 - I_h^*)^2} \right] \cos\left(\frac{q\pi}{2}\right); \\ a'_0 &= c_2^2 - c_3^2 = \delta^2(\mu + \gamma)^2 (\mathcal{R}_0 - 1) [1 + (1 - 2I_h^*)\mathcal{R}_0]. \end{aligned}$$

It is noted that $a'_3 > 0$ for $q \in (0, 1)$, whereas other coefficients may be positive or negative. Suppose that the condition $(ac + 2\delta)(\mu + \gamma) > a^2 bck$ holds, then it can be deduced that $a'_2 > 0$, $a'_1 > 0$, $a'_0 > 0$. Consequently, there is no purely imaginary root of the characteristic equation (14). If the condition $a^3 b^2 c k^2 > abk(\mu + \gamma)(ac + 3\delta) + \delta(\mu + \gamma)^2$ holds, we obtain $a'_0 < 0$, then at least one root is a positive root w_0 for Eq. (16). From Eq. (15), define

$$\tau^* = \frac{1}{w_0} [\arccos G(w_0) + 2k\pi], k = 0, 1, 2, \dots$$

where

$$G(w_0) = -\frac{k_3 \cos\left(\frac{q\pi}{2}\right) w_0^{3q} + (c_1 k_3 + c_3 \cos(q\pi)) w_0^{2q} + (c_1 c_3 + c_2 k_3) \cos\left(\frac{q\pi}{2}\right) w_0^q + c_2 c_3}{k_3^2 w_0^{2q} + 2k_3 c_3 \cos\left(\frac{q\pi}{2}\right) w_0^q + c_3^2}.$$

Let $s(\tau) = \varrho(\tau) + iw(\tau)$ be the root of Eq. (14) near $\tau = \tau^*$, satisfying $\varrho(\tau^*) = 0$, $w(\tau^*) = w_0$. The following transversality condition is obtained by calculation.

$$\operatorname{Re} \left[\frac{ds}{d\tau} \right]^{-1} \bigg|_{(s=iw_0)} > 0.$$

Similar to the analysis process in Case II, so we omit it. Summarizing the above discussion, the following stability and bifurcation results of the system (2) are given.

Theorem 4 For the fractional-order malaria epidemic system (2), considering $\tau > 0, \sigma = 0$, the following results hold:

- a) If the condition $(ac + 2\delta)(\mu + \gamma) > a^2bck$ holds, E^* is locally asymptotically stable for all $\tau > 0$.
- b) If the condition $a^3b^2ck^2 > abk(\mu + \gamma)(ac + 3\delta) + \delta(\mu + \gamma)^2$ holds, the critical value τ^* can be defined the stability boundary, that is, E^* is locally asymptotically stable for $\tau \in (0, \tau^*)$ and lose stability when $\tau > \tau_0$. Furthermore, the fractional-order system (2) occurs a Hopf bifurcation at E^* near $\tau = \tau^*$.

Remark 6 Many papers have ignored the incubation period of Plasmodium in humans and only study the incubation period of mosquitoes for simplicity. This is because many authors perceived that the incubation period of Plasmodium in humans is about $\tau = 10 - 100$ days (see [12]), accounting for a small part of human life. Here the effect of the incubation period of Plasmodium in the human body for the dynamic behavior of the disease is considered. Furthermore, the analysis of Case II and Case III facilitate the comparison of the influence of the incubation period of Plasmodium in humans and mosquitoes on the dynamic behavior of malaria.

4.4 Case IV: The time delays $\tau > 0, \sigma > 0$ and $\sigma \neq \tau$

For the characteristic equation (8), the time delay τ is fixed in its stable interval, and σ is taken as a parameter. Assume that $s = iw = w(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$ is a purely imaginary root of characteristic equation (8) with $w > 0$, if and only if w satisfies

$$\begin{cases} A_2'' \cos(w\sigma) + B_2'' \sin(w\sigma) = -A_1'' \\ B_2'' \cos(w\sigma) - A_2'' \sin(w\sigma) = -B_1'' \end{cases} \quad (17)$$

where

$$\begin{aligned} A_1'' &= w^{2q} \cos(q\pi) + k_1 w^q \cos\left(\frac{q\pi}{2}\right) + k_3 w^q \cos\left(\frac{q\pi}{2} - w\tau\right) + k_4 \cos(w\tau) + k_2; \\ B_1'' &= w^{2q} \sin(q\pi) + k_1 w^q \sin\left(\frac{q\pi}{2}\right) + k_3 w^q \sin\left(\frac{q\pi}{2} - w\tau\right) - k_4 \sin(w\tau); \\ A_2'' &= k_5 w^q \cos\left(\frac{q\pi}{2}\right) + k_7 \cos(w\tau) + k_6; \\ B_2'' &= k_5 w^q \sin\left(\frac{q\pi}{2}\right) - k_7 \sin(w\tau). \end{aligned}$$

