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COLLOCATION METHOD FOR FUZZY VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND WITH WEAKLY SINGULAR KERNELS

ZAHRA ALIJANI[†] * AND URVE KANGRO*

ABSTRACT. In this paper, we consider fuzzy Volterra integral equation of the second kind with a weakly singular kernel. We propose piecewise spline collocation methods with a graded mesh. By increasing the number of collocation points we show that the numerical solution exists and converges to the exact solution. We obtain exact convergence values depending on the smoothness of the solution and on the grading parameter of the mesh. We give sufficient conditions for the fuzziness of the approximate solution. The proposed method is illustrated by numerical examples which confirm the theoretical convergence estimates.

1. INTRODUCTION

Integral equations and their solutions play a significant role in science and engineering. Many important physical problems can be modeled by using integral or differential equations. Only a few of them can be solved explicitly, so it is necessary to engage numerical methods to obtain approximate solutions [7].

The theory of Volterra integral equations is thoroughly considered by Brunner [6]. Weakly singular integral equations have received considerable interest in the mathematical literature, because of their applications in many fields of science such as fractional differential equations and the physical problems with heredity and memory properties [6, 19].

Usually, Volterra integral equations can not be solved analytically. Hence numerical methods for these equations are needed. The simplest and at the same time robust methods which are all to deal with the nonsmooth solution are collocation methods with piecewise polynomials [2, 7].

Many natural phenomena are described by mathematical models. It is not always possible to find a deterministic model for such problems. In 1965, Zadeh introduced the concept of fuzzy sets. With time, this concept found interesting applications in the modeling of many real-world problems and it motivated many researchers to develop the area of fuzzy integral equations where the values of functions involved are fuzzy numbers. The

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topics related to fuzzy integral equations have received particular attention from the research community during the last few decades [4, 10, 17, 18].

A fuzzy Volterra integral equation of the second kind with the weakly singular kernel (FVIEW) is given by

$$(1.1) \quad g(t) = f(t) + \int_0^t K(t, s)g(s)ds, \quad t \in [0, T],$$

where $K: D_T \rightarrow \mathbb{R}$ is a weakly singular kernel with domain $D_T = \{(t, s): 0 \leq s < t \leq T\}$, $T \in \mathbb{R}$, f is a given fuzzy function and g is an unknown fuzzy function. The kernel K or its derivatives may have some singularities at $t = s$. If f is a real-valued function then equation (1.1) possesses a real-valued solution and if f is a fuzzy-number valued function then the solution is fuzzy.

In our previous paper [3] we investigated fuzzy Volterra integral equations with weakly singular kernels. We obtained the existence and uniqueness of solutions and then we showed that the corresponding solution is a fuzzy function that satisfies equation (1.1). We proved the smoothness of the solution, assuming that the sign of the kernel can change only along the horizontal and vertical lines.

In current work we use fully discretized collocation methods on piecewise polynomial spaces for solving the corresponding system of Volterra integral equations. We provide the conditions for the fuzziness of the numerical solutions. Based on smoothness results we obtain the convergence analysis.

The paper is organized as follows. Section 2 introduces some preliminaries about fuzzy functions which are necessary for the next sections. In section 3 we describe the parametric and operator form of the integral equation. Section 4 consists of existence and uniqueness results for the parametric form. In Section 5 we describe the collocation method and fully discretized collocation method on the discontinuous piecewise polynomial spaces. In section 6 we prove the convergence results. Some examples are presented in section 7 to illustrate our result.

2. PRELIMINARIES

In this section, we review the fundamental notions and definitions that will be used throughout the paper.

2.1. Fuzzy numbers and fuzzy functions. Fuzzy sets $\mathbb{R}$ (generally on $\mathbb{R}^n$, $n \geq 1$) were first introduced in [21]. Fuzzy numbers are particular fuzzy sets, identified with some additional properties. Since we only use the parametric form of the fuzzy numbers in this paper, we only introduce this form. The parametric form was introduced in [12], where also the equivalence of the standard definition and the parametric form was proved.

Definition 2.1. A fuzzy number u on $\mathbb{R}$ in parametric form is an ordered pair $u = (\underline{u}, \overline{u})$ of two real functions $\underline{u}, \overline{u} : [0, 1] \rightarrow \mathbb{R}$, that satisfy the following requirements:

- (1) $\underline{u}$ is bounded, non-decreasing, left continuous on $(0, 1]$ and right continuous at 0,
- (2) $\overline{u}$ is bounded, non-increasing, left continuous on $(0, 1]$ and right continuous at 0,
- (3) $\underline{u}(1) \leq \overline{u}(1)$.

For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ and $k \in \mathbb{R}$ the addition and multiplication by k are defined as

$$u + v = (\underline{u} + \underline{v}, \overline{u} + \overline{v}),$$

$$ku = \begin{cases} (k\underline{u}, k\overline{u}), & \text{if } k \geq 0, \\ (k\overline{u}, k\underline{u}), & \text{if } k < 0. \end{cases}$$

The set of all fuzzy numbers is denoted by E . Note that E is not a vector space, because $u + (-u) \neq 0$ in general.

Next we will define the metric D in E .

Definition 2.2. For arbitrary fuzzy numbers u, v , we use the distance

$$D(u, v) = \sup_{0 \leq r \leq 1} \max\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\}.$$

It is shown that (E, D) is a complete metric space [5].

Following Goetschel and Voxman [12] we define the integral of a fuzzy function using the Riemann integral concept.

Definition 2.3. Let $f : [a, b] \rightarrow E$. For each partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$ and for arbitrary $\xi_i \in [t_{i-1}, t_i]$, $1 \leq i \leq n$ suppose

$$R_P = \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}), \quad \Delta := \max\{t_i - t_{i-1}, i = 1, \dots, n\}.$$

The definite integral of $f(t)$ over $[a, b]$ is $\int_a^b f(t)dt = \lim_{\Delta \rightarrow 0} R_P$ provided this limit exists in metric D .

If the fuzzy function $f(t)$ is continuous in the metric D , its definite integral exists and

$$(2.1) \quad \int_a^b f(t)dt = \left(\int_a^b \underline{f}(t, r)dt, \int_a^b \overline{f}(t, r)dt \right),$$

where $(\underline{f}(t, r), \overline{f}(t, r))$ is the parametric form of $f(t)$. In the literature, weak singularity of the kernel K may have different definitions. We follow here the definition of [20], where it was introduced for Fredholm integral equations.

Definition 2.4. For given $m \in \mathbb{N}_0$, denote by $S^{m, \alpha} = S^{m, \alpha}(D_T)$ the set of m times continuously differentiable functions K on D_T that satisfy there for all $j, l \in \mathbb{N}_0$, $j + l \leq m$, the inequality

(2.2)

$$\left| \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^l K(t, s) \right| \leq C_{K, m} \begin{cases} 1 & \text{if } j + \alpha < 0, \\ 1 + |\log(t - s)| & \text{if } j + \alpha = 0, \\ (t - s)^{-j - \alpha} & \text{if } j + \alpha > 0. \end{cases}$$

A kernel $K \in S^{m,\alpha}$ is called weakly singular if $\alpha < 1$.

To describe the smoothness of the solution of (1.1) we need the following space of functions.

Definition 2.5. [20] For $m \in \mathbb{N}_0$, $\alpha < 1$, denote by $\mathcal{C}^{m,\alpha}(0, T]$, the space of functions $v \in \mathcal{C}^m(0, T]$, that satisfy the inequalities

$$(2.3) \quad |v^{(i)}(t)| \leq c \begin{cases} 1 & \text{if } i < 1 - \alpha, \\ 1 + |\log(t)| & \text{if } i = 1 - \alpha, \\ t^{1-\alpha-i} & \text{if } i > 1 - \alpha, \end{cases}$$

where $c = c(v)$, for all $t \in (0, T]$ and $i = 0, \dots, m$.

