General Formulation for Free Vibration of Elastic Solids with Static Loads and Application to Rotating Tapered Cantilever Beam Vibration

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Research Article

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Posted Date: June 18th, 2020

DOI: <https://doi.org/10.21203/rs.3.rs-35840/v1>

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General Formulation for Free Vibration of Elastic Solids with Static Loads and Application to Rotating Tapered Cantilever Beam Vibration

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Abstract: By solving the three one-dimensional (1D) nonlinear dynamic differential equations analytically, it has been proved that unless the nonlinear terms are in first order, a nonlinear dynamic system never has a vibration natural frequency. A simple nonlinear mass-spring system has been invented to demonstrate that vibration frequency is only calculated on a perturbation basis and to demonstrate that an external load may affect frequency too. Then for an elastic solid with nonlinear deformation and with static loads including a rotational angular velocity ϖ , a virtual small factor $\alpha \rightarrow 0$ has been introduced to ensure a small deformation, a general formulation to predict vibration frequencies has been developed, which proves that the strain energy and kinematic energy (or the work done by vibration inertial force) are calculated from the linear deformation terms while the work done by external loads is calculated from the second order nonlinear terms that cause stiffening/softening effects on vibration frequency. This may be different to the Rayleigh-Ritz method. Applying the developed formulation to rotating tapered cantilever beams, the simple analytical method has been developed. Validation against FE analysis has been carried out to show that the simple method can predict the out-of-plane vibration frequency of rotating tapered cantilever beams accurately.

Keywords: Virtual factor, vibration frequency, static loads, nonlinear dynamic system, elastic solid, general formulation, rotating tapered beam.

1. Introduction:

Vibration of solids/structures is a major subject to engineers and researchers, and it plays a vital role in engineering design. When a solid is subjected to a shock force, if a cyclically dynamic response occurs, then it is said that this solid vibrates and has a vibration response. However, this doesn't mean that this solid has a natural vibration frequency. When and only when a dynamic response repeats periodically over time with a fixed frequency, then it is said that this solid has a vibration response and has a natural frequency (therefore, in this paper vibration is equivalent to natural frequency, otherwise referred to as dynamic response instead). Due to simplicity, theories and numerical algorithms for vibration analysis of a linear system have been well established over centuries.

However, not every dynamic system is linear, for example, general motion formulation of a solid with large rigid rotational and translational displacement presented by Weng and Greenwood [1] in 1992 and nonlinear dynamics of elastic rods investigated by Cao and Tucker [2], they all are highly nonlinear. The author of this paper, Luo also presented work on transverse vibration of axial loaded beams, long offshore pipes and bladed disks [3-5]. For a nonlinear dynamic system, due to complicity, there never was a universal theory or algorithm available to tell whether it has a natural vibration frequency, to tell whether it has a unique solution, and to tell what the forced response is. Although the Rayleigh-Ritz method [6-12] was applied by scientists and engineers to compute eigenvalues or frequencies for specific individual problems successfully, it was never indicated explicitly how a nonlinear dynamic system should be assessed. Compared with a linear system, a nonlinear dynamic system is complicated.

Mathematically, for a nonlinear system, there is a fundamental difference between a natural frequency of a free vibration and a critical frequency in forced vibration, which makes the response reach a peak. This is because a critical frequency ω_0 in forced vibration may not be the same as a natural frequency

in free vibration. Often this critic frequency ω_0 was still defined as a natural frequency by scientists and the corresponding dynamic response may also be referred to as resonance. Because the resonant response of a nonlinear system to an external harmonic load is so complicated, the frequency domain-based methods were mainly applied to determine the transfer function implicitly by using Volterra series by scientists, such as Lang and Billings [13-15], Link [16] and Kuether et al. [17].

Fact is that for a nonlinear system, whether it has a natural frequency or not, on a perturbation/linearization basis, a frequency may be calculated when the system state is positive. In literature, many simple methods were developed to assess natural frequencies of a nonlinear system using perturbation technique, for example, Thomas et al [18], Bokain [19], Luo [3-5] and Naguleswaran [7] all investigated the vibration frequencies of the beam-type structures subjected to external static loads using different methods. With use of the von Karman nonlinear plate theory, Wu et al [8] proposed a method to predict the vibration of nonlinear moving printing membrane with a large deformation, but this method is invalid for a solid, while Marynowski [9] used a directly numerical solution to investigate the frequencies of moving paper web with pre-stress. Perturbation analysis was used to extract frequencies of every nonlinear dynamic system by all the current commercial software today.

Although in literature the static load effect on frequencies of a beam or a plate was investigated by scientists. So far, the author of this paper has not seen a general formulation which was reported to predict free vibration frequency of a three-dimensional continuous elastic solid subjected to any static loads. By determining analytical solutions of three proposed nonlinear 1D dynamical differential equations, this paper will investigate why a natural frequency may not occur to a nonlinear system and what makes a nonlinear system have a natural frequency. Free vibration of an elastic solid subjected to any static loads will be studied on a perturbation basis. By introducing a virtual small factor $\alpha \rightarrow 0$, a general formulation for free vibration of an elastic solid with any static loads will be developed, which proves that all the static loads will not affect vibration of a linear elastic solid. This formulation will provide a guidance in study on load effects on vibration frequency in the future. Application of the developed general formulation to rotating tapered cantilever beams and validation against FE analysis will be carried out.

2. Solution of Three 1D Nonlinear Dynamical Differential Equations

Before investigation into vibration of elastic solids with various static loads, efforts will be made to work out exact solutions of three 1D nonlinear dynamic differential equations to show how nonlinearity affects natural frequencies. Although there are many types of nonlinear terms in a dynamic system, without loss of generality, the following three 1D nonlinear differential equations shown in equations (1) to (3) were proposed and studied.

$$\frac{d^2 y}{dt^2} + cy^{-n} \left(\frac{dy}{dt} \right)^{n+1} + k \left(\frac{dy}{dt} \right)^{-m} y^{m+1} = 0 \quad (1)$$

$$\frac{d^2 y}{dt^2} + c \left(\frac{dy}{dt} \right)^2 + ky^2 = 0 \quad (2)$$

and

$$\frac{d^2 y}{dt^2} + c \left(\frac{dy}{dt} \right)^2 + ky = 0 \quad (3)$$

where n and m are integers and $k > 0$. The eigenvalue, λ of equation (1) may be determined from the eigen-equation, $\lambda^{m+2} + c\lambda^{n+m+1} + k = 0$. For any given n , m , c and k , λ can be calculated either analytically or numerically when it exists. In equations (1) to (3), three different relationships of the acceleration, velocity and displacement are defined, so they define three different dynamical systems. Like equation (3), when $n = 1$ and $m = 1$, equation (1) has an acceleration term, a damping-like term

and a stiffness term, which is very similar to a classic single mass-spring system. However, unlike a linear system they all have one or two nonlinear terms and they are the governing equations of three nonlinear dynamical systems with fundamental differences mathematically. In principle equation (3) can only be solved numerically unless $k = 0$ for which $y = y(0) + \ln(1 + cy'(0)t)/c$ while equation (1) may be solved analytically. For a given initial displacement $y(0)$ and velocity $y'(0)$ at time zero, equation (1) has a solution which is given by

$$\begin{cases} y = \alpha_i e^{\lambda_i t}, & i = 1, 2, \dots, n+m+1 \text{ for } \lambda^{m+2} + c\lambda^{n+m+1} \neq 0 \\ y = x(0)e^{\frac{1}{k(m+2)} \left(\left(\frac{y'(0)}{y(0)} \right)^{m+2} - \left(\left(\frac{y'(0)}{y(0)} \right)^{m+1} - (m+1)kt \right)^{\frac{m+2}{m+1}} \right)} & \text{for } \lambda^{m+2} + c\lambda^{n+m+1} = 0 \end{cases} \quad (4)$$

where λ_i , $i = 1, 2, \dots, n+m+1$, are constants which may be complex values and may and may not be fully determined from the initial conditions. This means that equation (1) may have multiple solutions, a typical example of the bifurcation theory. In particular when $c = n = 1$ and $m = 0$, one of the two eigenvalues of equation (1) is $i\sqrt{k/2}$ where $i = \sqrt{-1}$ and the other eigenvalue is $\frac{\beta}{2} + i\frac{\sqrt{3}}{2}\beta^2$ where $\beta = (k/(1+c))^{1/3}$ when $n = m = 1$ and $c \neq -1$, subsequently the solutions of equation (1) for the above two cases become

$$\begin{cases} y = \alpha_1 e^{-i\sqrt{\frac{k}{2}}t}, \alpha_2 e^{i\sqrt{\frac{k}{2}}t} & \text{for } c = n = 1 \text{ and } m = 0 \\ y = \alpha_1 e^{-\beta t}, \alpha_2 e^{-\beta t/2 + i\frac{\sqrt{3}}{2}\beta^2 t}, \alpha_3 e^{-\beta t/2 - i\frac{\sqrt{3}}{2}\beta^2 t} & \text{for } n = m = 1 \text{ and } c \neq -1 \end{cases} \quad (5)$$

