

# Three-body problem - from Newton to supercomputer plus machine learning

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## Research Article

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# Three-body problem – from Newton to supercomputer plus machine learning

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**Abstract** *The famous three-body problem can be traced back to Newton in 1687, but quite few families of periodic orbits were found in 300 years thereafter. In this paper, we propose an effective approach and roadmap to numerically gain planar periodic orbits of three-body systems with arbitrary masses by means of machine learning based on an artificial neural network (ANN) model. Given any a known periodic orbit as a starting point, this approach can provide more and more periodic orbits (of the same family name) with variable masses, while the mass domain having periodic orbits becomes larger and larger, and the ANN model becomes wiser and wiser. Finally we have an ANN model trained by means of all obtained periodic orbits of the same family, which provides a convenient way to give accurate enough predictions of periodic orbits with arbitrary masses for physicists and astronomers. It suggests that the high-performance computer and artificial intelligence (including machine learning) should be the key to gain periodic orbits of the famous three-body problem.*

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## 1 Times of Newton, Euler, Lagrange and Poinaré

How are the trajectories of three point masses  $m_1, m_2$  and  $m_3$  that are attracted each other by Newton's gravitational law? This so-called "three-body problem" can be traced back to Newton<sup>1</sup> in 1687. According to Newton's second law and gravitational law, the related governing equations about  $N$ -body problem read

$$m_k \frac{d^2 \mathbf{r}_k}{dt^2} = \sum_{j=1, j \neq k}^N \frac{m_k m_j}{|\mathbf{r}_j - \mathbf{r}_k|^2}, \quad 1 \leq k \leq N, \quad (1)$$

where  $\mathbf{r}_k$  and  $m_k$  are the position vector and mass of the  $k$ th-body,  $t$  denotes the time, respectively, with a given initial position  $\mathbf{r}_{0,k}$  and velocity  $\mathbf{v}_{0,k}$ , i.e.

$$\mathbf{r}_k(0) = \mathbf{r}_{0,k}, \quad \dot{\mathbf{r}}_k(0) = \mathbf{v}_{0,k}, \quad 1 \leq k \leq N. \quad (2)$$

Here the dot denotes the derivative with respect to  $t$ . Note that  $\mathbf{r}_k$ ,  $m_k$  and  $t$  are dimensionless using a characteristic length  $L$ , a characteristic mass  $M$  and the Newton's gravitational constant  $G$ . If the trajectory of each body at  $t = T$  exactly returns its initial status, say,

$$\mathbf{r}_k(T) = \mathbf{r}_k(0) = \mathbf{r}_{0,k}, \quad \dot{\mathbf{r}}_k(T) = \dot{\mathbf{r}}_k(0) = \mathbf{v}_{0,k}, \quad 1 \leq k \leq N, \quad (3)$$

one has a periodic solution of the  $N$ -body problem. The famous three-body problem corresponds to  $N = 3$ .

For the two-body problem, corresponding to  $N = 2$ , Newton gave a closed-form periodic solution. However, for a three-body problem ( $N = 3$ ), it becomes extremely difficult to find periodic orbits: no periodic orbits had been found until Euler reported one in 1740 and Lagrange

published one in 1772, both for a triple system with three equal masses. However, according to Montgomery's topological method<sup>2</sup> of classifying periodic orbits of three-body systems, they belong to the *same* family, namely the "Euler - Lagrange family". Thereafter, *no* new periodic orbits were reported in about two centuries.

Why is it so difficult to find periodic orbits of three-body systems? The mysterious reasons were revealed by Poinaré<sup>3</sup> in 1890, who proved that, unlike the two-body problem that is integrable and thus its solutions is completely understood, the three-body problem is *not* integrable: its first integral does *not* exist at all. This well explains why *only* the Euler - Lagrange family (in closed form) were found in more than two hundred years, since closed-form orbits do *not* exist at all in general. It implies that one generally *had* to use numerical algorithms to solve three-body problem, but unfortunately the electronic computer was even *not* invented in the times of Poinaré<sup>3</sup>. In addition, Poinaré<sup>3</sup> also found that trajectories of three-body system are generally rather sensitive to initial conditions (i.e. the butterfly-effect), which led to the foundation of a new field of modern science, i.e. chaotic dynamics. Nowadays, it is well-known that, due to the famous butterfly-effect, i.e. a hurricane in North America might be created by a flapping of wings of a distant butterfly in South America several weeks earlier, it is very difficult to gain reliable trajectories of chaotic systems, especially in a long interval of time. All of these explain why periodic orbits of three-body problem are so difficult to obtain and why it becomes one of the oldest open question in science.

In most cases, trajectories of three-body system are chaotic, i.e. non-periodic, as discovered by Poinaré<sup>3</sup> in 1890 and confirmed again by Lorenz<sup>4</sup> in 1963 and Stone & Leigh<sup>5</sup> in 2019. How-

ever, in some special cases, there indeed exist periodic orbits, i.e. the three bodies exactly return to their initial positions and initial velocities after a period  $T$ . The periodic orbits of the three-body problems are very important, since they are “the only opening through which we can try to penetrate in a place which, up to now, was supposed to be inaccessible”, as pointed out by Poinaré<sup>3</sup>. However, the question is: how can we find them *effectively*?

## 2 Times of supercomputer

The excellent work of Poinaré<sup>3</sup> made a historical turning point of searching for periodic orbits of three-body system. The non-existence of the uniform first integral of triple system reveals the impossibility of finding closed-form analytic solutions of three-body problems in general cases: it clearly indicates that we mostly *had to* (i.e. must) use numerical algorithms to solve this problem. This was indeed a revolutionary contribution of Poinaré<sup>3</sup> with great foresightness at that times when there was even *no* electronic computers at all: about a half century later, Turing<sup>6,7</sup> published his epoch-making papers that became the foundation of modern computer and artificial intelligence. Thanks to Von Neumann<sup>8</sup>, who proposed the so-called Von Neumann - Machine for modern computer, also to Jack S. Kilby, who won the Nobel prize in Physics 2000 for his taking part in the invention of the integrated circuit, and to many scientists, mathematicians, engineers and so on whose names we have no space to mention here, the performance of supercomputer becomes better and better, together with more and more advanced numerical algorithms. This provides us a strong support of hardware and software for discovering new periodic orbits of three-body problem now.

With the ceaseless progress of electronic computers, more and more researchers followed the way, which Poinaré<sup>3</sup> suggested, to numerically search for periodic orbits of three-body problem. According to Montgomery's topological method<sup>2</sup> of classifying periodic orbits of three-body systems, only three families of periodic orbits were found in 300 years after Newton, i.e. (1) the so-called "Euler - Lagrange family" found by Euler in 1740 and Lagrange in 1772; (2) the so-called "BHH family" for three equal masses, *numerically* found by Broucke<sup>9,10</sup> in 1975, Hadjidemetriou<sup>11</sup> in 1975 and Hénon<sup>12</sup> in 1976, respectively; (3) the so-called "figure-eight family" for three equal masses *numerically* found by Moore<sup>13</sup> in 1993, until 2013 when Šuvakov and Dmitrašinović<sup>14</sup> *numerically* gained 11 new families of periodic orbits of triple system with three equal masses. All of these suggest that finding periodic orbits of three-body problem by means of numerical methods should be a correct way.

Note that, among the three families of periodic orbits mentioned above, the BHH family and figure-8 family were found by numerical methods respectively in 1975s and 1993, when the performance of computer might be not good enough. However, in 2010s we had supercomputer with peak performance about 1,000 petaflops, i.e. several billion billion fundamental calculations per second, together with lots of powerful numerical algorithms. What prevents us from *effectively* finding thousands of new families of periodic orbits of three-body problem though we have so powerful supercomputer ?

