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Approximate Solutions of the Space-Fractional Diffusion Equations with additive noise

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Abstract

In this article we take into account a class of stochastic space diffusion equations with polynomials forced by additive noise. We derive rigorously limiting equations which define the critical dynamics. Also, we approximate solutions of stochastic fractional space diffusion equations with polynomial term by limiting equations, which are ordinary differential equations. Moreover, we address the effect of the noise on the solution's stabilization. Finally, we apply our results to Fisher's equation and Ginzburg-Landau models.

Keywords: Approximate solutions, fractional Laplacian, additive noise, limiting equation.

Mathematics Subject Classification: 35R60, 60H15.

1 Introduction

In the last few decades, fractional derivatives have drawn tremendous interest mainly because of their possible implementations in different areas for example in physics [1, 2, 3, 4], biology [5], finance [6, 7, 8], biochemistry and chemistry [9], hydrology [10, 11]. These fractional-order equations are better suited than equations with integer-order, because derivatives of fractional order are allowed the memory and hereditary properties of different substances to be represented [12].

In normal diffusion with time, the mean square displacement of an equation particle linearly increases, i.e. $\langle x^2(t) \rangle \propto t$. On the other side, anomalous diffusion is a diffusion process not following this linear relation. In some cases, they have a power-law scaling relation, $\langle x^2(t) \rangle \propto t^r$, that is present in various types of equations. r is defined as the anomalous exponent of diffusion, in the case $r = 1$ of normal diffusion, whereas $r = 2$, $r \in (0, 1)$ and $r \in (1, 2)$ correspond to a

ballistic diffusion, a sub-diffusion and a Levy super-diffusion, respectively [2]. By the transformation of Fourier, the anomalously diffusive operator $(-\Delta)^{\frac{r}{2}}$ is defined [6, 13] as:

$$\mathcal{L}\{(-\Delta)^{\frac{r}{2}}\varphi\}(\eta) = |\eta|^r \mathcal{L}\{\varphi\}(\eta),$$

where $\mathcal{L}\{\varphi\}$ is the Fourier transform of φ .

In this paper, we are concerned with fractional space-diffusion equation perturbed by additive noise on a bounded domain $G \subset \mathbb{R}$:

$$d\varphi = [-\varepsilon^{-2}D\mathcal{A}^{\frac{r}{2}}\varphi + \mathcal{P}(\varphi)]dt + \varepsilon^{-1}dW, \quad t \geq 0, \quad x \in G, \quad (1.1)$$

where $\varepsilon \ll 1$ is a small parameter, D is the diffusion coefficient, $\mathcal{A}^{\frac{r}{2}}$ is the fractional operator with $r \in (1, 2]$ (One standard example is fractional Laplacian $(-\Delta)^{\frac{r}{2}}$), $\mathcal{P}$ is a polynomial with degree m and it is representing reaction kinetics, W is a finite dimensional Wiener process. It is interesting to note that if we put $\mathcal{P}(\varphi) = \varphi(1 - \varphi)(\varphi - a)$, then Equation (1.1) becomes the stochastic fractional space Fitzhugh–Nagumo equation, which is used in the field of biology and population genetics in circuit theory [14], also is used to model nerve impulse transmission [15, 16]. While, if we set $\mathcal{P}(\varphi) = \varphi - \varphi^3$, we get to stochastic space-fractional heat equation, which is used in physics and describes the heat distribution within a given time interval in a given region. Furthermore, If $\mathcal{P}(\varphi) = \varphi(1 - \frac{\varphi}{N})$, then (1.1) is giving rise to stochastic space-fractional Fisher equation which is used as the spatial and temporal propagation model in an infinite medium of a virile gene. Also, it is used in chemical kinetics [17], auto catalytic chemical reaction [18], flame propagation [19], neurophysiology [20], nuclear reactor theory [21].

Recently, Equation (1.1) with $r = 2$ was studied analytically by [22, 23] in the deterministic case, i.e without noise. While in the stochastic case this equation with $r = 2$ was addressed by [24, 25, 26, 27]. Moreover, several numerical and analytical methods have recently been suggested to solve the space fractional partial differential Equation (1.1) without noise see for instance [28, 29, 30, 31]. Here, by perturbation method, we analytically approximate the solution of Equation (1.1), where this equation has not been solved with this method before.

We are not looking at the specifics of the existence and uniqueness of solutions in this paper, which is a well known issue. Always, we are assuming at least one local solution exists. We refer to monographs in [32, 33, 34, 35, 36, 37, 38] and the reference therein for the existence and uniqueness of solutions.

Our aim here is to show that the approximate solution of (1.1) is given by

$$\varphi(t, x) = \xi(t) + \chi(t, x) + \text{error}, \quad (1.2)$$

where ξ solves

$$d\xi = [\mathcal{P}(\xi) + G(\xi)]dt. \quad (1.3)$$

The polynomial $G(\xi)$, is defined later in (5.2), has degree $m - 2$. The stochastic process $\chi(t, x)$ in equation (1.2), will be defined later in (2.6), is called a fast Ornstein-Uhlenbeck process.