For $\alpha \in \mathbb{R}$ we define the weight function

$$(2.4) \quad |\omega_\alpha(t)| = \begin{cases} 1 & \text{if } \alpha < 0, \\ (1 + |\log(t)|)^{-1} & \text{if } \alpha = 0, \\ t^\alpha & \text{if } \alpha > 0. \end{cases}$$

Then $\mathcal{C}^{m,\alpha}(0, T]$, equipped with the norm

$$\|v\|_{m,\alpha} := \sum_{k=0}^m \sup_{0 < t \leq T} \omega_{k-1+\alpha}(t) |v^{(k)}(t)|,$$

becomes a Banach space and for $m \geq 1$.

3. PARAMETRIC AND OPERATOR FORM OF THE INTEGRAL EQUATION

Let $(\underline{f}(t, r), \overline{f}(t, r))$ and $(\underline{g}(t, r), \overline{g}(t, r))$, $(t, r) \in [0, T] \times [0, 1]$ be parametric forms of $f(t)$ and $g(t)$. Then equation (1.1) can be rewritten as a system of Volterra integral equations:

$$(3.1) \quad \begin{cases} \underline{g}(t, r) = \underline{f}(t, r) + \int_0^t (K_+(t, s)\underline{g}(s, r) - K_-(t, s)\overline{g}(s, r))ds, \\ \overline{g}(t, r) = \overline{f}(t, r) + \int_0^t (K_+(t, s)\overline{g}(s, r) - K_-(t, s)\underline{g}(s, r))ds, \end{cases}$$

where

$$K_+(t, s) = \begin{cases} K(t, s), & K(t, s) \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$K_-(t, s) = \begin{cases} -K(t, s), & K(t, s) \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We write this system as

$$(3.2) \quad G = F + \mathcal{K}G,$$

where $G = [g_1, g_2]^T$, $g_1 = \underline{g}$, $g_2 = \overline{g}$, $F = [f_1, f_2]^T$, $f_1 = \underline{f}$, $f_2 = \overline{f}$ and

$$(3.3) \quad \mathcal{K} = \begin{pmatrix} \mathcal{K}_{\alpha_+} & -\mathcal{K}_{\alpha_-} \\ -\mathcal{K}_{\alpha_-} & \mathcal{K}_{\alpha_+} \end{pmatrix}.$$

We also use the notation

$$(3.4) \quad \mathcal{K}G = \int_0^t \mathbf{K}(t, s)G(s, r)ds,$$

where

$$\mathbf{K}(t, s) = \begin{pmatrix} K_+(t, s) & -K_-(t, s) \\ -K_-(t, s) & K_+(t, s) \end{pmatrix}.$$

We call the vector G a fuzzy function if (g_1, g_2) is a fuzzy function.

In [3] we proved the existence, uniqueness and smoothness of the solution. Here we review them.

4. EXISTENCE, UNIQUENESS AND SMOOTHNESS OF THE SOLUTION

In [3] we suppose the kernel changes sign along the vertical and/or horizontal lines $s = a_i$ and/or $t = a_i$, $i = 1, \dots, n$, $0 < a_1 < a_2 < \dots < a_n < T$. Denote

$$D_{\{a_1, \dots, a_n\}} = D_T \setminus \bigcup_{i=1}^n (\{s = a_i\} \cup \{t = a_i\}).$$

Define $S^{m, \alpha}(D_{\{a_1, \dots, a_n\}})$ as the collection of m times continuously differentiable functions K on $D_{\{a_1, \dots, a_n\}}$ that satisfy inequality (2.2) for all $j, l \in \mathbb{N}_0$, $j + l \leq m$ and $(t, s) \in D_{\{a_1, \dots, a_n\}}$. Without loss of generality we assumed there is only one vertical and/or horizontal line of sign change of K . Denote $d = a_1$ and $D_d = D_T \setminus (\{s = d\} \cup \{t = d\})$. In order to characterize the growth rates of the derivatives of the function $u(t)$ as $t \rightarrow d$, Pedas et al. [15] introduced also the weight functions

$$(4.1) \quad |\omega_\alpha^{(d)}(t)| = \begin{cases} 1 & \text{if } \alpha < 0, \\ (1 + |\log(\rho_d)|)^{-1} & \text{if } \alpha = 0, \\ \rho_d(t)^\alpha & \text{if } \alpha > 0, \end{cases}$$

where $t \in \mathcal{G}_d$ and $\rho_d = |t - d|$.

We denote by $\mathcal{C}_d^{m, \alpha, p}(0, T]$ the Banach space of functions $u \in \mathcal{C}^m((0, T] \setminus \{d\}) \cap \mathcal{C}^p(0, T]$ such that

$$(4.2) \quad \|u\|_{d, m, \alpha, p} = \sum_{j=0}^m \sup_{t \in \mathcal{G}_d} e(t) \omega_{j+\alpha-1}(t) |u^{(j)}(t)| + \sum_{j=0}^m \sup_{t \in \mathcal{G}_d} (1 - e(t)) \omega_{j+\alpha-1-p}(t-d) |u^{(j)}(t)| < \infty,$$

where ω is defined in (2.4).

In [3] we provided some results for the case when the lines of sign change can only be vertical and/or horizontal lines.

Theorem 4.1. *Let $K_+, K_- \in S^{m, \alpha}(D_d) \cap \mathcal{C}^{p-1}(D_T)$ where $m, p \in \mathbb{N}$, $p \leq m$, $\alpha \in \mathbb{R}$, $\alpha < 1$. Let $F \in (\mathcal{C}_d^{m, \alpha, p}(0, T])^2$ be given. Then the equation (3.2) has a unique solution in $(\mathcal{C}_d^{m, \alpha, p}(0, T])^2$.*

5. COLLOCATION METHODS

5.1. Collocation method on the discontinuous piecewise polynomial spaces. Define a mesh on $[0, T]$ by

$$\triangle_h := \{t_n : 0 = t_0 < t_1 < \dots < t_N = T\}.$$

Let $\sigma_n := (t_n, t_{n+1}]$, $\bar{\sigma}_n := [t_n, t_{n+1}]$, $h_n = t_{n+1} - t_n$ ($n = 0, 1, \dots, N-1$) and let the diameter of the mesh be $h = \max\{h_n : 0 \leq n \leq N-1\}$. In the following we mainly use the graded mesh, where the mesh points are defined by

$$(5.1) \quad t_j := T \left(\frac{j}{N} \right)^\rho, \quad j = 0, \dots, N.$$

Here $\rho \geq 1$ is called the grading parameter.

Define the piecewise polynomial space which we use in this thesis as follows:

$$\mathcal{S}_{m-1}^{(-1)}(\triangle_h) := \{v : v|_{\sigma_n} \in \pi_{m-1} \ (n = 0, 1, \dots, N-1)\},$$

where π_{m-1} are polynomials of degree not exceeding $m-1$. Any $v_N \in (\mathcal{S}_{m-1}^{(-1)}(\triangle_h))$ can be determined by

$$(5.2) \quad v_N(t)|_{\sigma_n} = \sum_{j=1}^m L_j(\tau) V_{n,j}, \quad t = t_n + \tau h_n,$$

where $V_{n,i} := v_N(t_{n,i})$, $0 \leq c_1 < \dots < c_m \leq 1$ are the collocation parameters, $t_{n,i} := t_n + c_i h_n$ for $n = 0, \dots, N-1$, and

$$L_j(\tau) := \prod_{k=1, \dots, m, k \neq j} \frac{\tau - c_k}{c_j - c_k}, \quad j = 1, \dots, m, \quad \tau \in [0, 1],$$

are the Lagrange fundamental polynomials on $[0, 1]$.