The key of finding periodic orbits is to gain reliable computer-generated trajectories of three-body system under arbitrary initial conditions in a *long enough* interval of time. However, as

discovered by Poinaré<sup>3</sup> and rediscovered by Lorenz<sup>4</sup>, computer-generated trajectories of chaotic systems are sensitive to initial conditions, i.e. the famous butterfly-effect. In other words, a tiny difference on initial conditions might lead to a huge deviation of computer-generated simulation after a long time. In addition, Lorenz<sup>15,16</sup> further found that computer-generated trajectories of chaotic systems are also sensitive to algorithms: different numerical algorithms might give distinctly different computer-generated trajectories of chaotic systems after a long time. This is indeed a great obstacle! This kind of sensitive dependence on numerical algorithm (SDNA) for a chaotic system had been also observed and reported by many researchers<sup>17-19</sup>, which however unavoidably led to some intense arguments<sup>20</sup>: some researchers even suggested that “all chaotic responses should be simply numerical noises” and might “have nothing to do with differential equations”. Thus, by means of traditional numerical algorithms (mostly in double precision), it is rather difficult to gain reliable/convergent computer-generated trajectories of three-body system under arbitrary initial conditions in a long *enough* interval of time.

To overcome this obstacle, Liao<sup>21</sup> proposed the so-called “clean numerical simulation” (CNS) for chaotic systems. Unlike other traditional numerical algorithms, which mostly use double precision, the CNS can greatly reduce *not only* truncation error *but also* the round-off error to keep the total “numerical noises” in such a required tiny level that a reliable (or convergent) computer-generated simulation can be obtained in a long enough interval of time. In the frame of the CNS, the truncation error is decreased by means of numerical algorithms at high enough order in time and space, and the round-off error is reduced by using multiple-precision with many enough digits for all parameters and variables. The CNS has been successfully applied to many chaotic systems,

such as Lorenz equation, chaotic three-body systems, some spatio-temporal chaotic systems and so on<sup>22–25</sup>. As reported by Hu and Liao<sup>25</sup>, the use of double precision might lead to huge deviations of computer-generated simulations of spatio-temporal chaos *even* in statistics, not only quantitatively but also qualitatively, particularly in a long interval of time. This indicates that we must be very careful to numerically simulate chaotic systems. Fortunately, the CNS can provide us a guaranteed tool to gain reliable/convergent trajectories of chaotic systems (such as three-body systems with arbitrary initial conditions) during a long enough time.

Using the CNS as an integrator of the governing equations and combining the grid search method and the Newton-Raphson method<sup>26–28</sup>, Li and Liao<sup>29</sup> in 2017 successfully found 695 families of periodic planar collisionless orbits of three-body systems with *three* equal masses and zero angular momentum, including the figure-eight family, the 11 families found by Šuvakov and Dmitrašinović in 2013, and besides more than 600 new families that have never been reported. Similarly, Li, Jing and Liao<sup>30</sup> further found 1349 new families of periodic planar collisionless orbits of the three-body system with only *two* equal masses. In 2020, starting from a known periodic orbit with three equal masses and using the CNS to integrate the governing equations, Li, Li and Liao<sup>31</sup> successfully obtained 135445 new periodic orbits with arbitrarily *unequal* masses by means of combining the numerical continuation method<sup>32</sup> and the Newton-Raphson method<sup>26–28</sup>, including 13315 stable ones. Therefore, in only four years, using high-performance computer and our new strategy based on the CNS, we successfully increased the family number of the known periodic orbits of three-body systems by nearly *four* orders of magnitude! This strongly indicates that our numerical strategy is correct, powerful and rather *effective* for finding new periodic orbits

of three-body systems. It should be emphasized that this great progress is mainly due to the use of the new numerical strategy based on the CNS, since the performance of supercomputer is good enough for three-body systems even at the beginning of the 22nd century.

### 3 The times of machine learning

However, it is time-consuming to use the numerical continuation method<sup>32</sup> to find the 135445 periodic orbits (with *unequal* masses) reported by Li, Li and Liao<sup>31</sup>. Besides, these periodic orbits are in essence *discrete*, say, only for some *specific* values of  $m_1$  and  $m_2$  in an irregular domain (in case of  $m_3 = 1$  since we use the mass of the 3rd body as a characteristic mass  $M$ , without loss of generality). Can we gain a periodic orbit more *efficiently* for *arbitrary* values of masses  $m_1$  and  $m_2$ ? Thanks to the times of machine learning, the answer is positive and rather attractive, as described below.

For example, let us use a known BHH orbit as a starting point to illustrate this. Fifty-seven satellite periodic orbits of the BHH family of three-body systems with three equal masses were found<sup>33</sup> in 2016, and this number was extended to ninety-nine<sup>34</sup> in 2020. The initial configuration of the BHH satellite periodic orbits with zero angular momentum is described by

$$\mathbf{r}_1(0) = (x_1, 0), \mathbf{r}_2(0) = (1, 0), \mathbf{r}_3(0) = (0, 0),$$

and

$$\dot{\mathbf{r}}_1(0) = (0, v_1), \dot{\mathbf{r}}_2(0) = (0, v_2), \dot{\mathbf{r}}_3(0) = (0, -(m_1 v_1 + m_2 v_2)/m_3),$$

where  $\mathbf{r}_i$ ,  $\dot{\mathbf{r}}_i$  and  $m_i$  is the position vector, velocity vector and mass of the  $i$ -th body, respectively.

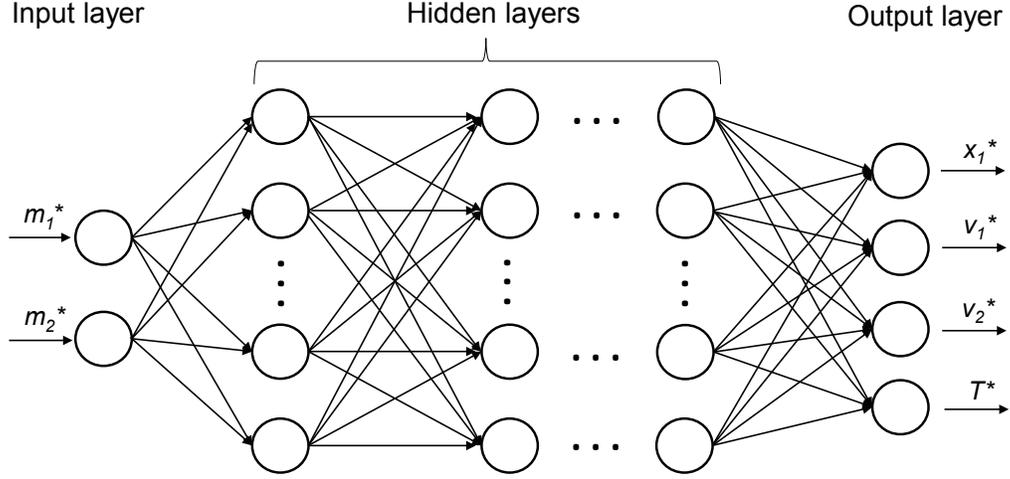


Figure 1: The artificial neural network (ANN) model, where the input  $(m_1^*, m_2^*)$  and output  $(x_1^*, v_1^*, v_2^*, T^*)$  are the normalized data of  $(m_1, m_2)$  and  $(x_1, v_1, v_2, T)$ , respectively.

Thus, for given  $m_1, m_2$  and  $m_3$  (we assume  $m_3 = 1$  thereafter), we should determine four unknown physical variables  $x_1, v_1, v_2$  and the period  $T$ . Note that these orbits are periodic in a rotating frame of reference, say, the frame of coordinates rotates an angle  $\theta_T$  in the corresponding period  $T$ .