We note that the ordinary differential Equation (1.3) contains the same polynomial $\mathcal{P}$ with a further polynomial G that exists due to interact between nonlinear term and additive noise. The following real-valued Ginzburg–Landau

equation, with Neumann boundary conditions on $[0, \pi]$, is considered for clarification of our results

$$d\varphi = [-\varepsilon^{-2}(-\Delta)^{\frac{r}{2}}\varphi + \varphi - \varphi^3]dt + \varepsilon^{-1}dW. \quad (1.4)$$

We demonstrate in approximation Theorem 17 the solution of Ginzburg–Landau Eq. (1.4) shall be of kind (1.2) where ξ is the solution of

$$d\xi = \left[\left(1 - \sum_{k=1}^N \frac{3\alpha_k^2}{2k^r} \right) \xi - \xi^3 \right] dt,$$

when $r = 2$, then we have the old result that obtained by [25].

In this paper, one great innovation of our approach is the explicit estimation of error in terms of arbitrarily high moments of error, while usually only weak convergence is handled against approximation. Moreover, this paper is the first paper, to our best knowledge, to find analytically the approximate solution for stochastic space-fractional partial differential equations.

The remainder of this article is set out as follows. In next section, we present some notations, assumptions and preliminaries, that we need in this paper, while we estimate an equation represent the high modes and give bounds on it in Section 3. We will state a general case of the averaging over OU-process In Section 4. After that we deduce the limiting equation and prove the main result 17 in Section 5. While in Section 6, there are two examples to clarify our results, such as the Ginzburg–Landau and the Fisher’s equations. Finally, we give the conclusion of this paper.

2 Preliminaries

Let $\mathcal{H} = \mathcal{L}^2([0, \pi])$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ norm.

Define $\mathcal{A} = -\Delta$, since the operator $\mathcal{A}$ is self-adjoint, there exists an complete orthonormal system $\{e_j\}_{j=0}^\infty$ and a sequence $\{\lambda_j\}_{j=0}^\infty$ such that

$$-\Delta e_j = \lambda_j e_j,$$

with

$$0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_j \leq \dots$$

Here, we consider $-\Delta$ with Neumann boundary condition on $[0, \pi]$, therefore

$$e_j(x) = \begin{cases} 1 & \text{if } j = 0, \\ \sqrt{\frac{2}{\pi}} \cos(jx) & \text{if } j \neq 0, \end{cases}$$

and

$$\lambda_j = j^2.$$

Define $\mathcal{H}^c$ and $\mathcal{H}^s$ as

$$\mathcal{H}^c := \text{kernel}\{\Delta\} = \text{span}\{1\} \quad \text{and} \quad \mathcal{H}^s = (\mathcal{H}^c)^\perp.$$

Define the projections

$$P_c := \mathcal{H} \rightarrow \mathcal{H}^c \quad \text{and} \quad P_s := \mathcal{I} - P_c,$$

where $\mathcal{I}$ is the identity operator on $\mathcal{H}$.

For $r > 0$, let the fractional space $\mathcal{H}^r$ be the domain of $\mathcal{A}^{\frac{r}{2}}$, which can be defined by

$$(-\Delta)^{\frac{r}{2}}\varphi = \sum_{j=0}^{\infty} \lambda_j^{\frac{r}{2}} \langle \varphi, e_j \rangle e_j,$$

$$\mathcal{H}^r = \mathcal{D}((-\Delta)^{\frac{r}{2}}) = \left\{ \varphi \in \mathcal{H} : \sum_{j=0}^{\infty} \lambda_j^r \varphi_j^2 < \infty \right\},$$

with the induced norms

$$\|\varphi\|_r^2 = \|(-\Delta)^{\frac{r}{2}}\varphi\|^2 = \sum_{k=0}^{\infty} \lambda_k^r \varphi_k^2.$$

Moreover, let $\mathbb{T}_r(t) = \exp(-t(-\Delta)^{\frac{r}{2}})$ for $t \geq 0$ be analytic semigroup generated by the fractional Laplacian $-(-\Delta)^{\frac{r}{2}}$ and satisfy

$$\|\mathbb{T}_r(t)\varphi\|_r \leq e^{-\omega t} \|\varphi\|_r \quad \text{for all } \varphi \in \mathcal{H}^r, \quad (2.1)$$

where a constant $\omega > 0$.

For the nonlinear $\mathcal{P}$ in Equation (1.1), we assume:

Assumption 1 Let $\mathcal{P} : \mathcal{H}^r \rightarrow \mathcal{H}^r$ satisfy, for all $\varphi \in \mathcal{H}^r$, that

$$\|\mathcal{P}(\varphi)\|_r \leq C(1 + \|\varphi\|_r^m), \quad (2.2)$$

where m is the degree of $\mathcal{P}$.