Taking square on the both sides of (17) and adding them, we have that

$$w^{4q} + a_3'' w^{3q} + a_2'' w^{2q} + a_1'' w^q + a_0'' = 0, \quad (18)$$

where

$$\begin{aligned} a_3'' &= 2 \left[k_3 \cos\left(\frac{q\pi}{2} + w\tau\right) + k_1 \cos\left(\frac{q\pi}{2}\right) \right], \\ a_2'' &= 2k_4 \cos(q\pi + w\tau) + 2k_1 k_3 \cos(w\tau) + 2k_2 \cos(q\pi) + k_1^2 + k_3^2 - k_5^2, \\ a_1'' &= 2 \left[(k_1 k_4 - k_5 k_7) \cos\left(\frac{q\pi}{2} + w\tau\right) + k_2 k_3 \cos\left(\frac{q\pi}{2} - w\tau\right) + k_1 k_2 + k_3 k_4 - k_5 k_6 \right] \cos\left(\frac{q\pi}{2}\right) \\ a_0'' &= k_2^2 + k_4^2 - k_6^2 - k_7^2 + 2(k_2 k_4 - k_6 k_7) \cos(w\tau). \end{aligned}$$

Suppose that $a_0'' < 0$, then Eq. (18) has at least one positive root w_0 . As long as all the parameters are given, it is easy to calculate the value of w_0 by means of numerical software Maple. From Eq. (17), we have

$$\sigma_* = \frac{1}{w_0} \left\{ \arccos\left(-\frac{\zeta_1}{\zeta_2}\right) + 2k\pi \right\}, k = 0, 1, 2, \dots \quad (19)$$

where

$$\begin{aligned}\zeta_1 &= k_5 w_0^{3q} \cos\left(\frac{q\pi}{2}\right) + [k_7 \cos(w_0\tau + q\pi) + k_3 k_5 \cos(w_0\tau) + k_6 \cos(q\pi) + k_1 k_5] w_0^{2q} \\ &\quad + \left[(k_1 k_7 + k_4 k_5) \cos\left(w_0\tau + \frac{q\pi}{2}\right) + k_3 k_6 \cos\left(w_0\tau - \frac{q\pi}{2}\right) + (k_1 k_6 + k_2 k_5 + k_3 k_7) \right. \\ &\quad \left. \cos\left(\frac{q\pi}{2}\right)\right] w_0^q + k_2 k_6 + k_4 k_7 + (k_2 k_7 + k_4 k_6) \cos(w_0\tau), \\ \zeta_2 &= k_5^2 w_0^{2q} + 2k_5 \left[k_7 \cos\left(w_0\tau + \frac{q\pi}{2}\right) + k_6 \cos\left(\frac{q\pi}{2}\right)\right] w_0^q + k_6^2 + k_7^2 + 2k_6 k_7 \cos(w_0\tau).\end{aligned}$$

By the implicit function theorem, the derivative with respect to σ for Eq. (8) yields

$$\left[\frac{ds}{d\sigma}\right]^{-1} = \frac{2qs^{2q} + k_1 qs^q + k_3 qs^q e^{-s\tau} + k_5 qs^q e^{-s\sigma} - \tau(k_3 s^q + k_4 + k_7 e^{-s\sigma}) e^{-s\tau} s}{s^2 [(k_5 s^q + k_6) e^{-s\sigma} + k_7 e^{-s(\tau+\sigma)}]} - \frac{\sigma}{s}$$