For equation (2), after complicated manipulations the analytical solution was also obtained and is given by

$$\begin{cases} y(t) = e^{\frac{1}{(1+c)} \ln \left(-\frac{1}{2} \sqrt{\frac{1+c}{k}} e^{-(2\sqrt{-(1+c)kt} + \beta_2)} + \beta_1 \right) + \sqrt{\frac{k}{1+c}} t} & \text{for } c \neq -1 \\ y(t) = e^{-\frac{1}{2}kt^2 + \beta_2 t + \beta_1} & \text{for } c = -1 \end{cases} \quad (6)$$

where β_1 and β_2 are two constants determined by initial conditions. Complementary functions (4) to (6) show that a solution to a nonlinear dynamical system may be very complicated with multiple solutions which cannot be superposed to form a new one due to nonlinearity. They also indicate that, when a function of $y(t) = \beta e^{\lambda t}$ is substituted into a governing equation like equations (1) to (3), when λ can be determined exactly with a complex value, then this dynamical system has a natural frequency. Otherwise like equation (3), it doesn't have one. The question is why equation (1) sometimes has a vibration frequency while equation (3) never has one. Inspection into the differences between equations (4) and (6) shows that all the nonlinear terms of equation (1) are in the first order, e.g. for the second term $cy^{-n} \left(\frac{dy}{dt} \right)^{n+1}$, n out of $n+1$ orders of the velocity are cancelled by the n orders of the displacement to make a first order. On the other hand, equation (3) has a second order term. This is the fundamental difference between equations (1) and (3), and it is the reason why for the above two nonlinear dynamical systems, one has a vibration response while the other doesn't.

Like Chauchy strain, equation (2) has second order terms. When $k / (1 + c) > 0$, solution (6) shows that $y(t)$ is a complex function which can be decomposed into the real and imaginary parts. A derivation shows that the solution can be simplified into the following equation.

$$\left\{ \begin{array}{l} y(t) = e^{\varphi(t, \omega_1, \beta_1, \beta_2) (\cos(\phi(t, \omega_1, \beta_1, \beta_2)) + i(\sin(\phi(t, \omega_1, \beta_1, \beta_2)) + \omega_2 t))} \\ \omega_1 = 2\sqrt{(1+c)k} \\ \omega_2 = \sqrt{\frac{k}{1+c}} \end{array} \right. \quad (7)$$

where φ and ϕ are the functions of t , ω_1 , β_1 and β_2 . The above equation clearly illustrates that this dynamical system will not have a vibration natural frequency and the modulus of function $y(t)$ changes with time. On the other hand, when $k / (1 + c) < 0$, solution (6) shows that the nonlinear dynamical system not only does not have a vibration natural frequency but also is unstable. This indicates that mathematically like a linear system, a nonlinear dynamic system can become unstable on certain conditions too. Only difference is that for a linear system, based on eigenvalues it can be instantly worked out whether it is stable or unstable while for a nonlinear dynamic system, often it is difficult to tell.

Because sometimes a dampened dynamic system may have a second-order term of displacement, so the following equation was also studied in this paper.

$$\frac{d^2 y}{dt^2} + cy + ky^2 = 0 \quad (8)$$

With the use of variable replacement technique, equation (8) was converted to

$$\frac{d^2 (y + \frac{c}{2k})}{dt^2} + k (y + \frac{c}{2k})^2 = \frac{c^2}{4k} \quad (9)$$

Comparing equation (9) with (2) shows that they both have a second order term except equation (9) is a forced dynamical system while equation (2) is unforced. Therefore, it can be proved that like equation (2), equation (8) will not have a vibration frequency too.

Summarising the above work, a **vibration existence theorem** was proposed, that is, **a nonlinear dynamical system with the first-order terms only may and may not have a natural frequency depending on whether a complex eigenvalue is available while a dynamical system with the second order or higher terms must not have a natural frequency** although the solution of a dynamical system with higher order terms was not investigated in this paper.

3. Vibration of a nonlinear mass-spring system

Although the above work indicates that mathematically when and only when a nonlinear dynamic system has first-order terms, it may have a natural frequency in vibration, it's often seen that a system with second or higher order nonlinear terms can have a natural frequency. This is because the observed vibration magnitude is small on a perturbation basis. For instance, a nonlinear mass-spring system is shown as below

$$\frac{d^2 y}{dt^2} + ky^2 = 0 \quad (10)$$

This equation represents a mass-spring system with a nonlinear stiffness of ky . When it is subjected to an initial force of $F = ky_0^2$, then in a small range of $(y_0 - \Delta y, y_0 + \Delta y)$, by ignoring the second order term of Δy , equation (10) becomes

$$\frac{d^2 \Delta y}{dt^2} + 2ky_0 \Delta y + ky_0^2 = 0 \quad (11)$$

Hence, when Δy is small and $y_0 > 0$ (a positive state), this nonlinear mass-spring system has a frequency of $\sqrt{2ky_0}$ while when $y_0 \leq 0$ (a negative state, note: a linear system with a negative state is unstable) there is no a natural frequency. Equation (11) tells us three things. First a nonlinear system may have been seen a vibration natural frequency; second the perturbation vibration frequency may be affected by external forces; third an observed frequency from a perturbation-based test may not reflect the true characteristics of a nonlinear system. These show that a nonlinear system is complicated.

4. Free vibration frequency of elastic solids with various loads

Unlike the equilibrium equations which are linear and element based for a linear elastic solid, the general motion formulation presented by Weng and Geenwood [1] in 1992 shows that when a solid has a large deformation with rigid translational and rotational displacement, dynamic equations are body based and are nonlinearly defined by generalised coordinates. Moreover, even for a solid with a small deformation but with a large strain, the corresponding dynamic equations are nonlinear too. By assuming a rectangular element remains rectangular after deformation, it can be proved that under a large strain hypothesis, the dynamic equations are approximately given by

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} \frac{1}{1+\varepsilon_x} + \frac{\partial \tau_{xy}}{\partial y} \frac{1}{1+\varepsilon_y} + \frac{\partial \tau_{xz}}{\partial z} \frac{1}{1+\varepsilon_z} + F_x = \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial \sigma_y}{\partial y} \frac{1}{1+\varepsilon_y} + \frac{\partial \tau_{xy}}{\partial x} \frac{1}{1+\varepsilon_x} + \frac{\partial \tau_{yz}}{\partial z} \frac{1}{1+\varepsilon_z} + F_y = \rho \frac{\partial^2 v}{\partial t^2} \\ \frac{\partial \sigma_z}{\partial z} \frac{1}{1+\varepsilon_z} + \frac{\partial \tau_{xz}}{\partial x} \frac{1}{1+\varepsilon_x} + \frac{\partial \tau_{yz}}{\partial y} \frac{1}{1+\varepsilon_y} + F_z = \rho \frac{\partial^2 w}{\partial t^2} \end{cases} \quad (12)$$

These equations are so complicated that an analytical solution to them is almost impossible. As discussed for nonlinear 1D dynamic differential equation before, the stiffening/softening effect of external static loads on vibration, therefore, can only be analysed on a perturbation basis.