For short, we are using $\mathcal{P}_c(\varphi) = P_c\mathcal{P}(\varphi)$ and $\mathcal{P}_s(\varphi) = P_s\mathcal{P}(\varphi)$.

Assumption 2 Let $\mathbb{F}(u) = \mathcal{P}_c(u) + G(u)$, where $G(u)$ is defined in (5.2). Assume for $u, w, v \in \mathcal{H}^c$ that

$$\langle \mathbb{F}(u + v + w) - \mathbb{F}(v), u \rangle \leq C(|u|^2 + |w|^2 + |w|^m). \quad (2.3)$$

We note that from the assumption 2 if we put $v = w = 0$, then we have

$$\langle \mathbb{F}(u), u \rangle \leq C|u|^2 \quad \text{for } u \in \mathcal{H}^c \quad (2.4)$$

For the noise in Equation (1.1):

Assumption 3 Suppose the Wiener process $W(t)$, for $t \geq 0$, is finite dimensional and acts only on $\mathcal{H}^s$. Corresponding to [39] one can write it as

$$W(t, x) = \sum_{j=1}^N \alpha_j \beta_j(t) e_j(x),$$

where $\alpha_j \in \mathbb{R}$ for all $j \in \{1, 2, \dots, N\}$ and $(\beta_j)_{j \in \{1, 2, \dots, N\}}$ are mutually independent real-valued Brownian motions.

Definition 4 *The fast OU-process (OU-process, for short) χ is defined as*

$$\chi(t) = \sum_{j=1}^N \chi_j(t) e_j, \quad (2.5)$$

with

$$\chi_j(t) = \alpha_j \varepsilon^{-1} \int_0^t e^{-\varepsilon^{-2} D \lambda_j^{\frac{r}{2}}(t-s)} d\beta_j(s). \quad (2.6)$$

In the following definition we assume that the solution of Equation (1.1) is not too large.

Definition 5 *(stopping time) For some $T_0 > 0$ and $\kappa \in (0, \frac{1}{2m})$. Define*

$$\tau^* := \inf \{ t > 0 : \|\varphi(t)\|_r > \varepsilon^{-\kappa} \} \wedge T_0, \quad (2.7)$$

3 High modes and Its Bounds

In this section we deduce an equation represent high modes and bound it. We start by split the solution φ of (1.1) into

$$\varphi(t, x) = \varphi_c(t) + \varphi_s(t, x), \quad (3.1)$$

where $\varphi_c \in \mathcal{H}^c$ and $\varphi_s \in \mathcal{H}^s$. Substituting (3.1) into (1.1) we have

$$d(\varphi_c + \varphi_s) = [-\varepsilon^{-2} D(-\Delta)^{\frac{r}{2}} \varphi_s + \mathcal{P}(\varphi_c + \varphi_s)] dt + \varepsilon^{-1} dW. \quad (3.2)$$

By projecting to $\mathcal{H}^s$ we obtain

$$d\varphi_s = [-\varepsilon^{-2} D(-\Delta)^{\frac{r}{2}} \varphi_s + \mathcal{P}_s(\varphi_c + \varphi_s)] dt + \varepsilon^{-1} \alpha dW. \quad (3.3)$$

This equations can be written in the integral form as

$$\varphi_s(t) = \mathbb{T}_r(\varepsilon^{-2} D t) \varphi_s(0) + \int_0^t \mathbb{T}_r(\varepsilon^{-2} D(t-\tau)) (\varphi_c + \varphi_s) d\tau + \chi(t), \quad (3.4)$$

where χ is defined in Definition 4.

In the following lemma we will show $\varphi_s(t)$ equals $\chi(t)$ plus a small term.

Lemma 6 *Assume that Assumption 1 satisfies. Then there exists $C > 0$ such that*

$$\mathbb{E} \sup_{t \in [0, \tau^*]} \left\| \varphi_s(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2} D t) \varphi_s(0) \right\|_r^p \leq C \varepsilon^{p-m p \kappa}, \quad (3.5)$$

for $p \geq 1$ and $\kappa > 0$ from the definition of τ^* .

Proof. Using the triangle inequality for (3.4), yields

$$\begin{aligned} \left\| \varphi_s(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2} D t) \varphi_s(0) \right\|_r &\leq \left\| \int_0^t \mathbb{T}_r(\varepsilon^{-2} D(t-s)) \mathcal{P}_s(\varphi_c + \varphi_s) ds \right\|_r \\ &\leq C \left\| \mathcal{P}_s(\varphi_c + \varphi_s) \right\|_r \int_0^t e^{-\varepsilon^{-2} \omega D(t-s)} ds \\ &\leq C \varepsilon^2 (1 + \|\varphi_c + \varphi_s\|_r^m). \end{aligned}$$

Taking $\mathbb{E} \sup_{t \in [0, \tau^*]}$ on both sides

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, \tau^*]} \left\| \varphi_s(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2} Dt) \varphi_s(0) \right\|_r &\leq C\varepsilon^2 (1 + \mathbb{E} \sup_{t \in [0, \tau^*]} \|\varphi_c + \varphi_s\|_r^m) \\ &\leq C\varepsilon^2, \end{aligned}$$

where we used (2.1), Assumption 1 and the definition of τ^* , respectively. $\square$

Now let us state without proof the uniform bounds on $\chi(t)$. For the proof, see Lemma 4.2 in [25].