For fixed $r \in [0, 1]$ we look for approximate solution of equation (3.2) as a spline $u_N \in (\mathcal{S}_{m-1}^{(-1)}(\triangle_h))^2$. We require that the equation is exactly satisfied at collocation points $t_{n,i}$. Then we get the linear system of equations

$$(5.3) \quad u_N(t_{n,i}, r) = F(t_{n,i}, r) + (\mathcal{K}u_N)(t_{n,i}, r).$$

for determining $u_N(t_{n,i}, r)$. By partitioning the integration interval in equation (3.2) we obtain

$$(5.4) \quad \begin{aligned} u_N(t_{n,i}, r) = & F(t_{n,i}, r) + \sum_{l=0}^{n-1} h_l \int_0^1 \mathbf{K}(t_{n,i}, t_l + zh_l) u_N(t_l + zh_l, r) dz \\ & + h_n \int_0^{c_i} \mathbf{K}(t_{n,i}, t_n + zh_n) u_N(t_n + zh_n, r) dz. \end{aligned}$$

Denote $U_{n,i}(r) = u_N(t_{n,i}, r)$. Note that we can solve the equations on each interval σ_n separately, so when solving for $U_{n,i}(r)$ for fixed n , we can consider

$U_{l,j}$ with $l < n$ as known. By substituting (5.2) into (5.4), we obtain

$$(5.5) \quad \begin{aligned} U_{n,i}(r) = & h_n \sum_{j=1}^m \int_0^{c_i} \mathbf{K}(t_{n,i}, t_n + zh_n) L_j(z) dz U_{n,j}(r) \\ & + F(t_{n,i}, r) + \sum_{l=0}^{n-1} \sum_{j=1}^m h_l \int_0^1 \mathbf{K}(t_{n,i}, t_l + zh_l) L_j(z) dz U_{l,j}(r). \end{aligned}$$

For fixed n we obtain a linear system of $2m$ equations and $2m$ unknowns

$$(5.6) \quad \begin{aligned} U_{n,i}(r) = & h_n \sum_{j=1}^m Q_{n,i,j} U_{n,j}(r) \\ & + F(t_{n,i}, r) + \sum_{l=0}^{n-1} \sum_{j=1}^m h_l R_{n,l,i,j} U_{l,j}(r), \quad i = 1, \dots, m, \end{aligned}$$

where

$$(5.7) \quad Q_{n,i,j} = \int_0^{c_i} \mathbf{K}(t_{n,i}, t_n + zh_n) L_j(z) dz$$

and

$$(5.8) \quad R_{n,l,i,j} = \int_0^1 \mathbf{K}(t_{n,i}, t_l + zh_l) L_j(z) dz.$$

5.2. The fully discretized collocation method. To describe the fully discretized collocation method we make an additional assumption:

$$(5.9) \quad K(t, s) = k(t, s) p_\alpha(t - s), \quad p_\alpha(t) = \begin{cases} t^{-\alpha}, & \alpha < 1, \alpha \neq 0, \\ \log(t), & \alpha = 0, \end{cases}$$

where $k \in \mathcal{C}^m(\overline{D_T})$ and $\alpha < 1$. Then $K \in S^{m,\alpha}$. Assume that the lines of sign change of K are only at horizontal and/or vertical lines $t = t_j$ or $s = t_j$ for some $j \in \{1, \dots, N-1\}$. Then all integrals in (5.4) are either zero or of the form

$$\int_0^a k(t_{n,i}, t_l + zh_l) p_\alpha(t_{n,i} - (t_l + zh_l)) v(t_l + zh_l) dz$$

with $a = 1$ or $a = c_i$. We approximate these integrals by product quadrature rule with the mesh $\{c_1, \dots, c_m\}$ as follows

$$\int_0^{c_i} k(t_{n,i}, t_n + zh_n) p_\alpha(t_{n,i} - (t_n + zh_n)) v(t_n + zh_n) dz \approx \sum_{j=1}^m w_{n,i,j} k(t_{n,i}, t_n + c_j h_n) V_{n,j}$$

and

$$\int_0^1 k(t_{n,i}, t_l + zh_l) p_\alpha(t_{n,i} - (t_l + zh_l)) v(t_l + zh_l) dz \approx \sum_{j=1}^m w_{n,l,i,j} k(t_{n,i}, t_l + c_j h_l) V_{l,j},$$

where

$$w_{n,i,j} := \int_0^{c_i} p_\alpha((c_i - z)h_n) L_j(z) dz$$

and

$$w_{n,l,i,j} := \int_0^1 p_\alpha(t_{n,i} - (t_l + zh_l)) L_j(z) dz.$$

For fixed r we look for approximate solution of equation (3.2) as a spline $\hat{u}_N(\cdot, r) \in (\mathcal{S}_{m-1}^{(-1)}(\Delta_h))^2$ and denote $\hat{U}_{n,i}(r) = \hat{u}_N(t_{n,i}, r)$. Then for determining $\hat{U}_{n,i}(r)$ we get the following linear system of equations

$$(5.10) \quad \begin{aligned} \hat{U}_{n,i}(r) = & h_n \sum_{j=1}^m \hat{Q}_{n,i,j} \hat{U}_{n,j}(r) \\ & + F(t_n + c_i h_n, r) + \sum_{l=0}^{n-1} \sum_{j=1}^m h_l \hat{R}_{n,l,i,j} \hat{U}_{l,j}(r), \quad i = 1, \dots, m, \end{aligned}$$

where

$$\hat{Q}_{n,i,j} = \mathbf{k}(t_n + c_i h_n, t_n + c_j h_n) w_{n,i,j}$$

and

$$\hat{R}_{n,l,i,j} = \mathbf{k}(t_n + c_i h_n, t_l + c_j h_l) w_{n,l,i,j},$$

where

$$\mathbf{k}(t, s) = \begin{pmatrix} k(t, s) & 0 \\ 0 & k(t, s) \end{pmatrix} \quad \text{or} \quad \mathbf{k}(t, s) = \begin{pmatrix} 0 & -k(t, s) \\ -k(t, s) & 0 \end{pmatrix},$$

depending on whether $K(t, s)$ is positive or negative in $\sigma_n \times \sigma_l$. Then the approximate solution can be written as

$$(5.11) \quad \hat{u}_N(t, r)|_{\sigma_n} = \sum_{j=1}^m L_j(z) \hat{U}_{n,j}(r), \quad t = t_n + zh_n, \quad z \in (0, 1].$$

6. CONVERGENCE

6.1. Convergence estimates for the collocation method. We denote by p_N the interpolation projector onto the set of all piecewise polynomial functions on $[0, T]$ which are real polynomials of degree not exceeding $m-1$ on every interval $[t_j, t_{j+1}]$, $0 \leq j \leq N-1$, where the interpolation points are defined by $t_{n,i} = t_n + c_i h_n$, $0 \leq c_1 < \dots < c_m \leq 1$. The approximation properties of $p_N u$ on graded mesh (5.1) are considered in [13, 16, 19]. These results can be summarized as follows.

Lemma 6.1. [16] *Assume that $u \in \mathcal{C}^{m,\alpha}(0, T]$ and the graded mesh (5.1) with grading parameter ρ is used. Then the following estimates hold where the constant C does not depend on N :*

$$(6.1) \quad \max_{t \in [0, T]} |u(t) - (p_N u)(t)| \leq C \|u\|_{m,\alpha} E(N, m, \rho, \alpha),$$

where

$$(6.2) \quad E(N, m, \rho, \alpha) = \begin{cases} N^{-m}, & \text{for } m < 1 - \alpha, \rho \geq 1, \\ N^{-m}(|\log N| + 1), & \text{for } m = 1 - \alpha, \rho = 1, \\ N^{-m}, & \text{for } m = 1 - \alpha, \rho > 1, \\ N^{-\rho(1-\alpha)}, & \text{for } m > 1 - \alpha, 1 \leq \rho < \frac{m}{1-\alpha}, \\ N^{-m}, & \text{for } m > 1 - \alpha, \rho \geq \frac{m}{1-\alpha}. \end{cases}$$

Remark 6.2. If $u \in \mathcal{C}_d^{m, \alpha, p}(0, T]$, then we can use different graded meshes on $[0, d]$ and $[d, T]$, possibly with different grading parameters, and use Lemma 6.1 separately on these intervals.