4.1 Work done by external loads

Assuming an elastic solid, defined in an inertial Cartesian coordinate system o-xyz, rotates with an angular velocity of ϖ about o'z'-axis that has an angle of θ with oz-axis as shown in Figure 1 in which line oo' was formed by rotating ox-axis about oz-axis by angle φ , and the solid is also assumed to be subjected to three static body forces per unit volume, F_x , F_y and F_z which are either constant or distributed. Under deformation, an element moves from position A to B with a displacement of u , v and w in the three axis directions, then the work W_e done by all the loads is given by

$$\begin{aligned} W_e = & \int_V \left(\int_0^u F_x du + \int_0^v F_y dv + \int_0^w F_z dw \right) dV + \rho \varpi^2 \int_V \left(\int_0^u (\Delta_{xu} \cos \theta \cos \varphi + \Delta_{yv} \sin \varphi) du \right) dV \\ & + \rho \varpi^2 \int_V \left(\int_0^v (\Delta_{xu} \cos \theta \sin \varphi + \Delta_{yv} \cos \varphi) dv \right) dV + \rho \varpi^2 \int_V \left(\int_0^w \Delta_{xu} \sin \theta dw \right) dV \end{aligned} \quad (13)$$

where

$$\begin{cases} \Delta_{xu} = \cos \theta \cos \varphi (x - x_0 + u) + \cos \theta \sin \varphi (y - y_0 + v) + \sin \theta (z + w) \\ \Delta_{yv} = \sin \varphi (x - x_0 + u) + \cos \varphi (y - y_0 + v) \end{cases} \quad (14)$$

Figure 1 a body rotates about an axis

where $dV = dxdydz$ and ρ refers to the material density per unit volume. The strain energy W_s and the work W_i done by the inertial force are given by

$$\begin{cases} W_i = -\rho \int_V \left(\int_0^u \frac{\partial^2 u}{\partial t^2} du + \int_0^v \frac{\partial^2 v}{\partial t^2} dv + \int_0^w \frac{\partial^2 w}{\partial t^2} dw \right) dV \\ W_s = \int_V \left(\int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \right) dV \end{cases} \quad (15)$$

where σ_{ij} and ε_{ij} refer to the stress and strain tensors respectively and the subscripts i and j that appears twice in the same term indicates a summation over $(1, 2, 3)$ or (x, y, z) .

4.2 Free vibration frequency

Because for a linear elastic solid, based on Hook's law and equilibrium equations, it can be proved that when $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$ are modal displacement shape functions, then for a small virtual factor α , $\alpha u(x, y, z)$, $\alpha v(x, y, z)$ and $\alpha w(x, y, z)$ (representing virtual displacement variation) must be modal displacement shape functions to the system too. However, when a large deformation or geometrical nonlinearity occurs, this rule no longer applies anymore. Therefore, for a nonlinear elastic solid, the displacement components can be given by

$$\begin{cases} u = \sum_{k=1}^n \alpha^k u_k \\ v = \sum_{k=1}^n \alpha^k v_k \\ w = \sum_{k=1}^n \alpha^k w_k \end{cases} \quad (16)$$

where $n > 1$ and u_1 , v_1 and w_1 can be denoted as the linear elastic solution of the solid. To study vibration, it is assumed that the solid vibrates harmonically with a frequency of ω for a given total

shape function $\alpha\phi(x, y, z)$ (which means a modal shape variation), that is, for a small factor α that ensures the analysis on a perturbation basis, the displacement modal shapes can be given by equation (16). By using the energy conservation principle gives

$$W_i + W_e = W_s \quad (17)$$

where

$$W_e = \int_V \left(\int_0^\alpha \left(F_x \frac{\partial u}{\partial \alpha} + F_y \frac{\partial v}{\partial \alpha} + F_z \frac{\partial w}{\partial \alpha} \right) d\alpha \right) dV + \rho \bar{\omega}^2 \int_V \left(\int_0^\alpha \left(\Delta_{xu} \cos \theta \cos \varphi + \Delta_{yv} \sin \varphi \right) \frac{\partial u}{\partial \alpha} d\alpha \right) dV \quad (18)$$

$$+ \rho \bar{\omega}^2 \int_V \left(\int_0^\alpha \left(\left(\Delta_{xu} \cos \theta \sin \varphi + \Delta_{yv} \cos \varphi \right) \frac{\partial v}{\partial \alpha} + \Delta_{xu} \sin \theta \frac{\partial w}{\partial \alpha} \right) d\alpha \right) dV$$

$$\begin{cases} W_i = \frac{1}{2} \alpha^2 \rho \omega^2 \int_V (u_1^2 + v_1^2 + w_1^2) dV + 0(\alpha^3) \\ W_s = \frac{1}{2} \alpha^2 \int_V (\sigma_{ij} \varepsilon_{ij}) dV + 0(\alpha^3) \end{cases} \quad (19)$$

and stress tensor σ_{ij} , and strain tensor ε_{ij} are calculated from u_1 , v_1 and w_1 , and $0(\alpha^3)$ refers to sum of third and higher terms of α . For the above equation, differentiating two sides of (17) with respect to α twice and in the limit when $\alpha \rightarrow 0$ gives

$$\omega^2 = \frac{W_1 - (\bar{\omega}^2 W_2 + W_{3x} + W_{3y} + W_{3z})}{W_0} \quad (20)$$

where

$$\begin{cases} W_0 = \rho \int_V (u_1^2 + v_1^2 + w_1^2) dx dy dz \\ W_1 = \int_V (\sigma_{ij} \varepsilon_{ij}) dx dy dz \\ W_2 = \rho \int_V \left(\left(\frac{\partial \Delta_{xu}}{\partial \alpha} \cos \theta \cos \varphi + \frac{\partial \Delta_{yv}}{\partial \alpha} \sin \varphi \right) u_1 + 2 \left(\Delta_{xu} \cos \theta \cos \varphi + \Delta_{yv} \sin \varphi \right) u_2 \right) dV \\ + \rho \int_V \left(\left(\left(\frac{\partial \Delta_{xu}}{\partial \alpha} \cos \theta \sin \varphi + \frac{\partial \Delta_{yv}}{\partial \alpha} \cos \varphi \right) v_1 + 2 \left(\Delta_{xu} \cos \theta \sin \varphi + \Delta_{yv} \cos \varphi \right) v_2 \right) \right) dV \\ + \rho \int_V \left(\left(\frac{\partial \Delta_{xu}}{\partial \alpha} \sin \theta w_1 + 2 \Delta_{xu} \sin \theta w_2 \right) \right) dV \\ W_{3x} = 2 \int_V (F_x u_2) dx dy dz + 2 \int_\Omega (F_{\Omega x} u_2) ds \\ W_{3y} = 2 \int_V (F_y v_2) dx dy dz + 2 \int_\Omega (F_{\Omega y} v_2) ds \\ W_{3z} = 2 \int_V (F_z w_2) dx dy dz + 2 \int_\Omega (F_{\Omega z} w_2) ds \end{cases} \quad (21)$$

and

$$\begin{cases} \frac{\partial \Delta_{xu}}{\partial \alpha} = 2 \cos \theta \cos \varphi u_2 + 2 \cos \theta \sin \varphi v_2 + 2 w_2 \sin \theta w_2 \\ \frac{\partial \Delta_{yv}}{\partial \alpha} = 2 \sin \varphi u_2 + 2 \cos \varphi v_2 \end{cases} \quad (22)$$

where Ω , $F_{\Omega x}$, $F_{\Omega y}$, and $F_{\Omega z}$ refer to the solid surface area, the surface forces in the x-axis, y-axis and z-axis direction, respectively. Equation (21) indicates that except the second order terms, the third and higher terms in equation (16), that is, the terms for $k \geq 3$, has no contributions to the

stiffening/softening effects caused by external loads on the solid. Therefore, it has no effects on vibration frequencies. Hence for nonlinear elastic solids, the third and higher terms of shape functions (with α^3 or higher terms) can be ignored in analysis.