Without loss of generality, let us consider such a known BHH periodic orbit with the initial condition

$$x_1 = -1.325626981682458, v_1 = -0.8933877752879044, v_2 = -0.2885702941263346,$$

the period  $T = 9.199307755830397$  and the rotation angel  $\theta_T = 0.383160887655628$  of the coordinate frame, where  $m_1 = m_2 = m_3 = 1$ . Using this periodic orbit as a starting point, we first follow Li, Li and Liao<sup>31</sup> to obtain only 36 periodic orbits for different masses in a small domain  $m_1 \in [0.95, 1.00], m_2 \in [1.00, 1.05]$  (marked by  $S_1$ , with the mass increment  $\Delta m_1 = \Delta m_2 = 0.01$ ) by means of the numerical continuation method<sup>32</sup> and the Newton-Raphson method<sup>26-28</sup>. For

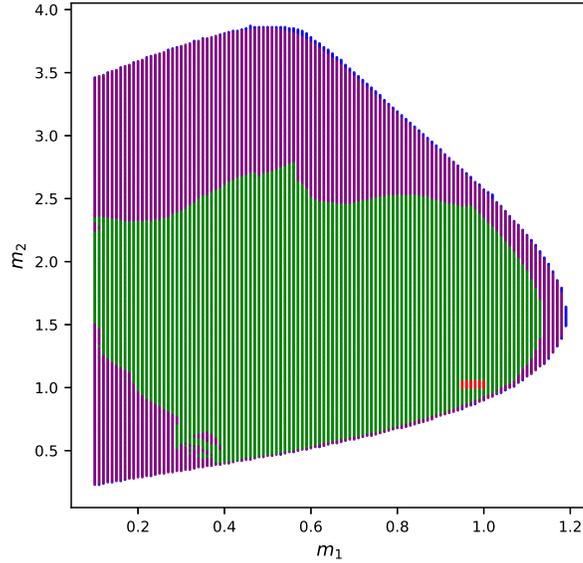


Figure 2: The periodic orbits found by expansion on the mass region. Red dot ( $S_1$ ): initial periodic orbits; green dot ( $S_2$ ): the first extrapolation/expansion; purple dot ( $S_3$ ): the second extrapolation/expansion; blue dot ( $S_4$ ): the third extrapolation/expansion.

details, please refer to Li, Li and Liao<sup>31</sup>. The return distance (deviation) of these periodic orbits is defined by

$$\delta_T = \sqrt{\sum_{i=1}^3 \left( |\mathbf{r}_i(T) - \mathbf{r}_i(0)|^2 + |\dot{\mathbf{r}}_i(T) - \dot{\mathbf{r}}_i(0)|^2 \right)}$$

in the rotating frame of reference, where  $T$  is the period. Note that all of the 36 periodic orbits satisfy the criteria  $\delta_T < 10^{-10}$ . Besides, they all have the same family name, which is defined by the so-called “free group element” according to Montgomery’s topological method<sup>2</sup>.

In the next step, we use an artificial neural network<sup>35–38</sup> (ANN) model to gain a relation between the input vector  $(m_1, m_2)$  and the output vector  $(x_1, v_1, v_2, T)$  of the periodic orbits. The

ANN model we used here consists of multiple fully connected layers, say, one input layer, six hidden layers and one output layer. The number of neurons for the input layer, the hidden layers and the output layer is 2, 1024 and 4, respectively, as shown in Figure 1. We use the optimization algorithm AMSGrad<sup>39</sup> as an optimizer to minimize the mean square error for training the ANN model. At beginning, we use the results of the 36 known periodic orbits gained by means of the numerical continuation method<sup>32</sup> and the Newton-Raphson method<sup>26-28</sup> as the training set to train the ANN model. The trained ANN model provides us a kind of relationship (expressed by  $\mathcal{F}_1$ ) between  $(m_1, m_2)$  and  $(x_1, v_1, v_2, T)$ , which can be further used to predict the initial conditions  $x_1, v_1, v_2$  and the period  $T$  of candidates (i.e. possible periodic orbits) for various masses  $(m_1, m_2)$  *outside* of the original small domain  $S_1 = \{(m_1, m_2) : m_1 \in [0.95, 1.00], m_2 \in [1.00, 1.05]\}$ . Using these predictions as the initial guesses for the Newton-Raphson method<sup>26-28</sup>, we indeed successfully obtain the 17421 new periodic orbits with  $\delta_T < 10^{-10}$  for various  $(m_1, m_2)$  *outside* of  $S_1$ , which are marked in green in Figure 2 and expressed by  $S_2$ . Then, in the same way, we further use the results of *all* (i.e.  $36 + 17421 = 17457$ ) known periodic orbits as a training set to further train our ANN model so as to gain a better relationship  $\mathcal{F}_2$  between  $(m_1, m_2)$  and  $(x_1, v_1, v_2, T)$  for extrapolation/expansion *outside* of the mass domain  $S_1 \cup S_2$ , which gives us 11473 new periodic orbits in a even larger domain, as marked in purple in Figure 2 and expressed by  $S_3$ . Similarly, we use the results of *all* (i.e.  $36 + 17421 + 11473 = 28930$ ) known periodic orbits as the training set to further train our ANN model so as to give a relationship  $\mathcal{F}_3$  between  $(m_1, m_2)$  and  $(x_1, v_1, v_2, T)$ , which however gives us only the 220 new periodic orbits *outside* of the mass domain  $S_1 \cup S_2 \cup S_3$ , as marked in blue in Figure 2 and expressed by  $S_4$ . This implies that we might find the nearly

largest mass domain  $S^* = S_1 \cup S_2 \cup S_3 \cup S_4$  for the existence of periodic orbits with a few of same properties. As shown in Figure 2, starting from the 36 periodic orbits in the original small domain  $(m_1, m_2) \in S_1$  (marked in red), we totally gain the 29150 periodic orbits in the mass domain  $(m_1, m_2) \in S^* = S_1 \cup S_2 \cup S_3 \cup S_4$  by the three times extrapolations/expansions. Finally, training our ANN model by using *all* (i.e. 29150) these known results as a training set, we obtain a relationship  $\mathcal{F}^*$  between  $(m_1, m_2) \in S^*$  and  $(x_1, v_1, v_2, T)$ , which can give predictions of periodic orbits for arbitrary values of  $(m_1, m_2) \in S^*$  in the accuracy level of  $10^{-4}$  for the return distance (deviation)  $\delta_T$ . Note that the mass domain  $(m_1, m_2)$  with the periodic orbits becomes larger and larger, i.e. from  $S_1$  to  $S_1 \cup S_2$  to  $S_1 \cup S_2 \cup S_3$  to  $S^* = S_1 \cup S_2 \cup S_3 \cup S_4$ , and the relationship between  $(m_1, m_2)$  and  $(x_1, v_1, v_2, T)$  becomes better and better, from  $\mathcal{F}_1$  to  $\mathcal{F}_2$  to  $\mathcal{F}_3$  to  $\mathcal{F}^*$ , implying that our ANN model becomes wiser and wiser!

Obviously, the smaller the return distance (deviation)  $\delta_T$  is, the more accurate the periodic orbit given by the numerical strategy. Note that  $\delta_T = 0$  exactly corresponds to a closed-form periodic orbit. But unfortunately, except the Euler - Lagrange family, nearly all known periodic orbits of three-body systems were gained by numerical methods, and this fact is consistent with Poinaré's famous proof of the non-existence of the uniform first integral of three-body systems<sup>3</sup>. It is found that, since the CNS and the MP (multiple precision) are used in our numerical strategy, the return distance (deviation)  $\delta_T$  of these periodic orbits can be reduced to *any* given value, say, the initial conditions  $(x_1, v_1, v_2)$  and the period  $T$  of these relatively periodic orbits can be at *arbitrary* level of precision, i.e. as accurate as one would like, as long as the performance of supercomputer is good enough.