Lemma 7 *Let $\chi(t)$ be defined in Definition 4. Then there is $C > 0$ such that*

$$\mathbb{E} \sup_{t \in [0, T_0]} \|\chi(t)\|_r^p \leq C\varepsilon^{-\kappa_0}, \quad (3.6)$$

for every $p \geq 1$ and $\kappa_0 > 0$.

The following corollary declare that $\varphi_s(t)$ is much smaller than $\varepsilon^{-\kappa}$ as stated in definition of stopping time τ^* .

Corollary 8 *Assume that the assumptions of Lemmas 6 and 7 are satisfying. Let $\varphi_s(0) = \mathcal{O}(1)$, then for, $p \geq 1$, $\kappa < \frac{1}{m}$ and $C > 0$,*

$$\mathbb{E} \sup_{t \in [0, \tau^*]} \|\varphi_s(t)\|_r^p \leq C\varepsilon^{-\kappa_0} \text{ for } p \geq 1. \quad (3.7)$$

Proof. Using Equation (3.5) and triangle inequality

$$\begin{aligned} \mathbb{E} \sup_{[0, \tau^*]} \|\varphi_s\|_r^p &\leq c\mathbb{E} \sup_{[0, \tau^*]} \left\| \varphi_s - \chi - \mathbb{T}_r(\varepsilon^{-2} Dt) \varphi_s(0) \right\|_r^p \\ &\quad + c\mathbb{E} \sup_{[0, \tau^*]} \left\| \mathbb{T}_r(\varepsilon^{-2} Dt) \varphi_s(0) \right\|_r^p + c\mathbb{E} \sup_{[0, \tau^*]} \|\chi\|_r^p. \end{aligned}$$

Using Lemmas 7 and 6 to finish the proof. $\square$

Lemma 9 *Let $\varphi_s(0) = \mathcal{O}(1)$, then*

$$\int_0^t \left\| \mathbb{T}_r(\varepsilon^{-2} Ds) \varphi_s(0) \right\|^n ds \leq C\varepsilon^2 \text{ for } n \geq 1.$$

Proof.

$$\int_0^t \left\| \mathbb{T}_r(\varepsilon^{-2} Ds) \varphi_s(0) \right\|^n ds \leq \int_0^t e^{-\varepsilon^{-2} \omega n D\tau} \left\| \varphi_s(0) \right\|^n ds = C\varepsilon^2.$$

$\square$

4 Averaging over OU-Process

Here we state a general case of Lemma 5.1 from [25] over the fast OU-process χ_j . This lemma declare that odd powers of χ_j are small of order $\mathcal{O}(\varepsilon^{1-\kappa_0})$, while even power of χ_j average to a constant.

Lemma 10 Assume that ϕ is a real-valued stochastic process with $\phi(0) = \mathcal{O}(\varepsilon^{-\gamma})$ for some $\gamma \geq 0$. If $d\phi = Fdt$ with $F = \mathcal{O}(\varepsilon^{-\gamma})$, then

$$\int_0^t \phi \prod_{i=1}^N \chi_i^{n_i} d\tau = \begin{cases} \mathcal{O}(\varepsilon^{1-\gamma-\kappa_0}) & \text{if one of the } n_i \text{ is odd,} \\ \sum_{i=1}^N \frac{n_i(n_i-1)\alpha_i^2}{2 \sum_{j=1}^N n_j D \lambda_j^{\frac{r}{2}}} \int_0^t \phi \prod_{j=1}^N \chi_j^{n_j} \chi_i^{-2} d\tau & \text{if all } n_i \text{ are even.} \end{cases} \quad (4.1)$$

In the following lemma, we apply the earlier Lemma 10 iteratively and check the outcome that we need later in our example.

Lemma 11 Assume that ϕ is as in Lemma 10. Then there is a constant C_{2k} for $k \in \mathbb{N}$ such that

$$\int_0^t \phi \chi^{2k} d\tau = C_{2k} \int_0^t \phi d\tau + \mathcal{O}(\varepsilon^{1-\gamma-\kappa_0}), \quad (4.2)$$

where χ is defined in Definition 4.

Proof. We address three cases as follows.