Consequently,

$$\operatorname{Re} \left[\frac{ds}{d\sigma} \right]^{-1} \Big|_{s=iw_0} = -\frac{1}{w_0^2} \frac{M_1 N_1 + M_2 N_2}{N_1^2 + N_2^2}$$

in which

$$\begin{aligned}N_1 &= k_5 w_0^q \cos\left(\frac{q\pi}{2} - w_0\sigma\right) + k_6 \cos(w_0\sigma) + k_7 \cos w_0(\tau + \sigma); \\ N_2 &= k_5 w_0^q \sin\left(\frac{q\pi}{2} - w_0\sigma\right) - k_6 \sin(w_0\sigma) - k_7 \sin w_0(\tau + \sigma); \\ M_1 &= q \left[2w_0^{2q} \cos(q\pi) + k_1 w_0^q \cos\left(\frac{q\pi}{2}\right) + k_3 w_0^q \cos\left(\frac{q\pi}{2} - w_0\tau\right) + k_5 w_0^q \cos\left(\frac{q\pi}{2} - w_0\sigma\right) \right] \\ &\quad - \tau w_0 \left[k_3 w_0^q \sin\left(w_0\tau - \frac{q\pi}{2}\right) + k_4 \sin(w_0\tau) + k_7 \sin w_0(\tau + \sigma) \right]; \\ M_2 &= q \left[2w_0^{2q} \sin(q\pi) + k_1 w_0^q \sin\left(\frac{q\pi}{2}\right) + k_3 w_0^q \sin\left(\frac{q\pi}{2} - w_0\tau\right) + k_5 w_0^q \sin\left(\frac{q\pi}{2} - w_0\sigma\right) \right] \\ &\quad - \tau w_0 \left[k_3 w_0^q \cos\left(w_0\tau - \frac{q\pi}{2}\right) + k_4 \cos(w_0\tau) + k_7 \cos w_0(\tau + \sigma) \right].\end{aligned}$$

Suppose that

$$(\mathbf{H1}) \quad M_1 N_1 + M_2 N_2 < 0.$$

Thus, by the Hopf bifurcation theorem for fractional-order dynamical systems in [13], we can draw the following theorem.

Theorem 5 Assume that $(\mathbf{H1})$ holds, $a_0'' < 0$ and $\tau \in [0, \tau^*)$ are satisfied, then E^* is locally asymptotically stable for all $\sigma \in (0, \sigma_*)$, and the fractional-order system (2) undergoes a Hopf bifurcation at E^* when $\sigma = \sigma_*$.

Remark 7 The above four Cases can be contrasted with each other to observe the influence of different time delays on the disease transmission steady-state by the real data simulation.

4.5 Case V: The time delays $\tau = \sigma = \tau > 0$

Also, the incubation period of Plasmodium in humans and mosquitoes may be equal. Consequently, it appears necessary to research the influence on disease spread for $\sigma = \tau$. In this case, we assume that the time delays $\sigma = \tau = \tau$, then the characteristic equation (8) becomes the following the form

$$(s^{2q} + k_1 s^q + k_2) e^{s\tau} + d_1 s^q + d_2 + k_7 e^{-s\tau} = 0, \quad (20)$$

where $d_1 = k_3 + k_5, d_2 = k_4 + k_6$. Let $s = iw (w > 0)$ be the root of Eq. (20) and substitute it into the above equation, we have

$$\begin{cases} (A_2 + k_7) \cos(w\tau) - B_2 \sin(w\tau) = -A_1 \\ (A_2 - k_7) \sin(w\tau) + B_2 \cos(w\tau) = -B_1 \end{cases} \quad (21)$$

where $A_1 = d_1 w^q \cos(\frac{q\pi}{2}) + d_2, B_1 = d_1 w^q \sin(\frac{q\pi}{2}), A_2 = w^{2q} \cos(q\pi) + k_1 w^q \cos(\frac{q\pi}{2}) + k_2, B_2 = w^{2q} \sin(q\pi) + k_1 w^q \sin(\frac{q\pi}{2})$. It is obvious that

$$\begin{cases} \cos(w\tau) = \frac{k_7 A_1 - A_1 A_2 - B_1 B_2}{A_2^2 + B_2^2 - k_7^2}, \\ \sin(w\tau) = \frac{A_1 B_2 - A_2 B_1 - k_7 B_1}{A_2^2 + B_2^2 - k_7^2}. \end{cases} \quad (22)$$

