Graded Rough Sets based on Neighborhood Operator Over two Different Universes and its Application

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Graded rough sets based on neighborhood operator over two different universes and its application

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Abstract The major concern of this paper is to present the notion of rough set based on neighborhood operator on universe set, along with its properties, and examples. Then, we generalize several notions of covering rough sets to neighborhood rough sets with respect to the graded n . Further, we present some notions such as probabilistic neighborhood rough approximations of X , (Type-I / Type-II) probabilistic neighborhood rough approximations of X with error α and β , and (Type-I / Type-II) probabilistic neighborhood rough approximations of X with respect to $\mathcal{N}$. The interesting properties of above notions are investigated in detail. On the other hand, we define the notion of rough set based on neighborhood operator over two different universes. Subsequently, we present some notions (Type-I / Type-II / Type-III) graded n -neighborhood rough sets and give a two approaches to decision-making problems based on the (Type-II / Type-III) grade n -neighborhood rough sets. Then, we construct the decision steps and give two algorithms of the decision methods. Also, we will give two illustrative examples to show the applicability of the rough set based on neighborhood operator over two different universes to solve the rough decision-making problems. Finally, we give a comparison between the Liu et al.'s approach and our approach.

Keywords Rough sets, covering rough sets, neighborhood operator, probabilistic neighborhood rough sets, graded n -neighborhood rough sets.

1 Introduction

In past few years, Pawlak [1] proposed the rough set theory which can be seen as a mathematical tool to handle with ambiguity and incomplete information systems. The basic notion of rough set, i.e. the (lower / upper) approximations are firstly described through the equivalence classes. In mild of this, such as the graded rough sets [2], similarity or tolerance relations [3-5], arbitrary binary relation [6, 7], and variable precision rough

sets [8, 9] in a few extensions of the classical rough sets. On the other hand, the notion of covering-based generalizations of the rough set of a universe are introduced in [10-15]. The expert researchers studied some notions related of covering rough sets in [16-19]. The notion of neighborhood covering of arbitrary set in an universe is proposed and the basic properties are also investigated in [11, 20-22]. Several types of neighborhood-related covering rough sets are defined in [23-27]. A new notion of complementary neighborhood and some basic of their properties is introduced in [28].

Wong et al. [29] proposed the notion of rough set model on two universes and several applications of the rough set model on two universes are discussed in [30-40]. Yao and Lin [41, 42] introduced the notion of graded rough sets on one universe. Zhang et al. [43] studied a comparative between the variable precision rough set model and the graded rough set model. Zhang and Miao [44] introduced two basic double-quantitative rough set models of precision and grade and their investigation using granular computing. Based on the work of Yao and Lin [41, 42], the notion of graded rough sets on two universes are presented by Liu et al. [45] and some interesting properties are also explored. Yu et al. [46] proposed a variable precision graded rough set model based on two universes. Yu and Wang [47] proposed a kind of graded rough set with variable precision which is over two inconsistent and related universes.

There are some shortcomings in the methods are used to solve the decision-making problem by using the grade rough sets on two universes in [45], as the motivation and objectives of this article, we will introduce a new notion of rough set based on neighborhood operator on universe set U . We will discuss the interesting properties of neighborhood rough sets and also generalized several notions of covering rough sets (i.e., Definitions 2.5 and 2.8-2.11) to neighborhood rough sets with respect to the graded n (see, Theorems 3.10-3.14), respectively. We also prove that graded rough sets based on two universes are neighborhood rough sets (see, Theorem 3.17). Furthermore, we will extend the classical probabilistic rough sets to probabilistic neighborhood rough sets and investigate several of their properties of probabilistic neighborhood rough sets. Meanwhile, the basic properties of the notions (Type-I / Type-II) probabilistic neighborhood rough approximations of X with error α and β are also proposed (see, Definition 3.32). At the same time, the

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notions (Type-I / Type-II) probabilistic neighborhood rough approximations of X with respect to $\mathcal{N}$ are defined (see, Definition 3.34). In other hand, we will present the notion of neighborhood rough sets on two different universes (see, Definition 4.1) and study the basic some of their properties of neighborhood rough set. In addition to, we will propose the notions of (Type-I / Type-II / Type-III) graded n -neighborhood rough set on two different universes (see, Definitions 4.4, 4.8, and 4.16), respectively. We then show the upper approximation can't contain the lower approximation (see, Definition 4.4) as illustrated by (Example 4.7). Consequently, according to (Definition 4.8 and Lemma 4.10), the upper approximation contain the lower approximation (see, Theorem 4.11 (1)). Finally, based on Definitions 4.8 and 4.16, we will construct the algorithm 1 (which how it is realize a computation to method of the grade and calculation all the approximations of the sets) and the algorithm 2 (which how it is realize a computation to method of the grade based on relation matrix and calculation all the approximations of the sets). Also, we will present two applications (see subsections 4.2 and 4.3) on (Type-II / Type-III) graded n -neighborhood rough set over two different universes by algorithms 1 and 2, respectively. Finally, we will give a comparison between the Liu et al.'s approach [45] and our approach.

The rest of this paper is organized as follows. In Section 2, we review some basic notions related to Pawlak's rough set, probabilistic rough set, and covering rough sets. In Section 3, we propose the notions of neighborhood operator, neighborhood rough set, and probabilistic neighborhood rough set. Some of their interesting properties of above notions are studied. In Section 4, we present the notions of neighborhood rough set and three types of graded n -neighborhood rough set over two different universes. Based on (Type-II / Type-III) graded n -neighborhood rough set over two different universes, the algorithms 1 and 2, respectively, are constructed and two applications on these notions are given. A comparison between the Liu et al.'s approach [45] and our approach is discussed. Finally, conclusions are given in Section 5.

2 Preliminaries

2.1. Pawlak's rough set

Let U be a set and $\mathcal{R}$ an equivalence relation on U (in this case $(U, \mathcal{R})$ is called an approximation space).¹ A rough approximation (briefly, a Pawlak rough set) of a subset $X \in 2^U$ (the set of all subsets of U) in $(U, \mathcal{R})$ is a pair $(\underline{\mathcal{R}}(X), \overline{\mathcal{R}}(X))$, defined by

$$\underline{\mathcal{R}}(X) = \{u \in U \mid [u]_{\mathcal{R}} \subseteq X\}$$

and

$$\overline{\mathcal{R}}(X) = \{u \in U \mid [u]_{\mathcal{R}} \cap X \neq \emptyset\},$$

where $[u]_{\mathcal{R}}$ is the equivalence class of $\mathcal{R}$ containing $u \in U$. The subsets $\text{pos}(X) = \underline{\mathcal{R}}(X)$, $\text{neg}(X) = U - \overline{\mathcal{R}}(X)$, and $\text{bn}(X) = \overline{\mathcal{R}}(X) - \underline{\mathcal{R}}(X)$ are called the positive region, the negative region, and the boundary region of X with respect to $(U, \mathcal{R})$, respectively (cf. [1]).

2.2. Probabilistic rough set

¹ The idea to define a Pawlak rough set of a subset X is to describe X by $U/\mathcal{R}$ (the family of all $\mathcal{R}$ -equivalence classes, so-called knowledge).

Let $(U, \mathcal{R})$ be an approximation space and P a probability measure defined on a σ -algebra of subsets of U (in this case $(U, \mathcal{R}, P)$ is called a probabilistic approximation space).² For $0 \leq \beta < \alpha \leq 1$. A rough approximation with error α and β (briefly, a probabilistic rough set) of a subset $X \in 2^U$ in $(U, \mathcal{R}, P)$ is a pair $(\underline{\mathcal{R}}_p^\alpha(X), \overline{\mathcal{R}}_p^\beta(X))$, defined by

$$\underline{\mathcal{R}}_p^\alpha(X) = \{u \in U \mid P(X|[u]_{\mathcal{R}}) \geq \alpha\}$$

and

$$\overline{\mathcal{R}}_p^\beta(X) = \{u \in U \mid P(X|[u]_{\mathcal{R}}) > \beta\}.$$

The subsets $\text{pos}_p(X) = \underline{\mathcal{R}}_p^\alpha(X)$, $\text{neg}_p(X) = U - \overline{\mathcal{R}}_p^\beta(X)$, and $\text{bn}_p(X) = \overline{\mathcal{R}}_p^\beta(X) - \underline{\mathcal{R}}_p^\alpha(X)$ are called the positive region, the negative region, and the boundary region of X with respect to $(U, \mathcal{R}, P)$, respectively (cf. [48]).

2.3. Covering rough sets

Definition 2.1 (cf. [10, 21]). Let U be a universe, and $\mathcal{C} = \{\theta_i \mid i \in I\}$ be a family of nonempty subsets of U . If $\bigcup_{i \in I} \theta_i = U$, then $\mathcal{C}$ is called a covering of U , and the ordered pair $(U, \mathcal{C})$ is called a covering approximation space.

Definition 2.2 (cf. [11, 20-22, 25]). Let $(U, \mathcal{C})$ be a covering approximation space. For any $u \in U$, we can define the neighborhood of u as

$$\mathcal{N}_{\mathcal{C}}(u) = \bigcap \{\theta \in \mathcal{C} \mid u \in \theta\}.$$

Based on Definition 2.2 we present the concept of strong neighborhood covering on a set U as follows:

Definition 2.3. Let $(U, \mathcal{C})$ be a covering approximation space. A strong neighborhood covering on U satisfying $\mathcal{N}_{\mathcal{C}}(u_1) \cap \mathcal{N}_{\mathcal{C}}(u_2) \neq \emptyset$ for all $u_1, u_2 \in U$ with $\mathcal{N}_{\mathcal{C}}(u_1) = \mathcal{N}_{\mathcal{C}}(u_2)$.

Lemma 2.4. Let $(U, \mathcal{C})$ be a covering approximation space. For all $u_1, u_2 \in U$ and $u_1 \in \mathcal{N}_{\mathcal{C}}(u_2)$. Then $\mathcal{N}_{\mathcal{C}}(u_1) \subseteq \mathcal{N}_{\mathcal{C}}(u_2)$ hold.

Proof. clear.

Next, we offer five different types of covering rough sets based on the neighborhood of u (see Definition 2.5) and the complementary neighborhood of u (see Definitions 2.8, 2.10 and 2.11) and adhesion of u (see Definition 2.9) as follows:

Definition 2.5 (cf. [25-27]). Let U be an universe set, $(U, \mathcal{C})$ be a covering approximation space and $X \in 2^U$. Then

$$\underline{\mathcal{C}}_1(X) = \{u \in U \mid \mathcal{N}_{\mathcal{C}}(u) \subseteq X\}$$

and

$$\overline{\mathcal{C}}_1(X) = \{u \in U \mid \mathcal{N}_{\mathcal{C}}(u) \cap X \neq \emptyset\}$$

are called the lower approximation and the upper approximation of X with respect to $\mathcal{C}$ respectively, $(\underline{\mathcal{C}}_1(X), \overline{\mathcal{C}}_1(X))$ is called the type-1 covering rough approximation of X (or a type-1 covering rough set).

From Definition 2.3, Lemma 2.4, and Definition 2.5 we introduce the following theorem.

² The idea to define a probabilistic rough set of X is to describe X by $U/\mathcal{R}$ based on the assumption that classification knowledge is supplemented with the probabilistic knowledge.

Theorem 2.6. Let $(U, \mathcal{C})$ be a covering approximation space, $\mathcal{C}$ is a strong neighborhood covering, and $X \in 2^U$. Then the followings hold:

- (1) $\underline{\mathcal{C}}_1(\emptyset) = \overline{\mathcal{C}}_1(\emptyset) = \emptyset$.
- (2) $\underline{\mathcal{C}}_1(\underline{\mathcal{C}}_1(X)) = \underline{\mathcal{C}}_1(X)$ and $\overline{\mathcal{C}}_1(\overline{\mathcal{C}}_1(X)) = \overline{\mathcal{C}}_1(X)$.
- (3) $\underline{\mathcal{C}}_1(\overline{\mathcal{C}}_1(X)) = \overline{\mathcal{C}}_1(X)$ and $\overline{\mathcal{C}}_1(\underline{\mathcal{C}}_1(X)) = \underline{\mathcal{C}}_1(X)$.
- (4) $\overline{\mathcal{C}}_1(\underline{\mathcal{C}}_1(X)) \subseteq X \subseteq \underline{\mathcal{C}}_1(\overline{\mathcal{C}}_1(X))$.

Proof. (1) Since $\mathcal{C}$ is a strong neighborhood covering of U . Then by Definition 2.2, $u \in \mathcal{N}_{\mathcal{C}}(u) \forall u \in U$, thus there does not exist $u \in U$, such that $\mathcal{N}_{\mathcal{C}}(u) \subseteq \emptyset$. Hence, $\underline{\mathcal{C}}_1(\emptyset) = \emptyset$. Similarity, $\overline{\mathcal{C}}_1(\emptyset) = \emptyset$.

(2) Obviously, $\underline{\mathcal{C}}_1(\underline{\mathcal{C}}_1(X)) \subseteq \underline{\mathcal{C}}_1(X)$. Conversely, let $u \in \underline{\mathcal{C}}_1(X)$. Then $\mathcal{N}_{\mathcal{C}}(u) \subseteq X$. Suppose that $u' \in \mathcal{N}_{\mathcal{C}}(u)$, then by Lemma 2.4 we have $\mathcal{N}_{\mathcal{C}}(u') \subseteq \mathcal{N}_{\mathcal{C}}(u) \forall u' \in \mathcal{N}_{\mathcal{C}}(u)$ and so $\mathcal{N}_{\mathcal{C}}(u') \subseteq X$. Consequently, $u' \in \underline{\mathcal{C}}_1(X)$. Hence $\mathcal{N}_{\mathcal{C}}(u) \subseteq \underline{\mathcal{C}}_1(X)$ and thus $u \in \underline{\mathcal{C}}_1(\underline{\mathcal{C}}_1(X))$. Hence, $\underline{\mathcal{C}}_1(X) = \underline{\mathcal{C}}_1(\underline{\mathcal{C}}_1(X))$. Similarity, $\overline{\mathcal{C}}_1(\overline{\mathcal{C}}_1(X)) = \overline{\mathcal{C}}_1(X)$.

(3) Obviously, $\underline{\mathcal{C}}_1(\overline{\mathcal{C}}_1(X)) \subseteq \overline{\mathcal{C}}_1(X)$. Conversely, let $u \notin \underline{\mathcal{C}}_1(\overline{\mathcal{C}}_1(X))$. Then $u \notin \mathcal{N}_{\mathcal{C}}(u) \subseteq \overline{\mathcal{C}}_1(X)$. Since $\mathcal{N}_{\mathcal{C}}(u) \not\subseteq \overline{\mathcal{C}}_1(X)$, i.e., there exists $u' \in U$ such that $u' \in \mathcal{N}_{\mathcal{C}}(u)$ and $u' \notin \overline{\mathcal{C}}_1(X)$, this implies that $u' \notin \mathcal{N}_{\mathcal{C}}(u')$, $\mathcal{N}_{\mathcal{C}}(u') \cap X = \emptyset$. Since $\mathcal{C}$ is strong neighborhood covering of U , then $\mathcal{N}_{\mathcal{C}}(u') = \mathcal{N}_{\mathcal{C}}(u)$. Consequently, $u \notin \mathcal{N}_{\mathcal{C}}(u)$, $\mathcal{N}_{\mathcal{C}}(u) \cap X = \emptyset$, i.e., $u \notin \overline{\mathcal{C}}_1(X)$. Consequently, $\overline{\mathcal{C}}_1(X) \subseteq \underline{\mathcal{C}}_1(\overline{\mathcal{C}}_1(X))$. Thus, $\underline{\mathcal{C}}_1(\overline{\mathcal{C}}_1(X)) = \overline{\mathcal{C}}_1(X)$. Similarity, $\overline{\mathcal{C}}_1(\underline{\mathcal{C}}_1(X)) = \underline{\mathcal{C}}_1(X)$.

(4) We know $\underline{\mathcal{C}}_1(X) \subseteq X \subseteq \overline{\mathcal{C}}_1(X)$ and from (3) above, we have $\overline{\mathcal{C}}_1(\underline{\mathcal{C}}_1(X)) \subseteq X \subseteq \underline{\mathcal{C}}_1(\overline{\mathcal{C}}_1(X))$.

The equality (4) of Theorem 2.6 does not hold as the following example.

Example 2.7. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $\mathcal{C} = \{\{x_1, x_2, x_5\}, \{x_3, x_4\}\}$. From Definition 2.2, we have $\mathcal{N}_{\mathcal{C}}(x_1) = \mathcal{N}_{\mathcal{C}}(x_2) = \mathcal{N}_{\mathcal{C}}(x_5) = \{x_1, x_2, x_5\}$ and $\mathcal{N}_{\mathcal{C}}(x_3) = \mathcal{N}_{\mathcal{C}}(x_4) = \{x_3, x_4\}$. This show $\mathcal{C}$ is a strong neighborhood covering of U . Let $X = \{x_1, x_2, x_3\}$. Thus $\underline{\mathcal{C}}_1(\underline{\mathcal{C}}_1(X)) = \emptyset \neq X \neq U = \underline{\mathcal{C}}_1(\overline{\mathcal{C}}_1(X))$.

Definition 2.8 (cf. [28]). Let $(U, \mathcal{C})$ be a covering approximation space, $\mathcal{M}_{\mathcal{C}}(u) = \{v \in U \mid u \in \mathcal{N}_{\mathcal{C}}(v)\}$, and $X \in 2^U$. Then

$$\underline{\mathcal{C}}_2(X) = \{u \in U \mid \mathcal{M}_{\mathcal{C}}(u) \subseteq X\}$$

and

$$\overline{\mathcal{C}}_2(X) = \{u \in U \mid \mathcal{M}_{\mathcal{C}}(u) \cap X \neq \emptyset\}$$

are called the lower approximation and the upper approximation of X with respect to $\mathcal{C}$ respectively, $(\underline{\mathcal{C}}_2(X), \overline{\mathcal{C}}_2(X))$ is called the type-2 covering rough approximation of X (or a type-2 covering rough set).

Definition 2.9 (cf. [11, 20-22, 25]). Let $(U, \mathcal{C})$ be a covering approximation space, $\mathcal{P}_{\mathcal{C}}(u) = \left\{v \in U \mid \forall \theta \in \mathcal{C} (u \in \theta \leftrightarrow v \in \theta)\right\}$, and $X \in 2^U$. Then

$$\underline{\mathcal{C}}_3(X) = \bigcup \{\mathcal{P}_{\mathcal{C}}(u) \mid \mathcal{P}_{\mathcal{C}}(u) \subseteq X\}$$

and

$$\overline{\mathcal{C}}_3(X) = \bigcup \{\mathcal{P}_{\mathcal{C}}(u) \mid \mathcal{P}_{\mathcal{C}}(u) \cap X \neq \emptyset\}$$

are called the lower approximation and the upper approximation of X with respect to $\mathcal{C}$ respectively, $(\underline{\mathcal{C}}_3(X), \overline{\mathcal{C}}_3(X))$ is called the type-3 covering rough approximation of X (or a type-3 covering rough set).

Definition 2.10 (cf. [28]). Let $(U, \mathcal{C})$ be a covering approximation space and $X \in 2^U$. Then

$$\underline{\mathcal{C}}_4(X) = \{u \in U \mid \mathcal{N}_{\mathcal{C}}(u) \subseteq X \vee \mathcal{M}_{\mathcal{C}}(u) \subseteq X\}$$

and

$$\overline{\mathcal{C}}_4(X) = \{u \in U \mid \mathcal{N}_{\mathcal{C}}(u) \cap X \neq \emptyset \wedge \mathcal{M}_{\mathcal{C}}(u) \cap X \neq \emptyset\}$$

are called the lower approximation and the upper approximation of X with respect to $\mathcal{C}$ respectively, $(\underline{\mathcal{C}}_4(X), \overline{\mathcal{C}}_4(X))$ is called the type-4 covering rough approximation of X (or a type-4 covering rough set).

Definition 2.11 (cf. [28]). Let $(U, \mathcal{C})$ be a covering approximation space and $X \in 2^U$. Then

$$\underline{\mathcal{C}}_5(X) = \{u \in U \mid \mathcal{N}_{\mathcal{C}}(u) \subseteq X \wedge \mathcal{M}_{\mathcal{C}}(u) \subseteq X\}$$

and

$$\overline{\mathcal{C}}_5(X) = \{u \in U \mid \mathcal{N}_{\mathcal{C}}(u) \cap X \neq \emptyset \vee \mathcal{M}_{\mathcal{C}}(u) \cap X \neq \emptyset\}$$

are called the lower approximation and the upper approximation of X with respect to $\mathcal{C}$ respectively, $(\underline{\mathcal{C}}_5(X), \overline{\mathcal{C}}_5(X))$ is called the type-5 covering rough approximation of X (or a type-5 covering rough set).

3 A neighborhood operator

In this section we will give a new notion it is called “a neighborhood operator”. Based on a neighborhood operator on a set U , we present some concepts of rough sets and then the basic properties.

We will begin defining the neighborhood operator on a set U , and will subsequently define neighborhood rough set.

3.1. Rough sets based on a neighborhood operator

Definition 3.1. A neighborhood operator on a set U is a mapping $\mathcal{N}$ from a subset $U_0 \subseteq U$ to 2^U (the set of all subsets of U) satisfying $u \in \mathcal{N}(u) (\forall u \in U_0)$. A strong neighborhood operator on a set U is a neighborhood operator $\mathcal{N}$ satisfying $\mathcal{N}(u) \supseteq \mathcal{N}(v)$ for all $u, v \in U$ with $v \in \mathcal{N}(u)$ ³.