First case when $k = 1$:

$$\begin{aligned} \int_0^t \phi \chi^2 d\tau &= P_c \int_0^t \phi \left(\sum_{j=1}^N \chi_j e_j \right)^2 d\tau \\ &= \sum_{j=1}^N (e_j^2) \int_0^t \phi \chi_j^2 d\tau + 2 \sum_{j \neq i} (e_j e_i) \int_0^t \phi \chi_j \chi_i d\tau. \end{aligned}$$

From Lemma 10 we have

$$\int_0^t \phi \chi^2 d\tau = \sum_{j=1}^N \frac{e_j^2 \alpha_j^2}{2D \lambda_j^{\frac{r}{2}}} \int_0^t \phi d\tau + \mathcal{O}(\varepsilon^{1-\gamma-\kappa_0}).$$

Thus

$$C_{2k} \stackrel{k=1}{=} C_2 = \sum_{j=1}^N \frac{e_j^2 \alpha_j^2}{2D \lambda_j^{\frac{r}{2}}}.$$

Second case when $k = 2$:

$$\begin{aligned} \int_0^t \phi \chi^4 d\tau &= \int_0^t \phi \left(\sum_{j=1}^N \chi_j e_j \right)^4 d\tau \\ &= \sum_{j=1}^N e_j^4 \int_0^t \phi \chi_j^4 d\tau + 6 \sum_{j \neq i} e_j^2 e_i^2 \int_0^t \phi \chi_j^2 \chi_i^2 d\tau + 4 \sum_{k \neq i} e_j e_i^3 \int_0^t \phi \chi_j \chi_i^3 d\tau. \end{aligned}$$

Again, from Lemma 10 we get

$$\int_0^t \phi \chi^4 d\tau = \left[\sum_{j=1}^N \frac{3e_j^4 \alpha_j^4}{4D^2 \lambda_j^r} + \sum_{j \neq i} \frac{3e_j^2 e_i^2 \alpha_j^2 \alpha_i^2}{2\lambda_j^{\frac{r}{2}} \lambda_i^{\frac{r}{2}}} \right] \int_0^t \phi d\tau + \mathcal{O}(\varepsilon^{1-\gamma-\kappa_0}).$$

Hence

$$C_{2k} \stackrel{k=2}{=} C_4 = \sum_{j=1}^N \frac{3e_j^4 \alpha_j^4}{4D^2 \lambda_j^r} + \sum_{j \neq i}^N \frac{3e_j^2 e_i^2 \alpha_j^2 \alpha_i^2}{2D^2 \lambda_j^{\frac{r}{2}} \lambda_i^{\frac{r}{2}}}.$$

Third case when $k > 2$: We can follow as the previous cases by expanding

$$\left(\sum_{j=1}^N \chi_j e_j \right)^{2k}.$$

□

5 Limiting Equation and Main Theorem

Here, the limiting equation is derived for Equation (1.1). Also, the main theorem of this paper is stated and proved.

Lemma 12 *Assume that the Assumptions 1, 2 and 3 are satisfying. If $\varphi_s(0) = \mathcal{O}(1)$, then*

$$\varphi_c(t) = \varphi_c(0) + \int_0^t \mathcal{P}_c(\varphi_c) d\tau + \int_0^t G(\varphi_c) d\tau + \mathcal{R}(t), \quad (5.1)$$

where

$$G(\varphi_c) = \sum_{k \geq 1} \frac{C_{2k}}{(2k)!} [\mathcal{P}_c(\varphi_c)]^{(2k)}, \quad (5.2)$$

and

$$\mathcal{R} = \mathcal{O}(\varepsilon^{1-(2m-1)\kappa}). \quad (5.3)$$

Proof. Recalling (3.2) and projecting to $\mathcal{H}^c$ we have

$$d\varphi_c = \mathcal{P}_c(\varphi_c + \varphi_s) dt. \quad (5.4)$$

Rewriting above equation in integral form as

$$\varphi_c(t) = \varphi_c(0) + \int_0^t \mathcal{P}_c(\varphi_c + \varphi_s) d\tau. \quad (5.5)$$

Recall Lemma 6, which states

$$\varphi_s(t) = \chi(t) + \psi(t) + z(t), \quad (5.6)$$

with

$$\psi(t) = \mathbb{T}_r(\varepsilon^{-2} Ds) \varphi_s(0) \text{ and } z(t) = \mathcal{O}(\varepsilon^{1-m\kappa}).$$

Substituting (5.6) into (5.5)

$$\varphi_c(t) = \varphi_c(0) + \int_0^t \mathcal{P}_c(\varphi_c + \chi + \psi + z)(\tau) d\tau. \quad (5.7)$$

Now, applying Taylor's expansion to $\mathcal{P}_c$ to get

$$\varphi_c(t) = \varphi_c(0) + \int_0^t \mathcal{P}_c(\varphi_c + \chi)(\tau) d\tau + \tilde{z}(t),$$

where

$$\tilde{z}(t) = \sum_{k=1}^m P_c \int_0^t \frac{[\mathcal{P}_c(\varphi_c + \chi)]^{(k)}}{k!} (\psi + z)^k d\tau. \quad (5.8)$$