Eliminating τ by $\sin^2(w\tau) + \cos^2(w\tau) = 1$, we have

$$w^{8q} + a_7 w^{7q} + a_6 w^{6q} + \cdots + a_1 w^q + a_0 = 0, \quad (23)$$

where

$$\begin{aligned} a_7 &= 4k_1 \cos(\frac{q\pi}{2}), \\ a_6 &= 2(k_1^2 + 2k_2) \cos(q\pi) + 4k_1^2 - d_1^2, \\ a_5 &= -2 \cos(\frac{q\pi}{2}) \left[(d_1^2 + 2k_2) k_1 - 2k_1^3 + d_1 d_2 - 8k_1 k_2 \cos^2(\frac{q\pi}{2}) \right], \\ a_4 &= k_1^4 + (4k_2 - d_1^2) k_1^2 - 2d_1 d_2 k_1 + 2d_1^2 k_7 - d_2^2 + 2k_2^2 - 2k_7^2 \\ &\quad + [8k_1^2 k_2 - 2d_1^2 k_2 - 2d_1 d_2 k_1 + 4k_2^2 \cos(q\pi)] \cos(q\pi), \\ a_3 &= -2 \cos(\frac{q\pi}{2}) \left[(k_1^2 + 2k_2 \cos(q\pi)) (d_1 d_2 - 2k_1 k_2) - 2d_1 d_2 k_7 \right. \\ &\quad \left. + k_1 (d_1^2 (k_2 - k_7) + d_2^2 + 2k_7^2 - 2k_2^2) \right], \\ a_2 &= (4k_2^2 - d_2^2 - 2k_7^2) k_1^2 + 2d_1 d_2 k_1 (2k_7 - k_2) - d_1^2 (k_2^2 + k_7^2) \\ &\quad + [2k_1 k_2 (k_1 k_2 - d_1 d_2) + 2k_2 k_7 (d_1^2 - 2k_7) + 2d_2^2 (k_7 - k_2)] \cos(q\pi), \\ a_1 &= 2(k_2 - k_7) [k_1 (2k_2^2 - d_2^2 + 2k_2 k_7) + d_1 d_2 (k_7 - k_2)] \cos(\frac{q\pi}{2}), \\ a_0 &= (k_2 - k_7)^2 [(k_2 + k_7)^2 - d_2^2]. \end{aligned}$$

Let $f(w)$ be defined as follows

$$f(w) = w^{8q} + a_7 w^{7q} + a_6 w^{6q} + \cdots + a_1 w^q + a_0.$$

In the above definition, we can deduce $\lim_{w \rightarrow +\infty} f(w) = +\infty$. Thus, if the parameter $a_0 < 0$ holds, Eq. (23) has at least one positive real root, denoted by w_0 .

From Eq. (22), one can get

$$\tau_* = \frac{1}{w_0} \left[\arccos \left(\frac{k_7 A_1 - A_1 A_2 - B_1 B_2}{A_2^2 + B_2^2 - k_7^2} \right) + 2k\pi \right], k = 0, 1, 2, \dots, \quad (24)$$

Let $s = \varrho(\tau) + iw(\tau)$ be the characteristic root of Eq. (20) near w_0 , satisfying $\varrho(\tau_*) = 0, w(\tau_*) = w_0$, then substituting s into Eq. (20) and taking the derivative with respect to τ on two sides, then it is easy derive that

$$\begin{aligned} \left[\frac{ds}{d\tau} \right]^{-1} &= \frac{q}{s^2} \frac{(2s^{2q} + k_1 s^q) e^{s\tau} + d_1 s^q}{k_7 e^{-s\tau} - (s^{2q} + k_1 s^q + k_2) e^{s\tau}} - \frac{\tau}{s} \\ &= \frac{q}{s^2} \frac{(s^{2q} - k_2) e^{s\tau} - k_7 e^{-s\tau} - d_2}{2k_7 e^{-s\tau} + d_1 s^q + d_2} - \frac{\tau}{s}. \end{aligned}$$