In principle, by using formulations (16) to (21) for any structures with external loads, such as beams, shells or beam-shell assembly or any structure assembled together from different components, their frequencies can be assessed accurately if modal shape functions can be given correctly.

For a linear elastic solid, without nonlinear terms shown in equation (16), that is $W_2 = W_{3x} = W_{3y} = W_{3z} = 0$, equation (20) becomes the Rayleigh-Ritz formulation presented in literature.

5. Derived from dynamic equation

The above formulation can also be worked out from the dynamic equations of the elastic solid with small deformation. To do so, assuming the total modal shape function is denoted as $\alpha\phi(x, y, z)$, when the solid vibrates harmonically with a frequency of ω , by ignoring third and higher order terms (note: even if these high order terms are included, it can be proved that they have no effects on frequencies by using the similar derivations as shown below), the displacement components can be given by

$$u_i = \sum_{k=1}^2 u_{ik} (\alpha \sin(\omega t))^k \quad (23)$$

where $i = 1, 2, 3$, refer to x , y , and z coordinates, respectively. Let us denote σ_{ij}^1 and σ_{ij}^2 are the stress components calculated from u_{i1} and u_{i2} respectively, the equilibrium equations of the solid in vibration on a perturbation basis are given by

$$\alpha \sin(\omega t) \sigma_{ij,j}^1 + (\alpha \sin(\omega t))^2 \sigma_{ij,j}^2 + F_i + \rho \omega^2 (\alpha \sin(\omega t) u_{i1} - 2\alpha^2 \cos(2\omega t) u_{i2}) = 0 \quad (24)$$

where F_i refers to sum of all the static body forces in i -axis direction and the notation $(\)_{,j}$ denotes differentiation with respect to coordinate j . Multiplying the two sides of the above equation by $u_{i1} \sin(\omega t) d\alpha + u_{i2} \sin^2(\omega t) d(\alpha^2)$ and performing integration over $[0, 2\pi]$ with respect to ωt , integration over $[0, \alpha]$ with respect to α and integration over solid volume gives

$$\int_V (\sigma_{ij,j}^1 u_{i1} + F_i u_{i2} + \rho \omega^2 u_{i1} u_{i1}) dv = 0 \quad (25)$$

The above equation was obtained in the limit when $\alpha \rightarrow 0$ because the frequency can only be extracted when deformation is small, that is, α must be infinitesimal. In the other words, in principle for any nonlinear dynamic system, frequency extraction can only be valid when and only when perturbation method is applied. Using Green's integration theorem equations, (25) and (20) can easily be proved to be equivalent. This equation indicates that for a linear elastic solid, the total work done by all the external static forces in a vibration cycle is zero, so any a static force has no effects on vibration frequencies.

Equations (21) and (25) show that the work done by external loads is determined only by α^2 terms, u_2 , v_2 and w_2 . For example, for a given transverse modal shape function $y(x)$, Luo [3] proved that the axial deformation of a beam with various loads is given by

$$u = -\frac{1}{2}\alpha^2 \int_0^x \left(\frac{dy(s)}{ds} \right)^2 ds \quad (26)$$

By applying this equation and energy conservation principle, a general formulation of transverse vibration of a uniform beam with various axial loads was presented and validated against finite element (FE) analysis.

6. Frequency calculation with uncertain modal shape functions

The above general formulation shown by equations (20) and (21) are well defined and correct when and only when the modal displacement shape functions u_{i1} and u_{i2} ($i = 1, 2, 3$) defined in equation (15) or (23) are accurately determined. If they are not, then as done in the Rayleigh-Ritz method, functions u_{i1} and u_{i2} can be constructed via a series of different functions with undefined coefficients α_k ($k = 1, \dots, m$), a potential energy function Π is defined by

$$\Pi = W_1 - \omega^2 W_0 - \varpi^2 W_2 - (W_{3x} + W_{3y} + W_{3z}) \quad (27)$$

When minimising the square of this function by letting derivatives with respect to coefficients α_k to be zero as given by

$$\frac{\partial \Pi^2}{\partial \alpha_k} = 2\Pi \frac{\partial \Pi}{\partial \alpha_k} = 0 \quad , \quad k = 1, \dots, m \quad (28)$$

The modal displacement shape functions u_{i1} and u_{i2} shall be estimated approximately. Therefore, by using the general formulation (20) together with the predicted shape functions, the free vibration frequency of the elastic solid with any static loads shall be predicted.

7. Application to cantilever beam vibration

For a tapered cantilever beam, because of the cross-section changes, an analytical vibration solution does not exist even for a load free case. In literature vibration frequencies for tapered cantilever beams usually were assessed using a series based analytical method such as done by Banerjee et al [20, 21] and by Wang and Li [22]. But these methods are complicated and are difficult to be applied to solve engineering problems by engineers. Therefore, in industry FE analysis is the main tool in solving this type of the problems. This is why efforts were made in this paper by using the developed general formulation to develop a simple analytical method to calculate the out-of-plane (the in-plane was ignored in this paper) vibration frequency of a rotating tapered cantilever beam and the subsequent validation was done against FE analysis reliably.

7.1 Geometry and modal shape function

For a tapered beam, it was assumed that the left end of a tapered cantilever beam is fixed on a hub disk with a radius R , as shown in Figure 2, where the thickness and width are denoted by h_0 and t_0 respectively. For the free end, the thickness and width are taken as h_1 and t_1 respectively. The beam has an axial span of L , and rotates at an angular velocity ϖ about the axis O_1Y_1 with an axial concentrated force F applied to the free end.

Figure 2 Sketch of a tapered cantilever beam.

In this paper, only the out-of-plane vibration was studied as the current method can be directly applied to the in-plane vibration with a tiny modification. The normalised dimensionless modal shape function $y(x)$, $x \in [0, 1]$ was assumed to be given by

$$y(x) = \left(\sum_{i=0}^n \alpha_i x^i \right) y_u(kx) \quad (29)$$

where $a_0 = 1$ and a_i , $i = 1, \dots, n$ are unknown constants, and $y_u(kx)$ is the modal shape function of a uniform cantilever beam and is given by

$$y_u(kx) = \cos kx - \cosh kx - \sin kx + \sinh kx \quad (30)$$

$$k_1 = 1.8751, k_2 = 4.694, k_3 = 7.855, k_4 = 10.966, k_5 = 14.137, k_6 = 17.279 \quad (31)$$

or

$$k_i \approx \left(i - \frac{1}{2} \right) \pi \quad (32)$$

where $i = 1, 2, 3, 4, \dots$

and

$$\left. \begin{aligned} \cosh x &= \frac{e^x + e^{-x}}{2} \\ \sinh x &= \frac{e^x - e^{-x}}{2} \end{aligned} \right\} \quad (33)$$

To meet the boundary conditions of a cantilever beam, $y(0) = y'(0) = y''(1) = y'''(1) = 0$, one has

$$\left\{ \begin{aligned} \left(\sum_{i=2}^n i(i-1) a_i \right) y_u(k) + 2 \left(\sum_{i=1}^n i a_i \right) \frac{dy_u(k)}{dx} &= 0 \\ \left(\sum_{i=3}^n i(i-1)(i-2) a_i \right) y_u(k) + 3 \left(\sum_{i=2}^n i(i-1) a_i \right) \frac{dy_u(k)}{dx} &= 0 \end{aligned} \right. \quad (34)$$

The above equation indicates that two out of n of a coefficients are not independent. Like all other series based approximate methods, will never be able to find constants, a_i , $i = 1, \dots, n$, to make this modal shape function $\sum_{i=0}^n a_i x^i y_u(x)$ to satisfy the following Euler-Bernoulli equation

$$E \frac{d^2}{dx^2} \left(I(x) \frac{d^2}{dx^2} \sum_{i=0}^n (a_i x^i) y_u(kx) \right) = \rho \omega^2 A(x) \sum_{i=0}^n (a_i x^i) y_u(kx) \quad (35)$$

where E , $A(x)$ and $I(x)$ refer to the material Young's modulus, the beam cross-section area and the second moment of cross-section area, respectively. Therefore, the only option is to work out a method to predict the frequency ω approximately.