We note that there is not relation between a neighborhood operator (i.e., $\mathcal{N}_1 : U_0 \rightarrow 2^U$ under condition $u \in \mathcal{N}(u) (\forall u \in U_0)$) as in Definition 3.1 and the neighborhood operator (i.e., $\mathcal{N}_2 : U \rightarrow 2^U$ without any condition) as given in [49], for example, let $U = \{x_1, x_2, x_3\}$ and $U_0 = \{x_2, x_3\} \subseteq U$, then we can define $\mathcal{N}_1(x_2) = \{x_1, x_2\}$, $\mathcal{N}_1(x_3) = \{x_1, x_3\}$ and $\mathcal{N}_2(x_1) = \{x_3\}$, $\mathcal{N}_2(x_2) = \{x_1, x_3\}$, $\mathcal{N}_2(x_3) = \{x_1, x_2\}$. We can see $\mathcal{N}_1(x_2) = \{x_1, x_2\} \neq \{x_1, x_3\} = \mathcal{N}_2(x_2)$ and $\mathcal{N}_1(x_3) = \{x_1, x_3\} \neq \{x_1, x_2\} = \mathcal{N}_2(x_3)$. This implies that $\mathcal{N}_1(U_0) \neq \mathcal{N}_2(U)$.

Definition 3.2. Let $\mathcal{N}$ be a neighborhood operator on a set U and $X \in 2^U$. Then the triplet $(U, U_0, \mathcal{N})$ is called a neighborhood approximation space,

$$\underline{\mathcal{N}}(X) = \{u \in U_0 \mid \mathcal{N}(u) \subseteq X\}$$

³ Notice that $\{\mathcal{N}(u)\}_{u \in U_0}$ is not necessarily a cover of U .

and

$$\overline{\mathcal{N}}(X) = \{u \in U_0 \mid \mathcal{N}(u) \cap X \neq \emptyset\}$$

are called the lower $\mathcal{N}$ -approximation and the upper $\mathcal{N}$ -approximation of X with respect to $\mathcal{N}$ respectively, $(\underline{\mathcal{N}}(X), \overline{\mathcal{N}}(X))$ is called the neighborhood rough approximation of X (or the $\mathcal{N}$ -rough approximation of X , briefly, a neighborhood rough set).

To illustrate this idea, let us consider the following example.

Example 3.3. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $U_0 = \{x_2, x_4, x_5\} \subseteq U$. We define a neighborhood operator $\mathcal{N}$ as follows: $\mathcal{N}(x_2) = \{x_1, x_2, x_3\}$, $\mathcal{N}(x_4) = \{x_2, x_4, x_5\}$, and $\mathcal{N}(x_5) = \{x_3, x_5\}$. Assume that $X = \{x_1, x_3, x_4, x_5\}$. Then $\underline{\mathcal{N}}(X) = \{x_5\}$ and $\overline{\mathcal{N}}(X) = U_0$.

Definition 3.4. An inclusion degree on U is a mapping $\mathcal{I}_D : (2^U \times 2^U) \rightarrow [0, 1]$ satisfying the following conditions:

(1) $\mathcal{I}_D(B, A) = 1$ if and only $B \supseteq A$, and $\mathcal{I}_D(B, A) = 0$ if and only if $U - B \supseteq A$,

$$(2) \mathcal{I}_D\left(\bigcap_{i \in I} B_i, C\right) = \inf_{i \in I} \mathcal{I}_D(B_i, C) \quad (\{B_i\}_{i \in I} \subseteq 2^U).$$

A commonly used inclusion degree $\mathcal{I}_D : (2^U \times 2^U) \rightarrow [0, 1]$ on a nonempty finite set U is defined by $\mathcal{I}_D(B, A) = 1$ if $A = \emptyset$, or $\frac{|B \cap A|}{|A|}$ otherwise, where $|X|$ is the cardinality of a set $X \in 2^U$.

Definition 3.5. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $\mathcal{I}_D : U^2 \rightarrow [0, 1]$ an inclusion degree on U , $\alpha \in [0, 0.5)$, and $X \in 2^U$. Then,

$$\underline{\mathcal{N}}^\alpha(X) = \{u \in U_0 \mid \mathcal{I}_D(X, \mathcal{N}(u)) \geq 1 - \alpha\}$$

and

$$\overline{\mathcal{N}}^\alpha(X) = \{u \in U_0 \mid \mathcal{I}_D(X, \mathcal{N}(u)) > \alpha\}$$

are called the lower $(\mathcal{N}, \mathcal{I}_D)$ -approximation and the upper $(\mathcal{N}, \mathcal{I}_D)$ -approximation of X with error α respectively, and $(\underline{\mathcal{N}}^\alpha(X), \overline{\mathcal{N}}^\alpha(X))$ is called the neighborhood rough approximation of X (or the $(\mathcal{N}, \mathcal{I}_D)$ -rough approximation of X , briefly, a neighborhood rough set) with error α .

The main results are as follows:

Theorem 3.6. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space. Then the followings hold:

$$(1) \underline{\mathcal{N}}(\emptyset) = \overline{\mathcal{N}}(\emptyset) = \emptyset \text{ and } \underline{\mathcal{N}}(U) = \overline{\mathcal{N}}(U) = U_0.$$

(2) $\underline{\mathcal{N}}(X) = \sim(\overline{\mathcal{N}}(\sim X))$, $\underline{\mathcal{N}}(X) \subseteq \overline{\mathcal{N}}(X)$ hold for any $X \in 2^U$, $\sim X = U - X$.

$$(3) \underline{\mathcal{N}}\left(\bigcap_{k \in K} X_k\right) = \bigcap_{k \in K} \underline{\mathcal{N}}(X_k) \quad (\{X_k\}_{k \in K} \subseteq 2^U);$$

$\overline{\mathcal{N}}\left(\bigcup_{k \in K} X_k\right) = \bigcup_{k \in K} \overline{\mathcal{N}}(X_k) \quad (\{X_k\}_{k \in K} \subseteq 2^U)$. Therefore, both $\underline{\mathcal{N}}$ and $\overline{\mathcal{N}}$ preserve orders, i.e. $\underline{\mathcal{N}}(X_1) \subseteq \underline{\mathcal{N}}(X_2) \quad ((X_1, X_2) \in 2^U \times 2^U, X_1 \subseteq X_2)$ and $\overline{\mathcal{N}}(X_1) \subseteq \overline{\mathcal{N}}(X_2) \quad ((X_1, X_2) \in 2^U \times 2^U, X_1 \subseteq X_2)$ hold.

$$(4) \underline{\mathcal{N}}\left(\bigcup_{k \in K} X_k\right) \supseteq \bigcup_{k \in K} \underline{\mathcal{N}}(X_k) \quad (\{X_k\}_{k \in K} \subseteq 2^U) \text{ and } \overline{\mathcal{N}}\left(\bigcap_{k \in K} X_k\right) \subseteq \bigcap_{k \in K} \overline{\mathcal{N}}(X_k) \quad (\{X_k\}_{k \in K} \subseteq 2^U).$$

Proof. Follows from Definition 3.2.

The equality (5) of Theorem 3.6 does not hold as the following example.

Example 3.7. (Continued from Example 3.3). Let $X = \{x_1, x_3, x_4, x_5\}$ and $X_1 = \{x_2\}$. Then $\underline{\mathcal{N}}(X) = \{x_5\}$, $\underline{\mathcal{N}}(X_1) = \emptyset$, $\overline{\mathcal{N}}(X) = U_0$, $\overline{\mathcal{N}}(X_1) = \{x_2, x_4\}$, and thus $\underline{\mathcal{N}}(X \cup X_1) = U_0 \neq \{x_5\} = \underline{\mathcal{N}}(X_1) \cup \underline{\mathcal{N}}(X_2)$ and $\overline{\mathcal{N}}(X \cap X_1) = \emptyset \neq \{x_2, x_4\} = \overline{\mathcal{N}}(X) \cap \overline{\mathcal{N}}(X_1)$.

Theorem 3.8. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $\mathcal{I}_D$ an inclusion degree on U , $\alpha, \alpha_1 \in [0, 0.5)$, and $X \in 2^U$.

(1) $\hat{\mathcal{N}}$ is strong neighborhood operator on U , and $\{\hat{\mathcal{N}}(u)\}_{u \in U_0}$ is a cover of U if $U_0 = U$, where $\hat{\mathcal{N}}(u) = \bigcap \{\mathcal{N}(v) \mid v \in U_0, u \in \mathcal{N}(v)\}$.

(2) $(\underline{\mathcal{N}}^0(X), \overline{\mathcal{N}}^0(X)) = (\underline{\mathcal{N}}(X), \overline{\mathcal{N}}(X))$, $\underline{\mathcal{N}}^\alpha(X) \subseteq \overline{\mathcal{N}}^\alpha(X)$, $\underline{\mathcal{N}}^\alpha(\emptyset) = \overline{\mathcal{N}}^\alpha(\emptyset) = \emptyset$, $\underline{\mathcal{N}}^\alpha(U) = \overline{\mathcal{N}}^\alpha(U) = U_0$, and both $\underline{\mathcal{N}}^\alpha$ and $\overline{\mathcal{N}}^\alpha$ preserves orders, i.e. $\underline{\mathcal{N}}^\alpha(X_1) \subseteq \underline{\mathcal{N}}^\alpha(X_2)$ and $\overline{\mathcal{N}}^\alpha(X_1) \subseteq \overline{\mathcal{N}}^\alpha(X_2) \quad ((X_1, X_2) \in 2^U \times 2^U, X_1 \subseteq X_2)$ hold.

(3) $\underline{\mathcal{N}}^\alpha(X) = \sim(\overline{\mathcal{N}}^\alpha(\sim X)) \quad (\forall X \in 2^U)$ if and only if $U = U_0$ and $\alpha = 0$, where $\sim X = U - X$. Moreover, $\underline{\mathcal{N}}^\alpha(X) \subseteq X \cap U_0 \subseteq \overline{\mathcal{N}}^\alpha(X)$ holds if $\alpha = 0$.

$$(4) \underline{\mathcal{N}}^\alpha\left(\bigcap_{k \in K} X_k\right) = \bigcap_{k \in K} \underline{\mathcal{N}}^\alpha(X_k) \quad (\{X_k\}_{k \in K} \subseteq 2^U).$$

(5) For any countable family $\{X_k\}_{k \in K}$ of disjoint subsets of U , there exists a countable family $\{\alpha_k\}_{k \in K} \subseteq [0, 0.5)$ such that

$$\sum_{k \in K} \alpha_k \geq \alpha \text{ and } \overline{\mathcal{N}}^\alpha\left(\bigcup_{k \in K} X_k\right) = \bigcap_{k \in K} \overline{\mathcal{N}}^{\alpha_k}(X_k).$$

(6) $\underline{\mathcal{N}}^\alpha(X) \subseteq \underline{\mathcal{N}}^{\alpha_1}(X)$ and $\overline{\mathcal{N}}^{\alpha_1}(X) \subseteq \overline{\mathcal{N}}^\alpha(X) \quad (0 \leq \alpha \leq \alpha_1 < 0.5)$ hold.

(7) $\underline{\mathcal{N}}^\alpha(\underline{\mathcal{N}}^\alpha(X)) = \underline{\mathcal{N}}^\alpha(X)$ and $\overline{\mathcal{N}}^\alpha(\overline{\mathcal{N}}^\alpha(X)) = \overline{\mathcal{N}}^\alpha(X)$ if $\alpha = 0$ and $\mathcal{N}$ is strong neighborhood operator on U .

Proof. (1) Obviously, $\hat{\mathcal{N}} : U_0 \rightarrow 2^U$ is a neighborhood operator on a set U . For any $u, v \in U$ with $v \in \hat{\mathcal{N}}(u)$, as $\hat{\mathcal{N}}(v) = \bigcap \{\mathcal{N}(w) \mid w \in U_0, v \in \mathcal{N}(w)\}$ and $\hat{\mathcal{N}}(u) = \bigcap \{\mathcal{N}(w) \mid w \in U_0, u \in \mathcal{N}(w)\}$, $\hat{\mathcal{N}}_u = \{\mathcal{N}(w) \mid w \in U_0, u \in \mathcal{N}(w)\} \subseteq \{\mathcal{N}(w) \mid w \in U_0, v \in \mathcal{N}(w)\} = \hat{\mathcal{N}}_v$, and thus $\hat{\mathcal{N}}(v) = \bigcap \hat{\mathcal{N}}_v \subseteq \bigcap \hat{\mathcal{N}}_u = \hat{\mathcal{N}}(u)$. Therefore, $\hat{\mathcal{N}}$ is also a strong neighborhood operator on U . Clearly, $\{\hat{\mathcal{N}}(u)\}_{u \in U_0}$ is a cover of U if $U_0 = U$.

(2) Follows from Definition 3.5.

(3) As $u \in \underline{\mathcal{N}}^\alpha(X) \iff \mathcal{I}_D(X, \mathcal{N}(u)) \geq 1 - \alpha > 0.5$. Notice also $u \in \sim(\overline{\mathcal{N}}^\alpha(\sim X)) \iff u \notin \overline{\mathcal{N}}^\alpha(\sim X) \iff u \in U - U_0$, or $u \in U_0$ but $\mathcal{I}_D(X, \mathcal{N}(u)) \leq \alpha < 0.5$. Therefore, the first half of (3) holds. As $u \in \underline{\mathcal{N}}(X) \iff u \in U_0$ and $\mathcal{N}(u) \subseteq X$, $\underline{\mathcal{N}}^\alpha(X) \subseteq X \cap U_0$ holds; As $u \in \overline{\mathcal{N}}(X) \iff u \in U_0$ and $\mathcal{N}(u) \cap X \neq \emptyset$ holds. Therefore, the second half of (3) holds too.

(4) As $\mathcal{I}_D\left(\bigcap_{k \in K} X_k, \mathcal{N}(u)\right) = \inf_{k \in K} \mathcal{I}_D(X_k, \mathcal{N}(u)) \geq 1 - \alpha \iff \mathcal{I}_D(X_k, \mathcal{N}(u)) \geq 1 - \alpha \quad (\forall k \in K)$, $\underline{\mathcal{N}}^\alpha\left(\bigcap_{k \in K} X_k\right) = \bigcap_{k \in K} \underline{\mathcal{N}}^\alpha(X_k) \quad (\{X_k\}_{k \in K} \subseteq 2^U)$ holds, i.e. $\underline{\mathcal{N}}^\alpha$ preserves intersections.

(5) Let $\{X_k\}_{k \in K}$ be a countable family of disjoint subsets of U . As $\mathcal{I}_D\left(\bigcup_{k \in K} X_k, \mathcal{N}(u)\right) = \sum_{k \in K} \mathcal{I}_D(X_k, \mathcal{N}(u)) > \alpha \iff \mathcal{I}_D(X_k, \mathcal{N}(u)) > \alpha_k \ (k \in K)$ for some countable family $\{\alpha_k\}_{k \in K} \subseteq [0, 0.5]$ with $\sum_{k \in K} \alpha_k \geq \alpha$, $\bar{\mathcal{N}}^\alpha\left(\bigcup_{k \in K} X_k\right) = \bigcap_{k \in K} \bar{\mathcal{N}}^{\alpha_k}(X_k)$ holds.

(6) Let $u \in \underline{\mathcal{N}}^\alpha(X)$ implies $\mathcal{I}_D(X, \mathcal{N}(u)) \geq 1 - \alpha \geq 1 - \alpha_1$ since $0 \leq \alpha \leq \alpha_1 < 0.5$. Thus $\underline{\mathcal{N}}^\alpha(X) \subseteq \underline{\mathcal{N}}^{\alpha_1}(X)$. Similarly, $u \in \bar{\mathcal{N}}^{\alpha_1}(X)$ implies $\mathcal{I}_D(X, \mathcal{N}(u)) > \alpha_1 \geq \alpha$. Hence $\bar{\mathcal{N}}^{\alpha_1}(X) \subseteq \bar{\mathcal{N}}^\alpha(X)$.

(7) $\underline{\mathcal{N}}^\alpha(\underline{\mathcal{N}}^\alpha(X)) \subseteq \underline{\mathcal{N}}^\alpha(X)$ follows from (3). Conversely, for each $u \in \underline{\mathcal{N}}^\alpha(X)$ (i.e. $u \in U_0$ and $\mathcal{N}(u) \subseteq X$), as $\mathcal{N}$ is strong neighborhood operator on U , $\mathcal{N}(v) \subseteq \mathcal{N}(u)$ (and thus $\mathcal{N}(v) \subseteq X$) for all $u, v \in U$ with $v \in \mathcal{N}(u)$, which implies $\mathcal{N}(u) \subseteq \underline{\mathcal{N}}^\alpha(X)$, and thus $u \in \underline{\mathcal{N}}^\alpha(\underline{\mathcal{N}}^\alpha(X))$. Therefore, $\underline{\mathcal{N}}^\alpha(\underline{\mathcal{N}}^\alpha(X)) = \underline{\mathcal{N}}^\alpha(X)$ holds. Similarly, $\bar{\mathcal{N}}^\alpha(\bar{\mathcal{N}}^\alpha(X)) = \bar{\mathcal{N}}^\alpha(X)$ holds.

The following Definition 3.9, as expected, will generalize⁴ almost definitions of covering rough sets.

Definition 3.9. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space and $X \in 2^U$. Then

$$\underline{\mathcal{N}}_n(X) = \{u \in U \mid |\mathcal{N}(u) - X| \leq n\}$$

and

$$\bar{\mathcal{N}}_n(X) = \{u \in U \mid |\mathcal{N}(u) \cap X| > n\}$$

are called the lower $\mathcal{N}$ -approximation and the upper $\mathcal{N}$ -approximation of X with respect to the graded n respectively, and $(\underline{\mathcal{N}}_n(X), \bar{\mathcal{N}}_n(X))$ is called the neighborhood rough approximation of X (or the $\mathcal{N}$ -rough approximation of X , briefly, a neighborhood rough set) with respect to the graded n ⁵.

Next, we discuss the connections between neighborhood rough sets with respect to the graded n in Definition 3.9 and covering rough sets presented in Definitions 2.5 and 2.8-2.11.

Theorem 3.10. Type-1 covering rough sets (particularly, Pawlak rough sets) are neighborhood rough sets. More precisely, for each cover $\mathcal{C}$ of U and each $X \in 2^U$, the pair $(\underline{\mathcal{C}}_1(X), \bar{\mathcal{C}}_1(X))$ defined as in Definition 2.5 is the same as $(\underline{\mathcal{N}}_1(X), \bar{\mathcal{N}}_1(X))$ defined in the neighborhood approximation space $(U, \mathcal{N}_1)$ (see Definition 3.9, thus $(\underline{\mathcal{C}}_1(X), \bar{\mathcal{C}}_1(X))$ is a neighborhood rough set with respect to the graded 0), where $\mathcal{N}_1 : U \rightarrow 2^U$ is define by $\mathcal{N}_1(u) = \mathcal{N}_{\mathcal{C}}(u) = \bigcap \{\theta \in \mathcal{C} \mid u \in \theta\}$.

Theorem 3.11. Type-2 covering rough sets are neighborhood rough sets. More precisely, for each cover $\mathcal{C}$ of U and each $X \in 2^U$, the pair $(\underline{\mathcal{C}}_2(X), \bar{\mathcal{C}}_2(X))$ defined as in Definition 2.8 is the same as $(\underline{\mathcal{N}}_2(X), \bar{\mathcal{N}}_2(X))$ defined in the neighborhood approximation space $(U, \mathcal{N}_2)$ (see Definition 3.9, thus $(\underline{\mathcal{C}}_2(X), \bar{\mathcal{C}}_2(X))$ is a neighborhood rough set with respect to the graded 0), where $\mathcal{N}_2 : U \rightarrow 2^U$ is define by $\mathcal{N}_2(u) = \mathcal{M}_{\mathcal{C}}(u) = \{v \in U \mid u \in \mathcal{N}_{\mathcal{C}}(v)\}$.

Theorem 3.12. Type-3 covering rough sets are Pawlak rough sets determined by the partition $\{\mathcal{P}_{\mathcal{C}}(u)\}_{u \in U}$ of U . Therefore,

Type-3 covering rough sets are also neighborhood rough sets, and have properties that Pawlak rough sets have.

Theorem 3.13. $(\underline{\mathcal{C}}_4(X), \bar{\mathcal{C}}_4(X)) = (\underline{\mathcal{C}}_1(X) \cup \underline{\mathcal{C}}_2(X), \bar{\mathcal{C}}_1(X) \cap \bar{\mathcal{C}}_2(X)) = (\underline{\mathcal{N}}_1(X) \cup \underline{\mathcal{N}}_2(X), \bar{\mathcal{N}}_1(X) \cap \bar{\mathcal{N}}_2(X)) \ (\forall X \in 2^U)$.

Theorem 3.14. $(\underline{\mathcal{C}}_5(X), \bar{\mathcal{C}}_5(X)) = (\underline{\mathcal{C}}_1(X) \cap \underline{\mathcal{C}}_2(X), \bar{\mathcal{C}}_1(X) \cup \bar{\mathcal{C}}_2(X)) = (\underline{\mathcal{N}}_1(X) \cap \underline{\mathcal{N}}_2(X), \bar{\mathcal{N}}_1(X) \cup \bar{\mathcal{N}}_2(X)) \ (\forall X \in 2^U)$.