Applying Taylor's expansion again to polynomial $\mathcal{P}_c$

$$\varphi_c(t) = \varphi_c(0) + \int_0^t \mathcal{P}_c(\varphi_c)(\tau) d\tau + \sum_{k=1}^m \int_0^t \frac{[\mathcal{P}_c(\varphi_c)]^{(k)}}{k!} \chi^k d\tau + \tilde{z}(t),$$

where m is the degree of $\mathcal{P}$. Using (4.2) we get

$$\varphi_c(t) = \varphi_c(0) + \int_0^t \mathcal{P}_c(\varphi_c) d\tau + \int_0^t G(\varphi_c) d\tau + \mathcal{R}(t),$$

where

$$\mathcal{R}(t) = \tilde{z}(t) + \mathcal{O}(\varepsilon^{1-\kappa_0}). \quad (5.9)$$

To bound the error $\mathcal{R}$, first we take $\mathbb{E} \sup_{t \in [0, \tau^*]} \|\cdot\|_r^p$ on both sides of (5.8) to get

$$\mathbb{E} \sup_{[0, \tau^*]} \|\tilde{z}\|_r^p \leq C \sum_{k=1}^m \mathbb{E} \sup_{[0, \tau^*]} \left\| \int_0^t \frac{[\mathcal{P}_c(\varphi_c + \chi)]^{(k)}}{k!} (\psi + z)^k d\tau \right\|_r^p.$$

Using Lemmas 6 and 7, Assumption 1, theorem of Burkholder-Davis-Gundy (cf. Theorem 1.2.4 in [40]), yields

$$\mathbb{E} \sup_{[0, \tau^*]} \|\tilde{z}\|_r^p \leq C \varepsilon^{1-(2m-1)\kappa}. \quad (5.10)$$

Substituting (5.10) into (5.9), yields (5.3). $\square$

Lemma 13 *Assume that Assumption 1 satisfies. Define $\xi(t)$ in $\mathcal{H}^c$ be a solution of (1.3) with $\mathbb{E} |\xi(0)|^p \leq C$. Then for all $T_0 > 0$, there is $C > 0$ such that*

$$\mathbb{E} \sup_{[0, T_0]} |\xi(t)|^p \leq C. \quad (5.11)$$

Proof. Taking the scalar product $\langle \cdot, \xi \rangle$ on both sides of (1.3) to get

$$\frac{1}{2} d_t |\xi|^2 = \langle \mathcal{P}_c(\xi) + G(\xi), \xi \rangle.$$

Using Equation (2.4), yields

$$\frac{1}{2} d_t |\xi|^2 \leq C |\xi|^2.$$

Integrating from 0 to t taking expectation

$$|\xi(t)|^2 \leq |\xi(0)|^2 + 2C \int_0^t |\xi(\tau)|^2 d\tau.$$

Using Gronwall's lemma to obtain for $t \in [0, T_0]$

$$|\xi(t)| \leq |\xi(0)| e^{CT_0}. \quad (5.12)$$

Taking expectation on both sides after supremum of equation (5.12) to obtain (5.11). $\square$

In fact we can not control of the error terms, that are defined in terms of φ_s or φ_c . Therefore we are limited to a sufficiently large subset of Ω , where all our estimates of errors are true.

Definition 14 Define the set $\Omega^* \subset \Omega$ such that all these estimates

$$\sup_{[0, \tau^*]} \|\varphi_s\|_r < \varepsilon^{-\kappa_0 - \frac{1}{2}\kappa}, \quad (5.13)$$

$$\sup_{[0, \tau^*]} \|\varphi_s - \chi - \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0)\|_r < \varepsilon^{1-m\kappa-\kappa}, \quad (5.14)$$

$$\sup_{[0, \tau^*]} |\xi| < \varepsilon^{-\frac{1}{2}\kappa}, \quad (5.15)$$

and

$$\sup_{[0, \tau^*]} \|\mathcal{R}\|_r < \varepsilon^{1-2m\kappa}, \quad (5.16)$$

are valid on Ω^* .

We see that the set Ω^* has probability close to one as follows.

Proposition 15 Assume that Assumptions 1 and 2 are satisfying, then Ω^* has probability

$$\mathbb{P}(\Omega^*) \geq 1 - C\varepsilon^p. \quad (5.17)$$

Proof. We notice that

$$\begin{aligned} \mathbb{P}(\Omega^*) &\geq 1 - \mathbb{P}\left(\sup_{[0, \tau^*]} \|\varphi_s\|_r \geq \varepsilon^{-\kappa_0 - \frac{1}{2}\kappa}\right) - \mathbb{P}\left(\sup_{[0, \tau^*]} \|\varphi_s - \chi - \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0)\|_r \geq \varepsilon^{1-\kappa(m+1)}\right) \\ &\quad - \mathbb{P}\left(\sup_{[0, \tau^*]} |\xi| > \varepsilon^{-\frac{1}{2}\kappa}\right) - \mathbb{P}\left(\sup_{[0, \tau^*]} \|\mathcal{R}\|_r \geq \varepsilon^{1-2m\kappa}\right). \end{aligned}$$