In the consequent theorems we present the convergence result for fuzzy weakly singular integral equation.

Define an interpolation projector

$$\mathbf{p}_N : (\mathcal{C}[0, T])^2 \mapsto (\mathcal{S}_{m-1}^{(-1)}(\Delta_h))^2, \quad m, N \in \mathbb{N},$$

by

$$(6.3) \quad (\mathbf{p}_N v)(t_{n,i}) = v(t_{n,i}), \quad i = 1, \dots, m, \quad n = 0, \dots, N,$$

for any continuous function $v \in (\mathcal{C}(0, T))^2$.

Let $r \in [0, 1]$ be fixed. Then the system (5.6) can be replaced by an operator equation of the form

$$(6.4) \quad u_N(t, r) = \mathbf{p}_N F(t, r) + \mathbf{p}_N \mathcal{K} u_N(t, r).$$

Proposition 6.3. Assume that $F(\cdot, r) \in (\mathcal{C}^{m, \alpha}(0, T))^2$ for all $r \in [0, 1]$ and $\|F(\cdot, r)\|_{m, \alpha}$ are uniformly bounded with respect to r . Let $K_+, K_- \in S^{m, \alpha}$ with $m \geq 1, \alpha < 1$. Let G be the unique solution of the system (3.2). Let the graded mesh (5.1) with grading parameter ρ be used. Then

$$(6.5) \quad \sup_{r \in [0, 1]} \|G(\cdot, r) - \mathbf{p}_N G(\cdot, r)\|_\infty \leq \text{const} E(N, m, \rho, \alpha).$$

Proof. By Theorem 4.6 in [3] the operator $(I - \mathcal{K})$ is invertible in $(\mathcal{C}^{m, \alpha}(0, T))^2$. Since $F(\cdot, r) \in (\mathcal{C}^{m, \alpha}(0, T))^2$, we get $\|G(\cdot, r)\|_{m, \alpha} \leq \|(I - \mathcal{K})^{-1}\| \|F(\cdot, r)\|_{m, \alpha}$. Then by Lemma 6.1 we get (6.5). $\square$

In general case when the kernel K changes sign, the assumptions of the previous proposition may be too restrictive. Then we have the following result.

Proposition 6.4. Assume that $F(\cdot, r) \in (\mathcal{C}^{m, \alpha}(0, T))^2$ for all $r \in [0, 1]$ and $\|F(\cdot, r)\|_{m, \alpha}$ are uniformly bounded with respect to r . Let $K_+, K_- \in S^{m, \alpha}(D_d) \cap \mathcal{C}^{p-1}(D_T)$ with $m \geq 1, \alpha < 1, 1 \leq p \leq m$. Let G be the unique solution of the system (3.2). Let two graded meshes on $[0, d]$ and on $[d, T]$

with numbers of intervals N_1 and N_2 and grading parameters ρ_1 and ρ_2 be used. Then

$$(6.6) \quad \sup_{r \in [0,1]} \|G(\cdot, r) - \mathbf{p}_N G(\cdot, r)\|_\infty \leq \text{const} \max\{E(N_1, m, \rho_1, \alpha), E(N_1, m, \rho_2, \alpha - p)\}.$$

Proof. The proof is similar to the proof of the previous proposition, only we use Theorem 4.9 in [3] and the space is $(\mathcal{C}_d^{m, \alpha, p}(0, T))^2$. $\square$

Remark 6.5. Similar results hold when there are more lines of sign change of K .

We use the following general theorem about interpolation operator p_N .

Lemma 6.6. [8] Let $\mathcal{T}: \mathcal{L}^\infty(0, T) \rightarrow \mathcal{C}[0, T]$ be a linear compact operator. Let $p_N: \mathcal{C}[0, T] \rightarrow \mathcal{S}_{m-1}^{(-1)}(\Delta_h)$ be the interpolation operator with graded mesh (5.1). Then

$$\|\mathcal{T} - p_N \mathcal{T}\|_{\mathcal{L}(\mathcal{L}^\infty(0, T), \mathcal{L}^\infty(0, T))} \rightarrow 0, \text{ as } N \rightarrow \infty.$$

To establish convergence order we can use following theorem.

Theorem 6.7. Let X, X' be Banach spaces and $X' \subset X$. Assume $\mathcal{T}: X \rightarrow X'$ is bounded and $I - \mathcal{T}: X \rightarrow X$, is bijective operator. Further, assume

$$\|\mathcal{T} - p_N \mathcal{T}\| \rightarrow 0, \text{ as } N \rightarrow \infty,$$

where $p_N: X' \rightarrow X, N = 1, 2, \dots$ are bounded linear operators. Then for all sufficiently large N (say $N > N_0$) the operator $(I - p_N \mathcal{T})$ is invertible in X and

$$\sup_{N > N_0} \|(I - p_N \mathcal{T})^{-1}\| < \infty.$$

For the solutions of $x_N = p_N \mathcal{T} x_N + p_N f$ and $x = \mathcal{T} x + f$,

$$c_1 \|x - p_N x\| \leq \|x - x_N\| \leq c_2 \|x - p_N x\|,$$

where c_1 and c_2 are positive constants.

Proof. This theorem is a generalization of Theorem 12.1.2 in [1] and the proof is similar. $\square$

Theorem 6.8. Assume that $F(\cdot, r) \in (\mathcal{C}^{m, \alpha}(0, T))^2$ for all $r \in [0, 1]$ and $\|F(\cdot, r)\|_{m, \alpha}$ are uniformly bounded with respect to r . Assume $K_+, K_- \in \mathcal{S}^{m, \alpha}$. Let G be the unique solution of the system (3.2). Assume that the collocation method (6.4) with collocation parameters $0 \leq c_1 < \dots < c_m \leq 1$, $m \in \mathbb{N}$ and with grading parameter $\rho \geq 1$ are used. Then there exists an integer N_0 such that for all $N \geq N_0$, operator equation (6.4) possesses a unique solution $u_N(\cdot, r) \in (\mathcal{S}_{m-1}^{(-1)}(\Delta_h))^2$ and

$$\sup_{r \in [0, 1]} \|G(\cdot, r) - u_N(\cdot, r)\|_\infty \rightarrow 0, \quad N \rightarrow \infty.$$

Furthermore, for $N \geq N_0$ the following error estimates hold:

$$(6.7) \quad \sup_{r \in [0,1]} \|G(\cdot, r) - u_N(\cdot, r)\|_\infty \leq \text{const} E(N, m, \rho, \alpha).$$

Proof. The conditions of Theorem 6.7 are satisfied with $X = (\mathcal{L}^\infty(0, T))^2$, $X' = (\mathcal{C}[0, T])^2$, $\mathcal{T} = \mathcal{K}$, $x = G(\cdot, r)$, $x_N = u_N$ and $p_N = \mathbf{p}_N$. Thus

$$\|G(\cdot, r) - u_N(\cdot, r)\|_\infty \leq c \|G(\cdot, r) - \mathbf{p}_N G(\cdot, r)\|_\infty.$$

Now Proposition 6.3 completes the proof. $\square$

We state separately the case when the kernel K changes sign.