Corollary 3.15. Each kind of rough sets (apart from Type-4 and Type-5 covering rough sets) defined as in Definition 2.10 and Definition 2.11, respectively, are neighborhood rough sets.

Definition 3.16. A graded rough set with respect to the grade n (where n is a nonnegative integer) based on two universes V and W ⁶ (see [45] for nonempty finite set case) is a pair $(\underline{\mathcal{R}}_n(Y), \bar{\mathcal{R}}_n(Y))$, where $Y \in 2^W$, $\underline{\mathcal{R}}_n(Y) = \{v \in V \mid |\mathcal{R}(v) - Y| \leq n\}$, $\bar{\mathcal{R}}_n(Y) = \{v \in V \mid |\mathcal{R}(v) \cap Y| > n\}$, $\mathcal{R} \subseteq V \times W$, and $\mathcal{R}(v) = \{w \in W \mid (v, w) \in \mathcal{R}\} \ (\forall v \in V)$.

Theorem 3.17. Graded rough sets based on two universes are neighborhood rough sets.

Proof. For each $\mathcal{R} \subseteq V \times W$ and each nonnegative integer n , take a point $\infty \notin W$, let $U = V \times (\{\infty\} \cup W)$, and $U_0 = V \times \{\infty\}$. Define $\mathcal{N} : U_0 \rightarrow 2^U$ by $\mathcal{N}(v, \infty) = (\{v\} \times \{\infty\}) \cup (\{v\} \times \mathcal{R}(v)) \ (\forall v \in V)$. Then $(\underline{\mathcal{R}}_n(Y), \bar{\mathcal{R}}_n(Y)) = (\underline{\mathcal{N}}_n(V \times (\{\infty\} \cup Y)), \bar{\mathcal{N}}_n(V \times (\{\infty\} \cup Y)))$ for each $Y \in 2^W$.

By the way, we show that the rough sets defined in Definitions 2.5 and 2.8-2.11 can be constructed in topology-like spaces (called co-knowledge spaces).

Definition 3.18. Let $(U, \mathcal{K})$ be a co-knowledge space (i.e. U is a set and $\mathcal{K} \subseteq 2^U$ is closed under the operation intersection), and $X \in 2^U$. $a \in U$ is called a interior point of X if there exists a $K \in \mathcal{K}$ such that $a \in K \subseteq X$. $a \in U$ is called an adherent point of X if $K \cap X \neq \emptyset$ for each $K \in \mathcal{K}$ satisfying $a \in K$. The set of all interior points (resp., all adherent points) of X is denoted by $X_{\mathcal{K}}^o$ (resp., $X_{\mathcal{K}}^-$).

Theorem 3.19. Let $(U, \mathcal{C})$ be a covering rough space and $X \in 2^U$. Then

(1) $(U, \mathcal{K}(\mathcal{C}))$ is a co-knowledge space, where

$$\mathcal{K}(\mathcal{C}) = \left\{ \bigcap_{H \in \mathcal{H}} H \mid \mathcal{H} \subseteq \mathcal{C} \right\}.$$

(2) $\mathcal{N}(\mathcal{C}) = \{\mathcal{N}_{\mathcal{C}}(u) \mid u \in U\}$, $\mathcal{M}(\mathcal{C}) = \{\mathcal{M}_{\mathcal{C}}(u) \mid u \in U\}$, and $\mathcal{P}(\mathcal{C}) = \{\mathcal{P}_{\mathcal{C}}(u) \mid u \in U\}$ are covers of U , $\mathcal{K}[\mathcal{N}(\mathcal{C})] \subseteq \mathcal{K}(\mathcal{C})$, and $\mathcal{K}(\mathcal{C}) = \mathcal{K}[\mathcal{N}(\mathcal{C})]$ does not hold in general.

$$(3) (\underline{\mathcal{C}}_1(X), \bar{\mathcal{C}}_1(X)) = (X_{\mathcal{K}(\mathcal{C})}^o, X_{\mathcal{K}(\mathcal{C})}^-).$$

$$(4) (\underline{\mathcal{C}}_2(X), \bar{\mathcal{C}}_2(X)) = (X_{\mathcal{K}[\mathcal{N}(\mathcal{C})]}^o, X_{\mathcal{K}[\mathcal{N}(\mathcal{C})]}^-).$$

$$(5) (\underline{\mathcal{C}}_3(X), \bar{\mathcal{C}}_3(X)) = (X_{\mathcal{K}[\mathcal{P}(\mathcal{C})]}^o, X_{\mathcal{K}[\mathcal{P}(\mathcal{C})]}^-).$$

$$(6) (\underline{\mathcal{C}}_4(X), \bar{\mathcal{C}}_4(X)) = (X_{\mathcal{K}(\mathcal{C})}^o \cup X_{\mathcal{K}[\mathcal{N}(\mathcal{C})]}^o, X_{\mathcal{K}(\mathcal{C})}^- \cap X_{\mathcal{K}[\mathcal{N}(\mathcal{C})]}^-).$$

$$(7) (\underline{\mathcal{C}}_5(X), \bar{\mathcal{C}}_5(X)) = (X_{\mathcal{K}(\mathcal{C})}^o \cap X_{\mathcal{K}[\mathcal{N}(\mathcal{C})]}^o, X_{\mathcal{K}(\mathcal{C})}^- \cup X_{\mathcal{K}[\mathcal{N}(\mathcal{C})]}^-).$$

⁶ We do not discuss graded rough sets based on a family $\{V\} \cup \{W_k \mid k \in K\}$ ($|K| > 0$) of universes (see [45] for a special case) because each of them is actually a graded rough set based on two universes.

⁴ Notice that U and V are arbitrary sets.

⁵ We will write $(\underline{\mathcal{N}}_0(X), \bar{\mathcal{N}}_0(X)) = (\underline{\mathcal{N}}(X), \bar{\mathcal{N}}(X)) \ (\forall X \in 2^U)$.

Proof. (1) For each family $\left\{ \bigcap_{H \in \mathcal{H}_i} H \mid \mathcal{H}_i \subseteq \mathcal{C}, i \in I \right\} \subseteq \mathcal{K}(\mathcal{C})$, let $\mathcal{H} = \bigcup_{i \in I} \mathcal{H}_i$. Then $\mathcal{H} \subseteq \mathcal{C}$, and $\bigcap_{i \in I} \bigcap_{H \in \mathcal{H}_i} H = \bigcap_{H \in \mathcal{H}} H \in \mathcal{K}(\mathcal{C})$, which means that $(U, \mathcal{K}(\mathcal{C}))$ is a co-knowledge space.

(2) $\mathcal{N}(\mathcal{C}) = \{\mathcal{N}_{\mathcal{C}}(u) \mid u \in U\}$ and $\mathcal{M}(\mathcal{C}) = \{\mathcal{M}_{\mathcal{C}}(u) \mid u \in U\}$ are covers of U because $u \in \mathcal{N}_{\mathcal{C}}(u) \cap \mathcal{M}_{\mathcal{C}}(u) (\forall u \in U)$. Clearly, $\mathcal{K}[\mathcal{N}(\mathcal{C})] \subseteq \mathcal{K}(\mathcal{C})$. Take $U = R$ (the set of all real numbers), $\mathcal{C} = \{(a, b) \mid a, b \in R, a < b\}$. Then $\mathcal{C}$ is a cover of R , $(0, 1) \in \mathcal{C} \subseteq \mathcal{K}(\mathcal{C})$, but $(0, 1) \notin \mathcal{N}(\mathcal{C}) = \{\{a\} \mid a \in R\} \subseteq \{\{a\} \mid a \in R\} \cup \{\emptyset\} = \mathcal{K}[\mathcal{N}(\mathcal{C})]$.

(3) Firstly, we show $\underline{\mathcal{C}}_1(X) = X_{\mathcal{K}(\mathcal{C})}^o$. For each $u \in \underline{\mathcal{C}}_1(X)$ (i.e. $\mathcal{N}_{\mathcal{C}}(u) \subseteq X$), as $u \in \mathcal{N}_{\mathcal{C}}(u) \in \mathcal{K}(\mathcal{C})$, $u \in X_{\mathcal{K}(\mathcal{C})}^o$, which means $\underline{\mathcal{C}}_1(X) \subseteq X_{\mathcal{K}(\mathcal{C})}^o$. Conversely, for each $u \in X_{\mathcal{K}(\mathcal{C})}^o$ (i.e. $u \in K \subseteq X$ for some $K \in \mathcal{K}(\mathcal{C})$), as $u \in \mathcal{N}_{\mathcal{C}}(u) \subseteq K$ by definition of $\mathcal{K}(\mathcal{C})$, $u \in \underline{\mathcal{C}}_1(X)$, which means $X_{\mathcal{K}(\mathcal{C})}^o \subseteq \underline{\mathcal{C}}_1(X)$.

Secondly, we show $\overline{\mathcal{C}}_1(X) = X_{\mathcal{K}(\mathcal{C})}^-$. For each $u \in \overline{\mathcal{C}}_1(X)$ (i.e. $\mathcal{N}_{\mathcal{C}}(u) \cap X \neq \emptyset$) and each $K \in \mathcal{K}(\mathcal{C})$ satisfying $u \in K$, as $\mathcal{N}_{\mathcal{C}}(u) \subseteq K$, $K \cap X \neq \emptyset$, which means $\overline{\mathcal{C}}_1(X) \subseteq X_{\mathcal{K}(\mathcal{C})}^-$. Conversely, for each $u \in X_{\mathcal{K}(\mathcal{C})}^-$ (i.e. $K \cap X \neq \emptyset$ for each $K \in \mathcal{K}(\mathcal{C})$ satisfying $u \in K$), as $u \in \mathcal{N}_{\mathcal{C}}(u) \in \mathcal{K}(\mathcal{C})$, $\mathcal{N}_{\mathcal{C}}(u) \cap X \neq \emptyset$, which means $X_{\mathcal{K}(\mathcal{C})}^- \subseteq \overline{\mathcal{C}}_1(X)$.

(4) $(\underline{\mathcal{C}}_2(X), \overline{\mathcal{C}}_2(X)) = (X_{\mathcal{K}[\mathcal{M}(\mathcal{C})]}^o, X_{\mathcal{K}[\mathcal{M}(\mathcal{C})]}^-)$ ($\forall X \in 2^U$). Firstly, we show $\underline{\mathcal{C}}_2(X) = X_{\mathcal{K}[\mathcal{M}(\mathcal{C})]}^o$. For each $u \in \underline{\mathcal{C}}_2(X)$ (i.e. $\mathcal{M}_{\mathcal{C}}(u) \subseteq X$), as $u \in \mathcal{M}_{\mathcal{C}}(u) \in \mathcal{K}[\mathcal{M}(\mathcal{C})]$, $u \in X_{\mathcal{K}[\mathcal{M}(\mathcal{C})]}^o$, which means $\underline{\mathcal{C}}_2(X) \subseteq X_{\mathcal{K}[\mathcal{M}(\mathcal{C})]}^o$. Conversely, for each $u \in X_{\mathcal{K}[\mathcal{M}(\mathcal{C})]}^o$ (i.e. $u \in K \subseteq X$ for some $K \in \mathcal{K}[\mathcal{M}(\mathcal{C})]$), as $u \in \mathcal{M}_{\mathcal{C}}(u) \subseteq K$ by definition of $\mathcal{M}_{\mathcal{C}}(u)$, $u \in \underline{\mathcal{C}}_2(X)$, which means $X_{\mathcal{K}[\mathcal{M}(\mathcal{C})]}^o \subseteq \underline{\mathcal{C}}_2(X)$.

Secondly, we show $\overline{\mathcal{C}}_2(X) = X_{\mathcal{K}[\mathcal{M}(\mathcal{C})]}^-$. For each $u \in \overline{\mathcal{C}}_2(X)$ (i.e. $\mathcal{M}_{\mathcal{C}}(u) \cap X \neq \emptyset$) and each $K \in \mathcal{K}[\mathcal{M}(\mathcal{C})]$ satisfying $u \in K$, as $\mathcal{M}_{\mathcal{C}}(u) \subseteq K$, $K \cap X \neq \emptyset$, which means $\overline{\mathcal{C}}_2(X) \subseteq X_{\mathcal{K}[\mathcal{M}(\mathcal{C})]}^-$. Conversely, for each $u \in X_{\mathcal{K}[\mathcal{M}(\mathcal{C})]}^-$ (i.e. $K \cap X \neq \emptyset$ for each $K \in \mathcal{K}[\mathcal{M}(\mathcal{C})]$ satisfying $u \in K$), as $u \in \mathcal{M}_{\mathcal{C}}(u) \in \mathcal{K}[\mathcal{M}(\mathcal{C})]$, $\mathcal{M}_{\mathcal{C}}(u) \cap X \neq \emptyset$, which means $X_{\mathcal{K}[\mathcal{M}(\mathcal{C})]}^- \subseteq \overline{\mathcal{C}}_2(X)$.

(5)–(7) Analogously.

2.2. Probabilistic rough sets based on neighborhood operator

In this subsection, we will extend the classical probabilistic rough sets to probabilistic neighborhood rough sets. We present two types of probabilistic neighborhood rough sets and then the basic properties.

We begin by proposing the definition of a probabilistic neighborhood rough sets followed by remark.

Definition 3.20. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $0 \leq \beta < \alpha \leq 1$, and $X \in 2^U$.

$$\underline{\mathcal{N}}_p^\alpha(X) = \left\{ u \in U_0 \mid P(X|\mathcal{N}(u)) \geq \alpha \right\}$$

and

$$\overline{\mathcal{N}}_p^\beta(X) = \left\{ u \in U_0 \mid P(X|\mathcal{N}(u)) > \beta \right\}$$

are called the lower $\mathcal{N}$ -approximation and the upper $\mathcal{N}$ -approximation of X with error α and β respectively, and $(\underline{\mathcal{N}}_p^\alpha(X), \overline{\mathcal{N}}_p^\beta(X))$

is called the probabilistic neighborhood rough approximation of X (or the $(\alpha\beta)_p$ -rough approximation of X , briefly, a probabilistic neighborhood rough set). The subset $\text{bn}_p^{\alpha\beta}(X) = \{u \in U_0 \mid \beta < P(X|\mathcal{N}(u)) < \alpha\}$ is called the boundary region of X with error α and β .

Remark 3.21. If $\alpha = 1$ and $\beta = 0$, then

- (1) $\underline{\mathcal{N}}_p^\alpha(X) = \{u \in U_0 \mid P(X|\mathcal{N}(u)) \geq 1\}$
 $= \{u \in U_0 \mid \mathcal{N}(u) \subseteq X\}$
 $= \underline{\mathcal{N}}(X).$
- (2) $\overline{\mathcal{N}}_p^\beta(X) = \{u \in U_0 \mid P(X|\mathcal{N}(u)) > 0\}$
 $= \{u \in U_0 \mid \mathcal{N}(u) \cap X \neq \emptyset\}$
 $= \overline{\mathcal{N}}(X).$

The main results are as follows:

Theorem 3.22. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $0 \leq \beta < \alpha \leq 1$, and $X, X_1 \in 2^U$. Then the followings hold:

- (1) $\underline{\mathcal{N}}_p^\alpha(\emptyset) = \overline{\mathcal{N}}_p^\beta(\emptyset) = \emptyset, \underline{\mathcal{N}}_p^\alpha(U) = \overline{\mathcal{N}}_p^\beta(U) = U_0.$
- (2) $\underline{\mathcal{N}}_p^\alpha(X) \subseteq \overline{\mathcal{N}}_p^\beta(X).$
- (3) $\underline{\mathcal{N}}_p^\alpha(X) = \sim \overline{\mathcal{N}}_p^{(1-\alpha)}(\sim X)$ and $\overline{\mathcal{N}}_p^\beta(X) = \sim \underline{\mathcal{N}}_p^{(1-\beta)}(\sim X)$, where $\sim X = U - X.$
- (4) If $X \subseteq X_1$, then $\underline{\mathcal{N}}_p^\alpha(X) \subseteq \underline{\mathcal{N}}_p^\alpha(X_1)$ and $\overline{\mathcal{N}}_p^\beta(X) \subseteq \overline{\mathcal{N}}_p^\beta(X_1).$
- (5) $\underline{\mathcal{N}}_p^\alpha\left(\bigcap_{k \in K} X_k\right) \subseteq \bigcap_{k \in K} \underline{\mathcal{N}}_p^\alpha(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$) and $\overline{\mathcal{N}}_p^\beta\left(\bigcap_{k \in K} X_k\right) \subseteq \bigcap_{k \in K} \overline{\mathcal{N}}_p^\beta(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$).
- (6) $\underline{\mathcal{N}}_p^\alpha\left(\bigcup_{k \in K} X_k\right) \supseteq \bigcup_{k \in K} \underline{\mathcal{N}}_p^\alpha(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$) and $\overline{\mathcal{N}}_p^\beta\left(\bigcup_{k \in K} X_k\right) \supseteq \bigcup_{k \in K} \overline{\mathcal{N}}_p^\beta(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$).
- (7) $\underline{\mathcal{N}}_p^{\alpha_1}(X) \subseteq \underline{\mathcal{N}}_p^\alpha(X)$ (if $0.5 < \alpha < \alpha_1 \leq 1$) and $\overline{\mathcal{N}}_p^{\beta_1}(X) \subseteq \overline{\mathcal{N}}_p^\beta(X)$ (if $0 \leq \beta < \beta_1 < 0.5$).

Proof. (1) For $\forall u \in U_0, 0 \leq \beta < \alpha \leq 1$, we have $P(\emptyset|\mathcal{N}(u)) = \frac{|\emptyset \cap \mathcal{N}(u)|}{|\mathcal{N}(u)|} = 0 \leq \beta < \alpha$, $u \notin \underline{\mathcal{N}}_p^\alpha(\emptyset)$, and $u \notin \overline{\mathcal{N}}_p^\beta(\emptyset)$. Therefore, $\underline{\mathcal{N}}_p^\alpha(\emptyset) = \overline{\mathcal{N}}_p^\beta(\emptyset) = \emptyset$. Also, $P(U|\mathcal{N}(u)) = \frac{|U \cap \mathcal{N}(u)|}{|\mathcal{N}(u)|} = 1 \geq \beta > \alpha$, $u \in \underline{\mathcal{N}}_p^\alpha(U)$, and $u \in \overline{\mathcal{N}}_p^\beta(U)$. Thus, $\underline{\mathcal{N}}_p^\alpha(U) = \overline{\mathcal{N}}_p^\beta(U) = U_0.$

(2) and (3) follows from Definition 3.32.

(4) For $\forall u \in U_0, X \subseteq X_1$, we can obtain easily $P(X|\mathcal{N}(u)) < P(X_1|\mathcal{N}(u))$. Let $u \in \underline{\mathcal{N}}_p^\alpha(X)$, when $X \subseteq X_1$, we have $P(X_1|\mathcal{N}(u)) \geq P(X|\mathcal{N}(u)) \geq \alpha$. Therefore, $u \in \underline{\mathcal{N}}_p^\alpha(X_1)$, and thus $\underline{\mathcal{N}}_p^\alpha(X) \subseteq \underline{\mathcal{N}}_p^\alpha(X_1)$ ($(X, X_1) \in 2^U \times 2^U, X \subseteq X_1$). Similarly, $\overline{\mathcal{N}}_p^\beta(X) \subseteq \overline{\mathcal{N}}_p^\beta(X_1)$ ($(X, X_1) \in 2^U \times 2^U, X \subseteq X_1$) hold.

(5) It is sufficient to show $\underline{\mathcal{N}}_p^\alpha(X \cap X_1) \subseteq \underline{\mathcal{N}}_p^\alpha(X) \cap \underline{\mathcal{N}}_p^\alpha(X_1)$. Let $u \in \underline{\mathcal{N}}_p^\alpha(X \cap X_1) (\forall u \in U_0)$. Then

$$\begin{aligned} P(X \cap X_1|\mathcal{N}(u)) &\geq \alpha \\ \implies P(X|\mathcal{N}(u)) &\geq \alpha \text{ and } P(X_1|\mathcal{N}(u)) \geq \alpha \\ \implies u &\in \underline{\mathcal{N}}_p^\alpha(X) \text{ and } u \in \underline{\mathcal{N}}_p^\alpha(X_1) \\ \implies u &\in \underline{\mathcal{N}}_p^\alpha(X) \cap \underline{\mathcal{N}}_p^\alpha(X_1) (\forall u \in U_0). \end{aligned}$$

Hence, $\underline{N}_p^\alpha(X \cap X_1) \subseteq \underline{N}_p^\alpha(X) \cap \underline{N}_p^\alpha(X_1)$ and $(\forall \{X_k\}_{k \in K} \subseteq 2^U)$, then $\underline{N}_p^\alpha\left(\bigcap_{k \in K} X_k\right) \subseteq \bigcap_{k \in K} \underline{N}_p^\alpha(X_k)$. Similarity, we can obtain $\overline{N}_p^\beta\left(\bigcap_{k \in K} X_k\right) \subseteq \bigcap_{k \in K} \overline{N}_p^\beta(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$).

(6) Analogously to (5).