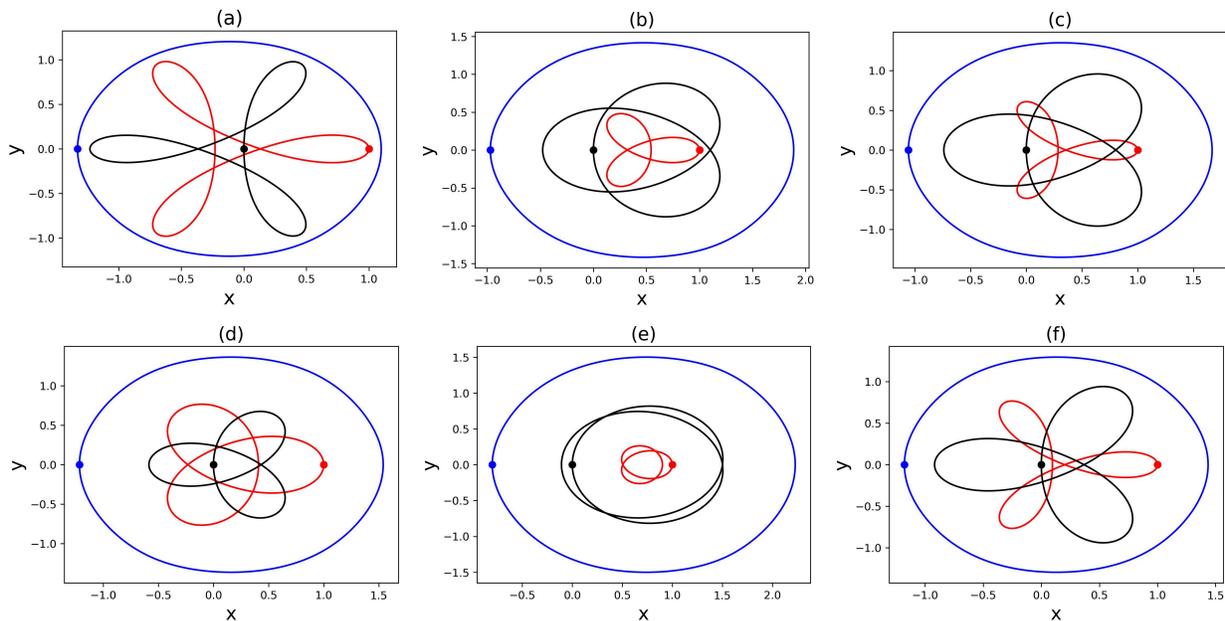


Figure 3: The relatively periodic BHH satellites orbits of the three-body system with various masses  $m_1$  and  $m_2$  in a rotating frame of reference. The corresponding physical parameters are given by ANN in Table 1. Blue line: body-1; red line: body-2; black line: body-3.

The relationship  $\mathcal{F}^*$  between  $(m_1, m_2) \in S^*$  and  $(x_1, v_1, v_2, T)$  given by the ANN model is fundamentally different from the set of the original 29150 periodic orbits in the following aspects.

(A) For *arbitrary* values of  $(m_1, m_2) \in S^*$ ,  $\mathcal{F}^*$  can always give a good enough prediction of the initial condition  $(x_1, v_1, v_2)$  and the period  $T$  of the corresponding periodic orbit, which can be used as a starting point to gain a more accurate periodic orbit with a tiny return distance (deviation)  $\delta_T < 10^{-10}$ . Therefore, in theory, our ANN model can provide us with an *infinite* number of periodic orbits.

(B) For each prediction of a periodic orbit given by our ANN model for arbitrary values of

$(m_1, m_2) \in S^*$ , we can always modify it by means of the Newton-Raphson method<sup>26-28</sup>, until a very accurate periodic orbit with  $\delta_T < 10^{-60}$  is gained. For examples, for the randomly chosen masses  $m_1 = 0.550073$  and  $m_2 = 1.738802$ , the initial condition and period of the periodic orbit predicted by  $\mathcal{F}^*$  is  $x_1 = -1.0509$ ,  $v_1 = -1.2291$ ,  $v_2 = -0.3751$  and  $T = 7.7189$ . The return distance  $\delta_T$  of this predicted periodic orbit is about  $10^{-3}$ , which can be reduced to  $10^{-60}$  by means of the Newton-Raphson method<sup>26-28</sup> and the CNS, with the initial condition and period in accuracy of hundred significant digits:

$$\begin{aligned} x_1 &= -1.0509175496041811604923698392786632425650672636140394304186656973, \\ v_1 &= -1.2291270518667041014360821708612407200218851479344331903064815068, \\ v_2 &= -0.37510727670051981894849679865106213490791658967297597038682248759, \\ T &= +7.7189475555888051714951891045174205372467192448577195610635970572. \end{aligned}$$

Note that the prediction given by our ANN model are in the accuracy of four significant digits. Similarly, we *randomly* check one thousand cases of  $(m_1, m_2) \in S^*$ , and found that, in every case, we indeed can always gain a periodic orbit with a quite tiny return distance (deviation)  $\delta_T < 10^{-60}$ . All of these suggest that, statistically speaking, for *arbitrary* mass values  $(m_1, m_2) \in S^*$ , every prediction given by our ANN model could lead to a periodic orbit that could be in *arbitrary* accuracy as long as the performance of computer is good enough.

- (C) It is found that the errors of the predictions given by our trained ANN model  $\mathcal{F}^*$  are mostly in the level of  $10^{-5}$  for the initial conditions  $(x_1, v_1, v_2)$  and  $10^{-4}$  for the period  $T$ . Although we indeed can further gain a periodic orbit in accuracy of one hundred significant digits by

means of the Newton-Raphson method<sup>26-28</sup> and the CNS, the periodic orbits predicted by our trained ANN model are good enough from practical viewpoint, since it is *unnecessary* to have so accurate trajectory in practice such as astronomical observation and so on. In practice, it is rather convenient to gain a periodic orbit for *arbitrary* masses  $(m_1, m_2) \in S^*$  by means of the initial conditions  $(x_1, v_1, v_2)$  and period  $T$  predicted by the trained ANN model, for example, as shown in Figure 3 for randomly chosen six different masses of  $m_1$  and  $m_2$ . Thus, our ANN model provides us great convenience in practice.

(D) The ANN model can ceaselessly learn and thus be further modified when some new periodic orbits are gained, say, the trained ANN model could become wiser and wiser.

In summary, we illustrate that the trained ANN model can provide us an *infinite* number of periodic orbits for *arbitrary* values of  $(m_1, m_2)$  in an irregular domain  $S^*$ .

How accurate is a periodic orbit with return distance (deviation)  $d_T < 10^{-60}$ ? Even if we use the diameter of universe  $d_u = 930$  light year  $= 8.8 \times 10^{18}$  meter as the characteristic length  $L$ , we have the corresponding inaccuracy of the dimensional initial position  $|\Delta x_1| d_u < 10^{-41}$  meter that is even smaller than the Planck length  $l_p \approx 1.62 \times 10^{-35}$  meter. Note that Planck length is a lower bound to physical proper length in any space-time: it is impossible to measure length scales smaller than Planck length, according to Padmanabhan<sup>40</sup>. Besides, it should be emphasized that all of these periodic orbits are *stable*, say, a tiny disturbance does not increase exponentially. So, from physical view-point, a stable periodic orbit with return distance (deviation)  $d_T < 10^{-60}$  gained by numerical method is *equivalent* to  $d_T = 0$  that corresponds to an *exact* solution of periodic orbit in

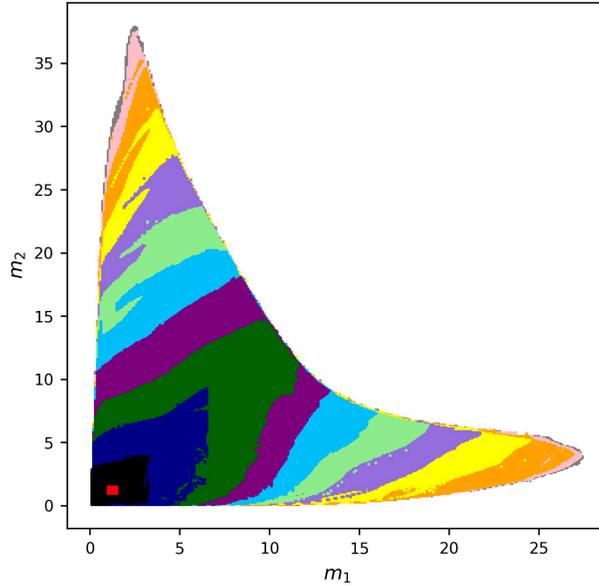


Figure 4: The relatively periodic orbits with the same rotation angle  $\theta = 0.0105056462558377$  of reference frame, found in each extrapolation/expansion on the various mass regions. Red dot: initial periodic orbits; black dot: 1st expansion; dark blue dot: 2nd expansion; dark green dot: the 3rd expansion. dark purple dot: the 4th expansion; light blue dot: the 5th expansion; light green dot: the 6th expansion; light purple dot: the 7th expansion; yellow dot: the 8th expansion; orange dot: the 9th expansion; pink dot: the 10th expansion; grey dot: the eleventh expansion.

a closed-form.