Theorem 6.9. Assume that $F(\cdot, r) \in (C^{m,\alpha}(0, T))^2$ for all $r \in [0, 1]$ and $\|F(\cdot, r)\|_{m,\alpha}$ are uniformly bounded with respect to r . Let $K_+, K_- \in S^{m,\alpha}(D_d) \cap \mathcal{C}^{p-1}(D_T)$ with $m \geq 1$, $\alpha < 1$, $1 \leq p \leq m$. Let G be the unique solution of the system (3.2). Let the collocation method (6.4) with collocation parameters $0 \leq c_1 < \dots < c_m \leq 1$, $m \in \mathbb{N}$ and with two graded meshes on $[0, d]$ and on $[d, T]$ with numbers of intervals N_1 and N_2 and grading parameters ρ_1 and ρ_2 be used. Then there exists an integer N_0 such that for all $N \geq N_0$, operator equation (6.4) possesses a unique solution $u_N(\cdot, r) \in (\mathcal{S}_{m-1}^{(-1)}(\Delta_h))^2$ and

$$\sup_{r \in [0,1]} \|G(\cdot, r) - u_N(\cdot, r)\|_\infty \rightarrow 0, \quad N \rightarrow \infty.$$

Furthermore, for $N \geq N_0$ the following error estimates hold:

$$(6.8) \quad \sup_{r \in [0,1]} \|G(\cdot, r) - u_N(\cdot, r)\|_\infty \leq \text{const} \max\{E(N_1, m, \rho_1, \alpha), E(N_1, m, \rho_2, \alpha - p)\}.$$

6.2. Convergence estimates for the fully discretized collocation method.

In deriving the fully discretized collocation method, we approximated the integrals by the product quadrature rule. The quadrature rule was obtained by substituting the smooth part under the integral sign by its interpolation polynomial. The next lemma estimates the error of quadrature rule.

Lemma 6.10. Let p_α be defined as in (5.9). Let p_N be the interpolation projector to spline space $S_{m-1}^{-1}(\Delta_N)$. Then the following estimates hold.

i) If $f \in \mathcal{C}^m[0, T]$ then

$$(6.9) \quad \left| \int_0^t f(s) p_\alpha(s) ds - \int_0^t (p_N f)(s) p_\alpha(s) ds \right| \leq C \max |f^{(m)}(s)| h^m.$$

ii) If $f \in C^{m,\alpha}(0, T]$ and the graded mesh with the grading parameter ρ is used, then

$$(6.10) \quad \left| \int_0^t f(s) p_\alpha(s) ds - \int_0^t (p_N f)(s) p_\alpha(s) ds \right| \leq C E(N, m, \rho, \alpha).$$

iii) If $k \in \mathcal{C}^m(\overline{D_T})$ and $v_N(s) \in S_{m-1}^{-1}(\Delta_h)$, then

$$(6.11) \quad \left| \int_0^t p_\alpha(t, s) [k(t, s) v_N(s) - p_N(kv)(t, s) ds] \right| \leq Ch \|v\|_\infty.$$

Proof. We use the standard estimate of the interpolation error:

$$\|f - p_N f\|_\infty \leq C \max_{s \in [0, t]} |f^{(m)}(s)| h^m.$$

Then

$$\begin{aligned} \left| \int_0^t f(s) p_\alpha(s) ds - \int_0^t (p_N f)(s) p_\alpha(s) ds \right| &\leq \int_0^t p_\alpha(s) ds \|f - p_N f\|_\infty \\ &\leq C \max_{s \in [0, t]} |f^{(m)}(s)| h^m. \end{aligned}$$

This proves the first estimate. Proof of the second estimate is similar, using Lemma 6.1.

For the proof of assertion (iii) we start as before

$$\begin{aligned} (6.12) \quad &\left| \int_0^t p_\alpha(t, s) [k(t, s) v_N(s) - p_N(kv)(t, s)] ds \right| \\ &\leq \int_0^t p_\alpha(t - s) ds \max_{s \in [0, t]} |k(t, s) v_N(s) - p_N(kv)(t, s)| \end{aligned}$$

We estimate the interpolation error in each subinterval $[t_l, t_{l+1}]$ separately. Since v_N is a polynomial of order $m - 1$ in each subinterval, we have

$$\max_{s \in [t_l, t_{l+1}]} |v_N^{(j)}(s)| \leq C h_l^{-j} \|v\|_\infty, \quad j = 0, 1, \dots, m - 1$$

and $v_N^{(m)}(s) = 0$. Hence we can estimate

$$\begin{aligned} (6.13) \quad &\max_{s \in [t_l, t_{l+1}]} |k(t_{n,i}, s) v_N(s) - p_N(kv)(t, s)| \\ &\leq C \max_{s \in [t_l, t_{l+1}]} \left| \frac{\partial^m}{\partial s^m} (k(t, s) v_N(s)) \right| h_l^m \leq C h_l^{-m+1} \cdot h_l^m \leq C h_l \leq Ch. \end{aligned}$$

Now using the fact that p_α is integrable, (6.12) and (6.13) give the desired estimate. $\square$

To prove the convergence of the fully discretized collocation method we use the following general theorem about convergence of projection methods, when the operator is first approximated. It is similar to Corollary 13.11 of [14], but since one of the assumptions is not satisfied in our case, we give a new proof. Similar results have also been proved in [7], but our results are more general.

Theorem 6.11. *Let X, X' be Banach spaces and $X' \subset X$. Assume $\mathcal{T}: X \rightarrow X'$ is bounded and $I - \mathcal{T}: X \rightarrow X$ is injective operator. Assume*

$$\|\mathcal{T} - p_N \mathcal{T}\| \rightarrow 0, \text{ as } N \rightarrow \infty,$$

where $p_N: X' \rightarrow X$, $N = 1, 2, \dots$ are bounded linear operators. Let $X_N = p_N(X')$. Let $\mathcal{T}_N: X_N \rightarrow X$ be an approximation of $\mathcal{T}$ such that

$$\sup_{v_N \in X_N, \|v_N\|=1} \|(p_N \mathcal{T}_N - p_N \mathcal{T})v_N\| \rightarrow 0, \quad N \rightarrow \infty.$$

Then for all sufficiently large N the operator $I - p_N \mathcal{T}_N$ is invertible in X_N and

$$\sup_{N > N_0} \|(I - p_N \mathcal{T}_N)^{-1}\|_{X_N} < \infty.$$

For the solutions of $\hat{x}_N = p_N \mathcal{T}_N \hat{x}_N + p_N f_N$ and $x = \mathcal{T}x + f$ we have the estimate

$$(6.14) \quad \|x - \hat{x}_N\| \leq C(\|x - p_N x\| + \|p_N(\mathcal{T}_N p_N - \mathcal{T})x\| + \|p_N(f_N - f)\|),$$

where C is a positive constant.

Proof. Since $I - p_N \mathcal{T}_N = (I - p_N \mathcal{T}) + (p_N \mathcal{T} - p_N \mathcal{T}_N)$, $I - p_N \mathcal{T}$ is invertible in X_N (see Theorem 6.7) and $\|p_N \mathcal{T}_N - p_N \mathcal{T}\|_{X_N} \rightarrow 0$, invertibility of $I - p_N \mathcal{T}_N$ in X_N and uniform boundedness of the inverse operators follows.