7.2 Frequency for free tapered beam

Before starting to figure out how to assess the vibration frequency of a tapered beam subjected to a centrifugal force and a concentrated force at the free end, the method for frequency calculation of a free tapered beam was developed first. The strain energy, W_1 and the work, $\omega^2 W_0$ done by inertial force are given by

$$\begin{cases} W_1 = \frac{1}{2} \sum_{i=0}^n \sum_{j=0}^n a_i a_j w_{1ij} \\ \omega^2 W_0 = \omega^2 \frac{1}{2} \sum_{i=0}^n \sum_{j=0}^n a_i a_j w_{0ij} \end{cases} \quad (36)$$

where

$$\begin{cases} w_{1ij} = \frac{E}{L^3} \int_0^1 I(xL) v_i(x) v_j(x) dx \\ v_i(x) = i(i-1)x^{i-2} y_u(kx) + 2ix^{i-1} \frac{dy_u(kx)}{dx} + x^i \frac{d^2 y_u(kx)}{dx^2} \end{cases} \quad (37)$$

and

$$w_{0ij} = \rho L \int_0^1 A(xL) x^{i+j} (y_u(kx))^2 dx \quad (38)$$

So based on the energy conservation principle, the vibration frequency of a free tapered beam is given by

$$\omega^2 W_0 = W_1 \quad (39)$$

Using equation (28) the unknown constants, a_i , $i = 1, \dots, n$, shall be determined by

$$\frac{\partial W_0}{\partial a_i} W_1 - W_0 \frac{\partial W_1}{\partial a_i} = 0 \quad (40)$$

From equations (34) and (40), it can be proved that equation (40) is a two order algebraic equation set about the unknown coefficients, a_i , $i = 1, \dots, n$. It is difficult to solve them. Investigation indicated that when $n = 1$, then one of four boundary conditions is not satisfied, that is, the first equation in equation (34) is not satisfied, which is equivalent to a small bending moment applied to the free end of the beam. In this case a_1 can be exactly determined by

$$a_1 = \frac{-(W_{011}W_{100} - W_{000}W_{111}) - \sqrt{(W_{011}W_{100} - W_{000}W_{111})^2 - 4(W_{101}W_{011} - W_{001}W_{111})(W_{100}W_{001} - W_{000}W_{101})}}{2(W_{101}W_{011} - W_{001}W_{111})} \quad (41)$$

Numerical studies show that this can give a very good prediction of the frequency. This will be demonstrated numerically later.

When $n = 2$, it was found out that one of four boundary conditions is not satisfied as well. Unlike for $n = 1$, the second equation in equation (34) is not satisfied, that is, equivalent to a small shear force applied to the free end. In this case the two unknown coefficients, a_1 and a_2 are determined by

$$\left\{ \begin{array}{l} a_1 = \frac{-(W_{12}W_{100} - W_{000}W_{02}) + \sqrt{(W_{12}W_{100} - W_{000}W_{02})^2 - (W_{01}W_{12} - W_{11}W_{02})(W_{100}W_{11} - W_{000}W_{01})}}{(W_{01}W_{12} - W_{11}W_{02})} \\ a_2 = -\frac{1}{y_u(k) + 2\frac{dy_u(k)}{dx}} \frac{dy_u(k)}{dx} a_1 \triangleq k_a a_1 \\ W_{01} = 2(w_{101} + k_a w_{102}) \\ W_{02} = (w_{111} + 2k_a w_{112} + k_a^2 w_{122}) \\ W_{11} = 2(w_{001} + k w_{002}) \\ W_{12} = (w_{011} + 2k_a w_{012} + k_a^2 w_{022}) \end{array} \right. \quad (42)$$

When $n = 3$, all the boundary conditions can be satisfied, and the three unknown coefficients are given by

$$\left\{ \begin{array}{l} a_1 = \frac{-(W_{12}W_{100} - W_{000}W_{02}) + \sqrt{(W_{12}W_{100} - W_{000}W_{02})^2 - (W_{01}W_{12} - W_{11}W_{02})(W_{100}W_{11} - W_{000}W_{01})}}{(W_{01}W_{12} - W_{11}W_{02})} \\ a_2 = -\frac{\left(\frac{dy_u(k)}{dx}\right)\left(y_u(kx) + 3\frac{dy_u(k)}{dx}\right)}{\left(y_u(k) + 2\frac{dy_u(k)}{dx}\right)\left(y_u(kx) + 3\frac{dy_u(k)}{dx}\right) - 3\frac{dy_u(k)}{dx}\left(y_u(k) + \frac{dy_u(k)}{dx}\right)} a_1 \triangleq k_{ka2} a_1 \\ a_3 = \frac{\left(\frac{dy_u(k)}{dx}\right)^2}{\left(y_u(k) + 2\frac{dy_u(k)}{dx}\right)\left(y_u(kx) + 3\frac{dy_u(k)}{dx}\right) - 3\frac{dy_u(k)}{dx}\left(y_u(k) + \frac{dy_u(k)}{dx}\right)} a_1 \triangleq k_{ka3} a_1 \\ W_{01} = 2\left(\sum_{j=1}^3 k_{aj} W_{10j}\right) \\ W_{02} = \sum_{i=1}^3 \sum_{j=1}^3 k_{ai} k_{aj} W_{1ij} \\ W_{11} = 2\sum_{j=1}^3 k_{aj} W_{00j} \\ W_{12} = \sum_{i=1}^3 \sum_{j=1}^3 k_{ai} k_{aj} W_{0ij} \end{array} \right. \quad (43)$$

where $k_{a1} = 1$. When $n \geq 4$, although all the boundary conditions can be satisfied, the constants, a_i , $i = 1, \dots, n$, cannot be fully determined analytically anymore. Instead they can only be solved numerically.

7.3 Case study and FE validation for free tapered beam

In the FE analysis, four different models were generated, which have the same material properties with a Young's modulus of 223GPa, a density of 7770kg/m³ and a Poisson's ratio of 0.3. For these models,

their geometrical dimensions are shown in Table 1. The FE models were meshed with 3D quadratic 10-node/20-node elements in ABAQUS with the fully fixed left end.

Table 1 Finite element model geometrical dimensions

Case	t_0 (mm)	t_1 (mm)	h_0 (mm)	h_1 (mm)	L (mm)
1	50	60	20	10	240
2	50	60	80	10	240
3	50	60	80	5	240
4	50	60	80	0	240

The out-of-plane natural frequencies obtained from the FE analysis and the analytical method for the above cases were normalised by $\omega_0 = k_i^2 \sqrt{\frac{EI_0}{\rho L^4}}$ where I_0 is the second moment of area at the fixed end, and they are shown in Table 2 below.