(7) For $0.5 < \alpha < \alpha_1 \leq 1$, let $u \in \underline{N}_p^{\alpha_1}(X)$ ($\forall u \in U_0$). Then $P(X|\mathcal{N}(u)) \geq \alpha_1$. Since $\alpha < \alpha_1$, then $P(X|\mathcal{N}(u)) \geq \alpha_1 > \alpha$ implies that $P(X|\mathcal{N}(u)) \geq \alpha$. Consequently, $u \in \underline{N}_p^\alpha(X)$. Hence $\underline{N}_p^{\alpha_1}(X) \subseteq \underline{N}_p^\alpha(X)$ ($\forall X \in 2^U, 0.5 < \alpha < \alpha_1 \leq 1$). Similarity, $\overline{N}_p^{\beta_1}(X) \subseteq \overline{N}_p^\beta(X)$ ($\forall X \in 2^U, 0 \leq \beta < \beta_1 < 0.5$).

The equality (5) and (6) of Theorem 3.22 does not hold as the following example.

Example 3.23 (Continued from Example 3.3). (1) Let $X = \{x_2, x_3\}$, $X_1 = \{x_1, x_3, x_4, x_5\}$, $X \cap X_1 = \{x_3\}$, and $0 \leq \beta = 0.2 < \alpha = 0.4 \leq 1$. Then we have

$$\begin{aligned} P(X|\mathcal{N}(x_2)) &= \frac{2}{3}, P(X_1|\mathcal{N}(x_2)) = \frac{2}{3}, P(X \cap X_1|\mathcal{N}(x_2)) = \frac{1}{3}, \\ P(X|\mathcal{N}(x_4)) &= \frac{1}{3}, P(X_1|\mathcal{N}(x_4)) = \frac{2}{3}, P(X \cap X_1|\mathcal{N}(x_4)) = 0, \\ P(X|\mathcal{N}(x_5)) &= \frac{1}{2}, P(X_1|\mathcal{N}(x_5)) = 1, P(X \cap X_1|\mathcal{N}(x_5)) = \frac{1}{2}. \end{aligned}$$

Thus, $\underline{N}_p^\alpha(X) = \{x_2, x_5\}$, $\underline{N}_p^\alpha(X_1) = \overline{N}_p^\beta(X) = \overline{N}_p^\beta(X_1) = U_0$, and hence $\underline{N}_p^\alpha(X \cap X_1) = \{x_5\} \neq \{x_2, x_5\} = \underline{N}_p^\alpha(X_1) \cap \underline{N}_p^\alpha(X_2)$ and $\overline{N}_p^\beta(X \cap X_1) = \{x_2, x_5\} \neq U_0 = \overline{N}_p^\beta(X) \cap \overline{N}_p^\beta(X_1)$.

(2) Let $X = \{x_2\}$, $X_1 = \{x_1, x_4\}$, $X \cup X_1 = \{x_1, x_2, x_4\}$, and $0 \leq \beta = 0.35 < \alpha = 0.5 \leq 1$. Then we have

$$\begin{aligned} P(X|\mathcal{N}(x_2)) &= \frac{1}{3}, P(X_1|\mathcal{N}(x_2)) = \frac{1}{3}, P(X \cup X_1|\mathcal{N}(x_2)) = \frac{2}{3}, \\ P(X|\mathcal{N}(x_4)) &= \frac{1}{3}, P(X_1|\mathcal{N}(x_4)) = \frac{1}{3}, P(X \cup X_1|\mathcal{N}(x_4)) = \frac{2}{3}, \\ P(X|\mathcal{N}(x_5)) &= 0, P(X_1|\mathcal{N}(x_5)) = 0, P(X \cup X_1|\mathcal{N}(x_5)) = 0. \end{aligned}$$

Thus, $\underline{N}_p^\alpha(X) = \underline{N}_p^\alpha(X_1) = \overline{N}_p^\beta(X) = \overline{N}_p^\beta(X_1) = \emptyset$, and hence $\underline{N}_p^\alpha(X \cup X_1) = \{x_2, x_4\} \neq \emptyset = \underline{N}_p^\alpha(X_1) \cup \underline{N}_p^\alpha(X_2)$ and $\overline{N}_p^\beta(X \cup X_1) = \{x_2, x_4\} \neq \emptyset = \overline{N}_p^\beta(X) \cup \overline{N}_p^\beta(X_1)$.

The relationship among the regulation for lower $\mathcal{N}$ -approximation, upper $\mathcal{N}$ -approximation, and $\mathcal{N}$ -boundary region with error α and β are discussed as the following theorems:

Theorem 3.24. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $0 < r < 1$, and $X \in 2^U$. Then the followings hold:

- (1) $\lim_{\alpha \rightarrow r^+} \underline{N}_p^\alpha(X) = \bigcup_{\alpha > r} \underline{N}_p^\alpha(X) = \overline{N}_p^r(X)$,
- (2) $\lim_{\beta \rightarrow r^-} \overline{N}_p^\beta(X) = \bigcap_{\beta < r} \overline{N}_p^\beta(X) = \underline{N}_p^r(X)$,
- (3) $\lim_{\beta \rightarrow r^-; \alpha \rightarrow r^+} \text{bn}_p^{\alpha\beta}(X) = \bigcap_{\beta < r < \alpha} (\overline{N}_p^\beta(X) \setminus \underline{N}_p^\alpha(X)) = \overline{N}_p^r(X) \setminus \underline{N}_p^r(X) = \{u \in U_0 \mid P(Y|\mathcal{N}(u)) = r\}$.

Proof. (1) From Definition 3.32, when $\alpha > r$ we have $\underline{N}_p^\alpha(X) = \{u \in U_0 \mid P(X|\mathcal{N}(u)) \geq \alpha\} \subseteq \{u \in U_0 \mid P(X|\mathcal{N}(u)) > r\} = \overline{N}_p^r(X)$ is hold. In addition, by Theorem 3.22 (7) we have $\underline{N}_p^\alpha(X)$ increase with error α decrease. Thus, $\lim_{\beta \rightarrow r^-} \overline{N}_p^\beta(X) = \bigcap_{\beta < r} \overline{N}_p^\beta(X) \subseteq \underline{N}_p^r(X)$. Conversely, if there exists $u_0 \in \overline{N}_p^r(X)$

$\setminus \bigcup_{\alpha > r} \underline{N}_p^\alpha(X)$ ($\forall u_0 \in U_0$), for any $\alpha > r$, $P(X|\mathcal{N}(u_0)) > r$ and $P(X|\mathcal{N}(u_0)) < \alpha$. That is, $P(X|\mathcal{N}(u_0)) \leq r$. It is contradict with $P(X|\mathcal{N}(u_0)) > r$, i.e., $u_0 \in \bigcup_{\alpha > r} \underline{N}_p^\alpha(X)$. Consequently, $\bigcup_{\alpha > r} \underline{N}_p^\alpha(X) = \overline{N}_p^r(X)$. Thus, $\lim_{\alpha \rightarrow r^+} \underline{N}_p^\alpha(X) = \bigcup_{\alpha > r} \underline{N}_p^\alpha(X) = \overline{N}_p^r(X)$ hold.

(2) From Definition 3.20, when $\beta < r$ we have $\overline{N}_p^\beta(X) = \{u \in U_0 \mid P(X|\mathcal{N}(u)) > \beta\} \supseteq \{u \in U_0 \mid P(X|\mathcal{N}(u)) \geq r\} = \underline{N}_p^r(X)$ is hold. In addition, by Theorem 3.22 (7) we know $\overline{N}_p^\beta(X)$ decrease with error β . Thus, $\lim_{\beta \rightarrow r^-} \overline{N}_p^\beta(X) = \bigcap_{\beta < r} \overline{N}_p^\beta(X) \supseteq \underline{N}_p^r(X)$. Conversely, if there exists $u_0 \in \bigcap_{\beta < r} \overline{N}_p^\beta(X) \setminus \underline{N}_p^r(X)$ ($\forall u_0 \in U_0$). For any $\beta < r$, $P(X|\mathcal{N}(u_0)) > \beta$ and $P(X|\mathcal{N}(u_0)) < r$ is satisfied. So, by $P(X|\mathcal{N}(u_0)) > \beta$ and $\beta < r$, we have $P(X|\mathcal{N}(u_0)) \geq r$. It is contradict with $P(X|\mathcal{N}(u_0)) < r$, i.e., $u_0 \in \underline{N}_p^r(X)$. Consequently, $\bigcap_{\beta < r} \overline{N}_p^\beta(X) = \underline{N}_p^r(X)$. Thus, $\lim_{\beta \rightarrow r^-} \overline{N}_p^\beta(X) = \bigcap_{\beta < r} \overline{N}_p^\beta(X) = \underline{N}_p^r(X)$.

(3) We know $\overline{N}_p^r(X) \setminus \underline{N}_p^r(X) = \{u \in U_0 \mid P(X|\mathcal{N}(u)) = r\} \subseteq \text{bn}_p^{\alpha\beta}(X)$ is holden. In addition, we know that when α decreased to r and β decreased to r , the boundary region $\text{bn}_p^{\alpha\beta}(X)$ will decrease. Then, we have

$$\begin{aligned} \overline{N}_p^r(X) \setminus \underline{N}_p^r(X) &= \{u \in U_0 \mid P(X|\mathcal{N}(u)) = r\} \\ &\subseteq \lim_{\beta \rightarrow r^-; \alpha \rightarrow r^+} \text{bn}_p^{\alpha\beta}(X) \\ &= \bigcap_{\beta < r < \alpha} (\overline{N}_p^\beta(X) \setminus \underline{N}_p^\alpha(X)). \end{aligned}$$

Conversely, if there exists $u_0 \in \bigcap_{\beta < r < \alpha} (\overline{N}_p^\beta(X) \setminus \underline{N}_p^\alpha(X)) \setminus \{u \in U_0 \mid P(X|\mathcal{N}(u)) = r\}$, then for any $\beta < r < \alpha$, we have $P(X|\mathcal{N}(u_0)) > \beta$, $P(X|\mathcal{N}(u_0)) < \alpha$, and $P(X|\mathcal{N}(u_0)) \neq r$. Furthermore, there is $P(X|\mathcal{N}(u_0)) \geq r$ by $P(X|\mathcal{N}(u_0)) > \beta$ and $\beta < r$. Similarity, there is $P(X|\mathcal{N}(u_0)) \leq r$ by $P(X|\mathcal{N}(u_0)) < \alpha$ and $\alpha > r$. This is a contradiction. Consequently, there is $P(X|\mathcal{N}(u_0)) = r$. That is, $\{u \in U_0 \mid P(X|\mathcal{N}(u)) = r\} = \bigcap_{\beta < r < \alpha} (\overline{N}_p^\beta(X) \setminus \underline{N}_p^\alpha(X))$.

Theorem 3.25. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $0 < r < 1$, and $X \in 2^U$. Then the followings hold:

- (1) $\lim_{\alpha \rightarrow r^-} \underline{N}_p^\alpha(X) = \bigcap_{\alpha < r} \underline{N}_p^\alpha(X) = \underline{N}_p^r(X)$,
- (2) $\lim_{\beta \rightarrow r^+} \overline{N}_p^\beta(X) = \bigcup_{\beta > r} \overline{N}_p^\beta(X) = \overline{N}_p^r(X)$.

Proof. (1) and (2) are similar to the proof of Theorem 3.24 (1) and (2) respectively.

Remark 3.26. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $0 < r < 1$, and $X \in 2^U$. Then the followings does not hold.

- (1) $\lim_{\alpha \rightarrow r^+} \underline{N}_p^\alpha(X) = \underline{N}_p^r(X)$
(i.e., the lower $\mathcal{N}$ -approximation $\underline{N}_p^\alpha(X)$ is not right continuous with error α).
- (2) $\lim_{\beta \rightarrow r^-} \overline{N}_p^\beta(X) = \overline{N}_p^r(X)$
(i.e., the upper $\mathcal{N}$ -approximation $\overline{N}_p^\beta(X)$ is not left continuous with error β).

Example 3.27. Assume that $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, and $U_0 = \{x_1, x_2, x_3, x_4, x_5\} \subseteq U$. We define a neighborhood operator $\mathcal{N}$ as follows: $\mathcal{N}(x_1) = \{x_1, x_2, x_4\}$, $\mathcal{N}(x_2) = \{x_2, x_3, x_5\}$, $\mathcal{N}(x_3) = \{x_3, x_4\}$, $\mathcal{N}(x_4) = \{x_1, x_4, x_5\}$ and $\mathcal{N}(x_5) = \{x_2, x_4, x_5\}$. Let $X = \{x_2, x_4\}$. Then

$$P(X|\mathcal{N}(x_1)) = \frac{2}{3}, P(X|\mathcal{N}(x_2)) = \frac{1}{3}, P(X|\mathcal{N}(x_3)) = \frac{1}{2}, \\ P(X|\mathcal{N}(x_4)) = \frac{1}{3}, P(X|\mathcal{N}(x_5)) = \frac{2}{3}.$$

Thus, we can obtain the following relations:

$$(1) \lim_{\alpha \rightarrow 0.5+} \underline{\mathcal{N}}_p^\alpha(X) = \{x_1, x_5\}, \underline{\mathcal{N}}_p^{0.5}(X) = \{x_1, x_3, x_5\}. \\ \text{Consequently, } \lim_{\alpha \rightarrow 0.5-} \underline{\mathcal{N}}_p^\alpha(X) \neq \underline{\mathcal{N}}_p^{0.5}(X), \\ (2) \lim_{\beta \rightarrow 0.5-} \overline{\mathcal{N}}_p^\beta(X) = \{x_1, x_3, x_5\}, \overline{\mathcal{N}}_p^{0.5}(X) = \{x_1, x_5\}. \\ \text{Consequently, } \lim_{\beta \rightarrow 0.5-} \overline{\mathcal{N}}_p^\beta(X) \neq \overline{\mathcal{N}}_p^{0.5}(X).$$

Definition 3.28. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $0 \leq \beta < \alpha \leq 1$, and $X \in 2^U$. Then the precision $(\rho_p)_\beta^\alpha(X)$ and rough degree $(\mu_p)_\beta^\alpha(X)$ of X with error α and β is defined by

$$(\rho_p)_\beta^\alpha(X) = 1 - \frac{|\underline{\mathcal{N}}_p^\alpha(X)|}{|\overline{\mathcal{N}}_p^\beta(X)|} \quad \text{and} \quad (\mu_p)_\beta^\alpha(X) = 1 - (\rho_p)_\beta^\alpha(X)$$

Obviously, $0 \leq (\rho_p)_\beta^\alpha(X) \leq 1$ and $0 \leq (\mu_p)_\beta^\alpha(X) \leq 1$.

Lemma 3.29. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $0 \leq \beta < \alpha \leq 1$, and $X \in 2^U$. Then the followings hold:

- (1) $(\rho_p)_\beta^\alpha(X)$ does not decrease with error α , and does not increase with error β .
- (2) $(\mu_p)_\beta^\alpha(X)$ does not increase with error α , and does not decrease with error β .

Proof. We only prove (1) and then the proof of (2) can be obtained using similar techniques.

(1) From Theorem 3.22 (7), we have $\underline{\mathcal{N}}_p^{\alpha_1}(X) \subseteq \underline{\mathcal{N}}_p^\alpha(X)$ ($0.5 < \alpha < \alpha_1 \leq 1$). Then there is $\frac{|\underline{\mathcal{N}}_p^{\alpha_1}(X)|}{|\overline{\mathcal{N}}_p^\beta(X)|} \leq \frac{|\underline{\mathcal{N}}_p^\alpha(X)|}{|\overline{\mathcal{N}}_p^\beta(X)|}$ implies $1 - \frac{|\underline{\mathcal{N}}_p^{\alpha_1}(X)|}{|\overline{\mathcal{N}}_p^\beta(X)|} \geq 1 - \frac{|\underline{\mathcal{N}}_p^\alpha(X)|}{|\overline{\mathcal{N}}_p^\beta(X)|}$. Therefore, $(\rho_p)_\beta^\alpha(X) \leq (\rho_p)_\beta^{\alpha_1}(X)$. Similarly, $(\rho_p)_\beta^{\alpha_1}(X) \leq (\rho_p)_\beta^\alpha(X)$ for $\overline{\mathcal{N}}_p^{\beta_1}(X) \subseteq \overline{\mathcal{N}}_p^\beta(X)$ ($0 \leq \beta < \beta_1 < 0.5$).

Theorem 3.30. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $0 \leq \beta < \alpha \leq 1$, $X, X_1 \in 2^U$ and $X \subseteq X_1$. Then the followings hold:

- (1) If $\overline{\mathcal{N}}_p^\beta(X) = \overline{\mathcal{N}}_p^\beta(X_1)$. There is $(\rho_p)_\beta^\alpha(X_1) \leq (\rho_p)_\beta^\alpha(X)$ and $(\mu_p)_\beta^\alpha(X) \leq (\mu_p)_\beta^\alpha(X_1)$,
- (2) If $\underline{\mathcal{N}}_p^\alpha(X) = \underline{\mathcal{N}}_p^\alpha(X_1)$. There is $(\rho_p)_\beta^\alpha(X) \leq (\rho_p)_\beta^\alpha(X_1)$ and $(\mu_p)_\beta^\alpha(X_1) \leq (\mu_p)_\beta^\alpha(X)$,
- (3) If $\underline{\mathcal{N}}_p^\alpha(X) = \underline{\mathcal{N}}_p^\alpha(X_1)$ and $\overline{\mathcal{N}}_p^\beta(X) = \overline{\mathcal{N}}_p^\beta(X_1)$. There is $(\rho_p)_\beta^\alpha(X) = (\rho_p)_\beta^\alpha(X_1)$ and $(\mu_p)_\beta^\alpha(X) = (\mu_p)_\beta^\alpha(X_1)$.

Proof. Follows from Theorem 3.22 (4) and Definition 3.28.

Theorem 3.31. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $0 \leq \beta < \alpha \leq 1$, and $X, X_1 \in 2^U$. Then the followings hold:

- (1) $(\rho_p)_\beta^\alpha(X \cup X_1) |\overline{\mathcal{N}}_p^\beta(X) \cup \overline{\mathcal{N}}_p^\beta(X_1)| \leq (\rho_p)_\beta^\alpha(X) |\overline{\mathcal{N}}_p^\beta(X)| + (\rho_p)_\beta^\alpha(X_1) |\overline{\mathcal{N}}_p^\beta(X_1)| - (\rho_p)_\beta^\alpha(X \cap X_1) |\overline{\mathcal{N}}_p^\beta(X) \cap \overline{\mathcal{N}}_p^\beta(X_1)|.$
- (2) $(\mu_p)_\beta^\alpha(X \cup X_1) |\overline{\mathcal{N}}_p^\beta(X) \cup \overline{\mathcal{N}}_p^\beta(X_1)| \geq (\mu_p)_\beta^\alpha(X) |\overline{\mathcal{N}}_p^\beta(X)| + (\mu_p)_\beta^\alpha(X_1) |\overline{\mathcal{N}}_p^\beta(X_1)| - (\mu_p)_\beta^\alpha(X \cap X_1) |\overline{\mathcal{N}}_p^\beta(X) \cap \overline{\mathcal{N}}_p^\beta(X_1)|.$

Proof. (1) From Definition 3.28, we have $(\rho_p)_\beta^\alpha(X \cup X_1) = 1 - \frac{|\underline{\mathcal{N}}_p^\alpha(X \cup X_1)|}{|\overline{\mathcal{N}}_p^\beta(X \cup X_1)|} = 1 - \frac{|\underline{\mathcal{N}}_p^\alpha(X \cup X_1)|}{|\overline{\mathcal{N}}_p^\beta(X) \cup \overline{\mathcal{N}}_p^\beta(X_1)|} \leq 1 - \frac{|\underline{\mathcal{N}}_p^\alpha(X) \cup \underline{\mathcal{N}}_p^\alpha(X_1)|}{|\overline{\mathcal{N}}_p^\beta(X) \cup \overline{\mathcal{N}}_p^\beta(X_1)|}.$

So, $(\rho_p)_\beta^\alpha(X \cup X_1) |\overline{\mathcal{N}}_p^\beta(X) \cup \overline{\mathcal{N}}_p^\beta(X_1)| \leq |\overline{\mathcal{N}}_p^\beta(X) \cup \overline{\mathcal{N}}_p^\beta(X_1)| - |\underline{\mathcal{N}}_p^\alpha(X) \cup \underline{\mathcal{N}}_p^\alpha(X_1)|$. Similarly, $(\rho_p)_\beta^\alpha(X \cap X_1) |\overline{\mathcal{N}}_p^\beta(X) \cap \overline{\mathcal{N}}_p^\beta(X_1)| \leq |\overline{\mathcal{N}}_p^\beta(X) \cap \overline{\mathcal{N}}_p^\beta(X_1)| - |\underline{\mathcal{N}}_p^\alpha(X) \cap \underline{\mathcal{N}}_p^\alpha(X_1)|.$