Let x_N be the solution of $x_N = p_N \mathcal{T}x + p_N f$ (note that the assumptions of Theorem 6.7 are satisfied). Subtracting the equations for x_N and $\hat{x}_N$ we get

$$x_N - \hat{x}_N = p_N \mathcal{T}_N(x_N - \hat{x}_N) - (p_N \mathcal{T}_N - p_N \mathcal{T})x_N + p_N f - p_N f_N,$$

hence

$$(6.15) \quad \|x_N - \hat{x}_N\| \leq \|(I - p_N \mathcal{T}_N)^{-1}\| (\|(p_N \mathcal{T}_N - p_N \mathcal{T})x_N\| + \|p_N f - p_N f_N\|).$$

We can estimate $\|(p_N \mathcal{T}_N - p_N \mathcal{T})x_N\|$ as follows, using Theorem 6.7:

$$\begin{aligned} \|(p_N \mathcal{T}_N - p_N \mathcal{T})x_N\| &\leq \|(p_N \mathcal{T}_N - p_N \mathcal{T})(x_N - p_N x)\| \\ &\quad + \|p_N \mathcal{T}(x - p_N x)\| + \|(p_N \mathcal{T}_N p_N - p_N \mathcal{T})x\| \\ &\leq \|p_N \mathcal{T}_N - p_N \mathcal{T}\|_{X_N} \|x_N - p_N x\| + \|p_N \mathcal{T}\| \|x - p_N x\| + \|(p_N \mathcal{T}_N p_N - p_N \mathcal{T})x\| \\ &\leq C \|p_N x - x\| + \|(p_N \mathcal{T}_N p_N - p_N \mathcal{T})x\| \end{aligned}$$

Estimate (6.14) now follows from (6.15), the inequality

$$\|x - \hat{x}_N\| \leq \|x_N - \hat{x}_N\| + \|x_N - x\|$$

and Theorem 6.7. $\square$

Now we can state the convergence result for fully discretized collocation method.

Theorem 6.12. Assume that $F(\cdot, r) \in (\mathcal{C}^{m, \alpha}(0, T])^2$ for all $r \in [0, 1]$ and $\|F(\cdot, r)\|_{m, \alpha}$ are uniformly bounded with respect to r . Assume $K_+, K_- \in \mathcal{S}^{m, \alpha}$. Let G be the unique solution of the system (3.2). Assume that the fully discretization collocation method (5.10) with collocation parameters $0 \leq c_1 < \dots < c_m \leq 1$, $m \in \mathbb{N}$ and with grading parameter $\rho \geq 1$ are used. Then

there exists an integer N_0 such that for all $N \geq N_0$, operator equation (6.4) possesses a unique solution $\hat{u}_N(\cdot, r) \in (\mathcal{S}_{m-1}^{(-1)}(\Delta_h))^2$ and

$$\sup_{r \in [0,1]} \|G(\cdot, r) - \hat{u}_N\|_\infty \rightarrow 0, \quad N \rightarrow \infty.$$

Furthermore, for $N \geq N_0$ the following error estimates hold:

$$(6.16) \quad \sup_{r \in [0,1]} \|G(\cdot, r) - \hat{u}_N\|_\infty \leq \text{const} E(N, m, \rho, \alpha).$$

Proof. The conditions of Theorem 6.11 are satisfied with $X = (\mathcal{L}^\infty(0, T))^2$, $X' = (\mathcal{C}[0, T])^2$, $\mathcal{T} = \mathcal{K}$, $x = G(\cdot, r)$, $\hat{x}_N = \hat{u}_N$, $p_N = \mathbf{p}_N$ and $\mathcal{T}_N$ the approximation of $\mathcal{K}$ by using the product quadrature rule introduced in Section 5.2. Furthermore the assumption of Theorem 6.11 about $\mathcal{T}_N$ is satisfied by iii) of Lemma 6.10. The estimate for the first term in right hand side of (6.14) is given in (6.5). By ii) of Lemma 6.10 we get the estimate of second right hand side term of (6.14). The last term is zero, because in our method $f_N = f$. $\square$

Similar results also holds when the kernel changes sign on vertical and/or horizontal lines and different graded meshes are used on subintervals where the kernel does not change sign.

6.3. Fuzziness of the approximate solution. The main question is whether the approximate solution is fuzzy. In this section, we propose sufficient conditions which guarantee fuzziness of approximate solution.

Definition 6.13. Suppose $F = [\underline{f}, \overline{f}]^T$ is a vector function. We say F is a strictly fuzzy function if $[\underline{f}, \overline{f}]$ is a fuzzy function and there is $\delta > 0$ such that

- (1) $\frac{\underline{f}(t, r_2) - \underline{f}(t, r_1)}{r_2 - r_1} > \delta$ for all $t \in [0, T]$ and $0 \leq r_1 < r_2 \leq 1$.
- (2) $\frac{\overline{f}(t, r_2) - \overline{f}(t, r_1)}{r_2 - r_1} < -\delta$ for all $t \in [0, T]$ and $0 \leq r_1 < r_2 \leq 1$.
- (3) $\underline{f}(t, 1) < \overline{f}(t, 1)$, $t \in [0, T]$.

Now, it is possible to prove the fuzziness of $\mathbf{u}_N = (\underline{u}_N, \overline{u}_N)$ for those F that are strictly fuzzy functions.

In the following theorem by adding some more assumptions on F we guarantee the fuzziness of approximate solution.

Theorem 6.14. Suppose that F is a strictly fuzzy vector function. Let for any $r \in [0, 1]$, $F(\cdot, r) \in (\mathcal{C}^{m, \alpha}(0, T))^2$ and $K_+, K_- \in S^{m, \alpha}$. Then the system (3.2) has a unique solution $G = [\underline{g}, \overline{g}]$, $G(\cdot, r) \in (\mathcal{C}^{m, \alpha}(0, T))^2$, and G is a strictly fuzzy function. Assume that a collocation method of the form (6.4) with collocation points $0 \leq c_1 < \dots < c_m \leq 1$, $m \in \mathbb{N}$ and with grading parameter $\rho \geq 1$ is used. Let $\|\frac{F(\cdot, r_2) - F(\cdot, r_1)}{r_2 - r_1}\|_{1, \alpha} \leq \text{const}$, where the constant does not depend on N and r . Then there exists an integer N_0 such that for all $N \geq N_0$, operator equation (6.4) possesses a unique fuzzy solution u_N .

Proof. Fuzziness of the exact solution is proved in Theorem 4.10 [3]. First we prove that if F is a strictly fuzzy function, then the exact solution G is also a strictly fuzzy function. Let $0 \leq r_1 < r_2 \leq 1$. Then by the first equation of the system (3.1)

$$\frac{\underline{g}(t, r_2) - \underline{g}(t, r_1)}{r_2 - r_1} > \frac{\underline{f}(t, r_2) - \underline{f}(t, r_1)}{r_2 - r_1} > \delta.$$

The second condition of strictly fuzziness follows similarly. To prove the third condition we take $r = 1$ in system (3.1) and subtract the two equations.

Next we prove the monotonicity of approximate solution (condition 1. of Definition 2.1). Let $0 \leq r_1 < r_2 \leq 1$. By Theorem 6.8

$$(6.17) \quad \left\| \frac{\underline{g}(\cdot, r_2) - \underline{g}(\cdot, r_1)}{r_2 - r_1} - \frac{\underline{u}_N(\cdot, r_2) - \underline{u}_N(\cdot, r_1)}{r_2 - r_1} \right\|_\infty \leq \left\| \frac{\underline{F}(\cdot, r_2) - \underline{F}(\cdot, r_1)}{r_2 - r_1} \right\|_{1, \alpha} E(N, 1, \rho, \alpha)$$

By assumption, $\left\| \frac{\underline{F}(\cdot, r_2) - \underline{F}(\cdot, r_1)}{r_2 - r_1} \right\|_{1, \alpha} \leq \text{const}$, where the constant does not depend on N and r . Then for sufficiently large N the right hand side of

(6.17) is less than $\delta/2$ and since $\frac{\underline{g}(\cdot, r_2) - \underline{g}(\cdot, r_1)}{r_2 - r_1} > \delta$, we get $\underline{u}_N(\cdot, r_2) - \underline{u}_N(\cdot, r_1) > \delta/2$. Similarly we can prove that $\overline{u}_N(\cdot, r_2) - \overline{u}_N(\cdot, r_1) < -\delta/2$.