First using Chebychev inequality and after then using Lemmas 6, 12, 13, Corollary 8 we get

$$\mathbb{P}(\Omega^*) \geq 1 - C[\varepsilon^{q\kappa} + \varepsilon^{q\kappa} + \varepsilon^{q\kappa}] \geq 1 - C\varepsilon^{q\kappa} \geq 1 - C\varepsilon^p.$$

□

Theorem 16 Let Assumption 2 hold. Assume that ξ and φ_c are solutions of (1.3) and (5.1), respectively, with $\xi(0) = \varphi_c(0) = \mathcal{O}(1)$. Then

$$\sup_{t \in [0, \tau^*]} \|\varphi_c(t) - \xi(t)\|_r \leq C\varepsilon^{1-2m\kappa}, \quad (5.18)$$

and

$$\sup_{t \in [0, \tau^*]} \|\varphi_c(t)\|_r \leq C\varepsilon^{-\frac{1}{2}\kappa}. \quad (5.19)$$

Proof. Subtracting (1.3) from (5.1) we get

$$\varphi_c(t) - \xi(t) = \int_0^t \mathcal{P}_c(\varphi_c) - \mathcal{P}_c(\xi) d\tau + \int_0^t G(\varphi_c) - G(\xi) d\tau + \mathcal{R}(t).$$

Let $\mathbb{F}(u) = \mathcal{P}_c(u) + G(u)$ to have

$$\varphi_c(t) - \xi(t) = \int_0^t [\mathbb{F}(\varphi_c) - \mathbb{F}(\xi)] d\tau + \mathcal{R}(t).$$

Now, define $\Theta = \varphi_c - \xi - \mathcal{R}$ to obtain

$$\Theta(t) = \int_0^t [\mathbb{F}(\Theta + \xi + \mathcal{R}) - \mathbb{F}(\xi)] d\tau.$$

Thus,

$$d_t \Theta(t) = \mathbb{F}(\Theta + \xi + \mathcal{R}) - \mathbb{F}(\xi).$$

Taking the scalar product $\langle \cdot, \Theta \rangle$ on both sides and using Assumption 2

$$\begin{aligned} \frac{1}{2} d_t |\Theta(t)|^2 &= \langle \mathbb{F}(\Theta + \xi + \mathcal{R}) - \mathbb{F}(\xi), \Theta \rangle \\ &\leq |\Theta(t)|^2 + |\mathcal{R}(t)|^2 + |\mathcal{R}(t)|^m \\ &\leq |\Theta(t)|^2 + C\varepsilon^{2-4m\kappa} \quad \text{on } \Omega^*. \end{aligned}$$

Using Gronwall's lemma, we obtain

$$\sup_{[0, \tau^*]} |\Theta| \leq C\varepsilon^{1-2m\kappa} \quad \text{on } \Omega^*.$$

We finish the first part by using

$$\sup_{[0, \tau^*]} |\varphi_c - \xi| \leq \sup_{[0, \tau^*]} |\Theta| + \sup_{[0, \tau^*]} |\mathcal{R}| \leq C\varepsilon^{1-2m\kappa}.$$

For the second part. Consider

$$\begin{aligned} \sup_{[0, \tau^*]} |\varphi_c| &\leq \sup_{[0, \tau^*]} |\varphi_c - \xi| + \sup_{[0, \tau^*]} |\xi| \\ &\leq C\varepsilon^{1-2m\kappa} + C\varepsilon^{-\frac{1}{2}\kappa} \\ &\leq C\varepsilon^{-\frac{1}{2}\kappa}, \end{aligned}$$

for $\kappa < \frac{1}{2m}$, where we used the first part and (5.15) □

Theorem 17 (*Approximation*) Assume that Assumptions 1, 2 and 3 are satisfying. Let φ be a solution of (1.1) with splitting $\varphi = \varphi_c + \varphi_s$ defined in (3.1). Also, let ξ be a solution of (1.3) with $\xi(0) = \varphi_c(0)$. Then for $T_0 > 0$ and all $\kappa \in (0, \frac{1}{2m})$, there exists $C > 0$ such that

$$\mathbb{P}\left(\sup_{t \in [0, T_0]} \left\| \varphi(t) - \xi(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0) \right\|_r > \varepsilon^{1-2m\kappa}\right) \leq C\varepsilon^p, \quad (5.20)$$

for all $p > 0$.

Proof. We notice that for τ^* :

$$\Omega \supset \{\tau^* = T_0\} \supseteq \left\{ \sup_{[0, \tau^*]} \|\varphi_c\|_r < \varepsilon^{-\kappa}, \sup_{[0, \tau^*]} \|\varphi_s\|_r < \varepsilon^{-\kappa} \right\} \supseteq \Omega^*.$$