Table 2 Frequency results obtained from FE analysis and analytical method

Case	Mode	FE (ω/ω_0)	Analytical results(ω/ω_0)				Relative Error(%)			
			n=1	n=2	n=3	n=4	n=1	n=2	n=3	n=4
1	1	1.071771	1.03019	1.067935	1.114868	1.058449	3.88	0.36	-4.02	1.24
	2	0.821959	0.769334	0.783191	0.8041	0.782671	6.40	4.72	2.17	4.78
	3	0.739872	0.719722	0.741093	0.772707	0.731588	2.72	-0.17	-4.44	1.12
	4	0.702038	0.687788	0.716035	0.761717	2.731673	2.03	-1.99	-8.50	
	5	0.66235	0.665052	0.697784	0.756302		-0.41	-5.35	-14.18	
	6	0.621779	0.650454	0.684918	0.753241		-4.61	-10.15	-21.14	
2	1	0.976037	0.954537	0.991445	1.029384	1.360392	2.20	-1.58	-5.47	-39.38
	2	0.801384	0.756264	0.766268	0.781792	0.606603	5.63	4.38	2.44	24.31
	3	0.733994	0.711515	0.731038	0.758413	3.797445	3.06	0.40	-3.33	-417.37
	4	0.699022	0.680609	0.707393	0.749512		2.63	-1.20	-7.22	
	5	0.674259	0.658373	0.689652	0.744933		2.36	-2.28	-10.48	
	6	0.630889	0.644144	0.677144	0.742267		-2.10	-7.33	-17.65	
3	1	1.022131	1.042439	1.073383	1.134253	1.057360	-1.99	-5.01	-10.97	-3.45
	2	0.678450	0.665889	0.685440	0.726029	0.677046	1.85	-1.03	-7.01	0.21
	3	0.585762	0.590185	0.627996	0.686027	3.730052	-0.76	-7.21	-17.12	-536.79
	4	0.551150	0.545870	0.593789	0.672026		0.96	-7.74	-21.93	
	5	0.528236	0.518134	0.570013	0.664965		1.91	-7.91	-25.88	
	6	0.506002	0.500947	0.553949	0.660899		1.00	-9.48	-30.61	
4	1	1.322872	1.357362	1.390466	1.483364	1.360392	-2.61	-5.11	-12.13	-2.84
	2	0.645531	0.600215	0.633268	0.702111	0.606603	7.02	1.90	-8.76	6.03
	3	0.463246	0.520637	0.571527	0.652321	3.797445	-12.39	-23.37	-40.82	-719.75
	4	0.395456	0.475397	0.535119			-20.21	-35.32		
	5	0.354312	0.447560	0.509499			-26.32	-43.80		
	6	0.328617	0.429595	0.491688			-30.73	-49.62		

The results show that when $n = 1$ the analytical method can give reasonable frequency predictions for all the cases except for the third and higher modes for case 4. This is because when the thickness of free end is zero, for the higher modes, the beam deformation is three- than one-dimensional, so it doesn't follow the beam theory anymore. Therefore, the corresponding vibration frequencies were not predicted accurately by the analytical methods. When $n = 2$ and $n = 4$, for case 1 for the first three modes, the frequency predictions are more accurate than those predicted by $n = 1$. However, for the higher modes, the error differences are higher than predicted by $n = 1$. Particularly, when $n = 4$, for case 2 the results are completely incorrect because the numerical calculations did not converge. The results show that for

a small tapered ratio, using $n = 2$ calculates frequencies for the first three modes with higher accuracy while for the higher models $n = 1$ can be applied to the rest of modes.

In theory when $n = 3$, all the four boundary conditions were satisfied and all the three coefficients were fully determined, the results should be better for all the cases. However, the above numerical results don't show this except for the first three modes for case 1. This is because the determined modal shape by equation (43) may be different from the real one, more so for some modes of some tapered ratios.

However, the above results illustrate that when $n = 1$ the analytical method is consistent for all the four cases, and can give a good prediction for a tapered ratio in a range of 0.25 to 1 while for a tapered ratio of 0 the first two modes can also be reasonably predicted. Hence, for axially loaded tapered beams, the out-of-plane vibration frequencies will be studied in the following section.

7.4 Frequency for axially loaded tapered beam

When a beam has a transverse vibration, the axial deformation at x relative to the fixed end is given by $\frac{1}{2} \int_0^x \left(\frac{dy(s)}{ds} \right)^2 ds$. For $n = 1$, this is given by

$$\int_0^x \left(\frac{dy(s)}{ds} \right)^2 ds = x \left(\alpha_1^2 + k^2 + \alpha_1 k^2 x \right) - \frac{5}{2} \alpha_1 + \left(\frac{1}{2} + \alpha_1 x \right) \left(e^{-2kx} (\alpha_1 - k) - (k \cos 2kx + \alpha_1 \sin 2kx) \right) + e^{-kx} \left(\frac{2}{k} - \alpha_1 \frac{1}{k} + 3\alpha_1 x \right) \alpha_1 \sin kx + e^{-kx} \left(k + 2\alpha_1 + \alpha_1 (3k - \alpha_1) x - \alpha_1^2 kx^2 \right) \cos kx \quad (44)$$

When the beam rotates about the hub disk with a given angular velocity ϖ and is subjected to a concentrated force F at the free end, then the negative work, W_e done by the centrifugal force and F is given by

$$\left\{ \begin{array}{l} W_e = -\varpi^2 W_2 - F W_3 \\ W_2 = \frac{1}{2} \rho L \int_0^1 \left(\frac{R}{L} + x \right) A(Lx) \left[\int_0^x \left(\frac{dy(s)}{ds} \right)^2 ds \right] dx \\ W_3 = \frac{1}{2} \frac{1}{L} \int_0^1 \left(\frac{dy(x)}{dx} \right)^2 dx \end{array} \right. \quad (45)$$

From equation (20) the out-of-plane vibration frequency, ω is given by

$$\omega = \sqrt{\frac{W_1}{W_0} + \frac{W_2}{W_0} \varpi^2 + \frac{W_3}{W_0} F} \quad (46)$$

Substituting equations (41) and (44) into (46), the frequency can be fully determined. For the in-plane vibration, ω can be given by

$$\omega = \sqrt{\frac{W_1}{W_0} (1 - \varpi^2) + \frac{W_2}{W_0} \varpi^2 + \frac{W_3}{W_0} F} \quad (47)$$

However, in literature it was noticed that Banerjee et al. [20, 21] presented the external work done by the centrifugal force (when without the end force F) as sum of the section force timing the square of the deflection gradient over the beam as shown below

$$W_e = \frac{1}{2} \rho \varpi^2 L^2 \int_0^1 \left(\int_x^1 A(Ls) \left(\frac{R}{L} + s \right) ds \right) \left(\frac{dy(x)}{dx} \right)^2 dx \quad (48)$$

Although the right side of the above equation shares the same unit as $\omega^2 W_2$ in equation (45), physically it has nothing to do with the work done by the centrifugal force. Because the work done by the centrifugal force is a key factor dominating the stiffening/softening effects to the rotating cantilever beam vibration, therefore, the results presented in [20, 21] are questionable.

7.5 FE validation and comparison for axially loaded tapered beam

As when the beam is subjected to axial forces such as a centrifugal force and an axial force F at the free end, the modal shape function is unknown. The accuracy of formulation (46) shall be validated against FE analysis as the FE results are reliable when the mesh density is small enough. Because the rotor speed of gas turbine engines is in a range of 40Hz to 210Hz, to validate formulation (46) and limit the paper size, the FE analysis was carried out only for representative cases 1 and 3 shown in Table 1 for the hub radius R of 0, 100mm and 200mm and the angular velocity ϖ of 30Hz, 60Hz, 100Hz and 200Hz while a free end force F in a range of 10kN to 400kN separately.

Let

$$\begin{cases} p = \frac{FL^2}{EI_0} \\ \eta^2 = \frac{\rho A_0 L^4}{EI_0} \varpi^2 \end{cases} \quad (48)$$

where A_0 is the cross-section area of the fixed end. The normalised results obtained from the FE analysis and analytical method for tapered case 1 are shown in Table 3 to Table 6 while the results are listed in Table 7 to Table 10 for tapered case 3.

Table 5 shows that all the errors are less than 7% for all the applied angular velocities. Table 6 shows that the errors are less than 5% for majority modes except for mode 2 which has a peak error up to 11.13% for $p = 0.1148$ (or 400kN). This is equivalent to a stress of 400MPa at fixed end and 667MPa at free end.

For tapered case 3,

Table 9 shows that the peak error value is about 7.13% but the majority is less than 3%. Table 10 also shows that for tapered case 3, the majority of the errors is less than 3% except for the mode 2 for $p = 0.1148$, where the peak error is up to 16.67%. As this peak force is equivalent to a stress of 400MPa at fixed end and 1000MPa at free end, which exceeds the ultimate tensile stress of most steels, so this will never happen in practice. However, when $p = 0.0287$ (or 100kN), which is equivalent to a stress of 250MPa at free end for case 3, the peak error is only 8.42%. This is still acceptable in engineering assessment.

Because the results shown in the previous section for the free loaded beams show that the large error of the analytical method also occurred to the mode 2 for tapered case 1, it is expected that the accuracy will be improved when the first term on the right side in equation (46) is calculated using $n = 2$ and the other two terms are still computed using $n = 1$.