Further,

$$\begin{aligned} \operatorname{Re} \left[\frac{ds}{d\tau} \right]^{-1} \Big|_{s=iw_0} &= \operatorname{Re} \left[\frac{q}{s^2} \frac{(s^{2q} - k_2) e^{s\tau} - k_7 e^{-s\tau} - d_2}{2k_7 e^{-s\tau} + d_1 s^q + d_2} \right] \Big|_{s=iw_0} \\ &= -\frac{q}{w_0^2} \left(\frac{P_1 Q_1 + P_2 Q_2}{P_1^2 + P_2^2} \right) \end{aligned}$$

where

$$\begin{aligned} P_1 &= 2k_7 \cos(w_0 \tau) + d_1 w_0^q \cos\left(\frac{q\pi}{2}\right) + d_2; \\ P_2 &= -2k_7 \sin(w_0 \tau) + d_1 w_0^q \sin\left(\frac{q\pi}{2}\right); \\ Q_1 &= \left(w_0^{2q} \cos(q\pi) - k_2 - k_7 \right) \cos(w_0 \tau) - w_0^{2q} \sin(q\pi) \sin(w_0 \tau) - d_2; \\ Q_2 &= \left(w_0^{2q} \cos(q\pi) - k_2 + k_7 \right) \sin(w_0 \tau) + w_0^{2q} \sin(q\pi) \cos(w_0 \tau); \end{aligned}$$

Hence, let us formulate the hypothesis

$$(\mathbf{H2}) \quad P_1 Q_1 + P_2 Q_2 < 0.$$

Combining the literature [13] and the above analysis results, the following theorem is obtained.

Theorem 6 *For time delays $\sigma = \tau$, assume that the conditions **(H2)** and $a_0 < 0$ are satisfied. As τ increases from zero to a critical value τ_* , E^* is locally asymptotically stable and loses its stability as τ_* continued to increase. The fractional-order system (2) undergoes a Hopf bifurcation at E^* when $\tau = \tau_*$.*

Remark 8 Although there have been many studies on the dynamical behavior of the malaria epidemic model, most authors only considered the integer-order models of the malaria epidemic. Still, the fractional-order delayed models for malaria transmission have been less research. This study combines time delays and fractional-order, enriching the malaria model dynamics to a certain extent. Besides this, this paper's analysis method is also suitable for the general fractional-order delayed model.

Remark 9 However, in the literature [3], the author has addressed two special cases of $I_h^* + I_m^* = 1, \tau = \sigma = \tau$ and $I_h^* + I_m^* - 1 = I_h^* I_m^*, \tau \neq \sigma$, which have its limitations. In this work, the local stability and bifurcation of the fractional-order malaria epidemic model for different time delays are researched.

5 Numerical simulations

In this section, numerical simulations using realistic parameter values (see Table 1) demonstrate and support the effectiveness of the analytical results. Given the parameter values:

$$a = 0.5, b = 0.5, c = 0.5, k = 2, \mu = 0.05, \gamma = 0.04, \delta = 0.05.$$

By calculating, malaria epidemic model (2) has a unique positive equilibrium point $E^* = (0.8169, 0.8033)$, and the basic reproduction number $\mathcal{R}_0 > 1$ implies that malaria persists in a certain range.

Table 1 Parameters values of malaria model (2).

Parameters	Values	References
a	$0.2 - 0.55 \text{ day}^{-1}$	[12, 14]
b	0.1-0.5	[12, 15, 16]
c	0.3-0.5	[17, 18]
k	2	[12, 16]
μ	0.05 day^{-1}	[19]
γ	$0.01 - 0.05 \text{ day}^{-1}$	[12]
δ	0.05 day^{-1}	[12, 15]
τ	$10 - 100 \text{ day}^{-1}$	[12, 20]
σ	$5 - 15 \text{ day}^{-1}$	[12, 21]

Case I: When $\sigma = \tau = 0$, the corresponding waveform plots and phase portraits show in Figure 1. As is evident from Fig. 1, the state trajectories of the system converge to the equilibrium point E^* for different values of the fractional-order q . At the same time, we observe that the fractional-order q has an impact on the peak of the waveform plots.

Case II: When $\tau = 0$, we can obtain $\sigma^* = 9.697197392$ for fractional-order $q = 0.95$. The corresponding waveform plots and phase portraits show in Fig. 2, which show that the state trajectories of system (2) converge to the equilibrium point when $\sigma < \sigma^*$, and lose its stability once $\sigma > \sigma^*$. Fig. 3 illustrates that the amplitude of the periodic solution gradually increases with the delay increases, which means that the instability becomes stronger as increasing the time delay.