Now, we obtain by using Equation (3.1) and the triangle inequality

$$\sup_{t \in [0, T_0]} \left\| \varphi(t) - \xi(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0) \right\|_r$$

$$\begin{aligned}
&= \sup_{t \in [0, \tau^*]} \|\varphi(t) - \xi(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0)\|_r \\
&\leq \sup_{[0, \tau^*]} \|\varphi_c - \xi\|_r + \sup_{[0, \tau^*]} \|\varphi_s - \chi - \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0)\|_r \\
&\leq C\varepsilon^{1-2m\kappa} \quad \text{on } \Omega^*.
\end{aligned}$$

where we used (5.14) and (5.18). Hence,

$$\mathbb{P}\left(\sup_{t \in [0, T_0]} \|\varphi(t) - \xi(t) - \chi(t) - \mathbb{T}_r(\varepsilon^{-2}Ds)\varphi_s(0)\|_r > C\varepsilon^{1-2m\kappa}\right) \leq 1 - \mathbb{P}(\Omega^*).$$

By using (5.17), yields (5.20). $\square$

6 Application

Throughout chemistry, physics, biology and other fields of reaction-diffusion equations with nonlinearities of polynomials, there are many models where the main theory of approximation is applied. For example, Fisher's and Fitzhugh–Nagumo equations in biology and real-valued Ginzburg–Landau equation in physics. We are looking at two models, one from physics and the other from biology, as follows:

6.1 Physical Example

The first example is Ginzburg–Landau equation [41]. The Ginzburg–Landau equations is used for modeling a wide variety of physical systems. Also, it was first formulated in the sense of pattern formation as a long-wave amplitude equation in the case of convection in binary mixtures close to the onset of instability. The fractional space Ginzburg–Landau equations with additive noise is

$$d\varphi = [-\varepsilon^{-2}(-\Delta)^{\frac{r}{2}}\varphi + \varphi - \varphi^3]dt + \varepsilon^{-1}dW \quad \text{for } t \geq 0, \quad (6.1)$$

where the variable $\varphi(t, x)$ is a real-valued function of t and x .

To check Assumption 1. We note that $\mathcal{P}(\varphi) = \varphi - \varphi^3$, then for $r > \frac{1}{2}$

$$\begin{aligned}
\|\mathcal{P}(\varphi)\|_r &= \|\varphi - \varphi^3\|_r \leq \|\varphi\|_r + \|\varphi^3\|_r \\
&\leq \frac{2}{3} + \frac{1}{3} \|\varphi\|_r^3 + \|\varphi\|_r^3 \\
&\leq \frac{4}{3}(1 + \|\varphi\|_r^3),
\end{aligned}$$

where we used Young inequality.

Moreover, we use (5.2) with $k = 1$ to obtain

$$G(\xi) = \sum_{j=1}^N \frac{3\alpha_j^2}{2j^r} \xi,$$

hence, the limiting equation is

$$d\xi = \left[\left(1 - \sum_{j=1}^N \frac{3\alpha_j^2}{2j^r}\right)\xi - \xi^3\right]dt. \quad (6.2)$$

Now, the solution of (6.1) by our main theorem approximates by

$$\varphi(t, x) \simeq \xi(t) + \chi(t, x),$$

where ξ is a solution of (6.2) and χ is defined in (4). If we suppose that the noise acts only in one mode, i.e $W(t) = \alpha_j \beta_j \cos(jx)$. Then Equation (6.2) takes the form

$$d\xi = \left[\left(1 - \frac{3\alpha_j^2}{2j^r} \right) \xi - \xi^3 \right] dt. \quad (6.3)$$

If we choose α_j such that $\alpha_j^2 < \frac{2j^r}{3}$ for $r \in (1, 2]$, then the term $(1 - \frac{3\alpha_j^2}{2j^r})$ is negative. We may say in this case that the dynamics of the dominant modes was stabilized by the degenerated additive noise.

6.2 Biological Example

The second example is the Fisher's equation [42]. The Fisher's equation becomes one of the most important types of nonlinear equations due to their existence in many chemical and biological processes. The Fisher's equation with fractional space and forced by additive noise takes form

$$d\varphi = [-\varepsilon^{-2} D(-\Delta)^{\frac{r}{2}} \varphi + A\varphi(1 - \frac{\varphi}{K})] dt + \varepsilon^{-1} dW, \quad (6.4)$$

where A, K are positive constant. Here, $\varphi(t, x)$ describes the evolution of the state over the spatial-temporal domain defined respectively by the coordinates t, x .

Our main theory displays that the approximate solution of Fisher's equation (6.4) is

$$\varphi(t, x) = \xi(t) + \chi(t, x) + error,$$

where ξ is the solution of

$$d\xi = \left[A\xi(1 - \frac{\xi}{K}) - \frac{A}{2DK} \sum_{j=1}^N \frac{\alpha_j^2}{j^r} \right] dt.$$

7 Conclusions

In this paper we approximated the solutions of stochastic fractional space diffusion equations via the solutions of ordinary differential equations which is called limiting equations. We illustrated our results by applying to Fisher's equation and Ginzburg–Landau models. We discussed the influence of degenerate additive noise on the stabilization of the solutions. These solutions are of considerable importance in understanding many important complex physical phenomena as fractional diffusion equations arise in the modeling of turbulent flow, contaminant transport in groundwater flow and chaotic dynamics of classical conservative systems.

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