We know $|A \cup B| = |A| + |B| - |A \cap B|$. Then,

$$(\rho_p)_\beta^\alpha(X \cup X_1) |\overline{\mathcal{N}}_p^\beta(X) \cup \overline{\mathcal{N}}_p^\beta(X_1)| \\ \leq |\overline{\mathcal{N}}_p^\beta(X) \cup \overline{\mathcal{N}}_p^\beta(X_1)| - |\underline{\mathcal{N}}_p^\alpha(X) \cup \underline{\mathcal{N}}_p^\alpha(X_1)| \\ = |\overline{\mathcal{N}}_p^\beta(X)| + |\overline{\mathcal{N}}_p^\beta(X_1)| - |\overline{\mathcal{N}}_p^\beta(X) \cap \overline{\mathcal{N}}_p^\beta(X_1)| - |\underline{\mathcal{N}}_p^\alpha(X)| \\ + |\underline{\mathcal{N}}_p^\alpha(X_1)| - |\underline{\mathcal{N}}_p^\alpha(X) \cap \underline{\mathcal{N}}_p^\alpha(X_1)| \\ = |\overline{\mathcal{N}}_p^\beta(X)| + |\overline{\mathcal{N}}_p^\beta(X_1)| - |\underline{\mathcal{N}}_p^\alpha(X)| - |\underline{\mathcal{N}}_p^\alpha(X_1)| - (|\overline{\mathcal{N}}_p^\beta(X) \cap \overline{\mathcal{N}}_p^\beta(X_1)| - |\underline{\mathcal{N}}_p^\alpha(X) \cap \underline{\mathcal{N}}_p^\alpha(X_1)|) \\ \leq |\overline{\mathcal{N}}_p^\beta(X)| + |\overline{\mathcal{N}}_p^\beta(X_1)| - |\underline{\mathcal{N}}_p^\alpha(X)| - |\underline{\mathcal{N}}_p^\alpha(X_1)| - (\rho_p)_\beta^\alpha(X \cap X_1) |\overline{\mathcal{N}}_p^\beta(X) \cap \overline{\mathcal{N}}_p^\beta(X_1)|.$$

Also, by the relation of $(\rho_p)_\beta^\alpha(X) = 1 - \frac{|\underline{\mathcal{N}}_p^\alpha(X)|}{|\overline{\mathcal{N}}_p^\beta(X)|}$ and $(\rho_p)_\beta^\alpha(X_1) = 1 - \frac{|\underline{\mathcal{N}}_p^\alpha(X_1)|}{|\overline{\mathcal{N}}_p^\beta(X_1)|}$, then $|\overline{\mathcal{N}}_p^\beta(X)| - |\underline{\mathcal{N}}_p^\alpha(X)| = (\rho_p)_\beta^\alpha(X) |\overline{\mathcal{N}}_p^\beta(X)|$ and $|\overline{\mathcal{N}}_p^\beta(X_1)| - |\underline{\mathcal{N}}_p^\alpha(X_1)| = (\rho_p)_\beta^\alpha(X_1) |\overline{\mathcal{N}}_p^\beta(X_1)|$. Therefore, we have $(\rho_p)_\beta^\alpha(X \cup X_1) |\overline{\mathcal{N}}_p^\beta(X) \cup \overline{\mathcal{N}}_p^\beta(X_1)| \leq |\overline{\mathcal{N}}_p^\beta(X)| + |\overline{\mathcal{N}}_p^\beta(X_1)| - |\underline{\mathcal{N}}_p^\alpha(X)| - |\underline{\mathcal{N}}_p^\alpha(X_1)| - (\rho_p)_\beta^\alpha(X \cap X_1) |\overline{\mathcal{N}}_p^\beta(X) \cap \overline{\mathcal{N}}_p^\beta(X_1)| = (\rho_p)_\beta^\alpha(X) |\overline{\mathcal{N}}_p^\beta(X)| + (\rho_p)_\beta^\alpha(X_1) |\overline{\mathcal{N}}_p^\beta(X_1)| - (\rho_p)_\beta^\alpha(X \cap X_1) |\overline{\mathcal{N}}_p^\beta(X) \cap \overline{\mathcal{N}}_p^\beta(X_1)|$. Hence (1) is holding.

(2) It can be easily proved by the relationship $(\mu_p)_\beta^\alpha(X) = 1 - (\rho_p)_\beta^\alpha(X)$.

Definition 3.32. Let U_0 is subset on a set U and $\mathcal{N} = \{\hat{\mathcal{N}}_i, i \in N\}$ is a family of neighborhood operators on U , where $\hat{\mathcal{N}}_i : U_0 \rightarrow 2^U$ satisfying $u \in \hat{\mathcal{N}}_i(u) (\forall u \in U_0, i \in N)$, $0 \leq \beta < \alpha \leq 1$, and $X \in 2^U$. Then

$$\underline{\mathcal{N}}_{\hat{\mathcal{N}}_i}^{I\alpha}(X) = \left\{ u \in U_0 \mid P(X|\hat{\mathcal{N}}_1(u)) \geq \alpha \wedge P(X|\hat{\mathcal{N}}_2(u)) \geq \alpha \right. \\ \left. \wedge \dots \wedge P(X|\hat{\mathcal{N}}_i(u)) \geq \alpha \right\}$$

and

$$\overline{\mathcal{N}}_{\hat{\mathcal{N}}_i}^{I\beta}(X) = \left\{ u \in U_0 \mid P(X|\hat{\mathcal{N}}_1(u)) > \beta \wedge P(X|\hat{\mathcal{N}}_2(u)) > \beta \right. \\ \left. \wedge \dots \wedge P(X|\hat{\mathcal{N}}_i(u)) > \beta \right\}$$

are called the Type-I lower $\hat{\mathcal{N}}_i$ -approximation and the Type-I upper $\hat{\mathcal{N}}_i$ -approximation of X with error α and β respectively, $(\underline{\mathcal{N}}_{\hat{\mathcal{N}}_i}^{I\alpha}(X), \overline{\mathcal{N}}_{\hat{\mathcal{N}}_i}^{I\beta}(X))$ is called the Type-I probabilistic neighborhood rough approximation of X (or the Type-I probabilistic $\hat{\mathcal{N}}_i$ -rough approximation of X) with respect to $\mathcal{N}$.

(2)

$$\underline{\mathcal{N}}_{\hat{\mathcal{N}}_i}^{II\alpha}(X) = \left\{ u \in U_0 \mid P(X|\hat{\mathcal{N}}_1(u)) \geq \alpha \vee P(X|\hat{\mathcal{N}}_2(u)) \geq \alpha \right. \\ \left. \vee \dots \vee P(X|\hat{\mathcal{N}}_i(u)) \geq \alpha \right\}$$

and

$$\overline{\mathcal{N}}_{\hat{\mathcal{N}}_i}^{II\beta}(X) = \left\{ u \in U_0 \mid P(X|\hat{\mathcal{N}}_1(u)) > \beta \vee P(X|\hat{\mathcal{N}}_2(u)) > \beta \right.$$

$$\vee \dots \vee P(X|\hat{\mathcal{N}}_i(u)) > \beta \Big\}$$

are called the Type-II lower $\hat{\mathcal{N}}_i$ -approximation and the Type-II upper $\hat{\mathcal{N}}_i$ -approximation of X with error α and β respectively, $(\underline{\mathcal{N}}_{\bigwedge_i^n \hat{\mathcal{N}}_i}^{II\alpha}(X), \overline{\mathcal{N}}_{\bigwedge_i^n \hat{\mathcal{N}}_i}^{II\beta}(X))$ is called the Type-II probabilistic neighborhood rough approximation of X (or the Type-II probabilistic $\hat{\mathcal{N}}_i$ -rough approximation of X) with respect to $\mathcal{N}$.

To illustrate Definition 3.32, let us consider the following example.

Example 3.33. Consider U and U_0 are given in Example 3.3. Let $\mathcal{N} = \{\hat{\mathcal{N}}_1, \hat{\mathcal{N}}_2, \hat{\mathcal{N}}_3\}$ is a family of neighborhood operators on U , where $\hat{\mathcal{N}}_1, \hat{\mathcal{N}}_2, \hat{\mathcal{N}}_3 : U_0 \rightarrow 2^U$ defined as follows: $\hat{\mathcal{N}}_1(x_2) = \{x_1, x_2, x_3\}, \hat{\mathcal{N}}_1(x_4) = \{x_2, x_4\}, \hat{\mathcal{N}}_1(x_5) = \{x_1, x_5\}, \hat{\mathcal{N}}_2(x_2) = \{x_2, x_5\}, \hat{\mathcal{N}}_2(x_4) = \{x_4\}, \hat{\mathcal{N}}_2(x_5) = \{x_1, x_3, x_4, x_5\}, \hat{\mathcal{N}}_3(x_2) = \{x_1, x_2\}, \hat{\mathcal{N}}_3(x_4) = \{x_3, x_4\}, \hat{\mathcal{N}}_3(x_5) = \{x_4, x_5\}$. Assume that $X = \{x_1, x_3, x_4\} \in 2^U$. Then we have

$$P(X|\hat{\mathcal{N}}_1(x_2)) = \frac{P(X \cap \hat{\mathcal{N}}_1(x_2))}{P(\hat{\mathcal{N}}_1(x_2))} = \frac{2}{3}.$$

Similarity,

$$P(X|\hat{\mathcal{N}}_1(x_4)) = \frac{1}{2}, P(X|\hat{\mathcal{N}}_1(x_5)) = \frac{1}{2}; P(X|\hat{\mathcal{N}}_2(x_2)) = 0, P(X|\hat{\mathcal{N}}_2(x_4)) = 1, P(X|\hat{\mathcal{N}}_2(x_5)) = \frac{3}{4}; P(X|\hat{\mathcal{N}}_3(x_2)) = \frac{1}{2}, P(X|\hat{\mathcal{N}}_3(x_4)) = 1, P(X|\hat{\mathcal{N}}_3(x_5)) = \frac{1}{2}.$$

If $\beta = 0.3$ and $\alpha = 0.7$. Thus, from Definition 3.24 we obtain

$$(1) \underline{\mathcal{N}}_{\hat{\mathcal{N}}_1 \wedge \hat{\mathcal{N}}_2 \wedge \hat{\mathcal{N}}_3}^{I\alpha}(X) = \emptyset \text{ and } \overline{\mathcal{N}}_{\hat{\mathcal{N}}_1 \wedge \hat{\mathcal{N}}_2 \wedge \hat{\mathcal{N}}_3}^{I\beta}(X) = \{x_4, x_5\}.$$

$$(2) \underline{\mathcal{N}}_{\hat{\mathcal{N}}_1 \vee \hat{\mathcal{N}}_2 \vee \hat{\mathcal{N}}_3}^{II\alpha}(X) = \{x_4, x_5\} \text{ and } \overline{\mathcal{N}}_{\hat{\mathcal{N}}_1 \vee \hat{\mathcal{N}}_2 \vee \hat{\mathcal{N}}_3}^{II\beta}(X) = U_0.$$

$$\text{Clearly, } \underline{\mathcal{N}}_{\bigwedge_i^n \hat{\mathcal{N}}_i}^{I\alpha}(X) \subseteq \underline{\mathcal{N}}_{\bigvee_i^n \hat{\mathcal{N}}_i}^{II\alpha}(X) \text{ and } \overline{\mathcal{N}}_{\bigwedge_i^n \hat{\mathcal{N}}_i}^{I\beta}(X) \subseteq \overline{\mathcal{N}}_{\bigvee_i^n \hat{\mathcal{N}}_i}^{II\beta}(X).$$

Definition 3.34. Let U_0 is subset on a set U and $\mathcal{N} = \{\hat{\mathcal{N}}_i, i \in N\}$ is a family of neighborhood operators on U , where $\hat{\mathcal{N}}_i : U_0 \rightarrow 2^U$ satisfying $u \in \hat{\mathcal{N}}_i(u) (\forall u \in U_0, i \in N)$, and $X \in 2^U$. Then

(1)

$$\underline{\mathcal{N}}_{\bigwedge_i^n \hat{\mathcal{N}}_i}^I(X) = \left\{ u \in U_0 \mid \hat{\mathcal{N}}_1(u) \subseteq X \wedge \hat{\mathcal{N}}_2(u) \subseteq X \wedge \dots \wedge \hat{\mathcal{N}}_i(u) \subseteq X \right\}$$

and

$$\overline{\mathcal{N}}_{\bigwedge_i^n \hat{\mathcal{N}}_i}^I(X) = \left\{ u \in U_0 \mid \hat{\mathcal{N}}_1(u) \cap X \neq \emptyset \wedge \hat{\mathcal{N}}_2(u) \cap X \neq \emptyset \wedge \dots \wedge \hat{\mathcal{N}}_i(u) \cap X \neq \emptyset \right\}$$

are called the Type-I lower $\hat{\mathcal{N}}_i$ -approximation and the Type-I upper $\hat{\mathcal{N}}_i$ -approximation of X with respect to $\mathcal{N}$ respectively, $(\underline{\mathcal{N}}_{\bigwedge_i^n \hat{\mathcal{N}}_i}^I(X), \overline{\mathcal{N}}_{\bigwedge_i^n \hat{\mathcal{N}}_i}^I(X))$ is called the Type-I probabilistic neighborhood rough approximation of X (or the Type-I probabilistic $\hat{\mathcal{N}}_i$ -rough approximation of X) with respect to $\mathcal{N}$.

(2)

$$\underline{\mathcal{N}}_{\bigvee_i^n \hat{\mathcal{N}}_i}^{II}(X) = \left\{ u \in U_0 \mid \hat{\mathcal{N}}_1(u) \subseteq X \vee \hat{\mathcal{N}}_2(u) \subseteq X \vee \dots \vee \hat{\mathcal{N}}_i(u) \subseteq X \right\}$$

and

$$\overline{\mathcal{N}}_{\bigvee_i^n \hat{\mathcal{N}}_i}^{II}(X) = \left\{ u \in U_0 \mid \hat{\mathcal{N}}_1(u) \cap X \neq \emptyset \vee \hat{\mathcal{N}}_2(u) \cap X \neq \emptyset \vee \dots \vee \hat{\mathcal{N}}_i(u) \cap X \neq \emptyset \right\}$$

are called the Type-II lower $\hat{\mathcal{N}}_i$ -approximation and the Type-II upper $\hat{\mathcal{N}}_i$ -approximation of X with respect to $\mathcal{N}$ respectively, $(\underline{\mathcal{N}}_{\bigvee_i^n \hat{\mathcal{N}}_i}^{II}(X), \overline{\mathcal{N}}_{\bigvee_i^n \hat{\mathcal{N}}_i}^{II}(X))$ is called the Type-II probabilistic neighborhood rough approximation of X (or the Type-II probabilistic $\hat{\mathcal{N}}_i$ -rough approximation of X) with respect to $\mathcal{N}$.

Remark 3.35. If $\alpha = 1$ and $\beta = 0$, then

(1)

$$\begin{aligned} \underline{\mathcal{N}}_{\bigwedge_i^n \hat{\mathcal{N}}_i}^{I\alpha}(X) &= \left\{ u \in U_0 \mid P(X|\hat{\mathcal{N}}_1(u)) \geq 1 \wedge P(X|\hat{\mathcal{N}}_2(u)) \geq 1 \wedge \dots \wedge P(X|\hat{\mathcal{N}}_i(u)) \geq 1 \right\} \\ &= \left\{ u \in U_0 \mid \hat{\mathcal{N}}_1(u) \subseteq X \wedge \hat{\mathcal{N}}_2(u) \subseteq X \wedge \dots \wedge \hat{\mathcal{N}}_i(u) \subseteq X \right\} \\ &= \underline{\mathcal{N}}_{\bigwedge_i^n \hat{\mathcal{N}}_i}^I(X) \end{aligned}$$

and

$$\begin{aligned} \overline{\mathcal{N}}_{\bigwedge_i^n \hat{\mathcal{N}}_i}^{I\beta}(X) &= \left\{ u \in U_0 \mid P(X|\hat{\mathcal{N}}_1(u)) > 0 \wedge P(X|\hat{\mathcal{N}}_2(u)) > 0 \wedge \dots \wedge P(X|\hat{\mathcal{N}}_i(u)) > 0 \right\} \\ &= \left\{ u \in U_0 \mid \hat{\mathcal{N}}_1(u) \cap X \neq \emptyset \wedge \hat{\mathcal{N}}_2(u) \cap X \neq \emptyset \wedge \dots \wedge \hat{\mathcal{N}}_i(u) \cap X \neq \emptyset \right\} \\ &= \overline{\mathcal{N}}_{\bigwedge_i^n \hat{\mathcal{N}}_i}^I(X). \end{aligned}$$

(2)

$$\begin{aligned} \underline{\mathcal{N}}_{\bigvee_i^n \hat{\mathcal{N}}_i}^{II\alpha}(X) &= \left\{ u \in U_0 \mid P(X|\hat{\mathcal{N}}_1(u)) \geq 1 \vee P(X|\hat{\mathcal{N}}_2(u)) \geq 1 \vee \dots \vee P(X|\hat{\mathcal{N}}_i(u)) \geq 1 \right\} \\ &= \left\{ u \in U_0 \mid \hat{\mathcal{N}}_1(u) \subseteq X \vee \hat{\mathcal{N}}_2(u) \subseteq X \vee \dots \vee \hat{\mathcal{N}}_i(u) \subseteq X \right\} \\ &= \underline{\mathcal{N}}_{\bigvee_i^n \hat{\mathcal{N}}_i}^{II}(X) \end{aligned}$$

and

$$\begin{aligned} \overline{\mathcal{N}}_{\bigvee_i^n \hat{\mathcal{N}}_i}^{II\beta}(X) &= \left\{ u \in U_0 \mid P(X|\hat{\mathcal{N}}_1(u)) > 0 \vee P(X|\hat{\mathcal{N}}_2(u)) > 0 \vee \dots \vee P(X|\hat{\mathcal{N}}_i(u)) > 0 \right\} \\ &= \left\{ u \in U_0 \mid \hat{\mathcal{N}}_1(u) \cap X \neq \emptyset \vee \hat{\mathcal{N}}_2(u) \cap X \neq \emptyset \vee \dots \vee \hat{\mathcal{N}}_i(u) \cap X \neq \emptyset \right\} \\ &= \overline{\mathcal{N}}_{\bigvee_i^n \hat{\mathcal{N}}_i}^{II}(X). \end{aligned}$$

The main results are as follows:

Theorem 3.36. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $\mathcal{N} = \{\hat{N}_i, i \in N\}$ is a family of neighborhood operators on U , $0 \leq \beta < \alpha \leq 1$, and $X, X_1 \in 2^U$. Then the followings hold:

- (1) $\mathcal{N}_{\hat{N}_i}^{I\alpha}(\emptyset) = \mathcal{N}_{\hat{N}_i}^{I\beta}(\emptyset) = \emptyset, \mathcal{N}_{\hat{N}_i}^{I\alpha}(U) = \mathcal{N}_{\hat{N}_i}^{I\beta}(U) = \overline{U_0}$.
- (2) $\mathcal{N}_{\hat{N}_i}^{I\alpha}(X) \subseteq \mathcal{N}_{\hat{N}_i}^{I\beta}(X)$.
- (3) If $X \subseteq X_1$, then $\mathcal{N}_{\hat{N}_i}^{I\alpha}(X) \subseteq \mathcal{N}_{\hat{N}_i}^{I\alpha}(X_1)$ and $\mathcal{N}_{\hat{N}_i}^{I\beta}(X) \subseteq \mathcal{N}_{\hat{N}_i}^{I\beta}(X_1)$.
- (4) $\mathcal{N}_{\hat{N}_i}^{I\alpha}\left(\bigcap_{k \in K} X_k\right) \subseteq \bigcap_{k \in K} \mathcal{N}_{\hat{N}_i}^{I\alpha}(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$) and $\mathcal{N}_{\hat{N}_i}^{I\beta}\left(\bigcap_{k \in K} X_k\right) \subseteq \bigcap_{k \in K} \mathcal{N}_{\hat{N}_i}^{I\beta}(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$).
- (5) $\mathcal{N}_{\hat{N}_i}^{I\alpha}\left(\bigcup_{k \in K} X_k\right) \supseteq \bigcup_{k \in K} \mathcal{N}_{\hat{N}_i}^{I\alpha}(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$) and $\mathcal{N}_{\hat{N}_i}^{I\beta}\left(\bigcup_{k \in K} X_k\right) \supseteq \bigcup_{k \in K} \mathcal{N}_{\hat{N}_i}^{I\beta}(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$).
- (6) $\mathcal{N}_{\hat{N}_i}^{I\alpha_1}(X) \subseteq \mathcal{N}_{\hat{N}_i}^{I\alpha}(X)$ (if $0.5 < \alpha < \alpha_1 \leq 1$) and $\mathcal{N}_{\hat{N}_i}^{I\beta_1}(X) \subseteq \mathcal{N}_{\hat{N}_i}^{I\beta}(X)$ (if $0 \leq \beta < \beta_1 < 0.5$).

Proof. By Definition 3.34, the result can be similarly proven as Theorem 3.22.

The equality (4) and (5) of Theorem 3.36 does not hold as the following example.