Case III: Similarly, we can get $\tau^* = 4.55659524$ for $q = 0.95$ and $\sigma = 0$ by calculating. Fig. 4 shows E^* is locally asymptotically stable with $\tau = 4.2 < \tau^*$, however, the system (2) generates a Hopf bifurcation at E^* for $\tau = 5 > \tau^*$.

Furthermore, the impacts of fractional-order q on the values of critical frequency and bifurcation point are illustrated in Fig. 5. It can be observed clearly from Fig. 5 (a) that the values of critical frequency w_0 increases slowly with the increasing of fractional-order q . Conversely, the increase of the order q goes along with a decrease of the bifurcation point σ^* or τ^* , as presented in Fig. 5 (b), which accounts for the stability interval of the system is inversely proportional to the fractional-order q .

Case IV: Then, by regarding σ as a parameter and considering the time delay $\tau = 4 \in [0, \tau^*)$ is fixed in its stable interval for $q = 0.95$, we can obtain $\sigma_* = 8.15647935$. It is clearly evident that the incubation period of Plasmodium in humans can narrow the stability interval of the system by comparing it with Case II. Thus, this is inaccurate to ignore the human incubation period of Plasmodium in human. The corresponding of Phase portraits are shown in Fig. 6.

Fig. 1 Waveform plots and phase portrait of the model (2) with different values of fractional-order q for $\sigma = \tau = 0$.

Case V: Finally, we can obtain $\tau_* = 4.024372059$ with $\sigma = \tau = \tau$ and $q = 0.95$. As can be seen from Fig. 7, E^* is local asymptotically stable when $\tau = 3.5 < \tau_*$ and becomes unstable when $\tau = 5 > \tau_*$, at which point a Hopf bifurcation occurs.

Following the above analysis, it can be seen that the numerical simulation results and the theoretical results match well with each other, validating the correctness of the theoretical analysis in Section 4.

6 Conclusions

In this paper, the nonlinear delayed Ross-Macdonald model for malaria transmission [3] is generalized to a novel fractional-order model, enriching the dynamics behavior and adding complexity to the model. The objective of the present paper is to systematically discuss the effect of time delays and fractional-order on the stability and bifurcation of this malaria model. The results of this study indicate the time delay can change the stable equilibrium point from stable to unstable, and a Hopf bifurcation appears. That is, the stability of the system remains virtually unchanged as the delay increases from zero to a critical value. However, the system loses its stability and undergoes a Hopf bifurcation when the delay is equal to or more than a critical value. It would indicate that the appropriate incubation periods (via control measures or drugs) are beneficial to control the disease at a stable level better and prevent the massive outbreak of malaria. Another interesting result is that the change of order influences stability interval size. This is because the

Fig. 2 Waveform plots and phase portrait of the fractional-order malaria model (2) with $q = 0.95$ and $\tau = 0$. (a)-(b) E^* is asymptotically stable for $\sigma = 8 < \sigma^* = 9.6972$. (c)-(d) The model (2) has a family of periodic solutions and undergoes a Hopf bifurcation for $\sigma = 9.8 > \sigma^* = 9.6972$.

change of order clearly impacts the critical frequency, further leading to the bifurcation point change. The numerical simulations show that stability interval can increase with the decrease of order. That is, the stability interval of the system is inversely proportional to the order. This result suggests that the order is significant in modeling epidemic diseases since the best value of order can better predict disease risk and progression.

The discussion method presented in this paper is also suitable for the delayed fractional-order system. The global stability of the positive equilibrium point in the fractional delayed Ross-Macdonald model for malaria transmission is the matter of our future work.

Fig. 3 Periods and amplitudes of periodic solutions for $\sigma = 10, 11, 12$ and $q = 0.95$.

Fig. 4 (a) E^* is locally asymptotically stable for $\tau = 4.2 < \tau^*, \sigma = 0$. (b) E^* loses stability and the Hopf bifurcation occurs for $\tau = 5 > \tau^*, \sigma = 0$.

Fig. 5 (a) Critical frequency versus the order q . (b) Bifurcation point versus the order q .

Fig. 6 Phase portraits show that E^* is locally asymptotically stable for $\sigma = 8 < \sigma_*$ (see (a)) and lose its stability for $\sigma = 8.5 > \sigma_*$ (see (b)) with $q = 0.95$ and $\tau = 4 \in [0, \tau^*)$.