Similarly we get

$$\|\overline{g}(\cdot, 1) - \underline{g}(\cdot, 1) - (\overline{u}_N(\cdot, 1) - \underline{u}_N(\cdot, 1))\| \leq E(N, m, \rho, \alpha),$$

therefore for sufficiently large N , $\underline{u}_N(t, 1) < \overline{u}_N(t, 1)$. Hence all conditions of Definition 2.1 are satisfied. In fact we have proved that for N large enough, $\mathbf{u}_N$ is a strictly fuzzy function. $\square$

Similar result holds also for the fully discretized collocation method.

If F does not satisfy condition 3 of Definition 6.13, we cannot guarantee that the approximate solution $\mathbf{u}_N$ satisfies condition 3 of Definition 2.1. In this case we can modify our approximate solution to make it fuzzy without spoiling the rate of convergence. Assume that for sufficiently large N , $\underline{u}_N(t, 1) > \overline{u}_N(t, 1)$ for some $t \in [0, T]$. Let $r_N = \inf\{r \in [0, 1]: \underline{u}_N(t, r) > \overline{u}_N(t, r)\}$. In this case, we propose to use new forms of approximating functions:

$$\underline{u}_N^{\text{new}}(t, r) = \begin{cases} \underline{u}_N(t, r), & \text{if } 0 \leq r < r_N, \\ \frac{\underline{u}_N(t, r_N) + \overline{u}_N(t, r_N)}{2}, & \text{if } r_N \leq r \leq 1 \end{cases}$$

and

$$\overline{u}_N^{\text{new}}(t, r) = \begin{cases} \overline{u}_N(t, r), & \text{if } 0 \leq r < r_N, \\ \frac{\underline{u}_N(t, r_N) + \overline{u}_N(t, r_N)}{2}, & \text{if } r_N \leq r \leq 1. \end{cases}$$

Obviously $\mathbf{u}_N^{\text{new}}$ is a fuzzy function.

Let $t \in [0, T]$ be such that $\underline{u}_N(t, 1) > \overline{u}_N(t, 1)$ for N large enough and let $r > r_N$. Then $\overline{u}_N(t, r) \leq \underline{u}_N(t, r)$, hence

$$\overline{u}_N(t, r) - \overline{g}(t, r) \leq \overline{u}_N(t, r) - \underline{g}(t, r) \leq \underline{u}_N(t, r) - \underline{g}(t, r).$$

Therefore $|\overline{u}_N(t, r) - \underline{g}(t, r)| \leq \max\{|\underline{u}_N(t, r) - \underline{g}(t, r)|, |\overline{u}_N(t, r) - \overline{g}(t, r)|\}$. Hence if N is large enough so that the convergence estimate of Theorem 6.8 (or Theorem 6.12) holds, we have for $r \in [r_N, 1]$,

$$\begin{aligned} |\underline{u}_N^{\text{new}}(t, r) - \underline{g}(t, r)| &= \left| \frac{\underline{u}_N(t, r) + \overline{u}_N(t, r)}{2} - \underline{g}(t, r) \right| \\ &\leq \left| \frac{\underline{u}_N(t, r) - \underline{g}(t, r)}{2} \right| + \left| \frac{\overline{u}_N(t, r) - \underline{g}(t, r)}{2} \right| \leq C E(N, m, \rho, \alpha). \end{aligned}$$

Similar estimate holds also for $\overline{u}_N^{\text{new}}(t, r)$ and the proof is also similar. Consequently the convergence estimates also hold for the modified solution.

7. NUMERICAL EXAMPLES

In this section, we illustrate the convergence of the fully discretized collocation method by some selected examples. In examples 7.1 and 7.2 kernels are non-negative, in example 7.3 the kernel changes the sign. Here we use the following approximations for errors

$$\underline{E}_N = \max_{\eta_k \in [0, T]} \{|\underline{u}_N(\eta_k, r) - \underline{g}(\eta_k, r)|\}$$

and

$$\overline{E}_N = \max_{\eta_k \in [0, T]} \{|\overline{u}_N(\eta_k, r) - \overline{g}(\eta_k, r)|\},$$

where $(g, \overline{g})$ and $(\underline{u}_N, \overline{u}_N)$ (for $N \in \mathbb{N}$) are exact and numerical solutions of the system (3.1), respectively, and $\eta_k = \frac{k}{10N}$, $k = 0, \dots, 10N$. The approximate order of the convergence can be obtained by using the formula

$$O_N = \log_2 \frac{E_N}{E_{2N}}.$$

Example 7.1. Consider the system of fuzzy Volterra integral equation with weakly singular kernel (FVIEW) on $[0, 1]$ with

$$\begin{aligned} K(t, s) &= \frac{(t-s)^{2.5} + 1}{(t-s)^{0.5}}, \\ \underline{f}(t, r) &= (t^{0.5} - \frac{\pi}{8}t^2 - \frac{\pi}{2}t)(r^2 + r), \\ \overline{f}(t, r) &= (t^{0.5} - \frac{\pi}{8}t^2 - \frac{\pi}{2}t)(4 - r^3 - r). \end{aligned}$$

The exact solution is

$$g(t, r) = (t^{0.5}(r^2 + r), t^{0.5}(4 - r^3 - r)).$$

Here $K \in S^{m, 0.5}$, $f \in \mathcal{C}^{m, 0.5}(0, 1]$ and according to Theorem 4.6 in [3], $g \in \mathcal{C}^{m, 0.5}(0, 1]$ for any $m \in \mathbb{N}$. We used fully discretized collocation method with

discontinuous linear splines with two collocation points, and with piecewise constant splines, and a graded mesh with grading parameter ρ .

Method (1): $m = 2$, $c_1 = 0.6$, $c_2 = 0.8$.

Method (2): $m = 1$, $c_1 = 0.4$.

By our convergence results we expect the order of convergence to be $\rho/2$ for $\rho < 2m$, and m for $\rho \geq 2m$. In Tables 1–5 we illustrate the error and order of convergence by applying Method (1), Method (2) on $r = 0.9$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	9.2334e-02	1.2803e-01		
128	6.4451e-02	8.9365e-02	0.51	0.51
256	4.5191e-02	6.2659e-02	0.51	0.51
512	3.1774e-02	4.4056e-02	0.51	0.51
1024	2.2381e-02	3.1032e-02	0.51	0.51

TABLE 1. The Errors and orders of Example 7.1 by Method (1) for $\rho = 1$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	1.1140e-02	1.5446e-02		
128	5.5576e-03	7.7059e-03	1.00	1.00
256	2.7758e-03	3.8488e-03	1.00	1.00
512	1.3871e-03	1.9233e-03	1.00	1.00
1024	6.9338e-04	9.6141e-04	1.00	1.00

TABLE 2. The Errors and orders of Example 7.1 by Method (1) for $\rho = 2$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	1.5303e-02	2.1211e-02		
128	3.8052e-03	5.2760e-03	2.00	2.00
256	9.4905e-04	1.3159e-03	2.00	2.00
512	2.3718e-04	3.2886e-04	1.98	1.98
1024	6.0028e-05	8.3232e-05	1.98	1.98

TABLE 3. The Errors and orders of Example 7.1 by Method (1) for $\rho = 4$.