Therefore, the high-performance computer and the machine learning play very important role in finding periodic orbits of three-body systems with arbitrary masses. It should be emphasized here that it is Turing<sup>6,7</sup> who laid the foundations of modern computer and artificial intelligence (including machine learning).

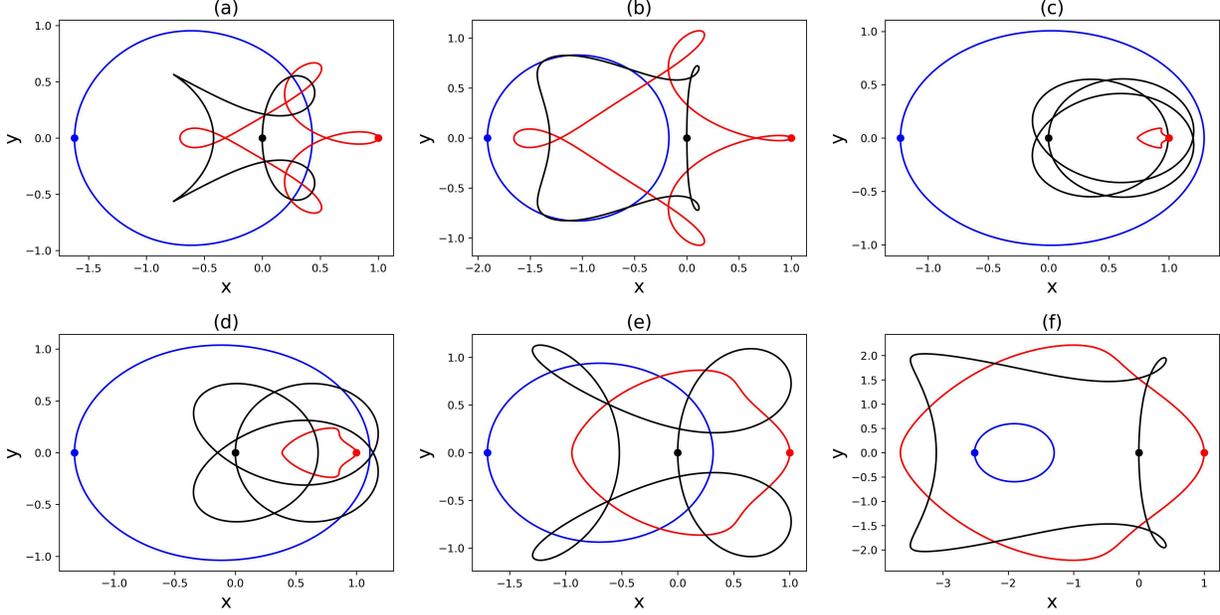


Figure 5: The relatively periodic BHH satellites orbits of the three-body system with various masses  $m_1$  and  $m_2$  in a rotating frame of reference. The corresponding physical parameters are given by ANN in Table 2. Blue line: body-1; red line: body-2; black line: body-3.

The above approach has general meaning. To show this point, let us further consider another BHH satellite periodic orbit with three equal masses  $m_1 = m_2 = m_3 = 1$ ,  $x_1 = -1.609965115714630$ ,  $v_1 = -0.6656909425824538$ ,  $v_2 = -0.1529561125709906$ , the period  $T = 6.879203007710456$  and the rotation angle  $\theta = 0.0105056462558377$  of the reference frame (for relatively periodic orbits). At first, using this known periodic orbit as a starting point, we can obtain 36 periodic orbits with various masses in a small mass domain  $m_1 \in [1.0, 1.5]$  and  $m_2 \in [1.0, 1.5]$  (with mass increment  $\Delta m_1 = \Delta m_2 = 0.1$ ) by combining the numerical continuation method<sup>32</sup> and the Newton-Raphson method<sup>26-28</sup>.

Similarly, we train a ANN model (with the same structure as mentioned above) by means of

the initial conditions and periods of these 36 known periodic orbits and then use the trained ANN model to predict the initial conditions and periods of some candidates of possible periodic orbits *outside* the previous mass domain, while the initial conditions and periods of each candidate are modified by the Newton-Raphson method<sup>26-28</sup> so as to confirm whether or not the candidate is a periodic one, say, its return distance can be reduced to the tiny level of  $10^{-10}$ . The same process repeats 11 times, while the number of the known relatively periodic orbits becomes more and more, the mass domain becomes larger and larger, and the ANN model becomes wiser and wiser! Finally, we totally obtain 36251 relatively periodic orbits in a mass domain  $\bar{S}^*$  whose area is about 1000 times larger than the initial one  $m_1 \in [1.0, 1.5]$  and  $m_2 \in [1.0, 1.5]$ , as shown in Fig. 4.

Similarly, we further train our ANN model by using the known 36251 relatively periodic orbits. It is found that, for randomly chosen one thousand cases of *arbitrary* mass  $(m_1, m_2) \in \bar{S}^*$ , our trained ANN model can always predict the initial conditions and periods that often provide us corresponding periodic orbits with a return distance in a tiny level of  $10^{-4}$ , and more importantly, that can be always further improved until a very small return distance  $10^{-60}$  is satisfied. Thus, statistically speaking, our trained ANN model can always give periodic orbits for arbitrary mass  $(m_1, m_2) \in S_2^*$ . It is found that our trained ANN model is accurate enough from practical viewpoint. For example, randomly choosing six cases with various masses  $m_1$  and  $m_2$ , we can use our ANN model to quickly predict their initial conditions and periods, as listed in Table 2, and give the corresponding trajectories in a satisfied level of accuracy, as shown in Fig 5. Thus, our approach based on the ANN model has indeed general meaning.

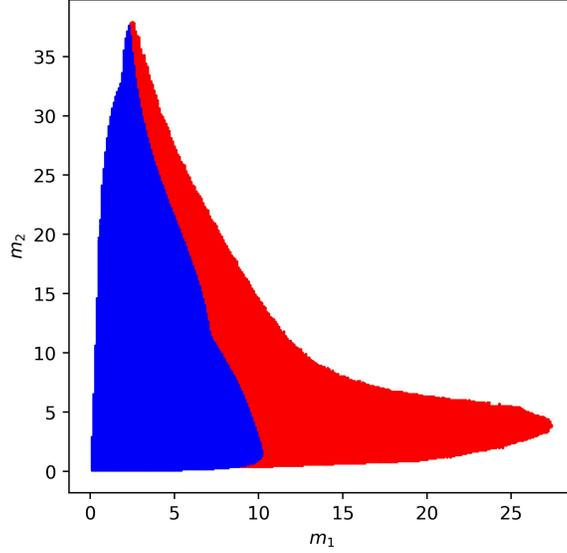


Figure 6: The linear stability of the relatively periodic orbits in the second case with the rotation angle  $\theta = 0.0105056462558377$ . Red domain: stable; blue domain: unstable.

Stability is an important property of periodic orbits, because only stable triple systems can be observed. We employ a theorem given by Kepela and Simó<sup>42</sup> to determine the linear stability of periodic orbits of three-body problem through the monodromy matrix<sup>41</sup>. It is found that all relatively periodic orbits for the first case with the rotation angle  $\theta = 0.3831608876556280$  are linearly stable. For the second case with the rotation angle  $\theta = 0.0105056462558377$ , there are 16739 linearly stable periodic orbits among the 35895 computer-generated periodic orbits, as shown in Fig 6. It should be emphasized that we can give an ANN classifier model to classify the periodicity and stability of the orbits for *arbitrary* masses  $m_1$  and  $m_2$ . The orbits are classified into three categories: stable periodic orbits, unstable periodic orbits and non-periodic orbits, expressed by  $[1, 0, 0]$ ,  $[0, 1, 0]$  and  $[0, 0, 1]$ , respectively. The ANN classifier model consists of eight fully connected layers with one input layer, six hidden layers and one output layer. The numbers of

neurons for input layer, hidden layers and output layer are 2, 256 and 3, respectively. Different from the above-mentioned ANN regression model, the activation function of the ANN classifier model is softmax function. The loss function is cross entropy. For each case, the whole dataset is randomly divided into two sets, the training set (90%) and test set (10%). The 90% of the entire data are used to train the ANN classifier model, and the remaining is used to verify the generalization capability of the model. We train the model twenty thousand epochs and choose the model with maximum accuracy on the training set. The accuracy of the ANN classifier model is about 99.9%. Finally, we use the ANN classifier model to predict the periodicity and stability of orbits for *any* given masses  $m_1$  and  $m_2$  in each case.