Fig. 7 Waveform plots show that E^* is locally asymptotically stable for $\tau = 3.5 < \tau_*$ (see (a)) and the Hopf bifurcation occurs for $\tau = 5 > \tau_*$ (see (b)) with $q = 0.95$.

Appendix A

$$\begin{aligned}
& \text{(i)} \quad e^{-Nt} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |x(s-\tau) - y(s-\tau)| \, ds \\
&= e^{-Nt} \left(\int_0^\tau \frac{(t-s)^{q-1}}{\Gamma(q)} |x(s-\tau) - y(s-\tau)| \, ds + \int_\tau^t \frac{(t-s)^{q-1}}{\Gamma(q)} |x(s-\tau) - y(s-\tau)| \, ds \right) \\
&= e^{-Nt} \int_\tau^t \frac{(t-s)^{q-1}}{\Gamma(q)} |x(s-\tau) - y(s-\tau)| \, ds \\
&= \int_0^{t-\tau} \frac{(t-\tau-\theta)^{q-1}}{\Gamma(q)} e^{-N(t-\theta)} e^{-N\theta} |x(\theta) - y(\theta)| \, d\theta \\
&\leq \|x - y\| \int_0^{t-\tau} \frac{(t-\tau-\theta)^{q-1}}{\Gamma(q)} e^{-N(t-\theta)} \, d\theta \\
&= \|x - y\| \int_0^{t-\tau} \frac{\mu^{q-1}}{\Gamma(q)} e^{-N(\mu+\tau)} \, d\mu \\
&\leq \frac{e^{-N\tau} \|x-y\|}{N^q} \int_0^{+\infty} \frac{(N\mu)^{q-1}}{\Gamma(q)} e^{-N\mu} \, dN\mu \\
&= \frac{e^{-N\tau}}{N^q} \|x - y\|
\end{aligned}$$

and

$$\begin{aligned}
& \text{(ii)} \quad e^{-Nt} \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |x(s-\tau) y(s-\tau) - \bar{x}(s-\tau) \bar{y}(s-\tau)| \, ds \\
&= e^{-Nt} \int_\tau^t \frac{(t-s)^{q-1}}{\Gamma(q)} |x(s-\tau) y(s-\tau) - \bar{x}(s-\tau) \bar{y}(s-\tau)| \, ds \\
&= e^{-Nt} \int_0^{t-\tau} \frac{(t-\tau-\theta)^{q-1}}{\Gamma(q)} |x(\theta) y(\theta) - \bar{x}(\theta) \bar{y}(\theta)| \, d\theta \\
&= e^{-Nt} \int_0^{t-\tau} \frac{(t-\tau-\theta)^{q-1}}{\Gamma(q)} |x(\theta) y(\theta) - \bar{x}(\theta) y(\theta) + \bar{x}(\theta) y(\theta) - \bar{x}(\theta) \bar{y}(\theta)| \, d\theta \\
&= e^{-Nt} \int_0^{t-\tau} \frac{(t-\tau-\theta)^{q-1}}{\Gamma(q)} |(x(\theta) - \bar{x}(\theta)) y(\theta) + \bar{x}(\theta) (y(\theta) - \bar{y}(\theta))| \, d\theta \\
&\leq \kappa e^{-Nt} \int_0^{t-\tau} \frac{(t-\tau-\theta)^{q-1}}{\Gamma(q)} |x(\theta) - \bar{x}(\theta)| \, d\theta + \kappa e^{-Nt} \int_0^{t-\tau} \frac{(t-\tau-\theta)^{q-1}}{\Gamma(q)} |y(\theta) - \bar{y}(\theta)| \, d\theta \\
&\leq \frac{\kappa e^{-N\tau}}{N^q} (\|x - \bar{x}\| + \|y - \bar{y}\|).
\end{aligned}$$

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Conflict of Interest: The authors declare that they have no conflict of interest.

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Figures

Figure 1

"Please see the Manuscript PDF file for the complete figure caption".

(a)

(b)

(c)

(d)

Figure 2

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Figure 3

Periods and amplitudes of periodic solutions for $\sigma = 10; 11; 12$ and $q = 0.95$.

Figure 4

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Figure 5

(a) Critical frequency versus the order q . (b) Bifurcation point versus the order q .

Figure 6

"Please see the Manuscript PDF file for the complete figure caption".

(a)

(b)

Figure 7

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