Example 3.37 (Continued from Examples 3.3 and 3.33). Let $X = \{x_1, x_5\}, X_1 = \{x_2, x_4\}, X \cap X_1 = \emptyset$, and $X \cup X_1 = \{x_1, x_2, x_4, x_5\}$. Then we have

$$\begin{aligned}
 P(X|\hat{N}_1(x_2)) &= \frac{1}{3}, P(X|\hat{N}_1(x_4)) = 0, P(X|\hat{N}_1(x_5)) = 1, \\
 P(X|\hat{N}_2(x_2)) &= \frac{1}{2}, P(X|\hat{N}_2(x_4)) = 0, P(X|\hat{N}_2(x_5)) = \frac{1}{2}, \\
 P(X|\hat{N}_3(x_2)) &= \frac{1}{2}, P(X|\hat{N}_3(x_4)) = 0, P(X|\hat{N}_3(x_5)) = \frac{1}{2}; \\
 P(X_1|\hat{N}_1(x_2)) &= \frac{1}{3}, P(X_1|\hat{N}_1(x_4)) = 1, P(X_1|\hat{N}_1(x_5)) = 0, \\
 P(X_1|\hat{N}_2(x_2)) &= \frac{1}{2}, P(X_1|\hat{N}_2(x_4)) = 1, P(X_1|\hat{N}_2(x_5)) = \frac{1}{4}, \\
 P(X_1|\hat{N}_3(x_2)) &= \frac{1}{2}, P(X_1|\hat{N}_3(x_4)) = \frac{1}{2}, P(X_1|\hat{N}_3(x_5)) = \frac{1}{2}, \\
 P(X \cup X_1|\hat{N}_1(x_2)) &= \frac{2}{3}, P(X \cup X_1|\hat{N}_1(x_4)) = 1, \\
 P(X \cup X_1|\hat{N}_1(x_5)) &= 1, P(X \cup X_1|\hat{N}_2(x_2)) = 1, \\
 P(X \cup X_1|\hat{N}_2(x_4)) &= 1, P(X \cup X_1|\hat{N}_2(x_5)) = \frac{3}{4}, \\
 P(X \cup X_1|\hat{N}_3(x_2)) &= 1, P(X \cup X_1|\hat{N}_3(x_4)) = \frac{1}{2}, \\
 P(X \cup X_1|\hat{N}_3(x_5)) &= 1.
 \end{aligned}$$

Take $\alpha = 0.3$ and $\beta = 0.2$. Then we obtain $\mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\alpha}(X) = \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\beta}(X) = \{x_2, x_5\}, \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\alpha}(X_1) = \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\beta}(X_1) = \emptyset$.

$\{x_2, x_4\}$, and hence $\mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\alpha}(X \cap X_1) = \emptyset \neq \{x_2\} = \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\alpha}(X) \cap \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\alpha}(X_1)$ and $\mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\beta}(X \cap X_1) = \emptyset \neq \{x_2\} = \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\beta}(X) \cap \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\beta}(X_1)$. Again, take $\alpha = 0.5$ and $\beta = 0.4$. Then we obtain $\mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\alpha}(X) = \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\beta}(X) = \{x_5\}, \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\alpha}(X_1) = \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\beta}(X_1) = \{x_4\}$, and hence $\mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\alpha}(X \cup X_1) = U_0 \neq \{x_4, x_5\} = \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\alpha}(X) \cup \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\alpha}(X_1)$ and $\mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\beta}(X \cup X_1) = U_0 \neq \{x_4, x_5\} = \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\beta}(X) \cup \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\beta}(X_1)$.

Remark 3.38. Let $X \in 2^U, \sim X = U - X$, and $0 \leq \beta < \alpha \leq 1$. Then $\mathcal{N}_{\hat{N}_i}^{I\alpha}(\sim X) = \sim \mathcal{N}_{\hat{N}_i}^{I(1-\alpha)}(X)$ and $\mathcal{N}_{\hat{N}_i}^{I\beta}(\sim X) = \sim \mathcal{N}_{\hat{N}_i}^{I(1-\beta)}(X)$ does not hold by the following example.

Example 3.39 (Continued from Examples 3.3 and 3.33). Then $\sim X = \{x_2, x_5\}$ and $P(\sim X|\hat{N}_1(x_2)) = \frac{1}{3}, P(\sim X|\hat{N}_1(x_4)) = \frac{1}{2}, P(\sim X|\hat{N}_1(x_5)) = \frac{1}{2}; P(\sim X|\hat{N}_2(x_2)) = 1, P(\sim X|\hat{N}_2(x_4)) = 0, P(\sim X|\hat{N}_2(x_5)) = \frac{1}{4}; P(\sim X|\hat{N}_3(x_2)) = \frac{1}{2}, P(\sim X|\hat{N}_3(x_4)) = 0, P(\sim X|\hat{N}_3(x_5)) = \frac{1}{2}$. If $\beta = 0.3$ and $\alpha = 0.7$. Thus $\mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\alpha}(X) = \emptyset \neq \{x_4\} = \sim \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I(1-\alpha)}(\sim X)$ and $\mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I\beta}(X) = \{x_4, x_5\} \neq U_0 = \sim \mathcal{N}_{\hat{N}_1 \wedge \hat{N}_2 \wedge \hat{N}_3}^{I(1-\beta)}(\sim X)$.

Corollary 3.40. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $\mathcal{N} = \{\hat{N}_i, i \in N\}$ is a family of neighborhood operators on U , and $X, X_1 \in 2^U$. Then the followings hold:

- (1) $\mathcal{N}_{\hat{N}_i}^I(\emptyset) = \mathcal{N}_{\hat{N}_i}^I(\emptyset) = \emptyset, \mathcal{N}_{\hat{N}_i}^I(U) = \mathcal{N}_{\hat{N}_i}^I(U) = \overline{U_0}$.
- (2) $\mathcal{N}_{\hat{N}_i}^I(X) \subseteq \mathcal{N}_{\hat{N}_i}^I(X)$.
- (3) If $X \subseteq X_1$, then $\mathcal{N}_{\hat{N}_i}^I(X) \subseteq \mathcal{N}_{\hat{N}_i}^I(X_1)$ and $\mathcal{N}_{\hat{N}_i}^I(X) \subseteq \mathcal{N}_{\hat{N}_i}^I(X_1)$.
- (4) $\mathcal{N}_{\hat{N}_i}^I\left(\bigcap_{k \in K} X_k\right) = \bigcap_{k \in K} \mathcal{N}_{\hat{N}_i}^I(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$) and $\mathcal{N}_{\hat{N}_i}^I\left(\bigcap_{k \in K} X_k\right) = \bigcap_{k \in K} \mathcal{N}_{\hat{N}_i}^I(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$).
- (5) $\mathcal{N}_{\hat{N}_i}^I\left(\bigcup_{k \in K} X_k\right) \supseteq \bigcup_{k \in K} \mathcal{N}_{\hat{N}_i}^I(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$) and $\mathcal{N}_{\hat{N}_i}^I\left(\bigcup_{k \in K} X_k\right) \supseteq \bigcup_{k \in K} \mathcal{N}_{\hat{N}_i}^I(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$).

Proof. Follows from Definition 3.34.

Theorem 3.41. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $\mathcal{N} = \{\hat{N}_i, i \in N\}$ is a family of neighborhood operators on U , $0 \leq \beta < \alpha \leq 1$, and $X, X_1 \in 2^U$. Then the followings hold:

- (1) $\mathcal{N}_{\hat{N}_i}^{II\alpha}(\emptyset) = \mathcal{N}_{\hat{N}_i}^{II\beta}(\emptyset) = \emptyset, \mathcal{N}_{\hat{N}_i}^{II\alpha}(U) = \mathcal{N}_{\hat{N}_i}^{II\beta}(U) = \overline{U_0}$.

- (2) $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\alpha}(X) \subseteq \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\beta}}(X)$.
- (3) If $X \subseteq X_1$, then $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\alpha}(X) \subseteq \mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\alpha}(X_1)$ and $\overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\beta}}(X) \subseteq \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\beta}}(X_1)$.
- (4) $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\alpha}\left(\bigcap_{k \in K} X_k\right) \subseteq \bigcap_{k \in K} \mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\alpha}(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$) and $\overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\beta}}\left(\bigcap_{k \in K} X_k\right) \subseteq \bigcap_{k \in K} \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\beta}}(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$).
- (5) $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\alpha}\left(\bigcup_{k \in K} X_k\right) \supseteq \bigcup_{k \in K} \mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\alpha}(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$) and $\overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\beta}}\left(\bigcup_{k \in K} X_k\right) \supseteq \bigcup_{k \in K} \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\beta}}(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$).
- (6) $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\alpha_1}(X) \subseteq \mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\alpha}(X)$ (if $0.5 < \alpha < \alpha_1 \leq 1$) and $\overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\beta_1}}(X) \subseteq \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\beta}}(X)$ (if $0 \leq \beta < \beta_1 < 0.5$).

Proof. Follows from Definition 3.32.

The equality (4) and (5) of Theorem 3.41 does not hold as the following example.

Example 3.42 (Continued from Example 37). Take $\alpha = 0.3$ and $\beta = 0.2$. Then we obtain $\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\alpha}(X) = \overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\beta}}(X)$
 $= \{x_2, x_5\}$, $\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\alpha}(X_1) = \overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\beta}}(X_1) = U_0$, and hence
 $\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\alpha}(X \cap X_1) = \emptyset \neq \{x_2, x_5\} = \mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\alpha}(X) \cap$
 $\overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\beta}}(X_1)$ and $\overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\beta}}(X \cap X_1) = \emptyset \neq \{x_2, x_5\} =$
 $\overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\beta}}(X) \cap \overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\beta}}(X_1)$. Again, take $\alpha = 0.6$ and
 $\beta = 0.5$. Then we obtain $\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\alpha}(X) = \overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\beta}}(X) =$
 $\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\alpha}(X_1) = \overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\beta}}(X_1) = \emptyset$, and hence $\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\alpha}(X \cup$
 $X_1) = U_0 \neq \emptyset = \mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\alpha}(X) \cup \mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\alpha}(X_1)$ and
 $\overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\beta}}(X \cup X_1) = U_0 \neq \emptyset = \overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\beta}}(X) \cup \overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\beta}}(X_1)$.

Remark 3.43. Let $X \in 2^U$, $\sim X = U - X$, and $0 \leq \beta < \alpha \leq 1$. Then $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\alpha}(X) = \sim \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II(1-\alpha)}}(\sim X)$ and $\overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\beta}}(X) = \sim \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II(1-\beta)}}(\sim X)$ does not hold by the following example.

Example 3.44 (Continued from Example 3.39). If $\beta = 0.3$ and $\alpha = 0.7$. Thus $\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\alpha}(X) = \{x_4, x_5\} \neq \emptyset = \sim$
 $\overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II(1-\alpha)}}(\sim X)$ and $\overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II\beta}}(X) = U_0 \neq \{x_4, x_5\} = \sim$
 $\overline{\mathcal{N}_{\hat{N}_1 \vee \hat{N}_2 \vee \hat{N}_3}^{II(1-\beta)}}(\sim X)$.

From Examples 3.39 and 3.44, we conclude the following corollary:

Corollary 3.45. Let $X \in 2^U$, $\sim X = U - X$, and $0 \leq \beta < \alpha \leq 1$. Then $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{I\alpha}(X) = \sim \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II(1-\alpha)}}(\sim X)$ and $\overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{I\beta}}(X) = \sim \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II(1-\beta)}}(\sim X)$.

Corollary 3.46. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $\mathcal{N} = \{\hat{N}_i, i \in N\}$ is a family of neighborhood operators on U , and $X, X_1 \in 2^U$. Then the followings hold:

- (1) $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}(\emptyset) = \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}}(\emptyset) = \emptyset$, $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}(U) = \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}}(U) = U_0$.
- (2) $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}(X) \subseteq \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}}(X)$.
- (3) If $X \subseteq X_1$, then $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}(X) \subseteq \mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}(X_1)$ and $\overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}}(X) \subseteq \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}}(X_1)$.
- (4) $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}\left(\bigcap_{k \in K} X_k\right) = \bigcap_{k \in K} \mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$) and $\overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}}\left(\bigcap_{k \in K} X_k\right) = \bigcap_{k \in K} \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}}(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$).
- (5) $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}\left(\bigcup_{k \in K} X_k\right) \supseteq \bigcup_{k \in K} \mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$) and $\overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}}\left(\bigcup_{k \in K} X_k\right) \supseteq \bigcup_{k \in K} \overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II}}(X_k)$ ($\{X_k\}_{k \in K} \subseteq 2^U$).

Proof. Follows from Definition 3.34.

Corollary 3.47. Let $X \in 2^U$ and $\sim X = U - X$. Then $\mathcal{N}_{\bigwedge_i^n \hat{N}_i}^{\alpha}(X) = \sim \overline{\mathcal{N}_{\bigwedge_i^n \hat{N}_i}^{II}}(\sim X)$ and $\overline{\mathcal{N}_{\bigwedge_i^n \hat{N}_i}^{\alpha}}(X) = \sim \overline{\mathcal{N}_{\bigwedge_i^n \hat{N}_i}^{II}}(\sim X)$.

Theorem 3.48. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $\mathcal{N} = \{\hat{N}_i, i \in N\}$ is a family of neighborhood operators on U , $0 \leq \beta < \alpha \leq 1$, and $X \in 2^U$. Then the followings hold:

- (1) $\mathcal{N}_{\bigwedge_i^n \hat{N}_i}^{I\alpha}(X) = \mathcal{N}_{\hat{N}_1}^{\alpha}(X) \cap \mathcal{N}_{\hat{N}_2}^{\alpha}(X) \cap \dots \cap \mathcal{N}_{\hat{N}_p}^{\alpha}(X)$,
- (2) $\overline{\mathcal{N}_{\bigwedge_i^n \hat{N}_i}^{I\beta}}(X) = \overline{\mathcal{N}_{\hat{N}_1}^{\beta}}(X) \cap \overline{\mathcal{N}_{\hat{N}_2}^{\beta}}(X) \cap \dots \cap \overline{\mathcal{N}_{\hat{N}_p}^{\beta}}(X)$,
- (3) $\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\alpha}(X) = \mathcal{N}_{\hat{N}_1}^{\alpha}(X) \cup \mathcal{N}_{\hat{N}_2}^{\alpha}(X) \cup \dots \cup \mathcal{N}_{\hat{N}_p}^{\alpha}(X)$,
- (4) $\overline{\mathcal{N}_{\bigvee_i^n \hat{N}_i}^{II\beta}}(X) = \overline{\mathcal{N}_{\hat{N}_1}^{\beta}}(X) \cup \overline{\mathcal{N}_{\hat{N}_2}^{\beta}}(X) \cup \dots \cup \overline{\mathcal{N}_{\hat{N}_p}^{\beta}}(X)$.

Proof. We only prove (3). It is sufficient to show $\mathcal{N}_{\bigvee_{i=1}^n \hat{N}_i}^{II\alpha}(X) = \mathcal{N}_{\hat{N}_1}^{\alpha}(X) \cup \mathcal{N}_{\hat{N}_2}^{\alpha}(X)$. Let $u \in \mathcal{N}_{\hat{N}_1}^{\alpha}(X)$ ($u \in U_0$), where $\mathcal{N}_{\hat{N}_1}^{\alpha}(X) = \{u \in U_0 \mid P(X|\hat{N}_1(u)) \geq \alpha\}$, but $u \in \mathcal{N}_{\bigvee_{i=1}^n \hat{N}_i}^{II\alpha}(X)$. So, $\mathcal{N}_{\hat{N}_1}^{\alpha}(X) \subseteq \mathcal{N}_{\bigvee_{i=1}^n \hat{N}_i}^{II\alpha}(X)$. Similarity, $\mathcal{N}_{\hat{N}_2}^{\alpha}(X) \subseteq \mathcal{N}_{\bigvee_{i=1}^n \hat{N}_i}^{II\alpha}(X)$. Consequently, $\mathcal{N}_{\hat{N}_1}^{\alpha}(X) \cup \mathcal{N}_{\hat{N}_2}^{\alpha}(X) \subseteq \mathcal{N}_{\bigvee_{i=1}^n \hat{N}_i}^{II\alpha}(X)$. Conversely, for $u \in \mathcal{N}_{\bigvee_{i=1}^n \hat{N}_i}^{II\alpha}(X)$ and by Definition 3.32 we have either $P(X|\hat{N}_1(u)) \geq \alpha$, then $u \in \mathcal{N}_{\hat{N}_1}^{\alpha}(X)$ or $P(X|\hat{N}_2(u)) \geq \alpha$, then $u \in \mathcal{N}_{\hat{N}_2}^{\alpha}(X)$. Hence $u \in \mathcal{N}_{\hat{N}_1}^{\alpha}(X) \cup \mathcal{N}_{\hat{N}_2}^{\alpha}(X)$, and thus $\mathcal{N}_{\bigvee_{i=1}^n \hat{N}_i}^{II\alpha}(X) \subseteq \mathcal{N}_{\hat{N}_1}^{\alpha}(X) \cup \mathcal{N}_{\hat{N}_2}^{\alpha}(X)$. Therefore, (3) is holding.

Corollary 3.49. Let $(U, U_0, \mathcal{N})$ be a neighborhood approximation space, $\mathcal{N} = \{\hat{N}_i, i \in N\}$ is a family of neighborhood operators on U , and $X \in 2^U$. Then the followings hold:

- (1) $\mathcal{N}_{\bigwedge_i^n \hat{N}_i}^I(X) = \mathcal{N}_{\hat{N}_1}(X) \cap \mathcal{N}_{\hat{N}_2}(X) \cap \dots \cap \mathcal{N}_{\hat{N}_i}(X)$,
- (2) $\overline{\mathcal{N}_{\bigwedge_i^n \hat{N}_i}^I}(X) = \overline{\mathcal{N}_{\hat{N}_1}}(X) \cap \overline{\mathcal{N}_{\hat{N}_2}}(X) \cap \dots \cap \overline{\mathcal{N}_{\hat{N}_i}}(X)$,

- (3) $\mathcal{N}_{\bigvee_i^n \mathcal{N}_i}^{II}(X) = \mathcal{N}_{\mathcal{N}_1}(X) \cup \mathcal{N}_{\mathcal{N}_2}(X) \cup \dots \cup \mathcal{N}_{\mathcal{N}_i}(X),$
 (4) $\mathcal{N}_{\bigvee_i^n \mathcal{N}_i}^{II}(X) = \overline{\mathcal{N}_{\mathcal{N}_1}}(X) \cup \overline{\mathcal{N}_{\mathcal{N}_2}}(X) \cup \dots \cup \overline{\mathcal{N}_{\mathcal{N}_i}}(X).$

4 Rough set based on a neighborhood operator over two different universes

In this section we will propose a new notion of rough set based on a neighborhood operator over two different universes U and V . Furthermore, we introduce three types of graded n -neighborhood rough set over two different universes and then study two algorithms and its applications.

We will define the neighborhood rough set over two different universes U and V followed by an example.

4.1. Rough set based on a neighborhood operator over two different universes

Definition 4.1. Let U_0 is subset on a set U , $\mathcal{N} : U_0 \rightarrow 2^V$ (where 2^V the set of all subsets of V) is a neighborhood operator over two different universes U and V satisfying $\mathcal{N}(u) \neq \emptyset (\forall u \in U_0)$, and $Y \in 2^V$. Then the triplet $(U^V, U_0, \mathcal{N})$ is called a neighborhood approximation space on two different universes U and V ,

$$\underline{\mathcal{N}}(Y) = \left\{ u \in U_0 \mid \mathcal{N}(u) \subseteq Y \right\}$$

and

$$\overline{\mathcal{N}}(Y) = \left\{ u \in U_0 \mid \mathcal{N}(u) \cap Y \neq \emptyset \right\}$$

are called the lower $\mathcal{N}$ -approximation and the upper $\mathcal{N}$ -approximation of Y with respect to $\mathcal{N}$ respectively, $(\underline{\mathcal{N}}(Y), \overline{\mathcal{N}}(Y))$ is called the neighborhood rough approximation of Y on U and V (or the $\mathcal{N}$ -rough approximation of Y , briefly, a neighborhood rough set on U and V).

Example 4.2. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a sets, $U_0 = \{x_2, x_4, x_5, x_6\} \subseteq U$, and $V = \{y_1, y_2, y_3, y_4\}$ be a sets. We define a neighborhood operator $\mathcal{N} : U_0 \rightarrow 2^V$ as follows: $\mathcal{N}(x_2) = \{y_4, y_5\}$, $\mathcal{N}(x_4) = \{y_2, y_3, y_6\}$, $\mathcal{N}(x_5) = \{y_1, y_4, y_5\}$, and $\mathcal{N}(x_6) = \{y_2, y_3, y_4, y_6\}$. Assume that $Y = \{y_1, y_2, y_4\}$. Then, $\underline{\mathcal{N}}(Y) = \overline{\mathcal{N}}(Y) = \{x_2, x_4, x_6\}$.

Theorem 4.3. Let $(U^V, U_0, \mathcal{N})$ is called a neighborhood approximation space over two different universes U and V . Then the followings hold:

- (1) $\underline{\mathcal{N}}(\emptyset) = \overline{\mathcal{N}}(\emptyset) = \emptyset$ and $\underline{\mathcal{N}}(V) = \overline{\mathcal{N}}(V) = U_0$ hold.
- (2) $\underline{\mathcal{N}}(Y) \subseteq \overline{\mathcal{N}}(Y)$ hold for any $Y \in 2^V$.
- (3) $\underline{\mathcal{N}}\left(\bigcap_{k \in K} Y_k\right) = \bigcap_{k \in K} \underline{\mathcal{N}}(Y_k)$ ($\{Y_k\}_{k \in K} \subseteq 2^V$) and $\overline{\mathcal{N}}\left(\bigcup_{k \in K} Y_k\right) = \bigcup_{k \in K} \overline{\mathcal{N}}(Y_k)$ ($\{Y_k\}_{k \in K} \subseteq 2^V$).
- (4) $\underline{\mathcal{N}}(Y) \subseteq \underline{\mathcal{N}}(Y_1)$ and $\overline{\mathcal{N}}(Y) \subseteq \overline{\mathcal{N}}(Y_1)$ hold for any $(Y, Y_1 \in 2^V, Y \subseteq Y_1)$.
- (5) $\underline{\mathcal{N}}\left(\bigcup_{k \in K} Y_k\right) \supseteq \bigcup_{k \in K} \underline{\mathcal{N}}(Y_k)$ ($\{Y_k\}_{k \in K} \subseteq 2^V$) and $\overline{\mathcal{N}}\left(\bigcap_{k \in K} Y_k\right) \subseteq \bigcap_{k \in K} \overline{\mathcal{N}}(Y_k)$ ($\{Y_k\}_{k \in K} \subseteq 2^V$).