Overall the results indicate that using the analytical method developed in this paper can predict out-of-plane vibration frequencies of a rotating tapered cantilever beam subjected to an end force with

reasonable accuracy. Due to its simplicity, the analytical method can be applied by engineers to perform assessment of blade vibration quickly without having to carry out sophisticated FE analysis, in particular at the design stage.

Table 3 Normalised frequency obtained from FE analysis for case 1 for different hub radii and angular velocities

R/L	0				0.42				0.83			
η	0.1170	0.2340	0.3900	0.7801	0.1170	0.2340	0.3900	0.7801	0.1170	0.2340	0.3900	0.7801
Mode	FE (ω/ω_0) for case 1											
1	1.076929	1.093045	1.130114	1.286125457	1.080151982	1.10465	1.161059	1.389918	1.083053	1.116254	1.191036	1.486297
2	0.822987	0.825045	0.831217	0.855906763	0.822987272	0.827102	0.835846	0.87391	0.823502	0.828645	0.840476	0.891398
3	0.740055	0.740974	0.742994	0.751076266	0.74023902	0.741525	0.744831	0.757321	0.740423	0.74226	0.746484	0.763567
4	0.702227	0.702604	0.70364	0.707410308	0.702226779	0.702981	0.704583	0.710426	0.702321	0.703263	0.705525	0.713536
5	0.66235	0.66235	0.663484	0.665185544	0.662350141	0.662917	0.664051	0.666887	0.66235	0.662917	0.664618	0.668588
6	0.621779	0.622159	0.622538	0.623297306	0.621778921	0.622159	0.622918	0.624436	0.622159	0.622159	0.622918	0.625195
7	0.598753	0.598753	0.599024	0.599839666	0.598752507	0.599024	0.599568	0.600383	0.598753	0.599024	0.59984	0.601199

Table 4 Normalised frequency obtained from analytical method for case 1 for different hub radii and angular velocities

R/L	0				0.42				0.83			
η	0.1170	0.2340	0.3900	0.7801	0.1170	0.2340	0.3900	0.7801	0.1170	0.2340	0.3900	0.7801
Mode	Analytical (ω/ω_0) for case 1											
1	1.03486	1.04851	1.08019	1.21773	1.03727	1.05800	1.10558	1.30570	1.03967	1.06740	1.13040	1.38810
2	0.77000	0.77199	0.77669	0.79836	0.77041	0.77360	0.78113	0.81550	0.77081	0.77521	0.78555	0.83229
3	0.71988	0.72036	0.72148	0.72675	0.71997	0.72073	0.72252	0.73085	0.72007	0.72110	0.72355	0.73492
4	0.68782	0.68788	0.68804	0.68877	0.68783	0.68794	0.68819	0.68936	0.68784	0.68799	0.68833	0.68994
5	0.66504	0.66499	0.66487	0.66433	0.66503	0.66495	0.66477	0.66394	0.66502	0.66492	0.66468	0.66355
6	0.65043	0.65036	0.65019	0.64940	0.65042	0.65031	0.65005	0.64882	0.65040	0.65026	0.64990	0.64824
7	0.64059	0.64052	0.64036	0.63960	0.64058	0.64047	0.64022	0.63904	0.64057	0.64042	0.64008	0.63848

Table 5 Relative error of results between FE analysis and analytical method for case 1 for different hub radii and angular velocities

R/L	0				0.42				0.83			
η	0.1170	0.2340	0.3900	0.7801	0.1170	0.2340	0.3900	0.7801	0.1170	0.2340	0.3900	0.7801
Mode	Relative error (%) for case 1											
1	3.91	4.07	4.42	5.32	3.97	4.22	4.78	6.06	4.01	4.38	5.09	6.61
2	6.44	6.43	6.56	6.72	6.39	6.47	6.55	6.68	6.40	6.45	6.54	6.63
3	2.73	2.78	2.90	3.24	2.74	2.80	3.00	3.50	2.75	2.85	3.07	3.75
4	2.05	2.10	2.22	2.63	2.05	2.14	2.33	2.97	2.06	2.17	2.44	3.31
5	-0.41	-0.40	-0.21	0.13	-0.40	-0.31	-0.11	0.44	-0.40	-0.30	-0.01	0.75
6	-4.61	-4.53	-4.44	-4.19	-4.61	-4.52	-4.36	-3.91	-4.54	-4.52	-4.33	-3.69
7	-6.99	-6.98	-6.90	-6.63	-6.99	-6.92	-6.78	-6.44	-6.98	-6.91	-6.71	-6.20

Table 6 Results obtained from FE analysis and analytical for case 1 for different concentrated forces at free end

Case1	p/mode	0.0115	0.0143	0.0172	0.0230	0.0287	0.0574	0.0861	0.1148
FE (ω/ω_0)	1	1.14881	1.166538	1.184267	1.21779	1.250024	1.392497	1.512729	1.617488
	2	0.839961	0.844076	0.848706	0.85745	0.86568	0.906315	0.943864	0.97884
	3	0.745566	0.746852	0.748321	0.751076	0.753831	0.767424	0.780833	0.793691
	4	0.704583	0.705243	0.705902	0.707128	0.708353	0.714667	0.720887	0.727013
	5	0.663484	0.664051	0.664051	0.665186	0.665753	0.669155	0.672558	0.675393
	6	0.622538	0.622918	0.622918	0.623297	0.623677	0.625575	0.627093	0.628991
Analytical (ω/ω_0)	1	1.100345	1.117177	1.133760	1.166218	1.197797	1.344613	1.476906	1.598286
	2	0.779980	0.782618	0.785247	0.790479	0.795677	0.821172	0.845900	0.869925
	3	0.722215	0.722837	0.723458	0.724699	0.725938	0.732102	0.738214	0.744276
	4	0.688126	0.688209	0.688291	0.688456	0.688621	0.689446	0.690269	0.691092
	5	0.664773	0.664704	0.664634	0.664494	0.664355	0.663657	0.662959	0.662260
	6	0.650060	0.649961	0.649862	0.649665	0.649468	0.648479	0.647490	0.646499
Relative Error(%)	1	4.22	4.23	4.26	4.23	4.18	3.44	2.37	1.19
	2	7.14	7.28	7.48	7.81	8.09	9.39	10.38	11.13
	3	3.13	3.22	3.32	3.51	3.70	4.60	5.46	6.23
	4	2.34	2.42	2.49	2.64	2.79	3.53	4.25	4.94
	5	-0.19	-0.10	-0.09	0.10	0.21	0.82	1.43	1.94
	6	-4.42	-4.34	-4.33	-4.23	-4.14	-3.66	-3.25	-2.78

Table 7 Normalised frequency obtained from FE analysis for case 3 for different hub radii and angular velocities

R/L	0				0.42				0.83			
η	0.1170	0.2340	0.3900	0.7801	0.1170	0.2340	0.3900	0.7801	0.1170	0.2340	0.3900	0.7801
Mode	FE (ω/ω_0) for case 3											
1	1.02793	1.04566	1.08628	1.25712	1.03116	1.05823	1.11980	1.36864	1.03438	1.07080	1.15236	1.47115
2	0.67948	0.68257	0.68925	0.71960	0.67999	0.68462	0.69440	0.74017	0.68051	0.68668	0.70005	0.76023
3	0.58613	0.58705	0.58925	0.59954	0.58631	0.58778	0.59109	0.60707	0.58650	0.58833	0.59311	0.61442
4	0.55134	0.55172	0.55285	0.55775	0.55143	0.55209	0.55379	0.56142	0.55143	0.55238	0.55464	0.56500
5	0.52829	0.52858	0.52920	0.53204	0.52835	0.52875	0.52971	0.53419	0.52841	0.52897	0.53028	0.53634
6	0.50600	0.50638	0.50676	0.50828	0.50600	0.50638	0.50714	0.50980	0.50600	0.50638	0.50752	0.51132

Table 8 Normalised frequency obtained from analytical method for case 3 for different hub radii and angular velocities