#### **4 A roadmap of searching for periodic orbits of three-body problem**

The successful examples mentioned above suggest us a general road map for finding new periodic orbits of three-body system (in case of  $m_3 = 1$ ) with the same “free group element” (word) given by Montgomery’s topological method<sup>2</sup>:

- (1) For a three-body system with three or two equal masses, first find candidates of the initial conditions for possible periodic orbits by means of the grid search method, and then modify these candidates by means of the Newton-Raphson method<sup>26–28</sup>, until a satisfied periodic orbit with a tiny enough return distance (deviation) is obtained;
- (2) Given a known periodic orbit, use it as a starting point to gain a few of new periodic orbits with various masses in a small domain of mass  $(m_1, m_2)$  by combining the numerical con-

tinuation method<sup>32</sup> and the Newton-Raphson method<sup>26–28</sup>. The initial conditions and periods of all these known periodic orbits form a training set for a ANN model.

(3) For a given training set in a mass domain of  $(m_1, m_2)$ , train the ANN model to predict initial conditions and periods so that some new periodic orbits *outside* of the previous mass domain of  $(m_1, m_2)$  could be found by modifying these predictions via the Newton-Raphson method<sup>26–28</sup>. Then, combining the results of these new periodic orbits with the previous training set, we further have a new training set with more elements, which could further provide us some new periodic orbits in a even larger mass domain of  $(m_1, m_2)$  in a similar way, *outside* of the previous one. The same process can repeat again and again, so that more and more periodic orbits are found in a larger and larger mass domain of  $(m_1, m_2)$ , and the ANN model becomes wiser and wiser, until quit few or no new periodic orbits can be found in a larger domain of  $(m_1, m_2)$ . Finally, we have a trained ANN model  $\mathcal{F}^*$  of all periodic orbits in the final mass domain  $(m_1, m_2) \in S^*$ .

(4) Randomly choose hundreds or thousands of *arbitrary* masses  $(m_1, m_2) \in S^*$ . For each case, check whether or not the trained ANN model  $\mathcal{F}^*$  could give an accurate enough prediction, and in addition whether or not the corresponding return distance (deviation) could be indeed reduced to  $10^{-60}$ . If yes, then statistically speaking, the ANN model  $\mathcal{F}^*$  can mostly provide a good enough prediction of periodic orbits in the domain  $(m_1, m_2) \in S^*$ , from practical viewpoint.

Note that it is better to use the CNS as integrator so as to guarantee the reliability and con-

vergence of computer-generated trajectories of chaotic three-body system under arbitrary initial conditions in a required long interval of time. Besides, the periodicity and stability of orbits for *arbitrary* masses  $m_1$  and  $m_2$  can be well predicted by an ANN classifier model.

## 5 Concluding remarks and discussions

The famous three-body problem can be traced back to Newton in 1687, but quite few families of periodic orbits were found in 300 years thereafter. As proved by Poincarè, the first integral does not exist for three-body systems, which implies that numerical approach had to be used in general. In this paper, we propose an effective approach and roadmap to numerically gain planar periodic orbits of three-body systems with arbitrary masses by means of machine learning based on an artificial neural network (ANN) model. Given any a known periodic orbit as a starting point, this approach can provide more and more periodic orbits (of the same family name) with variable masses, while the mass domain having periodic orbits becomes larger and larger, and the ANN model becomes wiser and wiser. Finally we have an ANN model trained by means of all obtained periodic orbits of the same family, which provides a convenient way to give accurate enough periodic orbits with *arbitrary* masses for physicists and astronomers. In addition, the periodicity and stability of orbits for *arbitrary* masses can be well predicted by an ANN classifier model.

It must be emphasized that high performance computer and artificial intelligence (including machine learning) play important roles in solving periodic orbits of triple systems. Today, nothing

can prevent us from obtaining massive periodic solutions of three-body problem. This is due to the great contributions of some great mathematicians, scientists and engineers in more than three hundred years! Especially, it is Poinaré<sup>3</sup> who made a historical turning point by proving the non-existence of the uniform first integral of triple system, which implies that we had to use numerical approach in general. It is Turing<sup>6,7</sup> who published his epoch-making papers that became the foundation of modern computer and artificial intelligence. It is Von Neumann<sup>8</sup> who proposed the so-called “Von Neumann - Machine” for modern computer. It is Jack S. Kilby, the winner of Nobel prize for physics in 2000, who took part in the invention of the integrated circuit. And so on. The famous three-body problem might be an excellent example to illustrate the importance of inventing new tools for human being to better understand and explore the nature.

The approach and roadmap mentioned in this paper has general meanings. Note that thousands of families of periodic orbits of triple systems with three or two equal masses have currently been found (for example, please visit the website <https://github.com/sjtu-liao/three-body>). Using any of them as a starting point, we can similarly gain a trained ANN model to give accurate enough predictions of periodic orbits of the same family of triple system in a corresponding mass domain. All of these might form a massive data base for periodic orbits of triple systems, which should be helpful to enrich and deepen our understandings about the famous three-body systems. Besides, hopefully, some physical laws, such as generalied Kepler’s law<sup>29</sup> for periodic orbits of triple systems with *arbitrary* masses, could be found in future by analysing these massive data by means of machine learning.

Note also that nearly all known periodic orbits of triple systems are planar, i.e. two-dimensional. In theory, the same ideas mentioned in this paper can be used to search for periodic orbits of triple systems in three dimensions.

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**Author contributions** S.J.L. provided the main ideas and wrote the manuscript. X.M.L. calculated the periodic orbits. Y.Y. analysed the results by means of machine learning based on an ANN model. All authors contributed to the discussion and revision of the final manuscript.

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Table 1: Initial conditions and periods  $T$  predicted by our trained ANN model for the six examples of BHH satellites with various masses  $m_1$  and  $m_2$  in case of  $\mathbf{r}_1(0) = (x_1, 0)$ ,  $\mathbf{r}_2(0) = (1, 0)$ ,  $\mathbf{r}_3(0) = (0, 0)$ ,  $\dot{\mathbf{r}}_1(0) = (0, v_1)$ ,  $\dot{\mathbf{r}}_2(0) = (0, v_2)$ ,  $\dot{\mathbf{r}}_3(0) = (0, -(m_1v_1 + m_2v_2)/m_3)$ , where  $m_i$ ,  $x_i$  and  $v_i$  are the mass, initial position and velocity of the  $i$ -th body, respectively, with the same rotation angle  $\theta = 0.38316088765562$  of the reference frame for relatively periodic orbits.

Case	$m_1$	$m_2$	$x_1$	$v_1$	$v_2$	$T$
(a)	1.0124	0.9968	-1.32962	-0.88963	-0.28501	9.2111
(b)	0.5312	2.2837	-0.97138	-1.37584	-0.34528	7.0421
(c)	0.8056	2.0394	-1.05795	-1.24044	-0.26723	7.4389
(d)	0.3916	0.8341	-1.21503	-1.00328	-0.53749	9.2807
(e)	0.1472	3.4219	-0.80027	-1.74811	-0.42176	5.8793
(f)	0.8413	1.4155	-1.17777	-1.05903	-0.29934	8.3444

Table 2: Initial conditions and periods  $T$  of six examples of BHH satellites with various masses  $m_1$  and  $m_2$ , predicted by our trained ANN model in case of  $\mathbf{r}_1(0) = (x_1, 0)$ ,  $\mathbf{r}_2(0) = (1, 0)$ ,  $\mathbf{r}_3(0) = (0, 0)$ ,  $\dot{\mathbf{r}}_1(0) = (0, v_1)$ ,  $\dot{\mathbf{r}}_2(0) = (0, v_2)$ ,  $\dot{\mathbf{r}}_3(0) = (0, -(m_1v_1 + m_2v_2)/m_3)$ , where  $m_i$ ,  $x_i$  and  $v_i$  are the mass, initial position and velocity of the  $i$ -th body, respectively, with the same rotation angle  $\theta = 0.0105056462558377$  of the reference frame for relatively periodic orbits.