Proof. Follows from Definition 4.1.

Definition 4.4. Let $(U^V, U_0, \mathcal{N})$ is called a neighborhood approximation space over two different universes U and V , $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers and $Y \in 2^V$. Then

$$\underline{\mathcal{N}}_n^I(Y) = \left\{ u \in U_0 \mid |\mathcal{N}(u) - Y| \leq n \right\}$$

and

$$\overline{\mathcal{N}}_n^I(Y) = \left\{ u \in U_0 \mid |\mathcal{N}(u) \cap Y| > n \right\}$$

are called the lower $\mathcal{N}$ -approximation and the upper $\mathcal{N}$ -approximation of Y with respect to the graded n respectively, and $(\underline{\mathcal{N}}_n^I(Y), \overline{\mathcal{N}}_n^I(Y))$ is called the Type-I graded n -neighborhood rough approximation of Y on U and V (or the $\mathcal{N}$ -Type-I graded n -neighborhood rough approximation of Y , briefly, Type-I graded n -neighborhood rough set on U and V).

Theorem 4.5. Let $(U^V, U_0, \mathcal{N})$ be a neighborhood approximation space on two different universes U and V and $Y \in 2^V$. Then the followings hold:

- (1) $(\underline{\mathcal{N}}_n^I(Y), \overline{\mathcal{N}}_n^I(Y)) = (\underline{\mathcal{N}}(Y), \overline{\mathcal{N}}(Y))$ when $n = 0$, $\overline{\mathcal{N}}_n^I(\emptyset) = \emptyset$, $\underline{\mathcal{N}}_n^I(V) = U_0$, and both $\underline{\mathcal{N}}_n^I$ and $\overline{\mathcal{N}}_n^I$ preserves orders, i.e. $\underline{\mathcal{N}}_n^I(Y) \subseteq \underline{\mathcal{N}}_n^I(Y_1)$ and $\overline{\mathcal{N}}_n^I(Y) \subseteq \overline{\mathcal{N}}_n^I(Y_1)$ ($(Y, Y_1) \in 2^V \times 2^V, Y \subseteq Y_1$) hold.
- (2) $\underline{\mathcal{N}}_n^I(Y) = \sim (\overline{\mathcal{N}}_n^I(\sim Y))$ and $\overline{\mathcal{N}}_n^I(Y) = \sim (\underline{\mathcal{N}}_n^I(\sim Y))$, where $\sim Y = V - Y$.
- (3) $\underline{\mathcal{N}}_n^I\left(\bigcap_{k \in K} Y_k\right) \subseteq \bigcap_{k \in K} \underline{\mathcal{N}}_n^I(Y_k)$ ($\{Y_k\}_{k \in K} \subseteq 2^V$) and $\overline{\mathcal{N}}_n^I\left(\bigcup_{k \in K} Y_k\right) \supseteq \bigcup_{k \in K} \overline{\mathcal{N}}_n^I(Y_k)$ ($\{Y_k\}_{k \in K} \subseteq 2^V$).
- (4) $\underline{\mathcal{N}}_n^I\left(\bigcup_{k \in K} Y_k\right) \supseteq \bigcup_{k \in K} \underline{\mathcal{N}}_n^I(Y_k)$ ($\{Y_k\}_{k \in K} \subseteq 2^V$) and $\overline{\mathcal{N}}_n^I\left(\bigcap_{k \in K} Y_k\right) \subseteq \bigcap_{k \in K} \overline{\mathcal{N}}_n^I(Y_k)$ ($\{Y_k\}_{k \in K} \subseteq 2^V$).
- (5) If $n \leq n_1$, then $\underline{\mathcal{N}}_n^I(Y) \subseteq \underline{\mathcal{N}}_{n_1}^I(Y)$ and $\overline{\mathcal{N}}_n^I(Y) \subseteq \overline{\mathcal{N}}_{n_1}^I(Y)$ hold.

Proof. (1) When $n = 0$. Then $\underline{\mathcal{N}}_n^I(Y) = \{u \in U_0 \mid |\mathcal{N}(u) - Y| \leq 0\} = \{u \in U_0 \mid \mathcal{N}(u) \subseteq Y\}$ and $\overline{\mathcal{N}}_n^I(Y) = \{u \in U_0 \mid |\mathcal{N}(u) \cap Y| > 0\} = \{u \in U_0 \mid \mathcal{N}(u) \cap Y \neq \emptyset\}$. Therefore, $(\underline{\mathcal{N}}_n^I(Y), \overline{\mathcal{N}}_n^I(Y)) = (\underline{\mathcal{N}}(Y), \overline{\mathcal{N}}(Y))$, and $\underline{\mathcal{N}}_n^I(Y) \subseteq \overline{\mathcal{N}}_n^I(Y)$, and directly from Definition 4.4, we have $\overline{\mathcal{N}}_n^I(\emptyset) = \emptyset$ and $\underline{\mathcal{N}}_n^I(V) = U_0$. Now, we only prove that $\underline{\mathcal{N}}_n^I(Y) \subseteq \underline{\mathcal{N}}_n^I(Y_1)$ ($(Y, Y_1) \in 2^V \times 2^V, Y \subseteq Y_1$) hold. Let $u \in \underline{\mathcal{N}}_n^I(Y)$, then we have $|\mathcal{N}(u) - Y| \leq n$. Since $Y \subseteq Y_1$, then $|\mathcal{N}(u) - Y_1| \leq |\mathcal{N}(u) - Y| \leq n$, that is $u \in \underline{\mathcal{N}}_n^I(Y_1)$. Thus, $\underline{\mathcal{N}}_n^I(Y) \subseteq \underline{\mathcal{N}}_n^I(Y_1)$ ($(Y, Y_1) \in 2^V \times 2^V, Y \subseteq Y_1$).

(2) $\sim (\overline{\mathcal{N}}_n^I(\sim Y)) = \sim \{u \in U_0 \mid |\mathcal{N}(u) \cap (\sim Y)| > n\} = \sim \{u \in U_0 \mid |\mathcal{N}(u) - Y| \leq n\} = \underline{\mathcal{N}}_n^I(Y)$ and $\sim (\underline{\mathcal{N}}_n^I(\sim Y)) = \sim \{u \in U_0 \mid |\mathcal{N}(u) - (\sim Y)| \leq n\} = \sim \{u \in U_0 \mid |\mathcal{N}(u) \cap Y| > n\} = \overline{\mathcal{N}}_n^I(Y)$.

(3) It is sufficient to show $\underline{\mathcal{N}}_n^I(Y \cap Y_1) \subseteq \underline{\mathcal{N}}_n^I(Y) \cap \underline{\mathcal{N}}_n^I(Y_1)$. Since $Y \cap Y_1 \subseteq Y$ and $Y \cap Y_1 \subseteq Y_1$. From (1) above, we have that $\underline{\mathcal{N}}_n^I(Y \cap Y_1) \subseteq \underline{\mathcal{N}}_n^I(Y)$ and $\underline{\mathcal{N}}_n^I(Y \cap Y_1) \subseteq \underline{\mathcal{N}}_n^I(Y_1)$.

Combining the two formulas, we obtain that $\underline{\mathcal{N}}_n^I(Y \cap Y_1) \subseteq \underline{\mathcal{N}}_n^I(Y) \cap \underline{\mathcal{N}}_n^I(Y_1)$.

(4) Analogue to (3).

(5) Let $u \in \underline{\mathcal{N}}_n^I(Y)$, then we have $|\mathcal{N}(u) - Y| \leq n$. Since $n \leq n_1$, then $|\mathcal{N}(u) - Y| \leq n \leq n_1$ implies that $|\mathcal{N}(u) - Y| \leq n_1$. Consequently, $u \in \underline{\mathcal{N}}_{n_1}^I(Y)$ and thus $\underline{\mathcal{N}}_n^I(Y) \subseteq \underline{\mathcal{N}}_{n_1}^I(Y)$. Similarity, if $n \leq n_1$, then $\overline{\mathcal{N}}_{n_1}^I(Y) \subseteq \overline{\mathcal{N}}_n^I(Y)$.

Remark 4.6. Let $Y \in 2^V$. Then $\underline{\mathcal{N}}_n^I(Y) \subseteq \overline{\mathcal{N}}_n^I(Y)$ does not hold by the following example.

Example 4.7. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$, $U_0 = \{x_2, x_4, x_5\} \subseteq U$, and $V = \{y_1, y_2, y_3, y_4\}$. A neighborhood operator $\mathcal{N} : U_0 \rightarrow 2^V$ defined as $\mathcal{N}(x_2) = \{y_1, y_3\}$, $\mathcal{N}(x_4) = \{y_4\}$, and $\mathcal{N}(x_5) = \{y_2, y_3, y_4\}$. Assume that $Y = \{y_1, y_2\}$ and $n = 2$. This show $\underline{\mathcal{N}}_n^I(Y) = U_0 \supseteq \emptyset = \overline{\mathcal{N}}_n^I(Y)$.

The above example show that $\overline{\mathcal{N}}_n^I(Y) \subseteq \underline{\mathcal{N}}_n^I(Y)$ implies that it is a contradiction with Pawlaks rough set theory. So, we present the definition of Type-II graded n -neighborhood rough set on U and V as follows:

Definition 4.8. Let $(U^V, U_0, \mathcal{N})$ is called a neighborhood approximation space over two different universes U and V , $n \in N$ s.t. $n \in [0, \min(|\mathcal{N}(u)|/2, |Y|)]$, and $Y \in 2^V$. Then

$$\underline{\mathcal{N}}_n^{II}(Y) = \left\{ u \in U_0 \mid |\mathcal{N}(u) - Y| \leq n \right\}$$

and

$$\overline{\mathcal{N}}_n^{II}(Y) = \left\{ u \in U_0 \mid |\mathcal{N}(u) \cap Y| > n \right\}$$

are called the lower $\mathcal{N}$ -approximation and the upper $\mathcal{N}$ -approximation of Y with respect to the graded n respectively, and $(\underline{\mathcal{N}}_n^{II}(Y), \overline{\mathcal{N}}_n^{II}(Y))$ is called the Type-II graded n -neighborhood rough approximation of Y on U and V (or the $\mathcal{N}$ -Type-II graded n -neighborhood rough approximation of Y , briefly, Type-II graded n -neighborhood rough set on U and V). The subsets $\underline{\mathcal{N}}_n^{II}(Y)$, $U_0 - \overline{\mathcal{N}}_n^{II}(Y)$, and $\overline{\mathcal{N}}_n^{II}(Y) - \underline{\mathcal{N}}_n^{II}(Y)$ are called the $\mathcal{N}$ -positive region ($\mathcal{N}$ -POS), the $\mathcal{N}$ -negative region ($\mathcal{N}$ -NEG), and the $\mathcal{N}$ -boundary region ($\mathcal{N}$ -BND), of Y with respect to n , respectively.

Lemma 4.9. Let $(U^V, U_0, \mathcal{N})$ is called a neighborhood approximation space over two different universes U and V . Then, $n \in [0, \min(|\mathcal{N}(u)|/2, |Y|)]$, where $|\mathcal{N}(u)|$ is the cardinality of $\mathcal{N}(u)$ ($\forall u \in U_0$) and $|Y|$ is the cardinality of Y .

Proof. For any $u \in U_0$, $(\mathcal{N}(u) - Y) \cup (\mathcal{N}(u) \cap Y) = \mathcal{N}(u)$. Then $|\mathcal{N}(u) - Y| + |\mathcal{N}(u) \cap Y| = |\mathcal{N}(u)|$ implies that $|\mathcal{N}(u) \cap Y| = |\mathcal{N}(u)| - |\mathcal{N}(u) - Y|$. From Definition 4.8, if $|\mathcal{N}(u) \cap Y| > n$, then $|\mathcal{N}(u)| - |\mathcal{N}(u) - Y| > n$ and if $|\mathcal{N}(u) - Y| \leq n$, then $|\mathcal{N}(u)| - n > n$. Consequently, $|\mathcal{N}(u)| > 2n$ and thus $n < |\mathcal{N}(u)|/2$. Since $n < |\mathcal{N}(u) \cap Y| \leq \min(|Y|, |\mathcal{N}(u)|)$ and $n < |\mathcal{N}(u)|/2$. Then $n < \min(|\mathcal{N}(u)|/2, |Y|)$. Therefore, $n \in [0, \min(|\mathcal{N}(u)|/2, |Y|)]$.

Lemma 4.10. Let $(U^V, U_0, \mathcal{N})$ is called a neighborhood approximation space over two different universes U and V . Then, we can obtain the limits of the grade easily

$$n = \max[0, \min(|\mathcal{N}(u)|/2, |Y|)], \text{ where } n \in N.$$

Proof. Obviously.

Theorem 4.11. Let $(U^V, U_0, \mathcal{N})$ be a neighborhood approximation space over two different universes U and V and $Y \in 2^V$. Then the followings hold:

- (1) $(\underline{\mathcal{N}}_n^{II}(Y), \overline{\mathcal{N}}_n^{II}(Y)) = (\underline{\mathcal{N}}(Y), \overline{\mathcal{N}}(Y))$ when $n = 0$, $\underline{\mathcal{N}}_n^{II}(Y) \subseteq \overline{\mathcal{N}}_n^{II}(Y)$, $\underline{\mathcal{N}}_n^{II}(\emptyset) = \overline{\mathcal{N}}_n^{II}(\emptyset) = 0$, $\underline{\mathcal{N}}_n^{II}(V) = \overline{\mathcal{N}}_n^{II}(V) = U_0$, and both $\underline{\mathcal{N}}_n^{II}$ and $\overline{\mathcal{N}}_n^{II}$ preserves orders, i.e. $\underline{\mathcal{N}}_n^{II}(Y) \subseteq \underline{\mathcal{N}}_n^{II}(Y_1)$ and $\overline{\mathcal{N}}_n^{II}(Y) \subseteq \overline{\mathcal{N}}_n^{II}(Y_1)$ ($(Y, Y_1) \in 2^V \times 2^V, Y \subseteq Y_1$) hold.
- (2) $\underline{\mathcal{N}}_n^{II}(Y) = \sim(\overline{\mathcal{N}}_n^{II}(\sim Y))$ and $\overline{\mathcal{N}}_n^{II}(Y) = \sim(\underline{\mathcal{N}}_n^{II}(\sim Y))$, where $\sim Y = V - Y$.
- (3) $\underline{\mathcal{N}}_n^{II}\left(\bigcap_{k \in K} Y_k\right) \subseteq \bigcap_{k \in K} \underline{\mathcal{N}}_n^{II}(Y_k)$ ($\{Y_k\}_{k \in K} \subseteq 2^V$) and $\overline{\mathcal{N}}_n^{II}\left(\bigcup_{k \in K} Y_k\right) \supseteq \bigcup_{k \in K} \overline{\mathcal{N}}_n^{II}(Y_k)$ ($\{Y_k\}_{k \in K} \subseteq 2^V$).
- (4) $\underline{\mathcal{N}}_n^{II}\left(\bigcup_{k \in K} Y_k\right) \supseteq \bigcup_{k \in K} \underline{\mathcal{N}}_n^{II}(Y_k)$ ($\{Y_k\}_{k \in K} \subseteq 2^V$) and $\overline{\mathcal{N}}_n^{II}\left(\bigcap_{k \in K} Y_k\right) \subseteq \bigcap_{k \in K} \overline{\mathcal{N}}_n^{II}(Y_k)$ ($\{Y_k\}_{k \in K} \subseteq 2^V$).
- (5) If $n \leq n_1$, then $\underline{\mathcal{N}}_n^{II}(Y) \subseteq \underline{\mathcal{N}}_{n_1}^{II}(Y)$ and $\overline{\mathcal{N}}_n^{II}(Y) \subseteq \overline{\mathcal{N}}_{n_1}^{II}(Y)$ hold.

Proof. We only prove that $\underline{\mathcal{N}}_n^{II}(Y) \subseteq \overline{\mathcal{N}}_n^{II}(Y)$ ($\forall Y \in 2^V$). Assume that $\exists u \in U_0$ and $u \in \underline{\mathcal{N}}_n^{II}(Y)$ but $u \notin \overline{\mathcal{N}}_n^{II}(Y)$. From Definition 4.8, $|\mathcal{N}(u) - Y| \leq n$, $|\mathcal{N}(u) \cap Y| > n$, and for any $u \in U_0$, $(\mathcal{N}(u) - Y) \cup (\mathcal{N}(u) \cap Y) = \mathcal{N}(u)$. Since $|\mathcal{N}(u) - Y| + |\mathcal{N}(u) \cap Y| = |\mathcal{N}(u)|$ and $|\mathcal{N}(u) - Y| = |\mathcal{N}(u)| - |\mathcal{N}(u) \cap Y|$. Then $|\mathcal{N}(u)| - |\mathcal{N}(u) \cap Y| \leq n$ and thus $|\mathcal{N}(u)| - n \leq n$. Consequently, $|\mathcal{N}(u)| > 2n$ and thus $n < |\mathcal{N}(u)|/2$. It is contradiction with the limits of n , so the element which only belongs to $\underline{\mathcal{N}}_n^{II}(Y)$ but does not belong to $\overline{\mathcal{N}}_n^{II}(Y)$ does not exist.

For more explanation to the assertion $\underline{\mathcal{N}}_n^{II}(Y) \subseteq \overline{\mathcal{N}}_n^{II}(Y)$ according to Definition 4.8 and Lemma 4.10 we invoke the following example:

Example 4.12 (Continued from Example 4.7). We know $n = \max[0, \min(|\mathcal{N}(u_i)|/2, |Y|)]$, $n \in N$. Then,

$k_1 = \max[0, \min(1, 2)] = \max[0, 1] = 0$. Similarity, we can obtain $k_2 = 0$, and $k_3 = 1$. According to k_i we have

$$|\mathcal{N}(x_2) - Y| = 1 > 0, |\mathcal{N}(x_4) - Y| = 1 > 0, |\mathcal{N}(x_5) - Y| = 2 > 1,$$

and

$$|\mathcal{N}(x_2) \cap Y| = 1 > 0, |\mathcal{N}(x_4) \cap Y| = 0 = 0, |\mathcal{N}(x_5) \cap Y| = 1 = 1.$$

Therefore, $\underline{\mathcal{N}}_n^{II}(Y) = \emptyset \subseteq \{x_2\} = \overline{\mathcal{N}}_n^{II}(Y)$.

Remark 4.13. From Example 4.12, we can conclude that the method based on Definition 4.8 and Lemma 4.10 is more efficient than that based on Definition 4.4.

Now, we explore by an algorithm how it is realize a computation to method of the grade and calculation all the approximations of the sets.

An algorithm 1 involves two closely integrated stages. The first stage (from line 3 to 5) is the core of the algorithm 1. It computes the grade n_i for any $u_i \in U_0$ by using Theorem 4.11. In the second stage (from line 6 to 10), it computes the approximations according to Definition 4.8.

Algorithm 1. An algorithm for computing approximations.

Input: $U_0 = \{u_1, u_2, u_3, \dots, u_p\} \subseteq U = \{u_1, u_2, u_3, \dots, u_n\}$,
 $V = \{v_1, v_2, v_3, \dots, v_m\}, \mathcal{N} : U_0 \rightarrow 2^V$
 is the neighborhood operator, $Y \in 2^V$ and $n = 0$;
Output: Approximations of Y with respect to $\mathcal{N}$;
 1. Let $\underline{\mathcal{N}}_n^{II}(Y) = \emptyset; \overline{\mathcal{N}}_n^{II}(Y) = \emptyset, \text{card}(Y) = |Y|$;
 2. **for** $i = 1$ to p **do**
 3. $\mathcal{N}(u_i) = \{v_i \in V \text{ s.t. } \mathcal{N}(u_i) \neq \emptyset\}$,
 4. $\text{card}(X_i) = |\mathcal{N}(u_i)|; n_i = \min(\text{card}(X_i)/2, \text{card}(Y))$;
 5. $n_i = \max[0, n_i]$;
 6. **if** $|\mathcal{N}(u_i) \cap Y| > n_i$ **then**
 7. $\underline{\mathcal{N}}_n^{II}(Y) = \underline{\mathcal{N}}_n^{II}(Y) \cup \{u_i\}$;
 8. **end**
 9. **if** $|\mathcal{N}(u_i) - Y| \leq n_i$ **then**
 10. $\overline{\mathcal{N}}_n^{II}(Y) = \overline{\mathcal{N}}_n^{II}(Y) \cup \{u_i\}$;
 11. **end**
 12. **end**
 13. **return** $\underline{\mathcal{N}}_n^{II}(Y), \overline{\mathcal{N}}_n^{II}(Y)$

4.2. An application on Type II graded n -neighborhood rough set over two different universes

In the fifth apartment, we supply the utility of our model with variable precision over two inconsistent and associated universes which be suggested inside the 2d apartment. In an emergency processing gadget, an emergency has a variety of attributes at the equal time and an answer of emergency may additionally be constituted by using numerous attributes. There is a relation between an emergency and the solution. It is hard for the supervisor to make a decision approximately which solution need to be chosen to process the emergency. By our model proposed on this paper issues may be solved within the emergency processing system. The model also may be used within the medical treatment system.