Example 7.2. Consider a FVIEW of the form (1.1) with

$$K(t, s) = \frac{1}{(t - s)^{0.5}},$$

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{Q}_N$	$\overline{Q}_N$
64	1.5871e-01	2.20050e-01		
128	1.0411e-01	1.4435e-01	0.60	0.60
256	6.8837e-02	9.5446e-02	0.51	0.51
512	3.4095e-02	4.7275e-02	0.51	0.51
1024	1.4045e-02	3.3340e-02	0.50	0.50

TABLE 4. The Errors and orders of Example 7.1 by Method (2) for $\rho = 1$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{Q}_N$	$\overline{Q}_N$
64	1.1202e-01	1.5532e-01		
128	6.2853e-02	8.7148e-02	0.91	0.91
256	3.3659e-02	4.6532e-02	0.94	0.94
512	1.7434e-03	2.4173e-02	0.97	0.97
1024	8.9182e-03	1.2365e-03	0.98	0.98

TABLE 5. The Errors and orders of Example 7.1 by Method (2) for $\rho = 2$.

$$\underline{f}(t, r) = \left(\frac{\sin(t)}{\sqrt{t}} - \pi \sin\left(\frac{t}{2}\right) J_0\left(\frac{t}{2}\right) \right)(r),$$

$$\overline{f}(t, r) = \left(\frac{\sin(t)}{\sqrt{t}} - \pi \sin\left(\frac{t}{2}\right) J_0\left(\frac{t}{2}\right) \right)(2 - r).$$

Here $J_\nu(z)$ is a Bessel function defined by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(-\frac{z}{4}\right)^k}{k! \Gamma(k + \nu + 1)}.$$

The exact solution of this system is $g(t, r) = (\underline{g}(t, r), \overline{g}(t, r))$ with

$$\underline{g}(t, r) = \frac{\sin(t)}{\sqrt{t}}(r), \quad \overline{g}(t, r) = \frac{\sin(t)}{\sqrt{t}}(2 - r)$$

Again $K \in S^{m,0.5}$, $f \in \mathcal{C}^{m,0.5}(0, 1]$ and $g \in \mathcal{C}^{m,0.5}(0, 1]$ for any $m \in \mathbb{N}$. We used fully discretized collocation method with discontinuous linear splines with different choices of the collocation points, and a graded mesh with grading parameter ρ . For this example we use following methods:

Method (1): $m = 2$ with $c_1 = 0.5$, $c_2 = 1$.

Method (2): $m = 2$ with $c_1 = \frac{3-\sqrt{3}}{6}$, $c_2 = \frac{3+\sqrt{3}}{6}$ (the roots of shifted Legendre polynomial of degree 2).

In Tables 6–11, we illustrate the error and order of convergence by applying Method (1) and Method (2) on $r = 0.9$. Convergence rates are the same when using different collocation points, but the roots of shifted Legendre

polynomial (Gauss points) give better results, since the approximation of the integrals is better.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	1.9364e-02	2.3667e-02		
128	1.3290e-02	1.6244e-02	0.54	0.54
256	9.2114e-03	1.1258e-02	0.52	0.52
512	6.4252e-03	7.8530e-03	0.51	0.51
1024	4.5008e-03	5.5010e-03	0.51	0.51

TABLE 6. The Errors and orders of Example 7.2 by Method (1) for $\rho = 1$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	2.3215e-03	2.8374e-03		
128	1.1507e-03	1.4064e-03	1.01	1.01
256	5.7291e-04	7.0022e-04	1.00	1.00
512	2.8585e-04	3.4937e-04	1.00	1.00
1024	1.4278e-04	1.7450e-04	1.00	1.00

TABLE 7. The Errors and orders of Example 7.2 by Method (1) for $\rho = 2$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	2.6126e-03	3.1932e-03		
128	6.8685e-04	8.3948e-04	1.92	1.92
256	1.7534e-04	2.1431e-04	1.96	1.96
512	4.4197e-05	5.4018e-05	1.98	1.98
1024	1.1079e-05	1.3541e-05	1.99	1.99

TABLE 8. The Errors and orders of Example 7.2 by Method (1) for $\rho = 4$.

Example 7.3. Let us consider a FVIEW of the form (1.1) on $[0, 2]$ with

$$K(t, s) = \frac{t - 1}{(t - s)^{\frac{1}{3}}},$$

$$\underline{f}(t, r) = (1 + t^4)(r^2 + r),$$

$$\overline{f}(t, r) = (1 + t^4)(4 - r^3 - r).$$

Here $\alpha = \frac{1}{3}$. This time the kernel changes sign along the line $t = 1$. We don't know about exact solution. By Theorem 4.9 in [3] the exact solution belongs to $(\mathcal{C}_d^{m, \alpha, p}(0, T))^2$. In this case we should use graded meshes with different grading parameters on $[0, 1]$ and on $[1, 2]$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	7.6802e-03	9.3869e-03		
128	5.2434e-03	6.4086e-03	0.55	0.55
256	3.6196e-03	4.4240e-03	0.53	0.53
512	2.5173e-03	3.0767e-03	0.52	0.52
1024	1.7595e-03	2.1505e-03	0.51	0.51

TABLE 9. The Errors and orders of Example 7.2 by Method (2) for $\rho = 1$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	1.0624e-03	1.2985e-03		
128	5.2787e-04	6.4517e-04	1.00	1.00
256	2.6312e-04	3.2159e-04	1.00	1.00
512	1.3136e-04	1.6055e-04	1.00	1.00
1024	6.5628e-05	8.0211e-05	1.00	1.00

TABLE 10. The Errors and orders of Example 7.2 by Method (2) for $\rho = 2$.

N	$\underline{E}_N$	$\overline{E}_N$	$\underline{O}_N$	$\overline{O}_N$
64	2.6425e-04	3.2297e-04		
128	6.2779e-05	7.6730e-05	2.07	2.07
256	1.5120e-05	1.8480e-05	2.05	2.05
512	3.6791e-06	4.4966e-06	2.03	2.03
1024	9.0214e-07	1.1026e-06	2.07	2.07

TABLE 11. The Errors and orders of Example 7.2 by Method (2) for $\rho = 4$.

We use the fully discretized collocation method with $m = 2$ and $c_1 = 0.5$, $c_2 = 1$ for numerical approximation of the solution. The optimal grading parameters which give the convergence order $O(h^2)$, are $\rho = 3$ on $[0, 1]$ and $\rho = \frac{6}{5}$ on $[1, 2]$. Our numerical result should provide fuzzy numbers for every t . Since we do not have the exact solution, we use the difference of the approximate solutions with N and $2N$ as an error estimation. We report the numerical solutions on $r = 0.5$ with various values of N . In Table 12 the first columns show the estimated errors of the method and the last columns show the order of convergence which is 2.

N	$ \underline{U}_N - \underline{U}_{2N} $	$ \overline{U}_N - \overline{U}_{2N} $	$\underline{Q}_N$	$\overline{Q}_N$
32	-	-	-	-
64	3.2350e-02	1.4558e-01	-	-
128	8.1722e-03	3.6775e-02	1.98	1.98
256	2.0536e-03	9.2411e-03	1.99	1.99
512	5.1490e-04	2.3171e-03	1.99	1.99
1024	1.2895e-04	5.8030e-04	1.99	1.99

TABLE 12. The Errors and orders of Example 7.3 for the first and second components.

CONCLUSION

In this paper, we investigated a fuzzy Volterra integral equation with a weakly singular kernel. The existence, regularity, and fuzziness of the exact solution have been studied in [3]. Collocation methods on discontinuous piecewise polynomial space are proposed. Convergence analysis and fuzziness of the approximate solution have been considered. Both the analysis and numerical methods show that graded mesh is better than uniform mesh for this problem. The numerical method provided to support the theoretical.

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DECLARATIONS

Conflict of interest. The authors declare that they have no conflict of interest.

Human participants or animals performed. This article does not contain any studies with human participants or animals performed by any of the authors.

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