No.	$m_1$	$m_2$	$x_1$	$v_1$	$v_2$	$T$
(a)	1.0283	0.9879	-1.6206	-0.6597	-0.1479	6.9198
(b)	1.5142	0.4968	-1.9082	-0.5028	-0.0861	8.2919
(c)	2.9216	31.9067	-1.2289	-2.3723	0.09481	1.6822
(d)	4.4143	18.6575	-1.3288	-1.8610	0.2550	2.3305
(e)	10.3501	10.4522	-1.6980	-1.2278	0.8618	3.5604
(f)	18.7011	4.2388	-2.5159	-0.5326	1.4974	5.9615

# Figures

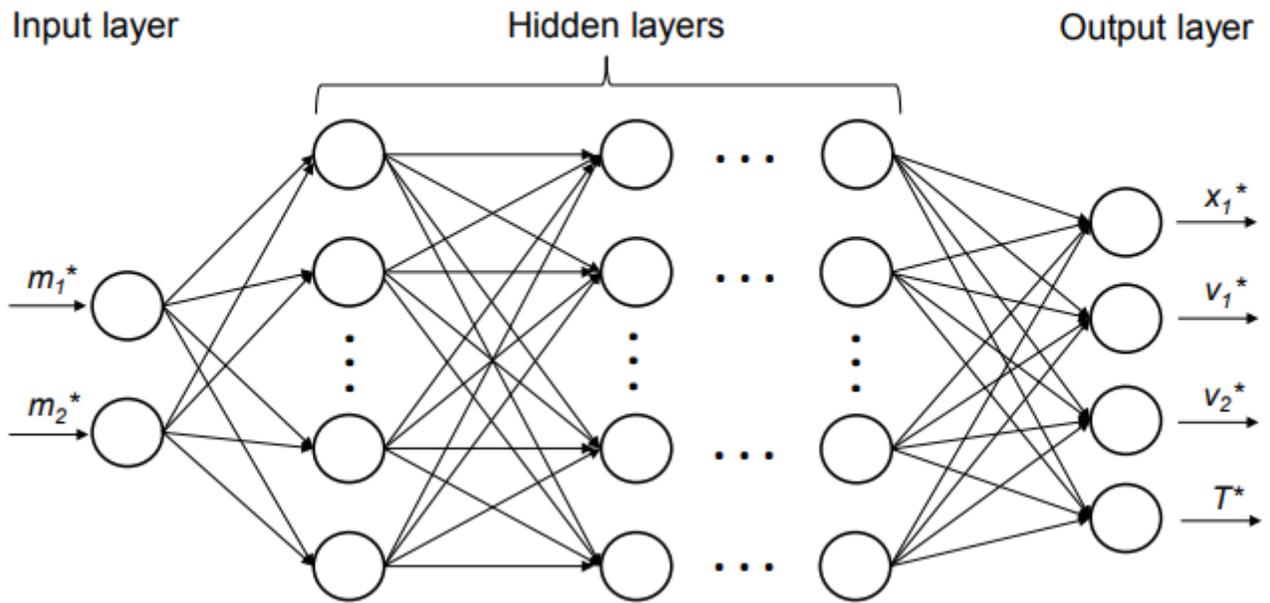


Figure 1

[Please see the manuscript file to view the figure caption.]

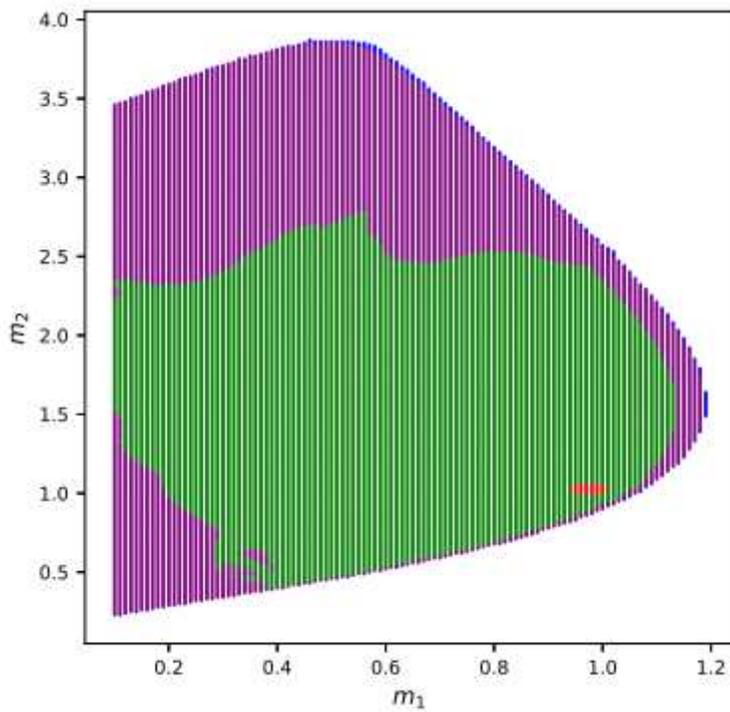
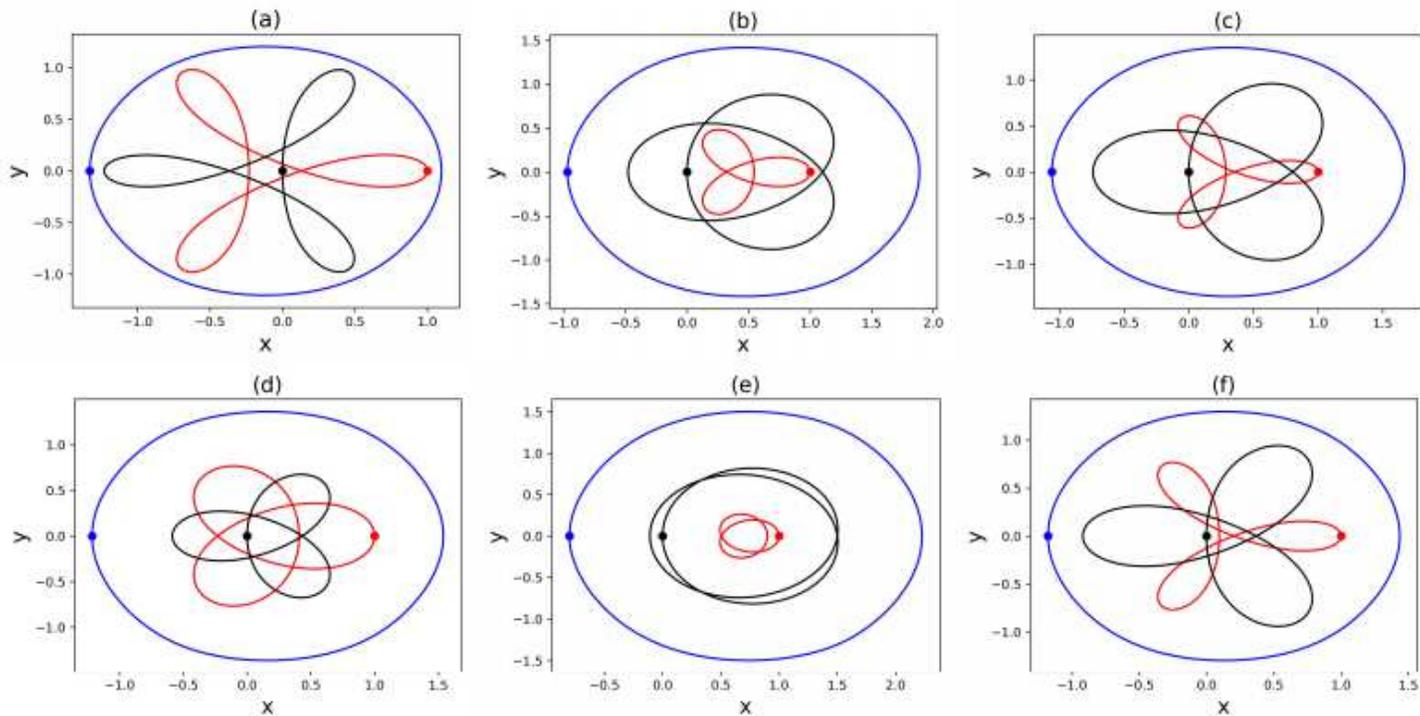