Assume that U_0 is a subset of U is a set about emergencies and V is the attributes set. For any emergency $u \in U$, some attributes in V and u are associated. For an emergency, if it belongs to the set $\mathcal{N}\text{-POS}(Y)$, the solution Y is suitable for the emergency. If it belongs to the set $\mathcal{N}\text{-BND}(Y)$, the solution Y is needed to be improved. If it belongs to set $\mathcal{N}\text{-NEG}(Y)$, the solution Y is completely not proper for the emergency. The manager can formulate an appropriate decision corresponding to the conclusion.

In the following example, we will show the above discussion:

Example 4.14. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of five emergencies, $U_0 = \{x_1, x_2, x_3\} \subseteq U$, $V = \{y_1, y_2, y_3, y_4\}$ be a set of four solutions, and $\mathcal{N} : U_0 \rightarrow 2^V$ defined as $\mathcal{N}(x_2) = \{y_1, y_4\}$, $\mathcal{N}(x_4) = \{y_1, y_3, y_4\}$, and $\mathcal{N}(x_5) = \{y_1, y_2, y_3\}$. Assume that $Y = \{y_1, y_2\}$ be denote two solutions and by Definition 4.8 and Lemma 4.10 (which $n = \max[0, \min(|\mathcal{N}(u_i)|/2, |Y|)]$, $n \in N$), we have

$\underline{\mathcal{N}}_n^{II}(Y) = \{x_2, x_3\}, \overline{\mathcal{N}}_n^{II}(Y) = U_0, \mathcal{N}\text{-POS}(Y) = \{x_2, x_3\}$,
 $\mathcal{N}\text{-NEG}(Y) = \emptyset$, and $\mathcal{N}\text{-BND}(Y) = \{x_1\}$.

Thus, we can obtain the subsequent conclusions based on the above demonstrations:

- (1) The manager can use the solution Y to process the emergency x_2 and x_3 .
- (2) If the manager want to use the solution Y to process the emergency x_1 , then solution Y is needed to be amended.

4.3. A matrix based on a neighborhood operator over two different universes

It is well-known that the computation of lower and upper approximations is not an easy task. To deal with this issue in the graded rough set model over two universes, we develop a novel method on the basis of relation matrix defined as following:

Definition 4.15. Let $U_0 = \{u_1, u_2, u_3, \dots, u_p\}$ is subset on a set $U = \{u_1, u_2, u_3, \dots, u_n\}$, $V = \{v_1, v_2, v_3, \dots, v_m\}$, (where p, n , and m are natural numbers), $X \in 2^U$, and $\mathcal{N} : U_0 \rightarrow 2^V$ is a neighborhood operator on two different universes U and V . Then

(1) The matrix $M_{\mathcal{N}} = (a_{ij})_{p \times m}$ induced by $\mathcal{N}$ is called the neighborhood matrix of $\mathcal{N}$, where for any i and j , $1 \leq i \leq p$ and $1 \leq j \leq m$. If $v_j \in \mathcal{N}(u_i) (\forall u_i \in U_0, \forall v_j \in V)$, then $a_{ij} = 1$, otherwise $a_{ij} = 0$.

(2) The summation of the i th row of $M_{\mathcal{N}}$ is defined as $\text{sum}(i) = \sum_{j=1}^m a_{ij}$, $1 \leq i \leq p$.

(3) The corresponding matrix of set X is defined as $X = (x_1, x_2, \dots, x_n)^T$, where T is the transposition of matrix.

Based on above definition we can present anew definition with respect to the graded n as follows:

Definition 4.16. Let $U_0 = \{u_1, u_2, u_3, \dots, u_p\}$ is subset on a set $U = \{u_1, u_2, u_3, \dots, u_n\}$, $V = \{v_1, v_2, v_3, \dots, v_m\}$ (where p, n , and m are natural numbers), $\mathcal{N} : U_0 \rightarrow 2^V$ is a neighborhood operator on two different universes U and V , and $M_{\mathcal{N}} = (a_{ij})_{p \times m}$ be the neighborhood matrix of $\mathcal{N}$. Given for any $Y \in 2^V$ (let $(x_1, x_2, \dots, x_m)^T$ be the matrix of Y) and $n \in N$, where N be the set of natural numbers. Suppose that $W = M_{\mathcal{N}} \cdot Y = (w_1, w_2, w_3, \dots, w_m)^T$, where $w_i = \sum_{j=1}^m a_{ij} \cdot u_j$ and $\cdot$ represents standard multiplication,

$$\underline{\mathcal{N}}_n^{III}(Y) = \left\{ u_i \in U_0 \mid \frac{\text{sum}(i) - w_i}{n} > n \right\}$$

and

$$\overline{\mathcal{N}}_n^{III}(Y) = \left\{ u_i \in U_0 \mid w_i - n \leq n \right\}$$

are called the lower $\mathcal{N}$ -approximation and the upper $\mathcal{N}$ -approximation of Y with respect to the graded n respectively, and $(\underline{\mathcal{N}}_n^{III}(Y), \overline{\mathcal{N}}_n^{III}(Y))$ is called the Type-III graded n -neighborhood rough approximation of Y on U and V (or the $\mathcal{N}$ -Type-III graded n -neighborhood rough approximation of Y , briefly, Type-III graded n -neighborhood rough set on two different universes U and V) with respect to the graded n . The subsets $\underline{\mathcal{N}}_n^{III}(Y) \cap \overline{\mathcal{N}}_n^{III}(Y)$, $\sim (\underline{\mathcal{N}}_n^{III}(Y) \cup \overline{\mathcal{N}}_n^{III}(Y))$, and $\sim [(\underline{\mathcal{N}}_n^{III}(Y) \cap \overline{\mathcal{N}}_n^{III}(Y)) \cup \sim (\underline{\mathcal{N}}_n^{III}(Y) \cup \overline{\mathcal{N}}_n^{III}(Y))]$ are called the $\mathcal{N}$ -positive region ($\mathcal{N}\text{-POS}$), the $\mathcal{N}$ -negative region ($\mathcal{N}\text{-NEG}$), and the $\mathcal{N}$ -boundary region ($\mathcal{N}\text{-BND}$), of Y with respect to n , respectively.

As a summary, we can present algorithm describe above definition as follows:

An algorithm 2 involves two closely integrated stages. In the first stage (from lines 1 to 2), it computes $M_{\mathcal{N}}$, sum and y_i according to Definition 4.15. This stage is the basis for the next stage. In the second stage (from lines 3 to 10), it computes approximations according to Definition 4.16.

Next, we apply the following example to show the Algorithm 2.

Algorithm 2. An algorithm for computing approximations.

Input: $U_0 = \{u_1, u_2, u_3, \dots, u_p\} \subseteq U = \{u_1, u_2, u_3, \dots, u_n\}$,
 $V = \{v_1, v_2, v_3, \dots, v_m\}$, $\mathcal{N} : U_0 \rightarrow 2^V$
 is the neighborhood operator, $Y \in 2^V$ ($x_1, x_2, \dots, x_m$)^T
 is the matrix of Y , and $n = 0$;

Output: Approximations of Y with respect to $\mathcal{N}$;

1. Let $\underline{\mathcal{N}}_n^{III}(Y) = \emptyset$; $\overline{\mathcal{N}}_n^{III}(Y) = \emptyset$;
2. Compute $M_{\mathcal{N}} = (a_{ij})_{p \times m}$ of $\mathcal{N}$, sum and W ;
3. **for** $i = 1$ to m **do**
4. **if** $\frac{sum(i) - w_i}{n} > n$ **then**
5. $\underline{\mathcal{N}}_n^{III}(Y) = \underline{\mathcal{N}}_n^{III}(Y) \cup \{u_i\}$;
6. **end**
7. **if** $w_i - n \leq n$ **then**
8. $\overline{\mathcal{N}}_n^{III}(Y) = \overline{\mathcal{N}}_n^{III}(Y) \cup \{u_i\}$;
9. **end**
10. **end**
11. **return** $\underline{\mathcal{N}}_n^{III}(Y), \overline{\mathcal{N}}_n^{III}(Y)$

Example 4.17. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $U_0 = \{x_1, x_2, x_3, x_4, x_5, x_6\} \subseteq U$, and $V = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}$. A neighborhood operator $\mathcal{N} : U_0 \rightarrow 2^V$ is defined as follows: $\mathcal{N}(x_1) = \{y_2, y_4, y_5\}$, $\mathcal{N}(x_2) = \{y_2, y_3, y_6, y_7\}$, $\mathcal{N}(x_3) = \{y_1, y_4, y_6, y_8\}$, $\mathcal{N}(x_4) = \{y_3, y_8\}$, $\mathcal{N}(x_5) = \{y_1, y_4, y_5, y_7\}$, and $\mathcal{N}(x_6) = \{y_2, y_5, y_8\}$. Consequently, $M_{\mathcal{N}} = (a_{ij})_{p \times m}$ be the neighborhood matrix of $\mathcal{N}$ is written as follows:

$$M_{\mathcal{N}} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

From $M_{\mathcal{N}}$, we can obtain on $sum(i) = \sum_{j=1}^m a_{ij}$ ($1 \leq i \leq 6, 1 \leq j \leq 8$) as $sum(1) = 3, sum(2) = 4, sum(3) = 4, sum(4) = 2, sum(5) = 4, sum(6) = 3$, and $sum = (3, 4, 4, 2, 4, 3)^T$.

Assume that $Y = \{y_1, y_3, y_4, y_5, y_8\}$. Then,

$$W = M_{\mathcal{N}} \cdot Y = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \\ 3 \\ 2 \end{pmatrix} = (w_1, w_2, w_3, w_4, w_5, w_6)^T.$$

Thus, $sum(1) - w_1 = 3 - 2 = 1$. Similarity, $sum(2) - w_2 = 3, sum(3) - w_3 = 1, sum(4) - w_4 = 0, sum(5) - w_5 = 1$ and $sum(6) - w_6 = 1$. Let $n = 2$. Then, we can obtain that $\underline{\mathcal{N}}_n^{III}(Y) = \emptyset$ and $\overline{\mathcal{N}}_n^{III}(Y) = U_0$.

4.4. An application on neighborhood matrix over two different universes

Assume that we are considering a certain group of sufferers in a medical analysis system. A sufferer may show numerous symptoms on the same time and a sickness may be observed by

means of numerous symptoms, so as to set up a relation among a sufferer and a sickness. In the current situation, how a specialized doctor could determine what kind of remedies that a patient must be taken based on the prevailing symptoms? This question can be responded through applying the neighborhood matrix model primarily founded on two universes to the medical analysis system.

Suppose that U and V are representing the set of patients and the set of symptoms, respectively, and U_0 is a subset of a set U . Then for any patient $u \in U_0$, there exist some symptoms in V correspond to u . For any $Y \in 2^V$, Y is a positive disorder which contains a few symptoms in V . Afterwards, given a patient, if she or he belongs to set $\mathcal{N}\text{-POS}(Y)$, then he or she is really suffered from the sickness denoted by using Y . Hence, all the sufferers belonging to $\mathcal{N}\text{-POS}(Y)$, are on appropriate for the remedy to Y right after diagnosis. In other words, if he or she belongs to set $\mathcal{N}\text{-BND}(Y)$, then his or her sickness is linked with Y ; however the connection isn't always sure. As a result, the doctor ought to further examine the pathogeny for the current sufferer and undertake the suitable treatment. Moreover, if he or she belongs to set $\mathcal{N}\text{-NEG}(Y)$, then his or her sickness has no connection with Y , and the specialized doctor should adopt other techniques.

In the following example, we will show the above discussion:

Example 4.18. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of five patients, $U_0 = \{x_1, x_2, x_4\} \subseteq U$, and $V = \{y_1, y_2, y_3, y_4\}$ be a set of four symptoms. Suppose that a neighborhood operator $\mathcal{N} : U_0 \rightarrow 2^V$ is defined as follows: $\mathcal{N}(x_1) = \{y_1, y_4\}$, $\mathcal{N}(x_2) = \{y_2, y_3, y_4\}$, and $\mathcal{N}(x_4) = \{y_1, y_2, y_3\}$. Then, $M_{\mathcal{N}} = (a_{ij})_{p \times m}$ (where if patient i has symptom j then $a_{ij} = 1$ else $a_{ij} = 0$) be the neighborhood matrix of $\mathcal{N}$ is written as follows:

$$M_{\mathcal{N}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

From $M_{\mathcal{N}}$, we can obtain on $sum(i) = \sum_{j=1}^m a_{ij}$ ($i = 1, 2, 4, 1 \leq j \leq 4$) as

$$sum(1) = 2, sum(2) = 3, sum(4) = 3,$$

and $sum = (2, 3, 3)^T$. Assume that $Y = \{y_2, y_4\} \in 2^V$ denote a certain disease, that is, Y shows two symptoms in the clinic. Then,

$$W = M_{\mathcal{N}} \cdot Y = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = (w_1, w_2, w_4)^T.$$

Thus,

$sum(1) - w_1 = 2 - 1 = 1$. Similarity, $sum(2) - w_2 = 1, sum(4) - w_4 = 2$. Let $n = 1$. Then, from Definition 4.16 we can obtain that

$$\underline{\mathcal{N}}_n^{III}(Y) = \{x_4\}, \overline{\mathcal{N}}_n^{III}(Y) = U_0, \mathcal{N}\text{-POS}(Y) = \{x_4\}, \mathcal{N}\text{-NEG}(Y) = \emptyset, \text{ and } \mathcal{N}\text{-BND}(Y) = \{x_1, x_2\}.$$

Thus, we can obtain the subsequent conclusions based on the above demonstrations:

- (1) It is certain that patient x_4 has sickness Y and the doctor ought to take the corresponding treatment immediately.

(2) The doctor cannot decide whether patients x_1 and x_2 have sickness Y or not based to the symptoms at present. The sufferers should be examined further.

(3) None of the three patients are healthy after diagnosis.

4.5. Comparison between our approach (i.e., Algorithms 1 and 2) and Liu et al.'s approach [45]

In this subsection, we compare the proposed approach (i.e., Algorithms 1 and 2) based on (Type II \ Type III) graded n-neighborhood rough set over two different universes and Liu et al.'s approach [45] in decision-making problem.

(a) Liu et al. [45] presented an approach using the graded rough set model based on two universes. We compare our approach with that of Liu et al.'s approach [45, Example 4].

Example 4.19 ([45, Example 4]). Let $U = \{x_1, x_2, x_3\}$ be the set of patients, $V = \{y_1, y_2, y_3, y_4\}$ the set of symptoms and $\mathcal{R} \subseteq U \times V$ is defined as follows:

$$r(x_2) = \{y_1, y_4\}, r(x_2) = \{y_1, y_2\}, \text{ and } r(x_3) = \{y_2, y_3, y_4\}.$$

By Algorithm 1, we can define $U_0 = \{x_2, x_3\} \subseteq U$ and $\mathcal{N} : U_0 \rightarrow 2^V$ defined as

$$\mathcal{N}(x_2) = \{y_1, y_2\} \text{ and } \mathcal{N}(x_3) = \{y_2, y_3, y_4\}.$$

Assume that $Y = \{y_2, y_4\}$ be denote a certain disease, that is, Y shows two symptoms in the clinic. By Definition 4.8 and Lemma 4.10 (which $n = \max[0, \min(|\mathcal{N}(u_i)|/2, |Y|)]$, $n \in \mathbb{N}$), we have

$$\underline{\mathcal{N}}_n^{II}(Y) = \{x_2\}, \overline{\mathcal{N}}_n^{II}(Y) = U_0, \mathcal{N}\text{-POS}(Y) = \{x_2\}, \mathcal{N}\text{-NEG}(Y) = \emptyset, \text{ and } \mathcal{N}\text{-BND}(Y) = \{x_3\}.$$

Thus, we can obtain the subsequent conclusions based on the above demonstrations:

(1) It is certain that patient x_2 has disease Y and the doctor should take the corresponding treatment immediately.

(2) The doctor cannot decide whether patient x_3 has disease Y or not according to the symptoms at present. The patient should be examined further.

(3) None of the three patients is healthy after diagnosis.

From above discussion, we can find the the conclusions (1)-(3) by our developed approach (i.e., based on Type II graded n-neighborhood rough set) are in difference with the decision results of proposed by Liu et al.'s approach in [45], which the Example 4 in [45] depend on the following property $\overline{\mathcal{R}}(Y) = \{x_3\} \subseteq \{x_1, x_2, x_3\} = \mathcal{R}(Y)$ (i.e., it's mean the upper approximation can't contain the lower approximation) which implies that it is a contradiction with Pawlak's rough set theory [1] (i.e., $\underline{\mathcal{R}}(Y) \subseteq \overline{\mathcal{R}}(Y)$). This mean there is an error in the approach proposed in [45] based on graded rough set model, which led to the difference in the decision-making results. Therefore, our proposed decision-making approach can successfully avoid and solve the above issue.

(b) Liu et al. [45] developed a novel method on the basis of relation matrix as defined in [45, Definitions 5 and 6]. We compare our approach with that of Liu et al.'s approach [45, Example 2].

Example 4.20 ([45, Example 2]). Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $V = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$. By Algorithm 2,

we can define $U_0 = \{x_1, x_2, x_3, x_4, x_5\} \subseteq U$, A neighborhood operator $\mathcal{N} : U_0 \rightarrow 2^V$ is defined as follows: $\mathcal{N}(x_1) = \{y_2, y_4, y_5\}$, $\mathcal{N}(x_2) = \{y_2, y_3, y_6, y_7\}$, $\mathcal{N}(x_3) = \{y_1, y_4, y_6\}$, $\mathcal{N}(x_4) = \{y_3, y_6\}$, and $\mathcal{N}(x_5) = \{y_1, y_4, y_5, y_7\}$. Consequently, $M_{\mathcal{N}} = (a_{ij})_{p \times m}$ be the neighborhood matrix of $\mathcal{N}$ is written as follows:

$$M_{\mathcal{N}} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

From $M_{\mathcal{N}}$, we can obtain on $sum(i) = \sum_{j=1}^m a_{ij}$ ($1 \leq i \leq 5$, $1 \leq j \leq 7$) as $sum(1) = 3$, $sum(2) = 4$, $sum(3) = 3$, $sum(4) = 2$, $sum(5) = 4$, and $sum = (3, 4, 3, 2, 4)^T$.

Assume that $Y = \{y_2, y_3, y_4\}$. Then,

$$W = M_{\mathcal{N}} \cdot Y = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} = (w_1, w_2, w_3, w_4, w_5)^T.$$

Thus, $sum(1) - w_1 = 3 - 2 = 1$. Similarity, $sum(2) - w_2 = 2$, $sum(3) - w_3 = 2$, $sum(4) - w_4 = 1$, and $sum(5) - w_5 = 3$. Let $n = 2$. Then, we can obtain that $\underline{\mathcal{N}}_n^{III}(Y) = \emptyset$ and $\overline{\mathcal{N}}_n^{III}(Y) = U_0$.

From above example, we can find the $\underline{\mathcal{N}}_n^{III}(Y) = \emptyset \subseteq U_0 = \overline{\mathcal{N}}_n^{III}(Y)$ by our developed approach (i.e., based on Type III graded n-neighborhood rough set) is in difference with the decision results of proposed by Liu et al.'s approach in [45], which the Example 2 in [45] depend on the following property $\overline{\mathcal{R}}(Y) = \{u_i \in U \mid w_i > 2\} = \{x_2, x_4\} \subseteq \{x_1, x_2, x_4\} = \{u_i \in U \mid sum(i) - w_i \leq 2\} = \mathcal{R}(Y)$ (i.e., it's mean the upper approximation can't contain the lower approximation) which implies that it is a contradiction with Pawlak's rough set theory [1] (i.e., $\underline{\mathcal{R}}(Y) \subseteq \overline{\mathcal{R}}(Y)$). This mean there is an error in the approach proposed in [45] based on relation matrix of $\mathcal{R}$, which led to the difference in the decision-making results. Therefore, our proposed decision-making approach can successfully avoid and solve the above issue.

5 Conclusions

This study aims to propose a new notion of neighborhood rough set on universe set and the interesting properties of neighborhood rough sets are discussed. Several notions of covering rough sets to neighborhood rough sets with respect to the graded n are generalized. We also proved a graded rough sets based on two universes are neighborhood rough sets. Furthermore, we extend the classical probabilistic rough sets to probabilistic rough sets based on neighborhood operator and investigate some basic of their properties of the probabilistic neighborhood rough sets.

Additionally, we propose some notions such as (Type-I / Type-II) probabilistic neighborhood rough approximations of X with error α and β and (Type-I / Type-II) probabilistic neighborhood rough approximations of X with respect to $\mathcal{N}$. Besides defining the notion of rough set based on neighborhood operator over two different universes U and V , three notions (Type-I / Type-II / Type-III) graded n -neighborhood rough sets so far are proposed. The basic properties of above notions are then explored. Finally, we give two applications based on (Type-II / Type-III) graded n -neighborhood rough sets over two different universes for a rough decision-making problem.

In the future, we can extend the results of this study as new notions to define the properties of the previous studies [13-15, 21-25, 32, 34, 38-40, 50-56]. In addition to, one may investigate further based on neighborhood operator with some links to topology.

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Compliance with ethical standards

Conflict of interest The authors declare that they has no conflict of interest.

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