R/L	0				0.42				0.83			
η	0.1170	0.2340	0.3900	0.7801	0.1170	0.2340	0.3900	0.7801	0.1170	0.2340	0.3900	0.7801
Mode	Analytical (ω/ω_0) for case 3											
1	1.04688	1.05920	1.08784	1.21310	1.04900	1.06756	1.11029	1.29192	1.05112	1.07585	1.13231	1.36621
2	0.66524	0.66678	0.67042	0.68722	0.66556	0.66805	0.67392	0.70080	0.66588	0.66932	0.67741	0.71412
3	0.59022	0.59045	0.59100	0.59357	0.59027	0.59064	0.59153	0.59570	0.59031	0.59083	0.59207	0.59782
4	0.54585	0.54577	0.54560	0.54478	0.54583	0.54572	0.54546	0.54423	0.54582	0.54567	0.54532	0.54367
5	0.51809	0.51797	0.51767	0.51627	0.51807	0.51787	0.51741	0.51524	0.51805	0.51778	0.51715	0.51421
6	0.50091	0.50079	0.50051	0.49920	0.50089	0.50070	0.50027	0.49822	0.50086	0.50061	0.50002	0.49724

Table 9 Relative error between FE analysis and analytical method for case 3 for different hub radii and angular velocities

R/L	0				0.42				0.83			
η	0.1170	0.2340	0.3900	0.7801	0.1170	0.2340	0.3900	0.7801	0.1170	0.2340	0.3900	0.7801
Mode	Relative error(%) for case 3											
1	-1.84	-1.29	-0.14	3.50	-1.73	-0.88	0.85	5.61	-1.62	-0.47	1.74	7.13
2	2.10	2.31	2.73	4.50	2.12	2.42	2.95	5.32	2.15	2.53	3.23	6.07
3	-0.70	-0.58	-0.30	1.00	-0.67	-0.49	-0.08	1.87	-0.65	-0.43	0.18	2.70
4	1.00	1.08	1.31	2.32	1.02	1.15	1.50	3.06	1.02	1.21	1.68	3.78
5	1.93	2.01	2.18	2.96	1.95	2.06	2.32	3.55	1.96	2.12	2.47	4.13
6	1.01	1.10	1.23	1.79	1.01	1.12	1.36	2.27	1.02	1.14	1.48	2.75

Table 10 Results obtained from FE analysis for case 3 for different concentrated forces at free end

Case 3	p/mode	0.0029	0.0057	0.0086	0.0115	0.0143	0.0172	0.0230	0.0287	0.0574	0.0861	0.1148
FE (ω/ω_0)	1	1.05211	1.08080	1.10787	1.13366	1.15816	1.18169	1.22553	1.26646	1.43762	1.57397	1.69034
	2	0.68719	0.69491	0.70314	0.71086	0.71857	0.72577	0.74017	0.75406	0.81579	0.86928	0.91557
	3	0.58870	0.59146	0.59440	0.59715	0.59991	0.60266	0.60817	0.61350	0.63866	0.66199	0.68348
	4	0.55247	0.55388	0.55520	0.55652	0.55784	0.55926	0.56180	0.56444	0.57716	0.58923	0.60072
	5	0.52897	0.52971	0.53045	0.53113	0.53186	0.53260	0.53402	0.53549	0.54253	0.54944	0.55614
	6	0.50638	0.50676	0.50752	0.50790	0.50828	0.50866	0.50980	0.51056	0.51511	0.51929	0.52346
Analytical (ω/ω_0)	1	1.06418	1.08520	1.10581	1.12605	1.14593	1.16547	1.20361	1.24057	1.41092	1.56281	1.70119
	2	0.66736	0.66998	0.67259	0.67519	0.67779	0.68037	0.68550	0.69060	0.71554	0.73965	0.76299
	3	0.59050	0.59086	0.59121	0.59157	0.59193	0.59229	0.59300	0.59371	0.59727	0.60080	0.60431
	4	0.54573	0.54559	0.54545	0.54531	0.54517	0.54503	0.54475	0.54447	0.54306	0.54165	0.54023
	5	0.51792	0.51770	0.51748	0.51727	0.51705	0.51683	0.51640	0.51596	0.51378	0.51159	0.50940
	6	0.50075	0.50055	0.50035	0.50015	0.49995	0.49975	0.49935	0.49895	0.49695	0.49494	0.49292
Relative Error(%)	1	-1.15	-0.41	0.19	0.67	1.06	1.37	1.79	2.04	1.86	0.71	-0.64
	2	2.89	3.59	4.34	5.02	5.68	6.26	7.39	8.42	12.29	14.91	16.67
	3	-0.31	0.10	0.54	0.93	1.33	1.72	2.49	3.22	6.48	9.24	11.58
	4	1.22	1.50	1.76	2.01	2.27	2.54	3.04	3.54	5.91	8.08	10.07
	5	2.09	2.27	2.44	2.61	2.79	2.96	3.30	3.65	5.30	6.89	8.40
	6	1.11	1.23	1.41	1.53	1.64	1.75	2.05	2.27	3.53	4.69	5.83

8. Conclusions & Discussion

By studying the exact solutions of three proposed 1D nonlinear dynamic differential equations, it was demonstrated that in principle unless the nonlinear terms are in first order, a dynamic system will not have a vibration natural frequency. A simple nonlinear mass-spring dynamic system was presented to demonstrate that a vibration natural frequency may be calculated based on perturbation basis and external load can have effects on frequencies of a nonlinear system. With the use of these simple cases, why and how nonlinearity has effects on vibration frequencies were presented.

By introducing a virtual factor α to ensure a small deformation of an elastic solid with various external static loads, a general vibration frequency formulation (20) for the solid was presented. It was proved that the corresponding displacement modal shape functions of a nonlinear elastic solid with various loads must be functions with second order or higher order terms of α as opposed to linear functions of α . It was also proved that only the α^2 terms, such as u_2 , v_2 and w_2 contribute to the external load effects on vibration frequencies. This is the reason why a static load on a linear elastic solid never affects vibration frequencies.

Comparison with the Rayleigh-Ritz method shows that the general formulation (20) was developed also based on the energy conservation principle. However, this paper proved that the strain energy and kinematic energy were calculated from the linear deformation terms. It also proved that the work done by external static loads was calculated from the nonlinear deformation terms that cause the stiffening/softening effect to vibration frequencies while the Rayleigh-Ritz method doesn't.

Applying the developed general formulation, the simple analytical method for rotating tapered cantilever beams was developed. The validation against the FE analysis for tapered cases shows that the out-of-plane vibration frequencies can be predicted with reasonable accuracy for tapered ratios in a range of 0.25 to 1. For the tapered ratio of zero (zero thickness at the free end), only the two first modes can be predicted correctly while the other higher modes cannot be assessed. It permits engineers to assess blade vibration in a simple way without having to perform sophisticated FE analysis.

Using the general formulation presented in this paper, for any other structures, such as the beams, the shells, the solids, or combination of them, when they are subjected to external static loads and have large deformation, their free vibration frequencies can be formulated and studied analytically by engineers and scientists in the future. As an application, the simple analytical method was developed for rotating tapered cantilever beams, which can be applied directly by engineers to assess blade vibrations.

Overall, with work of this paper, complicated issues arising in nonlinear dynamic vibration were presented in a concise way. How nonlinearity affects the vibration natural frequency was explored and the presented work indicates that the effects of the third and higher order nonlinear terms on vibration frequency are negligible. Unlike understood by some people, the presented natural frequency existence theorem indicates that a nonlinear dynamic system may have a natural frequency when all the nonlinear terms are in the first order. The developed general formulation can be a guideline for engineers and researchers when working on nonlinear vibration investigation in future. The simple analytical method for the rotating tapered cantilever beam shows that a tapered beam vibration can be studied approximately by using the modal shape functions of the corresponding uniform beam.

9. Funding

This research received no specific grant from any funding agency in public, commercial or not-for-profit sectors.

10. Conflict of interest

The work was completed in the author's personal interests. It does not conflict with any other parts/organisations.

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Figures

Figure 1

a body rotates about an axis

(a)

(b)

Figure 2

Sketch of a tapered cantilever beam.