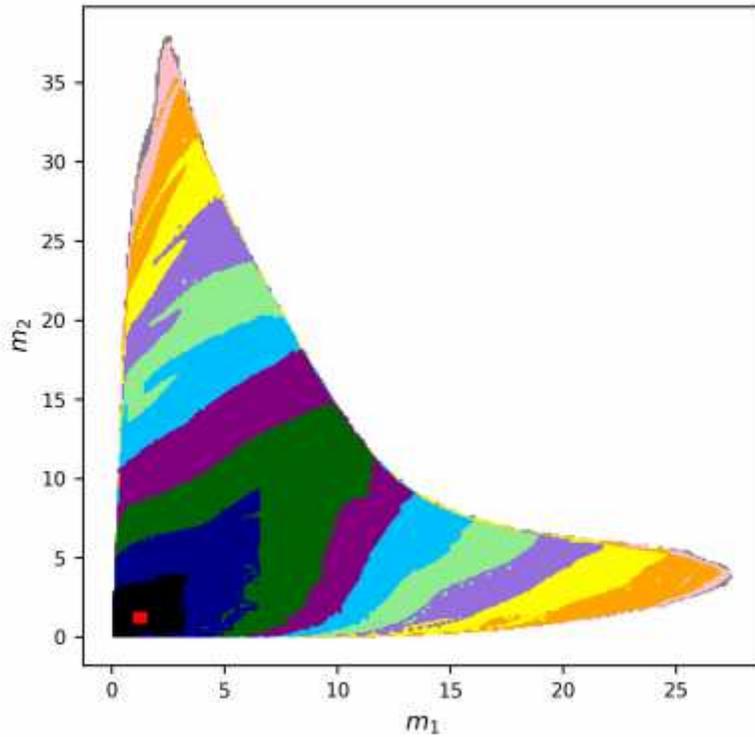
Figure 2

The periodic orbits found by expansion on the mass region. Red dot (S1): initial periodic orbits; green dot (S2): the first extrapolation/expansion; purple dot (S3): the second extrapolation/expansion; blue dot (S4): the third extrapolation



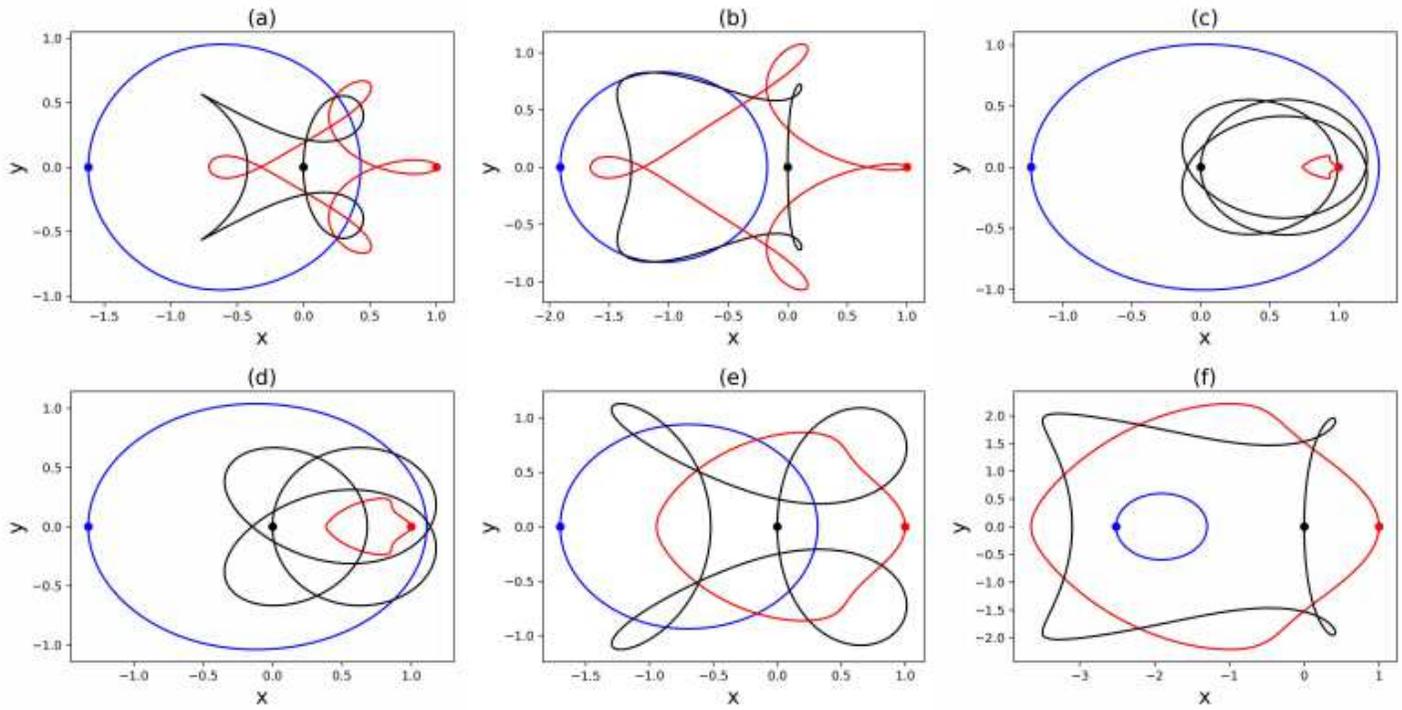
**Figure 3**

The relatively periodic BHH satellites orbits of the three-body system with various masses  $m_1$  and  $m_2$  in a rotating frame of reference. The corresponding physical parameters are given by ANN in Table 1. Blue line: body-1; red line: body-2; black line: body-3.



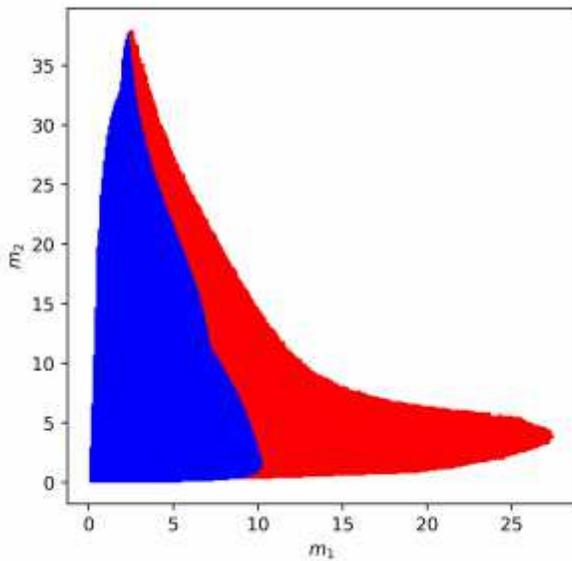
**Figure 4**

The relatively periodic orbits with the same rotation angle  $\theta = 0.0105056462558377$  of reference frame, found in each extrapolation/expansion on the various mass regions. Red dot: initial periodic orbits; black dot: 1st expansion; dark blue dot: 2nd expansion; dark green dot: the 3rd expansion. dark purple dot: the 4th expansion; light blue dot: the 5th expansion; light green dot: the 6th expansion; light purple dot: the 7th expansion; yellow dot: the 8th expansion; orange dot: the 9th expansion; pink dot: the 10th expansion; grey dot: the eleventh expansion.



**Figure 5**

The relatively periodic BHH satellites orbits of the three-body system with various masses  $m_1$  and  $m_2$  in a rotating frame of reference. The corresponding physical parameters are given by ANN in Table 2. Blue line: body-1; red line: body-2; black line: body-3.



**Figure 6**

The linear stability of the relatively periodic orbits in the second case with the rotation angle  $\theta = 0.0105056462558377$ . Red domain: stable; blue domain: unstable.